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HYPERGRAPH FORMATION GAME*

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Abstract

We define a hypergraph by a set of associations which consist of nonexclusive two or more players. It is a generalization of a graph (or a network) in the sense that an association, the counterpart of a link in a hypergraph, connects any number of nodes, not simply a pair of nodes. We characterize the efficient hypergraphs and stable hypergraphs for the linear variable cost of associations. The efficient hypergraph is either the empty hypergraph or the grand hypergraph consisting of a single grand association. The stable hypergraph can be a grand hypergraph, a star hypergraph or a line hypergraph. If a star hypergraph is stable, it must have a singleton center. Generally, a hypergraph can be underconnected, but cannot be overconnected.

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I. *Introduction*

Economic agents often share information only with some group of people by forming an informal or formal organization (or association) such as academic associations, social clubs,

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research joint ventures etc. Various kinds of associations engage in many other activities in addition to sharing information all of which could be understood as benefiting the members through their collaborations.

Such associations are, however, neither exclusive nor comprehensive. Some agents may join in several associations, while others may join in no association. Many associations have overlapping members. The members overlapped in more than one association may play the role of mediating information between associations. Thus, through the mediator, an agent can get indirect benefits from the association to which he does not belong. Joining in an association does not only yield (direct and indirect) benefits but also incurs some costs. For example, a member should pay the membership fee and perform some duty to sustain his association.

We will call a set of associations a hypergraph. It is a generalization of a graph (or a network) in graph theory. While a link in a network directly connects only a pair of nodes, an association in a hypergraph connects any number of nodes. While a link in a network can be formed by the joint decision of two players, an association in a hypergraph can be formed by the joint decision of any number of players more than one.

In a hypergraph, a player can share the value of people who join in the same association, thereby getting direct benefits from joining in an association. In addition, he enjoys indirect benefits from members in different associations who are indirectly connected through the member of his association. Indeed, the value from indirect connection is discounted. On the other hand, the cost of maintaining an association consists of the fixed cost and the variable cost proportional to its size. The total maintaining cost of an association is shared equally by its members.

In this paper, we will define the efficiency and the stability of a hypergraph by extending the concept of efficiency and stability of a network by Jackson and Wolinsky (1996). The efficient hypergraph is defined by the one that maximizes the sum of the net benefit of all the players. A hypergraph is defined to be stable if no player has an incentive to exit *unilaterally* from any of his associations, and no coalition of players has *joint* incentives to form a new association.

We define a hierarchical hypergraph by the hypergraph that contains an association with its subassociation. Then, we can show that a hierarchical hypergraph can be neither efficient nor stable. Intuitively, this is because any player in a subassociation can be made better off by exiting from the subassociation without affecting the payoff of players outside the subassociation. Then, we mainly characterize the efficient hypergraphs and stable hypergraphs for the linear variable cost of associations. The efficient hypergraph is either the empty hypergraph or the grand hypergraph consisting of a single grand association. The stable hypergraph can be a grand hypergraph, a star hypergraph or a line hypergraph. A circle hypergraph cannot be stable if the number of players is more than three. We also show that a stable star hypergraph must have a singleton center. The intuitive reason is that the loss of a center from exiting from one of his association is small as far as he maintains indirect connections with the association members via different associations. Since a player's gain from joining in the grand association is larger than his loss from exiting out of his association, there is no cost structure preventing both incentives to form a grand association and to exit from an association. The tension between efficiency and stability that is identified by Jackson and Wolinsky still exists in hypergraph formation, but only in one direction. In other words, we show that a stable hypergraph can be underconnected but not overconnected, which is contrasted with Jackson and

Wolinsky. The main intuition is that efficient hypergraphs are all symmetric, so that there is no coordination problem which is the main source of the overconnected network. Then, we consider a stronger concept of stability referred to as strong stability following the spirit of Dutta and Mutuswami (1997) and Jackson and Nouweland (2005). A restrictive feature of the stability is that it allows only a deviation to exit by a single player and a deviation to form a new association with maintaining current associations. Strong stability allows more deviations. Roughly speaking, it allows any kind of deviation by any coalition. We then demonstrate that the efficient hypergraph coincides with the strongly stable hypergraph.

There are closely related papers. Myerson (1980) was the first to introduce the concept of hypergraph into economics. However, he did not consider the problem of forming a hypergraph. Moreover, he interpreted a conference (or association) in a hypergraph as a group of people who can collaborate with one another only if all of them are present. Aumann and Drèze (1974), and Hart and Kurz (1983) studied a game with coalitional structures. An association in a hypergraph is similar to a coalition in a coalition structure in the sense that its members can communicate with one another as if it were exactly a complete network, but the one differs from the other because associations can overlap with each other unlike coalitions in a coalition structure. Slikker *et al*. (2000) also consider the problem of hypergraph formation. However, they do not take the cost into account, just as Aumann and Myerson, and adopt Myerson's interpretation of hypergraphs as a group of players who can communicate only when all of them are present, which leads to the different architecture of a hypergraph.

The paper is organized as follows. In Section II, we introduce some definitions in graph theory. In Section III, we set up the model. In Section IV, we define the efficiency and the stability of a hypergraph and characterize efficient hypergraphs and stable hypergraphs in the case that the variable cost of forming an association is linear in its size. In Section V, we introduce a more refined stability concept, strong stability. In Section VI, we examine how our results can be affected in the case of the convex variable cost. Concluding remarks follow in Section VII.

II. *Definitions*

Let N be a set of players with $|N|=n<\infty$. A set of nodes $S(\subset N)$ is called a coalition of N. A hypergraph H is defined by a family of coalitions (subsets) of N, $\{A\}$, with $|A|\geq 2$.[1] We will denote the set of all possible hypergraphs on N by H. An element A of a hypergraph is called an association.[2] The size of an association A is defined by $|A|$. All members in an association can communicate with one another without friction. Associations are not mutually exclusive, so that a player can participate in more than one association. If $A\subset B$ for some A, $B\in H$, we call A a subassociation of B. A hypergraph H is called hierarchical if some association $A\in H$ has a subassociation.

Many concepts for graphs can be extended to hypergraphs. We can define a path in H between players i and j by a sequence $(i,i_1,i_2,\cdots,i_k,j)$ such that $i,i_1\in A_0$, $i_1,i_2\in A_1,\cdots,i_k,j\in A_k$

[1] A hypergraph is a generalized concept of a graph (or a network) because it includes A with $|A|=2$.

[2] Note that an association is defined in a manner specific to a given hypergraph, whereas a coalition is defined independently of a hypergraph.

for some $A_0, \cdots, A_k \in H$, and say that the path has the length of k. If there is a path between i and j, we say that i and j are connected. In particular, if $i, j \in A$ for some $A \in H$ so that the path between i and j has the length of 0, we say that they are directly connected. The distance between two players i and j is defined by the length of the shortest path between them and denoted by $t(i, j)$. If i and j are directly connected, $t(i, j) = 0$. We define $t(i, j) = \infty$ if i and j are not connected.

We define the degree of player i by the number of players to whom player i is directly connected, and denote it by $d(i)$. We will call a hypergraph complete if $d(i) = n-1$ for all $i \in N$. Note that the complete hypergraph is not unique. We will call a hypergraph $H = \{N\}$ the grand hypergraph and denote it by H^n.[3] A hypergraph H is called connected if there is a path for any distinct players $i, j \in N$. If a hypergraph is not connected, the set N is partitioned into several disjoint connected components.[4] We will call a component consisting of a single player a trivial component. If all components in a hypergraph are trivial, it is called the empty hypergraph and denoted by H^0.

We can define a star, a line and a circle as follows. Let $H = \{A_k \mid 1 \leq k \leq m(\geq 2)\}$. We will call a hypergraph H a star and denote it by H^* if there exists a nonempty subset $R \subset N$ such that $A_i \cap A_j = R$ and $R \subset A_k$ for all i, j, k. The set R will be called centers of H^* and the set $H^* \backslash R$ called peripheries. We will call a hypergraph H a line and denote it by H^l if $A_k \cap A_{k+1} = L_k \neq \emptyset$, $L_k \subset A_k$, A_{k+1}, and $A_k \cap A_j = \emptyset$ for $j \neq k-1, k+1$ and for $1 \leq k \leq m-1$. Finally, a circle will be defined by $H^c \equiv H^l \cup A_{m+1}$ where $A_j \cap A_{m+1} \neq \emptyset$ for $j = 1, m$ and $A_j \cap A_{m+1} = \emptyset$ for $j \neq 1, m$. Note that hierarchical hypergraphs are excluded from stars, lines and circles by the conditions that $R \subset A_k$ and $L_k \subset A_k$, A_{k+1}.

We will denote by $H+A$ the hypergraph obtained by adding a coalition A to H as a new association and by $H-A$ the hypergraph obtained by eliminating an association A from H. Also, if there is no chance of confusion, we will use the notation of $H-i(A)$ to mean the hypergraph obtained by player i's exit from the association A. In other words, $H-i(A)$ and $H-(A-A\backslash\{i\})$ are equivalent.

III. *Model*

We consider the connections model developed by Jackson and Wolinsky (1996) with a modification of replacing links by associations.

Each player has a value normalized to one. Players can share their values by organizing an association. Players can also get indirect benefit from indirectly connected players.

Let A_i be an association to which player i belongs and A_i be the set of such associations, i.e., $\mathrm{A}_i = \{A_i\}$. Also, define $\overline{\mathrm{A}}_i = \cup A_i$. The total benefit of player i from hypergraph H is then

$$B_i(H) = 1 + d(i) + \sum_{j \notin \overline{\mathrm{A}}_i} \delta^{t(i,j)},$$

where $\delta \in (0,1)$ is the discount factor.

[3] In fact, it is the *minimal* complete hypergraph in the sense that it contains the minimum number of elements among all complete hypergraphs.

[4] A component C can be defined by a set of players in N such that $i, j \in C$ if and only if i and j are connected.

On the other hand, it is costly to form an association. We assume that the cost of organizing an association A, which is denoted by $C(A)$, is increasing in the size of A, $|A|$. We also assume that this cost is shared equally by the members of A. The cost of player i from the hypergraph H is then

$$C_i(H)=\sum_{A_i\in \mathrm{A}_i}\frac{C(A_i)}{|A_i|}.$$

Thus, the payoff (net benefit) of player i is $\pi_i(H)=B_i(H)-C_i(H)$. We can also define the value from hypergraph H by $V(H)=\sum_{i\in N}\pi_i(H)$.

This model is general in the sense that it includes a variety of network formation models as special cases. Note that the model corresponds to the connections model by Jackson and Wolinsky in the case that $C(A)=2c$ if $|A|=2$, and $C(A)=\infty$ if $|A|\geq 3$ where $c\in(0,\infty)$ is the connection cost incurred by a linking party.

In this paper, we will focus our attention to the case of the linear variable cost, $C(A)=c_0+c|A|$, where c_0, $c>0$.[5] We can think of c_0 the cost of installing the hub (dummy player, coordinator, secretary etc.) of the association, and c as the cost that the coordinator disseminates information to each member. This simple linear cost structure will help us to obtain clear analytic results.

IV. *Efficiency and Stability*

We can generalize two central concepts, efficiency and stability, by Jackson and Wolinsky (1996) to the formation of hypergraphs.

The hypergraph H is efficient if it maximizes the sum of net benefits, i.e., $V(H)\geq V(H')$ for all $H'\neq H$. On the other hand, the hypergraph H is stable if (I) for any $A\in H$ and for any $i\in A$, $\pi_i(H)\geq\pi_i(H-i(A))$, and (II) for any $S\notin H$ such that $|S|\geq 2$ and for any $i\in S$, $\pi_i(H+S)>\pi_i(H)$ implies $\pi_j(H+S)<\pi_j(H)$ for some $j\in S$. In words, the stability of a hypergraph requires that no player has an incentive to exit from an association *unilaterally*, and that no coalition of more than one player has the incentive to form a new association *collectively*.

1. Efficient Hypergraph

A series of lemmas are in order.

Lemma 1 *A hierarchical hypergraph cannot be efficient for any $C(A)$ and δ.*

Proof. See the appendix.

Lemma 1 can be strengthened by the following lemma.

[5] If $c_0=0$, much of the analysis is made trivial, while the results themselves remains unaffected. Moreover, the case that $c_0=0$ is somewhat unrealistic. An association could be interpreted as an alternative to a (complete) subnetwork among a group of players which works in the way that it reduces the per capita cost of communication by incurring some extra fixed cost.

Lemma 2 *If $S(\neq \emptyset) \subset N$ is contained in more than one association in H, H cannot be efficient for any $C(A)$ and δ.*

Proof. See the appendix.

This lemma implies that the efficient hypergraph must have $k(\geq 1)$ disjoint subsets and each of the associations constitutes a component. Thus, the only possible efficient hypergraph among complete hypergraphs is the grand hypergraph, and the efficient hypergraph other than the grand hypergraph must be disconnected.

Lemma 3 *The efficient hypergraph cannot contain more than one nontrivial component for any $C(A)$ and δ.*

Proof. See the appendix.

This lemma characterizes efficient hypergraphs.

Proposition 1 *The unique efficient hypergraph is H^n if $\tilde{c}(n) \equiv c + \frac{c_0}{n} < n-1$ and is H^0 if $\tilde{c}(n) > n-1$.*

Proof. See the appendix.

We can interpret $\tilde{c}(n)$ and $n-1$ as an increase in per player cost and benefit respectively when the hypergraph is changed from H^0 to H^*. Proposition 1 says that the efficient hypergraph must be either the empty hypergraph if the increase in the cost exceeds the increase in the benefit, or the grand hypergraph otherwise.

It deserves comparing this proposition with the result of Jackson and Wolinsky. Two striking differences are in order. First, the star structure cannot be an efficient hypergraph, while it can be an efficient network in the model of Jackson and Wolinsky. This contrasted feature is the direct consequence of Lemma 2 which has the implication that an indirect connection can never be efficient in hypergraph formation. This result comes mainly from the assumption of cost structures. The crucial intuition for the efficient star in network formation is that an indirect link between a pair of agents can be more efficient than a direct link between them, i.e., $1-c>\delta$ where c is the cost of forming a direct link.[6] In our model, it is not possible, that is, an indirect connection is always less efficient than a direct connection made by forming one large association encompassing all the agents involved in the connection structure. For example, a structure with two links between player 1 and 2 and between player 2 and 3 is inferior to one with the association $\{1,2,3\}$. By including all the involved players into one association, one could save the fixed cost and reduce the variable cost as well. Note that the variable connection cost increases with the size of an association, while the connection cost of a complete network increases geometrically with the size of the network. Second, the efficient hypergraph does not depend on δ, unlike the efficient network identified by Jackson and Wolinsky. This observation is a direct corollary of the first observation. Since Lemma 2 holds regardless of the size of δ, overlapping associations implying indirect connections cannot be constituent of the efficient hypergraph; hence, no indirect benefit in the efficient hypergraph.

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[6] Here, we are abusing notation c.

2. Stable Hypergraph

While the efficient hypergraph is expected to emerge in a centralized environment, a stable hypergraph can be formed in a decentralized environment as a consequence of the decision of each player maximizing his own payoff. After characterizing stable hypergraphs, we will compare them with the efficient hypergraph.

Lemma 4 *A hierarchical hypergraph cannot be stable for any* $C(A)$ *and* δ.

Proof. See the appendix.

Due to Lemma 4, we can restrict our attention to non-hierarchical hypergraphs for stability. Nonetheless, it is still burdensome to check whether a given hypergraph satisfies condition (II) of stability, since the number of all possible hypergraphs, $|\mathrm{H}|=2^{\mathrm{N}}$ where $\mathrm{N}=2^{|N|}-|N|-1$, is tremendously large. The following lemma provides a sufficient condition for condition (II).

Lemma 5 *Let* $i_0=\arg\min_{i\in N} B_i(H)$ *for a non-hierarchical hypergraph* $H\neq H^n$. *Then*, H *satisfies condition (II) if* $\pi_{i_0}(H)\geq\pi_{i_0}(H+N)$.

Proof. See the appendix.

Lemma 5 will turn out to be useful in characterizing stable hypergraphs. We have

Proposition 2 *(i) The grand hypergraph is stable if and only if* $\tilde{c}(n)<n-1$.[7] *(ii) Any star hypergraph with* $|R|=1$ *is stable if* $\tilde{c}(2)<1$ *and* $\tilde{c}(n)>(1-\delta)(n-2)$. *(iii) Any line hypergraph with* $|L_1|=|L_m|=1$ *is stable if* $\tilde{c}(2)<1$ *and* $\tilde{c}(n)>\sum_{t=1}^{n-2}(1-\delta^t)$. *(iv) If* $n\geq 4$, *a circle hypergraph cannot be stable. A circle can be stable if* $n=3$ *and* $\tilde{c}(2)<1-\delta$.

Proof. See the appendix.

Proposition 2 (i) says that the grand hypergraph is stable if and only if it is efficient. Proposition 2 (ii) and (iii) suggest that the hypergraph can be underconnected in the sense that a star hypergraph or a line can be stable when $\tilde{c}(n)<n-1$, i.e., the grand hypergraph is the unique efficient hypergraph. The intuition for this is that although the grand hypergraph is efficient, players cannot break up the status quo associations and reorganize the grand association. It is less beneficial for players to organize the grand association with maintaining their current associations than to move from the empty hypergraph to the grand hypergraph. The intuitive reason for Proposition 2 (iv) is that the cost high enough to discourage players from organizing the grand association cannot prevent the incentive to exit from a small association. Then, why can a line be stable although a circle cannot be? This is because it is more tempting for an agent to exit in a circle than in a line. He loses the benefit much less in a circle, as he still maintains an indirect connection with other agents even after he exits from one association in a circle.

Proposition 2 (ii) only provides sufficient conditions for the stability of a star hypergraph, but indeed a star hypergraph with $|R|\geq 2$ cannot be stable.

Proposition 3 *A star hypergraph with* $|R|\geq 2$ *cannot be stable.*

[7] We are assuming that a tie in payoffs is resolved in favor of deviation.

Proof. See the appendix.

The main reason for this proposition is that the loss in benefits occurring when an agent exits from the center with $|R|\geq 2$ is much smaller than when $|R|=1$. Let us elaborate on the intuition. For the grand association to be unprofitable, the *per capita* formation cost should be high enough that $\tilde{c}(n)>(1-\delta)(n-2)$. However, in that case, an agent in R with $|R|\geq 2$ does have an incentive to exit from one of his association, because $\tilde{c}(m)>\frac{\tilde{c}(n)}{n-2}>1-\delta$ for any $m\geq 3$. If the center of a star is a singleton, however, it can be stable if discounting is large, because the loss from the center's exit out of an association which does not depend on the discount factor can be larger than the gain from joining in the grand association which gets smaller as discounting is larger.

Proposition 4 *The stable hypergraph cannot be overconnected, i.e., no other hypergraph than the empty hypergraph is stable if* $\tilde{c}(n)>n-1$.

Proof. See the appendix.

The reason for the possibility of underconnectivity is crystal clear. The usual intuition applies; players do not take into account the positive externality that they could generate by forming an association. The possibility of overconnectivity in network formation by Jackson and Wolinsky was due to the possible coordination failure.[8] Since coordination by more than two people is allowed in a hypergraph, it is difficult that a hypergraph is overconnected due to coordination failure. In fact, all efficient hierarchies are symmetric among players. So, coordination among them cannot be a problem.

V. *Strong Stability*

Our concept of stability in a hypergraph has common with the concept of strong stability in a network proposed by Dutta and Mutuswami (1997) and Jackson and Nouweland (2005), in the sense that both possibly allow joint deviations by more than two players. In this section, we will briefly discuss how they are related and how they differ.

Jackson and Nouweland (2005) define a network to be strongly stable, rougly speaking, if for any coalition $S\subset N$, (i) any player in S has no incentive to break his link and (ii) any number of pairs in S has no incentive to form new links.[9] If we rephrase our definition in terms of the network, a hypergraph is stable if no coalition S has an incentive to form a complete network among them. Moreover, they allow more than one player to sever their links or a player to sever his link and simultaneously form a new link with another. None of them is allowed in our definition. Since fewer deviations are allowed in our definition, a strongly stable network implies a stable hypergraph but not vice versa. For example, consider a star H when $n=4$ and let player 1 be the center. If we assume that $c=c_0$ for simplicity, the stability of the

[8] When the star network is efficient, coordination failure may occur if no one wants to be the center who will get the lowest payoff.

[9] The definition of strong stability by Dutta and Mutuswami is almost the same, except an inequality in the definition.

hypergraph requires that (i) $c+c/4>2(1-\delta)$ and (ii) $c+c/2<1$. Now, to check the strong stability, suppose that player 4 exits from his current association with player 1 and makes a new association with player 2 and player 3. Then, the gains of each player in the new association is

$$\Delta B_4(H)=c+\frac{c}{2}+2(1-\delta)-(1-\delta)-\left(c+\frac{c}{3}\right)=\frac{c+6(1-\delta)}{6}>0,$$

$$\Delta B_2(H)=\Delta B_3(H)=2(1-\delta)-\left(c+\frac{c}{3}\right)=\frac{2(3-2c-3\delta)}{3}>0, \textit{ if } c<\frac{3}{2}(1-\delta).$$

Thus, the strong stability requires that $c<\frac{3}{2}(1-\delta)$. Therefore, if $\frac{3}{2}(1-\delta)<c<\frac{8}{5}(1-\delta)$, a star is not a strongly stable network, although it is a stable hypergraph.

By incorporating the spirit of the strongly stable network, we can formally define the strongly stable hypergraph. The following concepts will be used in defining it.

We say that H_S is a transformation of H by a coalition $S(\subset N)$ if (i) a coalition S exit from some $A\in H$, (ii) S form a new association $A\notin H$, or (iii) any pair of processes (i) and (ii) simultaneously occur. A transformation can involve a variety of forms. For example, a group of players in $A\in H$ exit and then, each of them may remain as a singleton or some/all players form a new association by (ii). Some coalition $S_1\subset A_1(\in H)$ and another coalition $S_2\subset A_2(\in H)$ may simultaneously exit and form a new association. Especially, two associations may merge into one association. We also say that H is subverted by a coalition S if for all $i\in S$, $\pi_i(H_S)\geq\pi_i(H)$ with inequality for at least one $i\in S$. Then, H is a strongly stable hypergraph if there exists no coalition S subverting H.

If we replace the concept of stability by the strongly stable hypergraph, all of Lemma 1, 2 and 3 stated in terms of efficiency hold for strong stability, whereas some of them did not hold for stability. We summarize the result by the following lemma.

Lemma 6 *(i)A hierarchical hypergraph cannot be strongly stable for any $C(A)$ and δ. (ii) If $S(\neq\emptyset)\subset N$ is contained in more than one association in H, H cannot be strongly stable for any $C(A)$ and δ. (iii) The strongly stable hypergraph cannot contain more than one nontrivial component for any $C(A)$ and δ.*

Proof. The proofs are immediate by considering transformations defined in the proof of Lemma 1, 2 and 3.

This lemma characterizes strongly stable hypergraphs.

Proposition 5 *The unique strongly stable hypergraph is H^n if $\tilde{c}(n)<n-1$ and is H^0 if $\tilde{c}(n)>n-1$.*

Proof. See the appendix.

This proposition implies that the tension between efficiency and stability disappears if we use the concept of strong stability.

In the setting of network formation, Jackson and Nouweland (2005) obtains a similar, quite general result that if the allocation rule is component-wise egalitarian, the set of strongly stable networks and the set of efficient networks coincide as long as strongly stable networks exist.

Roughly speaking, the component-wise egalitarian allocation rule means that the value should be split equally among all the coalition members. Since the payoff assumed in this paper clearly does not correspond to the component-wise egalitarian allocation rule,[10] our result is not a special case of theirs.

VI. *Convex Variable Cost*

Suppose that the variable cost of forming an association is convex in its size rather than linear. In particular, we assume that $C(A)=c_0+c|A|^k$ for $k\geq 2$. Also, let $\tilde{c}(m)=cm^{k-1}+\frac{c_0}{m}$ where $m=|A|$.

With the convex cost structure, Lemma 2 does not hold any more while Lemma 1 is still valid. The intuitive reason is that if the variable cost increases too rapidly with the size of an association, it may be more efficient to separate the members into several associations even though some of the members may overlap in more than one association. This suggests that the efficient hypergraph is not necessarily of an extreme form, either empty or grand. As an example, take the case that $k=2$ and $n=3$. A star H^* is more efficient than H^n which dominates H^c, if $\pi(H^n)-\pi(H^*)=2(1-\delta)+c_0-c<0$, i.e., if $c>2(1-\delta)+c_0$. Also, H^* is more efficient than H^0 which dominates the hypergraph with only one association of size two, if $c<\frac{2+\delta-c_0}{4}$. Therefore, when $c_0=\frac{1}{2}$ and $\delta\approx 1$, H^* will be efficient for $c\in(\frac{1}{2},\frac{5}{8})$.

Proposition 3 is also not robust to a variation to the convex cost function. For example, let $|R|=r\geq 2$ and $|A|=m$ in a star H^*. Then, player $i\in A\backslash R$ has no incentive to form a grand association if $\tilde{c}(n)>(1-\delta)(n-m)$. Now, since we know that one of the centers is more likely to deviate than any peripheral player, we will consider the incentive of player $j\in R$ to exit from his association A. We have

$$\Delta\pi_j(H^*)=\pi_j(H^*-A)-\pi_j(H^*)=\tilde{c}(m)-(1-\delta)(m-r),$$

and thus

$$\frac{d\Delta\pi_i(H^*)}{dm}=-\frac{(1-\delta-c)m^2+c_0}{m^2}.$$

If $c<1-\delta$, we have $\frac{d\Delta\pi_i(H^*)}{dm}<0$ and so $\Delta\pi_i(H^*)$ attains its maximum at $m=3$.[11] Take $r=2$. Then, player j would not exit if $\tilde{c}(3)<1-\delta$. If $k=3$ and $n=4$, there is $\delta\in(0,1)$ such that $\tilde{c}(3)<1-\delta<\frac{\tilde{c}(n)}{n-3}$ because $\tilde{c}(3)<\frac{\tilde{c}(n)}{n-3}$. In this case, $H^*=\{\{1,2,3\},\{1,2,4\}\}$ is stable. The main intuition is that if the cost function is convex, a center's exit from a small association of a star may not lead to a reduction of the *per capital* cost while it reduces the benefit, i.e., he will not exit, thus implying that a star can be stable. Note that the exit of a center from an

[10] The payoff to each player after forming a hypergraph is not equal in our model.

[11] Also, note that $\Delta\pi_i(H^*)$ increases with r, in other words, that the larger r is, the smaller the decrease in the benefit from the exit, and thus the more likely player j is to deviate.

association always reduce its *per capital* cost if the cost function is linear.

VII. *Conclusion*

In this paper, we defined the efficiency and the stability of the hypergraph and characterized efficient hypergraphs and stable hypergraphs for the linear cost function.

The hypergraph is a general concept encompassing the concept of network, and can be applied to economics in a more flexible manner. For example, the formation of free trade agreements (FTA) is the outcome of multilateral negotiations among possibly more than two countries. Although many authors model this process as network formation allowing only bilateral decisions,[12] we believe that it will be more relevant to view FTAs as associations and the process of forming them as the formation of a hypergraph allowing multilateral decisions. We look forward to a richer variety of economic applications in the near future.

Appendix

Proof of Lemma 1:

Suppose that there are A, $A' \in H$ such that $A' \subset A$. Define $H' = H - A'$. Then, for any $i \in A'$, $C_i(H') < C_i(H)$ but $B_i(H') = B_i(H)$, thus $\pi_i(H') > \pi_i(H)$. Also, it is clear that $\pi_i(H') = \pi_i(H)$ for any $i \notin A'$. Therefore, $V(H') > V(H)$. This implies that H cannot be efficient.

Proof of Lemma 2:

Let A, $A' \in H$ $(A \neq A')$ be two associations such that $S \subset A$, A'. Define H' by $H' = H - (A + A') + (A \cup A')$. Then, $C(A \cup A') < C(A) + C(A')$ since $|A \cup A'| < |A| + |A'|$. Therefore, $V(H') > V(H)$.

Proof of Lemma 3:

Suppose that A_1, $A_2 \in H$ with $A_1 \cap A_2 = \emptyset$, $|A_1| = n_1$ and $|A_2| = n_2$ for n_1, $n_2 \geq 2$. Take any node $i \in A_2$ and define $H' = H - (A_1 + A_2) + A_1 \cup A_2$. Then, it is clear that $\sum_{i \in A_1 \cup A_2} B_i(H') > \sum_{i \in A_1 \cup A_2} B_i(H)$ and that $\sum_{i \in A_1 \cup A_2} C_i(H') < \sum_{i \in A_1 \cup A_2} C_i(H)$. Therefore, $V(H') > V(H)$.

Proof of Proposition 1:

By Lemma 3, there must be at most one nontrivial component in the efficient hypergraph. Also, it must consist of one association by Lemma 2. Let H^m be the hypergraph with a nontrivial component association of size $m \geq 2$ and let $H^1 \equiv H^0$. Then, the efficient hypergraph must be H^m for some $m \geq 1$. Let the possible nontrivial component be $A \in H^m$. Now, consider $H' = H^m - A + A \cup \{i\}$ for some $i \notin A$. Then, we have

$$\Delta B(H^m) \equiv \sum_{i \in N} B_i(H') - \sum_{i \in N} B_i(H^m) = (m+1)^2 - (m^2+1) = 2m, \tag{1}$$

[12] See, for example, Furusawa and Konishi (2007), and Goyal and Joshi (2006).

$$\Delta C(H^m) \equiv \sum_{i\in N} C_i(H') - \sum_{i\in N} C_i(H^m) = \begin{cases} c_0 + 2c & \text{if } m=1 \\ c & \text{if } m \geq 2. \end{cases} \tag{2}$$

Note that $\Delta B(H^m)=2m$ is increasing in m. By comparing equation (1) and (2), we can see that $V(H^2)>V(H^1)$ if and only if $c_0+2c<2$, and that $V(H^m)$ is increasing in m for all $m\geq 2$ if and only if $c<4$. Thus, if $c<4$, the efficient hypergraph is either H^1 or H^n. If $c>4$, $V(H^m)$ is decreasing in m for all $m\geq 2$. Hence, the efficient hypergraph is either H^1 or H^2. In particular, H^2 would be efficient if and only if $c_0+2c<2$, but it is not possible as far as $c>4$. Hence, the only possible efficient hypergraph in this case is H^1. Comparing the values $V(H^1)$ and $V(H^n)$ directly shows that $V(H^1)\leq V(H^n)$ if and only if $\tilde{c}(n)\geq n-1$.

Proof of Lemma 4:

For any A, $B\in H$ such that $A\varnothing B$, we have $\pi_i(H-i(A))>\pi_i(H)$ for any $i\in A$, since $C_i(H-i(A))<C_i(H)$ and $B_i(H-i(A))=B_i(H)$.

Proof of Lemma 5:

For any association $A\notin H$ with $|A|=m$ and for any $i\in A$, we have

$$\Delta C_i(H) \equiv C_i(H+A) - C_i(H) = c + \frac{c_0}{m}.$$

Since $\Delta C_i(H)$ is decreasing in m, it is smallest when $m=n$. Also, it is clear that $\max_A B_i(H+A)=B_i(H+N)$. Thus, condition (II) is satisfied if any player $i\in N$ has no incentive to join in the grand association N. In fact, this is the case if $\pi_{i_0}(H)\geq \pi_{i_0}(H+A)$, because

$$\pi_i(H+N)-\pi_i(H)=B_i(H+N)-B_i(H)-\Delta C_i(H)\leq n-B_{i_0}(H)-(c+\frac{c_0}{n})=\pi_{i_0}(H+N)-\pi_{i_0}(H)\leq 0$$

for all $i\in N$.

Proof of Proposition 2:

(i) By Lemma 4, we only need to check whether the grand hypergraph is stable. Define $H'=H^n-i(N)$ for any $i\in N$. Then, $B_i(H^n)-B_i(H')=n-1$ and $C_i(H^n)-C_i(H')=\tilde{c}(n)$. Therefore, player $i\in N$ will not exit if and only if $\tilde{c}(n)<n-1$. It is clear that H^n satisfies the second condition of stability.

(ii) If player $i\in R$ exits from any A_k, his loss in benefits is

$$\Delta B_i(H^*) = \begin{cases} |A_k|-1 & \text{if } |R|=2 \\ (1-\delta)|A_k \backslash R| & \text{if } |R|\geq 2, \end{cases}$$

and his cost saving is $\Delta C_i(H^*)=c+\dfrac{c_0}{|A_k|}$.

If $|R|=1$, $\Delta\pi_i(H^*)=\Delta C_i(H^*)-\Delta B_i(H^*)=c+\dfrac{c_0}{|A_k|}(|A_k|-1)$ has the maximal value of $c+\dfrac{c_0}{2}-1$ when $|A_k|=2$. In this case, player $i\in R$ has no incentive to exit from A_k if

$$c+\frac{c_0}{2}<1. \tag{3}$$

If $|R|\geq 2$, $\Delta\pi_i(H^*)=c+\frac{c_0}{|A_k|}-(1-\delta)(|A_k|-|R|)$ has the maximal value of $c+\frac{c_0}{3}-(1-\delta)$ when $|A_k|=3$ and $|R|=2$. In this case, player $i\in R$ does not exit from A_k if

$$c+\frac{c_0}{3}<1-\delta, \tag{4}$$

and otherwise, he exits.

Next, consider the incentive of player $j\notin R$ to exit. If he exits from some A_k, $\Delta B_j(H^*)>\Delta B_i(H^*)$ and $\Delta C_j(H^*)=\Delta C_i(H^*)$. Therefore, if a player in R does not exit, neither does he.

Now, consider the incentive to form a new association. By Lemma 5, we only need to find player $i_0=\arg\min_i B_i(H^*)$. Clearly, $i_0\in A_k$ for some A_k with $|A_k|=2$ and $i_0\notin R$, and thus, $B_{i_0}(H^*)=2+\delta(n-2)$. Hence, we have $\Delta B_{i_0}(H^*)=n-(2+\delta(n-2))=(1-\delta)(n-2)$ and $\Delta C_{i_0}(H^*)=c+\frac{c_0}{n}$. Therefore, if $\Delta C_{i_0}(H^*)>\Delta B_{i_0}(H^*)$, i.e.,

$$\tilde{c}(n)\equiv c+\frac{c_0}{n}>(1-\delta)(n-2)\ , \tag{5}$$

no player will join in any new association by Lemma 5. Note that inequality (4) and (5) are not compatible with each other. Therefore, H^* is stable if $\tilde{c}(2)<1$ and $\tilde{c}(n)>(1-\delta)(n-2)$, and in this case it must be that $|R|=1$.

(iii) Consider the exit incentive. Suppose a player $i\in L_k$ exits from his association(s), A_k or A_{k+1}. The loss in his benefits has the minimal value of 1 when player $i\in L_1$ exits from A_1 with $|A_1|=2$. In this case, his cost saving is maximal, i.e., $\Delta C_i(H^l)=c+\frac{c_0}{2}$. Thus, no player will have an incentive to exit if $c+c_0/2<1$ (inequality (3)). If $|L_k|\geq 2$, the minimal loss is $1-\delta$ when player $i\in L_1$ exits from A_1 with $|A_1|=3$ and $|L_1|=2$. In this case, his cost saving is maximal, $\Delta C_i(H^l)=c+\frac{c_0}{3}$. Thus, no player will have an incentive to exit if $c+c_0/3<1-\delta$ (inequality (4)). Also, we know that player $j\notin L_k$ does not exit from his association for any k if player $i\in L_k$ does not exit for any k.

Finally, consider the incentive to form the grand association. By Lemma 5, we only need to find player i with minimal $B_i(H^l)$. It is easy to see that player $i\in A_1$ has the minimal $B_i(H^l)$ when $|A_k|=2$ for all k. Therefore, no player will join in any new association if

$$\tilde{c}(n)>\sum_{t=1}^{n-2}(1-\delta^t). \tag{6}$$

Again, inequality (4) and (6) are not compatible. Therefore, any H^l with $|L_1|=|L_m|=1$ is stable if $\tilde{c}(2)<1$ and $\tilde{c}(n)>\sum_{t=1}^{n-2}(1-\delta^t)$.

(iv) Consider the incentive to exit from some A_k. Clearly, $\Delta C_i(H^c)=c+\frac{c_0}{|A_k|}$. It is also clear that the loss in benefits has the minimal value of $1-\delta$ when player $i \in A_1 \cap A_3$ exits from either association where $H^c = \{A_1, A_2, A_3\}$ and $|A_1| = |A_2| = |A_3| = 2$. In this case, $\Delta C_i(H^c)$ is maximal, i.e., $\tilde{c}(2)$. Thus, no player will exit if

$$\tilde{c}(2) < 1-\delta. \tag{7}$$

Also, we know that no player will join in the grand association if

$$\tilde{c}(n) > \begin{cases} 2\sum_{t=1}^{\frac{n}{2}-1}(1-\delta^t) & \textit{if } n \textit{ is even} \\ 2\sum_{t=1}^{\frac{n-1}{2}}(1-\delta^t) & \textit{if } n \textit{ is odd,} \end{cases} \tag{8}$$

provided that $n \geq 4$. Note that inequality (7) and inequality (8) are not compatible, implying that a circle hypergraph cannot be stable. If $n=3$, it is clear that no player will join in the grand association. Therefore, in this case, a circle is stable if (7) is satisfied.

Proof of Proposition 3:

Consider a star $H^* = \{A_k \mid k=1,\cdots,m\}$ for $m \geq 2$. Suppose H^* is stable. Then, by condition (i) of stability, it must be that no peripheral player $i \in A_k$ has an incentive to exit from A_k for any k. This requires that

$$c + \frac{c_0}{|A_k|} < (|A_k| - |R|)(1-\delta). \tag{9}$$

Also, by condition (ii) of stability, there must be no new association to be formed. Note that any center has no incentive to join in a new association, because his gain from it is zero. This means that a profitable new association must consist only of peripheries. Consider a new association consisting of all the peripheral nodes. Since the number of peripheries is $n-|R|$, it requires that

$$(|A_k| + m - 2)(1-\delta) + \alpha < c + \frac{c_0}{n-|R|}. \tag{10}$$

We have $|A_k| - |R| < |A_k| + (m-2)$ for all k. Also, we have $|A_k| < n - |R|$ for some k. This is because $\sum_k (|A_k| - |R|) > \sum_k n - |R|$. Since $c + \frac{c_0}{A_k} > c + \frac{c_0}{n-|R|}$ and $|A_k| - |R| < |A_k| + m - 2$, inequalities (9) and (10) are contradictory, which implies that H^* cannot be stable.

Proof of Proposition 4:

Suppose $H \neq H^0$ is stable. If a player i exits from some $A \in H$, the cost reduction is $\Delta C_i(H) \geq \tilde{c}(n)$ and a decrease in the benefit is $\Delta B_i(H)$ cannot exceed $n-1$ which is maximal. Therefore, we have $\Delta C_i(H) \geq \tilde{c}(n) > n-1 \geq \Delta B_i(H)$. This means that player i always has an incentive to exit from his association. Hence, $H \neq H^0$ cannot be stable.

Proof of Proposition 5:

By Lemma 6(iii), the strongly stable hypergraph must be H^m for some $m \geq 1$. Consider H^m and let A_m be the unique nontrivial component of H^m. Take $S = A_m \cup \{j\}$ for any $j \notin A_m$. Then, H^{m+1} is a transformation of H^m by S. We can see that any $i \in A$ gets a higher payoff, since

$$\Delta\pi_i = \Delta B_i - \Delta C_i = 1 - \left(\frac{c_0}{m+1} - \frac{c_0}{m}\right) > 0.$$

On the other hand, player j is benefited by the merger if $m > \tilde{c}(m+1)$. Thus, H^m is subverted by S if $m > \tilde{c}(m+1)$. Note that $\tilde{c}(m+1)$ decreases in m. So, if there is $m = m^*$ such that $m^* \geq \tilde{c}(m+1)$, $m \geq \tilde{c}(m+1)$ for all $m \geq m^*$. If there is no such m^*, H^1 is the unique candidate for the strongly stable hypergraph. Suppose there is such m^*. Then, H^m cannot be strongly stable for all m such that $m^* \leq m \leq n-1$. Next, consider H^m where $m < m^*$. If any $i \in A_m$ exits and remains as a singleton, the change in his payoff is $\Delta\pi_i = \tilde{c}(m) - (m-1) > 0$, since $m < \tilde{c}(m)$ for all $m < m^*$. Thus, for any $m\,(\neq 1) < m^*$, H^m cannot be strongly stable. Therefore, the only possible candidate for the strongly stable hypergraph is either H^n or H^1. Now, consider H^n. No single player will exit if $\tilde{c}(n) < n-1$. Also, a deviation by S with $|S| = m \neq 1$ is most profitable when they form a new association after exiting. Clearly, they have no incentive to deviate since the change in the payoff is

$$\Delta\pi_j = -\left[(n-m) + \left(\frac{c_0}{m} - \frac{c_0}{n}\right)\right] < 0.$$

Therefore, H^m is strongly stable if $\tilde{c}(n) < n-1$. Finally, consider H^1. If player $i \in N$ forms an association with S with $|S| = m$ where $1 \leq m \leq n-1$, the change in the payoff is $\Delta\pi_i = m - \tilde{c}(m+1)$. Hence, H^1 is strongly stable if $\Delta\pi_i < 0$ for all m, i.e., $\tilde{c}(n) > n-1$.

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