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ON SOME PROPERTIES OF HOLOMORPHIC DIFFUSION PROCESSES

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I. *Introduction*

On a complex manifold we can well-define the notion of a *holomorphic martingale* (cf. Schwartz [7]), although on a manifold, there is no intrinsic notion of martingales unless some additional structures like connections are introduced (cf. Meyer [5]). Namely, we say that a continuous stochastic process on a complex manifold of dimension n is a holomorphic martingale if its coordinate process with respect to a holomorphic chart is a part of a C^n-valued holomorphic martingale and this notion is independent of a choice of the local chart. If a diffusion process on a complex manifold is a holomorphic martingale, we say simply that it is a *holomorphic diffusion.* The works by P. Lévy and S. Kakutani on one-dimensional holomorphic diffusions, i.e. holomorphic diffusions on Riemann surfaces are classical: In particular they are transformed each other by random time change and they are all *symmetric* i.e. the transition semigroups are symmetric with respect to some measure on the manifold. Note that, for a diffusion process having the invariant measure, it is symmetric if and only if stationary diffusion process under the invariant measure is time reversible. In the case of higher dimensions, however, the situation will change considerably: They are no longer transformed each other by random time change and they are not necessarily symmetric so that the time reversion with respect to the invariant measure does not necessarily coincide with the original one.

Purpose of this note is to study the symmetry of higher dimensional holomorphic diffusion processes: In particular, we show that, for the existence of symmetric holomorphic diffusion processes on a manifold, there exists generally a topological obstruction on the manifold.

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II. *Basic Notions and Notations*

Notations

$$\frac{\partial}{\partial z^\alpha}=\frac{1}{2}\left(\frac{\partial}{\partial x^{2\alpha-1}}-\sqrt{-1}\,\frac{\partial}{\partial x^{2\alpha}}\right),\quad dz^\alpha=dx^{2\alpha-1}+\sqrt{-1}dx^{2\alpha},$$

$$\frac{\partial}{\partial \bar{z}^\alpha}=\frac{1}{2}\left(\frac{\partial}{\partial x^{2\alpha-1}}+\sqrt{-1}\frac{\partial}{\partial x^{2\alpha}}\right),\quad d\bar{z}^\alpha=dx^{2\alpha-1}-\sqrt{-1}dx^{2\alpha},$$

$$\partial f=\frac{\partial f}{\partial z^\alpha}dz^\alpha,\quad \bar{\partial} f=\frac{\partial f}{\partial \bar{z}^\alpha}d\bar{z}^\alpha,\quad df=\partial f+\bar{\partial} f,\quad d^c f=\sqrt{-1}(\bar{\partial} f-\partial f)$$

where $z^\alpha=x^{2\alpha-1}+\sqrt{-1}x^{2\alpha}$ $(z_\alpha\in C,\ x^{2\alpha-1},\ x^{2\alpha}\in R)$.

Let some probability space and a filtration on it be fixed; All the martingales below are refered to this system.

Definition 2.1 A stochastic process $Z_t=(z_t^1, \ldots, z_t^n)$ taking values in C^n is called an n-dimensional *holomorphic martingale* if it is continuous and if z_t^i, $z_t^i\cdot z_t^j$, $i, j=1, 2, \ldots n$, are all complex valued local martingales.

Thus Z_t is a holomorphic martingale if z_t^i, $i=1, 2, \ldots, n$ are continuous local martingales with $<z^i,z^j>_t=0$, $(i,\ j=1,\ 2, \ldots, n)$ (by following the notation of Ikeda-Watanabe [4], $dz_t^i\cdot dz_t^j=0$). If $z_t^j=x_t^j+\sqrt{-1}y_t^j$, it is equivalent to say that $\{x_t^i,y_t^i\}$ are system of real continuous local martingales with $<x^i,x^j>_t=<y^i,y^j>_t$, $<x^i,y^j>_t=0$, $i,i=1,\ 2, \ldots, n$ $(dx_t^i\cdot dx_t^j=dy_t^i\cdot dy_t^j,\ dx_t^i\cdot dy_t^j=0)$.

Example 2.1 (Complex Brownian Motion)

Let $(b_t^1,\ b_t^2, \ldots, b_t^{2n})$ be a standard Brownian motion on R^{2n} and set $z_t^i=b_t^{2j-1}+\sqrt{-1}b_t^{2j}$, $j=1, 2, \ldots, n$. Then clearly $Z_t=(z_t^1, \ldots, z_t^n)$ is a holomorphic martingale which will be called *n-dimensional complex Brownian motion.*

If Z_t is an n-dimensional holomorphic martingale then, because of $dz_t^j\cdot dz_t^i=0$, the Itô formula is given in the following form: If u is a smooth C^∞ function on C^n,

$$(2.1)\quad du(Z_t)=\frac{\partial u}{\partial z^i}dz_t^i+\frac{\partial u}{\partial \bar{z}^i}d\bar{z}_t^i+\frac{1}{2}\frac{\partial^2 u}{\partial z^i\partial \bar{z}^j}d<z_t^i,\bar{z}_t^j>.$$

In particular, if $\vec{u}$ is a C^∞-valued holomorphic function on C^n,

$$(2.2)\quad d\vec{u}(Z_t)=\frac{\partial \vec{u}}{\partial z^i}dz_t^i$$

which shows that $\vec{u}(Z_t)$ is also an m-dimensional holomorphic martingale.

Let M be a manifold. By *diffusion* $X=(X_t,P_x)$, we mean as usual a time homogeneous. continuous and strong Markov process on M:P_x denotes the probability law governing the paths starting at $x\in M$. X is called *smooth* if the infinitesimal generator L restricted to smooth functions is a differential operator with smooth coefficients. X is called *non-degenerate* if L is strictly elliptic. Finally X is called *conservative* if its life time is infinite.

Definition 2.2 Let M be a complex manifold and $X=(X_t,P_x)$ be a conservative diffusion on M. We call X a *holomorphic diffusion* on M if, for any holomorphic chart $(U_\alpha,\varphi_\alpha)$ of M, $\{\varphi_\alpha(X_{t\wedge\tau_\alpha}),P_x\}$ is an n-dimensioanl holomorphic martingale for any $x\in U_\alpha$ where $\tau_\alpha=\inf\{t\,|\,X_t\notin U_\alpha\}$.

By (2.2) it is clear that this definition is well-defined independently of a particular choice of holomorphic charts.

In the following, we assume, for simplicity, that *M is a compact complex manifold of dimension n and we consider only smooth, non-degenerate and conservative holomorphic diffusions on M*. For such X, its time change is defined as follows; let $c(x)$ be a smooth, everywhere positive function on M and $x(t)=X(A(t))$ where $A(t)$ is the inverse function of $t \to \int_0^t c(X(s))ds$. Then $(x(t),P_x)$ is also a smooth, non-degenerate and conservative diffusion on M which is denoted by X^c and is called the *diffusion obtained from X by a time change determined by the function c*. It is obvious from Doob's optional sampling theorem that X^c is also a holomorphic diffusion on M. Note that if X is generated by the differential operator L, then X^c is generated by $\frac{1}{c}L$.

Let $X=(X(t),P_x)$ be a holomorphic diffusion on M. Then, since M is compact and X is nondegenerate, the unique invariant probability measure $m(dx)$ of X exists and also the unique diffusion $\hat{X}=(\hat{X}(t),\hat{P}_x)$ exists such that $\int_M E_x[f(X(t))]g(x)m(dx)=\int_M \hat{E}_x[g(\hat{X}(t))]f(x)m(dx)$, for any continuous functions f and g on M (cf. Ikeda-Watanabe [4]). $\hat{X}$ is called *the dual process of X with respect to the invariant measure m(dx)* or *the time-reversed process of X with respect to the invariant measure m(dx)*. Indeed if $P(\cdot)=\int P_x(\cdot)m(dx)$ and $\hat{P}(\cdot)=\int \hat{P}_x(\cdot)m(dx)$ then, for any $T>0$ and $0<t_1<t_2<\ldots<t_m<T$ $\{X(t_1),\ldots,X(t_m);P\}$ and $\{\hat{X}(T-t_1),\ldots,\hat{X}(T-t_m);\hat{P}\}$ are equally distributed.

Definition 2.3 X is called *symmetric* if X and $\hat{X}$ coincide.

It is easy to see that X is symmetric if and only if the transition probability $p(t,x,y)$ with respect to the invariant measure $m(dx)$, which always exists and is smooth in $t>0$ and $x,y\in M$, satisfies that $p(t,x,y)=p(t,y,x)$ for all $t>0$, $x,y\in M$. It is also clear that if X is a symmetric conformal diffusion then its time change X^c is also a symmetric holomorphic diffusion. Note that if $m(dx)$ is the invariant measure of X then $a\cdot c(x)m(dx)$ (a: the normalization constant) is the invariant measure of X^c.

III. *Symmetry of Holomorphic Diffusions*

As stated in section 2, we only consider smooth, non-degenerate and conservative holomorphic diffusions X_t on a compact complex manifold M.

Proposition 3.1 There is a one-to-one correspondence between a holomorphic diffusion X_t and a Hermitian metric $g_{\alpha\bar{\beta}}$ on M in the sense that X_t is generated by the differential operator $L=\frac{1}{2}g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}$. Here, as usual, $(g^{\alpha\bar{\beta}})$ is the inverse of $(g_{\alpha\bar{\beta}})$.

Proof is almost obvious from the Itô formula (cf. Schwartz [7] for details).

Corollary 3.2 There always exists a holomorphic diffusion on every compact complex manifold M.

It is not always true, however, that a symmetric holomorphic diffusion exists on M. In order to study this problem, we shall introduce the following subclass of symmetric holomorphic diffusions on M.

Definition 3.3 A holomorphic diffusion X is called a *holomorphic Brownian motion* if it is generated by the half Laplace-Beltrami operator $\frac{1}{2}\Delta(g)$ corresponding to the Hermitian metric $(g_{\alpha\bar{\beta}})$.

Thus if X is a holomorphic diffusion and if $(g_{\alpha\bar{\beta}})$ is the Hermitian metric corresponding to X by Prop. 3.1, then X is a conformal Brownian motion ifand only if $g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}$ is the Laplace-Beltrami operator for the metric $(g_{\alpha\bar{\beta}})$. Note that a holomorphic Brownian motion is symmetric and its invariant measure is given by the normalized Riemannian volume.

Let $(g_{\alpha\bar{\beta}})$ be a Hermitian metric and let Ω be the *real fundamental form* corresponding to $(g_{\alpha\bar{\beta}})$:

$$\Omega=\frac{\sqrt{-1}}{2}g_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta.$$

The Hermitian metric $(g_{\alpha\bar{\beta}})$ is called a *Kähler metric* if $d\Omega=0$.

Definition 3.4 A holomorphic diffusion X is called a *Kähler diffusion* if the Hermitian metric $(g_{\alpha\bar{\beta}})$ corresponding to X by Prop. 3.1 is a Kähler metric.

It is well-known that if $(g_{\alpha\bar{\beta}})$ is a Kähler metric, the corresponding Laplace-Beltrami operator is of the form $\Delta(g)=g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}$ (cf. Morrow-Kodaira [6]) and consequently, a Kähler diffusion is always a holomorphic Brownian motion. Thus we have the following diagram:

{Kähler diffusions (K.D.)} $\subset$ {Holomorphic Brownian motions (H.B.M.)} $\subset$ {Symmetric holomorphic diffusions (S.H.D.)} $\subset$ {Holomorphic diffusions (H.D.)}. If M is of complex dimension 1, it is well known that {K.D.}={H.B.M.}={S.H.D.}={H.D.} because every Hermitian metric is a Kähler metric.

From now on we consider the higher dimensional case. Then we need following propositions.

Proposition 3.5 Let the complex dimension of M be greater than 1. If X is a symmetric holomorphic diffusion on M, er can find uniquely the function c such that X^c is a holomorphic Brownian motion.

Proof Let $(g_{\alpha\bar{\beta}})$ be the Hermitian metric corresponding to X and $m(dx)$ be the invariant measure of X. Then there exists a smooth positive function $r(x)$ such that $m(dx)=r(x)v(g)(dx)$, where $v(g)$ is the Riemannian volume with respect to the metric $(g_{\alpha\bar{\beta}})$. The generator of X^c is $\frac{1}{c}g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}$ and the invariant measure of X^c is $c{\cdot}m=c{\cdot}r{\cdot}v(g)$. Then we can determine c such that $c{\cdot}r{\cdot}v(g)=v(c^{-1}g)$, i.e. $c=r^{\frac{1}{n-1}}$, and noting the fact that a symmetric $\Delta(g)+b$-diffusion with the invariant Riemannian volume must be the Brownian motion with respect to (g) (cf. Ikeda-Watanabe [3], p. 280), X^c is the Brownian motion with respect to $(c^{-1}g)$. (Q.E.D.)

Corollary 3.6 If the complex dimension of M is greater than 1,

$$\{\text{H.B.M.}\}\subseteq\{\text{S.H.D.}\}$$

Proposition 3.7 For every Hermitian metric $(g_{\alpha\bar\beta})$ on M, the following formula holds:

$$(3.1)\quad g^{\alpha\bar\beta}\frac{\partial^2}{\partial z^\alpha\partial\bar z^\beta}-\Delta(g)=g^{\alpha\bar\beta}(\delta\Omega)_{\bar\beta}\frac{\partial}{\partial z^\alpha}+g^{\alpha\bar\beta}(\delta\Omega)_\alpha\frac{\partial}{\partial\bar z^\beta}$$

where $\partial\Omega=(\partial\Omega)_\alpha dz^\alpha+(\partial\Omega)_{\bar\beta}d\bar z^\beta$.

Proof First we prove the following integration by parts formula (3.2);

$$(3.2)\quad (f\cdot d^ch,\ \partial\Omega)_1=\int_M df\wedge d^ch\wedge*\Omega+2\int_M f\cdot Lhdv(g)$$

where $L=\dfrac{1}{2}g^{\alpha\bar\beta}\dfrac{\partial^2}{\partial z^\alpha\partial\bar z^\beta}$.

Since $d(fd^ch)\wedge*\Omega=df\wedge d^ch\wedge*\Omega+f\cdot dd^ch\wedge*\Omega$, we get

$$(3.3)\quad \int_M d(fd^ch)\wedge*\Omega=\int_M df\wedge d^ch\wedge*\Omega+\int_M f\cdot dd^ch\wedge*\Omega.$$

Here, L.H.S. of (3.3)$=(d(fd^ch),\Omega)_2=(f\cdot d^ch,\delta\Omega)_1$ where we denote the inner product on differential forms of degree k by $(\ ,\ )_k$. From now, we calculate R.H.S. of (3.3). It is well known that a Hermitian metric $(g_{\alpha\bar\beta})$ can be decomposed by the following; $g_{\alpha\bar\beta}=\sum_k\sigma_{\alpha k}\sigma_{k\bar\beta}$ where $\bar\sigma_{k\bar\beta}=\overline{\sigma_{k\bar\beta}}$, $\sigma_{k\bar\beta}=\sigma_{\bar\beta k}$. If we set $\omega_k=\sigma_{\alpha\bar k}dz^\alpha$, then $\bar\omega_k=\overline{\sigma_{\beta\bar k}}d\bar z^\beta=\sigma_{\bar\beta k}d\bar z^\beta=\sigma_{k\bar\beta}d\bar z^\beta$. So $g_{\alpha\bar\beta}dz^\alpha\wedge d\bar z^\beta=\sum_k\sigma_{\alpha\bar k}\sigma_{k\bar\beta}dz^\alpha\wedge d\bar z^\beta=\sum\omega_k\wedge\bar\omega_k$ (cf. Goldberg [3], p. 159, p. 165).

$$\begin{aligned}\int_M f\cdot dd^ch\wedge*\Omega&=\left(2\sqrt{-1}\cdot f\cdot\frac{\partial^2h}{\partial z^\alpha\partial\bar z^\beta}dz^\alpha\wedge d\bar z^\beta,\ \Omega\right)_2\\&=\sum_{k,l,m}\left(\sqrt{-1}\cdot f\cdot\frac{\partial^2h}{\partial z^\alpha\partial\bar z^\beta}(\sigma_{\alpha\bar l})^{-1}(\sigma_{m\bar\beta})^{-1}\omega_l\wedge\bar\omega_m,\ \sqrt{-1}\,\omega_k\wedge\bar\omega_k\right)_2\\&=\sum_{k,l,m}\int_M\sqrt{-1}\cdot f\cdot\frac{\partial^2h}{\partial z^\alpha\partial\bar z^\beta}(\sigma_{\alpha\bar l})^{-1}(\sigma_{m\bar\beta})^{-1}\omega_l\wedge\bar\omega_m*(\sqrt{-1}\,\omega_k\wedge\bar\omega_k)\\&=\sum_{k,l,m}\int_M f\cdot\frac{\partial^2h}{\partial z^\alpha\partial\bar z^\beta}(\sigma_{\alpha\bar l})^{-1}(\sigma_{m\bar\beta})^{-1}\delta_{kl}\delta_{km}dv(g)\\&=\sum_k\int_M f\cdot\frac{\partial^2h}{\partial z^\alpha\partial\bar z^\beta}(\sigma_{\alpha\bar k}\sigma_{k\bar\beta})^{-1}dv(g)\\&=\int_M f\cdot\frac{\partial^2h}{\partial z^\alpha\partial\bar z^\beta}(g_{\alpha\bar\beta})^{-1}dv(g)\\&=\int_M f\cdot g^{\alpha\bar\beta}\frac{\partial^2h}{\partial z^\alpha\partial\bar z^\beta}dv(g)=2\int_M f\cdot Lhdv(g).\end{aligned}$$

Here we used the fact: $(\sqrt{-1}\,\omega_1\wedge\bar\omega_1)\wedge(\sqrt{-1}\,\omega_2\wedge\bar\omega_2)\wedge\ \ldots\ \wedge(\sqrt{-1}\,\omega_n\wedge\bar\omega_n)=dv(g)$ i.e. $(\sqrt{-1}\,\omega_l\wedge\bar\omega_m)\wedge*(\sqrt{-1}\,\omega_k\wedge\bar\omega_k)=\delta_{kl}\delta_{km}dv(g)$.
This completes the proof of (3.2).

Since

$$df\wedge d^c h\wedge *\Omega=\left(\frac{\partial f}{\partial z^\alpha}dz^\alpha+\frac{\partial f}{\partial \bar z^\alpha}d\bar z^\alpha\right)\wedge\sqrt{-1}\left(\frac{\partial h}{\partial \bar z^\alpha}d\bar z^\alpha-\frac{\partial h}{\partial z^\alpha}dz^\alpha\right)\wedge *\left(\frac{\sqrt{-1}}{2}g^{\alpha\beta}dz^\alpha\wedge d\bar z^\beta\right)$$
$$=\frac{1}{2}g^{\alpha\beta}\left(\frac{\partial f}{\partial z^\alpha}\frac{\partial h}{\partial \bar z^\beta}+\frac{\partial f}{\partial \bar z^\beta}\frac{\partial h}{\partial z^\alpha}\right)dv(g),$$

(3.2) is equivalent to

$$(f\cdot d^c h,\,\delta\Omega)_1=\int_M\frac{1}{2}g^{\alpha\beta}\left(\frac{\partial f}{\partial z^\alpha}\frac{\partial h}{\partial \bar z^\beta}+\frac{\partial f}{\partial \bar z^\beta}\frac{\partial h}{\partial z^\alpha}\right)dv(g)+2\int_M f\cdot Lh\,dv(g)$$

Combining this with Green's formula:

$$\int_M\frac{1}{2}g^{\alpha\beta}\left(\frac{\partial f}{\partial z^\alpha}\frac{\partial h}{\partial \bar z^\beta}+\frac{\partial f}{\partial \bar z^\beta}\frac{\partial h}{\partial z^\alpha}\right)dv(g)+\int_M f\cdot\Delta(g)(h)dv(g)=0,$$

we immediately obtain (3.1). (Q.E.D.)

Corollary 3.8 X is a holomorphic Brownian motion on M if and only if $\delta\Omega=0$ where Ω is the real fundamental form corresponding to $(g_{\alpha\beta})$.

Remark 3.9 If X is a holomorphic Brownian motion, the corresponding Dirichlet form is given by $\int_M df\wedge d^c h\wedge\frac{1}{2}*\Omega$. Fukushima and Okada (cf. [2]) obtained a similar expression for Dirichlet forms corresponding to symmetric conformal diffusions on C^n by generalizing $\frac{1}{2}*\Omega$ to a closed positive current of type $(n-1,n-1)$.

Theorem 3.10 On every compact manifold of the complex dimension 2, {H.B.M.}={K.D.}, {H.B.M.}$\subseteq${S.H.D.} hold. And for every symmetric holomorphic diffusion X, there uniquely exists a function c such that X^c is a holomorphic Brownian motion.

Proof If $n=2$, we get $*(\omega_1\wedge\bar\omega_1)=\omega_2\wedge\bar\omega_2$, $*(\omega_2\wedge\bar\omega_2)=\omega_1\wedge\bar\omega_1$. Then recalling $\Omega=\frac{\sqrt{-1}}{2}(\omega_1\wedge\bar\omega_1+\omega_2\wedge\bar\omega_2)$, $*\Omega=\Omega$ is easily obtained. Hence $\delta=-*\cdot d\cdot *$ shows that $\delta\Omega=0$ if and only if $d\Omega=0$. By Proposition 3.7, if X is a holomorphic Brownian motion on M, X must be a Kähler diffusion. The second statement of this proposition follows immediately from Proposition 3.5 and Corollary 3.6.

Corollary 3.11 If {H.C.D.}$\neq\phi$, then {K.D.}$\neq\phi$ for every compact complex manifold M of the complex dimension 2.

Corollary 3.12 There exists a compact complex manifold of the complex dimension 2 which has no symmetric holomorphic diffusion.

Proof By Corollary 3.11, every non Kähler manifold of the complex dimension 2 gives this example. (Q.E.D.)

Indeed the next example is well known.

Example 3.13 (Hopf manifold: An example of non Kähler manifold)
$M=C^2-\{(0,0)\}/\sim$ where $(z_1,z_2)\sim(w_1,w_2)$ means $z_1=2^n w_1$, $z_2=2^n w_2$ for certain integer n. It is well known that this manifold M is non Kähler because its second Betti number vanishes. For more details refer to (Chern [1]).
Finally we consider the case of the complex dimension 3.

Proposition 3.14 There exists a non Kähler compact complex manifold of the complex dimension 3 which had a holomorphic Brownian motion, i.e. M exists for which $\{\text{K.D.}\}\neq\phi$ but $\{\text{H.B.M.}\}\neq\phi$.

Proof By Prop. 3.8, it is sufficient that we give an example of a non Kähler manifold with the real fundamental form Ω which satisfies $\delta\Omega=0$. Indeed the next example shows this one.

Example 3.15 (Iwasawa manifold (cf. Chern [1], Morrow-Kodaira [6]))
Let

$$G=\left\{\begin{pmatrix}1, & z_1, & z_2\\ 0, & 1, & z_3\\ 0, & 0, & 1\end{pmatrix}\middle|\ z_1, z_2, z_3\in C\right\} \quad\text{and}\quad D=\left\{\begin{pmatrix}1, & w_1, & w_2\\ 0, & 1, & w_3\\ 0, & 0, & 1\end{pmatrix}\middle|\ w_1, w_2, w_3\in Z+\sqrt{-1}Z\right\}$$

The quotient manifold $M=G/D$ (the quotient by an equivalence relation $g_1\sim g_2$ defined by $g_1=g_2\cdot h$, for some $h\in D$) is a compact complex manifold called Iwasawa manifold. It is well known that M is non Kähler because M has a holomorphic form φ $(\varphi=dz_2-z_3dz_1)$ which satisfies $d\varphi\neq 0$ (cf. [6]). Let $\Omega=\sqrt{-1}dz_1\wedge d\bar{z}_1+\sqrt{-1}(dz_2-z_3dz_1)\wedge(d\bar{z}_2-\bar{z}_3d\bar{z}_1)+\sqrt{-1}dz_3\wedge d\bar{z}_3$. By $\Omega\wedge\Omega=2*\Omega$ we can get easily $\delta\Omega=0$. So a holomorphic diffusion X corresponding to Ω is a holomorphic Brownian motion. Concretely X is given by

$$X_t=\pi\left\{\begin{pmatrix}1, & z_t^1, & z_t^2-\int_0^t z_s^3dz_s^1\\ 0, & 1, & z_t^3\\ 0, & 0, & 1\end{pmatrix}\right\}$$

where $(z_t^1, {}_tz^2, z_t^3)$ is a three dimensional complex Brownian motion (EX.2.1) and π is the natural map from G to M.

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