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**Rationality and the Nash Solution to Non-convex Bargaining Problems**

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**Abstract.** Conditions  $\alpha$  and  $\beta$  are two well-known rationality conditions in the theory of rational choice. This paper examines the implication of weaker versions of these two rationality conditions in the context of solutions to non-convex bargaining problems. It is shown that, together with the standard axioms of efficiency, anonymity and scale invariance, they characterize the Nash solution. This result makes a further connection between solutions to non-convex bargaining problems and rationalizability of choice functions in the theory of rational choice.

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# 1 Introduction

In this paper, we study the Nash solution to non-convex bargaining problem by examining its connection to two well-known rationality conditions, namely conditions  $\alpha$  and  $\beta$ , in the theory of rational choice (see, for example, Sen (1971)). Condition  $\alpha$  says that, when a set  $A$  contracts to another set  $B$ , and if an option  $x$  chosen from  $A$  continues to be available in  $B$ , then  $x$  must be chosen from  $B$ . Condition  $\beta$ , on the other hand, says that, when two options  $x$  and  $y$  are chosen from a set  $A$  and when  $A$  expands to another set  $B$ , then, *either* both  $x$  and  $y$  are chosen from  $B$  *or* neither  $x$  nor  $y$  is chosen from  $B$ .

In the literature on non-convex bargaining problems, a stronger version of condition  $\alpha$ , often called *contraction independence*, has been used for characterizing the Nash solution (see, for example, Xu and Yoshihara (2006)). Contraction independence requires that, when a bargaining problem  $A$  shrinks to another bargaining problem  $B$  and if  $B$  contains some options of the solution to  $A$ , then the solution to  $B$  coincides with the intersection of  $B$  and the solution to  $A$ . This version of contraction independence can also be regarded as a natural generalization of Nash's independence of irrelevant alternatives (Nash (1950)) introduced for convex bargaining problems where a solution picks a single option from a bargaining problem.

Building on the intuitions of conditions  $\alpha$  and  $\beta$ , in this paper, we consider weaker versions of conditions  $\alpha$  and  $\beta$ , to be called *binary condition  $\alpha$*  and *binary condition  $\beta$* , respectively. Binary condition  $\alpha$  requires that, for any two options  $x$  and  $y$ , if either  $x$  or  $y$  is part of the solution to a bargaining problem  $A$ , then the solution to the bargaining problem given by the comprehensive hull (see Section 2 for a formal definition) of  $x$  and  $y$  must contain the intersection of  $\{x, y\}$  and the solution to  $A$ . Binary condition  $\beta$  requires that, if two options  $x$  and  $y$  are both chosen from the problem of the comprehensive hull of  $x$  and  $y$ , then when the problem is enlarged, either both belong to the solution to the enlarged problem or neither does not belong to the solution to the enlarged problem. Together with the standard axioms of efficiency, scale invariance and anonymity, we then show that binary condition  $\alpha$  and binary condition  $\beta$  characterize the Nash solution to non-convex bargaining problems. Note that the contraction independence discussed in non-convex bargaining problems implies binary condition  $\alpha$  and binary condition  $\beta$ , but the converse relation does not hold. Our result therefore strengthens the existing characterization of the Nash solution to non-convex bargaining problems (see Xu and Yoshihara (2006)) and makes

a close connection between solutions to non-convex bargaining problems and rationality conditions in the theory of rational choice.

The remainder of the paper is organized as follows. In the following section, Section 2, we present notation and definitions. Section 3 introduces our axioms and studies their implications on non-convex problems. We conclude in Section 4.

## 2 Notation and definitions

Let  $N = \{1, 2, \dots, n\}$  be the set of all individuals in the society. Let  $\mathbf{R}_+$  be the set of all non-negative real numbers, and  $\mathbf{R}_{++}$  be the set of all positive numbers. Let  $\mathbf{R}_+^n$  (resp.  $\mathbf{R}_{++}^n$ ) be the  $n$ -fold Cartesian product of  $\mathbf{R}_+$  (resp.  $\mathbf{R}_{++}$ ). For any  $x, y \in \mathbf{R}_+^n$ , we write  $x \geq y$  to mean  $[x_i \geq y_i \text{ for all } i \in N]$ ,  $x > y$  to mean  $[x_i \geq y_i \text{ for all } i \in N \text{ and } x \neq y]$ , and  $x \gg y$  to mean  $[x_i > y_i \text{ for all } i \in N]$ . For any  $x \in \mathbf{R}_+^n$  and any non-negative number  $q$ , we write  $z = (q; \mathbf{x}_{-i}) \in \mathbf{R}_+^n$  to mean that  $z_i = q$  and  $z_j = x_j$  for all  $j \in N \setminus \{i\}$ .

For any subset  $A \subseteq \mathbf{R}_+^n$ ,  $A$  is said to be (i) *non-trivial* if there exists  $a \in A$  such that  $a \gg 0$ , and (ii) *comprehensive* if for all  $x, y \in \mathbf{R}_+^n$ ,  $[x \geq y \text{ and } x \in A]$  implies  $y \in A$ . For all  $A \subseteq \mathbf{R}_+^n$ , define the *comprehensive hull* of  $A$ , to be denoted by  $\text{comp}A$ , as follows:

$$\text{comp}A \equiv \{z \in \mathbf{R}_+^n \mid z \leq x \text{ for some } x \in A\}.$$

Let  $\Sigma$  be the set of all non-trivial, compact and comprehensive subsets of  $\mathbf{R}_+^n$ . Elements in  $\Sigma$  are interpreted as (normalized) bargaining problems. A *bargaining solution*  $F$  assigns a nonempty subset  $F(A)$  of  $A$  for every bargaining problem  $A \in \Sigma$ .

Let  $\pi$  be a permutation of  $N$ . The set of all permutations of  $N$  is denoted by  $\Pi$ . For all  $x = (x_i)_{i \in N} \in \mathbf{R}_+^n$ , let  $\pi(x) = (x_{\pi(i)})_{i \in N}$ . For all  $A \in \Sigma$  and any permutation  $\pi \in \Pi$ , let  $\pi(A) = \{\pi(a) \mid a \in A\}$ . For any  $A \in \Sigma$ , we say that  $A$  is *symmetric* if  $A = \pi(A)$  for all  $\pi \in \Pi$ .

**Definition 1:** A bargaining solution  $F$  over  $\Sigma$  is the *Nash solution* if for all  $A \in \Sigma$ ,  $F(A) = \{a \in A \mid \prod_{i \in N} a_i \geq \prod_{i \in N} x_i \text{ for all } x \in A\}$ .

Denote the Nash solution by  $F^N$ . Note that, for non-convex bargaining problems, the Nash solution is typically multi-valued.

### 3 Basic axioms and their implications

In this section, we present several standard axioms, introduce weaker versions of two well-known rationality conditions, and examine their implications for solutions to bargaining problems. In particular, we show that the Nash solution to non-convex bargaining problems is characterized by the standard axioms and the weaker versions of two well-known rationality conditions. We begin by introducing three axioms, Efficiency, Anonymity and Scale Invariance, which are standard in the literature on Nash bargaining problems.

**Efficiency (E):** For any  $A \in \Sigma$  and any  $a \in F(A)$ , there is no  $x \in A$  such that  $x > a$ .

**Anonymity (A):** For any  $A \in \Sigma$ , if  $A$  is symmetric, then  $[a \in F(A) \Rightarrow \pi(a) \in F(A)]$  for all  $\pi \in \Pi$ .

**Scale Invariance (SI):** For all  $A \in \Sigma$  and all  $t \in \mathbf{R}_{++}^n$ , if  $tA = \{(t_i a_i)_{i \in N} \mid a \in A\}$  then  $F(tA) = \{(t_i a_i)_{i \in N} \mid a \in F(A)\}$ .

In the literature on Nash bargaining problems and on rational choice theory, various *contraction independence* properties have been proposed. The idea behind a contraction independence property is the following: given two bargaining problems,  $A$  and  $B$ , in which  $A$  is a subset of  $B$ , and suppose that a point  $x$  chosen from  $B$  as a solution to  $B$  continues to be available in  $A$ , then  $x$  should continue to be a solution to  $A$  provided certain restrictions are satisfied. Nash's independence of irrelevant alternatives represents a natural property of contraction independence in convex bargaining problems, and it, in conjunction with some other conditions such as Pareto efficiency and continuity in Peters and Wakker (1991) and Bossert (1994), and Monotonicity and a weaker continuity in Sánchez (2000), implies the rationalizability of solutions to two-person convex bargaining problems.

The following axiom is a weaker version of contraction independence used in non-convex bargaining problems (see, for example, Xu and Yoshihara (2006)). It requires that, for any two points  $x$  and  $y$  in a bargaining problem  $A$ , if either  $x$  or  $y$  is part of the solution to  $A$ , then the common points in  $\{x, y\}$  and the solution to  $A$  must be contained in the solution to the problem given by the comprehensive hull of  $x$  and  $y$ . Clearly, this weaker

version of contraction independence is also a weaker version of condition  $\alpha$ . The axiom is formulated as below.

**Binary Condition  $\alpha$  (BC $\alpha$ ):** For all  $A \in \Sigma$  and all  $x, y \in A$ , if  $\{x, y\} \cap F(A) \neq \emptyset$  then  $F(A) \cap \{x, y\} \subseteq F(\text{comp}\{x, y\})$ .

It may be noted that (BC $\alpha$ ) is specific to non-convex bargaining problems and is not applicable to convex bargaining problems. We next introduce a weaker version of condition  $\beta$  (see Sen (1971)).

**Binary Condition  $\beta$  (BC $\beta$ ):** For all  $A \in \Sigma$  and all  $x, y \in A$ , if  $\{x, y\} = F(\text{comp}\{x, y\})$ , then  $[x \in F(A) \Leftrightarrow y \in F(A)]$ .

Thus, (BC $\beta$ ) requires that, whenever  $x$  and  $y$  are both chosen as solutions to the problem  $\text{comp}\{x, y\}$ , then, for any problem  $A$  containing both  $x$  and  $y$ , either  $[x$  and  $y$  are both chosen as solutions to  $A]$  or neither  $x$  nor  $y$  is chosen as a solution to  $A$ .

We now explore implications of the above axioms to be imposed on a solution to bargaining problems. Our first result shows that, when a solution satisfies (E), (A), (SI), (BC $\alpha$ ) and (BC $\beta$ ), then, for any  $x, y \in \mathbf{R}_+^n$ , the solution to the problem  $A = \text{comp}\{x, y\}$  must be such that  $F(A) = \{x, y\}$  if  $\prod_{i \in N} x_i = \prod_{i \in N} y_i > 0$  and  $F(A) = \{x\}$  if  $\prod_{i \in N} x_i > \prod_{i \in N} y_i$ . That is, for a specific, simple problem given by  $A$ , these axioms imply that the solution to  $A$  must be given by the Nash solution. After establishing the above result, our next result, Proposition 2, shows that, when a solution satisfies (E), (A), (SI), (BC $\alpha$ ) and (BC $\beta$ ), then the solution to any bargaining problem must be the Nash solution.

**Proposition 1.** *Let a solution  $F$  satisfy (E), (A), (SI), (BC $\alpha$ ) and (BC $\beta$ ). Then, for all  $x, y \in \mathbf{R}_+^n$ ,*

$$(1.1) \quad \prod_{i \in N} x_i = \prod_{i \in N} y_i > 0 \Rightarrow F(\text{comp}(x, y)) = \{x, y\}, \text{ and}$$

$$(1.2) \quad \prod_{i \in N} x_i > \prod_{i \in N} y_i \geq 0 \Rightarrow F(\text{comp}(x, y)) = \{x\}.$$

**Proof.** (1.1). Let  $\prod_{i \in N} x_i = \prod_{i \in N} y_i > 0$ . Consider an appropriate  $t \in \mathbf{R}_{++}^n$  such that  $tx = (t_1x_1, \dots, t_ix_i, \dots, t_nx_n)$  and  $ty = (t_1y_1, \dots, t_iy_i, \dots, t_ny_n)$  are permutations of each other (this is always possible due to the fact that  $x$  and  $y$  have the same Nash product). Let  $S \equiv \text{comp}\{tx, ty\}$ . Then, let  $T \equiv \cup_{\pi \in \Pi} \pi(S)$ . By construction,  $T$  is symmetric, and  $\{\pi(tx), \pi(ty) \mid \pi \in \Pi\} \subseteq T$  is the set of all efficient outcomes in  $T$ . Thus,  $F(T) \subseteq \{\pi(tx), \pi(ty) \mid \pi \in \Pi\}$ ,

and let  $tx \in F(T)$ . Then, by (A),  $\{\pi(tx) \mid \pi \in \Pi\} \subseteq F(T)$ . Also, since  $tx$  and  $ty$  are permutations of each other,  $ty \in F(T)$  by (A). Then, again by (A),  $\{\pi(ty) \mid \pi \in \Pi\} \subseteq F(T)$ . Thus,  $F(T) = \{\pi(\alpha x), \pi(ty) \mid \pi \in \Pi\}$ . Thus,  $tx, ty \in F(T)$ . Then, by (E) and (BC $\alpha$ ),  $\{tx, ty\} = F(S)$ . Thus, by (SI),  $F(\text{comp}\{x, y\}) = \{x, y\}$ .

(1.2). Let  $\prod_{i \in N} x_i > \prod_{i \in N} y_i \geq 0$ . Then, by choosing an appropriate  $\varepsilon \in \mathbf{R}_+^n$  with  $\varepsilon_j > 0$  for some  $j$ , we can have  $\prod_{i \in N} x_i = \prod_{i \in N} z_i$  for  $z \equiv y + \varepsilon$ . Then, by (1.1),  $F(\text{comp}\{x, z\}) = \{x, z\}$ . Note that, by the construction,  $y \in \text{comp}\{x, z\}$ , and  $F(\text{comp}\{x, z\}) \cap \{x, y\} = \{x\}$ . Therefore,  $x \in F(\text{comp}\{x, y\})$  follows from (BC $\alpha$ ). If  $y \in F(\text{comp}\{x, y\})$ , then (BC $\beta$ ) would imply that  $y \in F(\text{comp}\{x, z\})$ , a contradiction. Therefore,  $y \notin F(\text{comp}\{x, y\})$ . ■

**Proposition 2.** *Let a solution  $F$  satisfy (E), (A), (SI), (BC $\alpha$ ) and (BC $\beta$ ). Then, for any  $A \in \Sigma$ ,  $F(A) = F^N(A)$ .*

**Proof.** Take any  $A \in \Sigma$  and  $x \in F(A)$ . Suppose  $x \notin F^N(A)$ . Then, there exists  $y \in F^N(A)$  such that  $\prod_{i \in N} y_i > \prod_{i \in N} x_i \geq 0$ . Then, by Proposition 1.2,  $F(\text{comp}\{x, y\}) = \{y\}$ . On the other hand, note that  $x \in F(A)$ . By (BC $\alpha$ ),  $x \in F(\text{comp}\{x, y\})$ , a contradiction. Therefore,  $F(A) \subseteq F^N(A)$ . By (BC $\beta$ ) and from Proposition 1.1,  $F(A) = F^N(A)$  then follows immediately. ■

Note that  $F^N$  satisfies (E), (A), (SI), (BC $\alpha$ ) and (BC $\beta$ ). The following result then follows from Proposition 2 immediately.

**Proposition 3.** *A solution  $F$  satisfies (E), (A), (SI), (BC $\alpha$ ) and (BC $\beta$ ) if and only if  $F = F^N$ .*

Proposition 3 thus gives an alternative characterization of the Nash solution to non-convex bargaining problems. From the characterization result of the Nash solution to non-convex bargaining problems in Xu and Yoshihara (2006), it is clear that, in the presence of (E), (A) and (SI), Contraction Independence is equivalent to (BC $\alpha$ ) and (BC $\beta$ ). Moreover, it is easy to check that Contraction Independence implies (BC $\alpha$ ) and (BC $\beta$ ), but the converse does not hold. Thus, Proposition 3 is a strengthening of the characterization result of the Nash solution to non-convex bargaining problems in Xu and Yoshihara (2006).



## 4 Conclusion

In this paper, we have examined the implications of two weaker versions of conditions  $\alpha$  and  $\beta$  in the context of solutions to non-convex bargaining problems. In particular, we have shown that, together with other standard axioms in this context, they restrict a solution to be the Nash solution. Conditions  $\alpha$  and  $\beta$ , together, characterize rationalizability of a choice function defined over the set of all non-empty subsets of a finite universal set in terms of an ordering. It is therefore interesting to note that, in non-convex bargaining problems,  $(BC\alpha)$  and  $(BC\beta)$  are associated with “rationalizability” of a solution to bargaining problems. Our results thus make a further connection between two widely used rationality conditions in rational choice theory and solutions to non-convex bargaining problems.

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