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THE MODIGLIANI-MILLER THEOREM IN A DYNAMIC ECONOMY

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Abstract

A dynamic economy with markets of equities and bonds is considered. The rational expectations equilibrium is defined in an asset pricing model and a condition under which the Modigliani-Miller theorem holds is shown. In an aggregate model the existence of a rational expectations equilibrium is proved.

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I. *Introduction*

In this paper, a general model of a dynamic economy is presented and the equilibrium of rational expectations for the economy is defined. In the model, we re-examine the Modigliani-Miller theorem, which asserts that the value of a firm is independent of its debt-equity ratio. In the context of a dynamic general equilibrium model, we will show that the M-M theorem holds in a much more general framework. The validity of the theorem depends heavily on the rationality of consumers' expectations.

In the proof of the M-M theorem, which originates in the paper by Modigliani-Miller (1958), it is usually assumed that the gross returns of a firm depend only on the state of the economy, since the theorem is based on static equilibrium rather than dynamic analysis. In a dynamic economy, the profits of firms are determined depending on the behaviors of all economic agents, especially their expectations.

The purpose of this paper is to show that M-M results are still valid in a dynamic equilibrium of rational expectations. Also, in an aggregate model of the economy where all consumers are identical, we prove the existence of rational expectations equilibria. Our model is a generalization of the asset pricing model presented by Lucas (1978). Asset pricing models have become established tools and have been applied to various analyses by many authors, e.g., Constantinides and Duffie (1996), Brav et al. (2002), and Kocherlakota and Pistaferri (2009).

The M-M theorem was originally proved in a static framework and was extended to the general equilibrium model of Stiglitz (1969). Also, the theorem was considered by Diamond (1967) and DeMarzo (1988) in dynamic economies in which expectations are not incorporated. In this paper, the theorem will be reconsidered and proved in a general model with rational expectations.

This paper is formulated in the following fashion. In section II, a general model of a dynamic economy is presented and in section III, the equilibrium for the economy is defined. In section IV, a condition for the equilibrium under which the Modigliani-Miller theorem holds is shown. In section V, an aggregate model in which all consumers are identical is presented. In section VI, the existence of equilibria for an aggregate economy is proved and the M-M theorem is shown to hold in the equilibrium.

II. *A General Model*

In this section, we consider a general dynamic economy in which there are infinitely many consumers and finitely many firms. The set of consumers is denoted by an atomless measure space $(A, \mathscr{A}, \nu)$, where A is the set of all consumers, $\mathscr{A}$ is a σ-field consisting of some subsets of A, and ν is a measure defined on $\mathscr{A}$ so that $\nu(A)=1$. On the other hand, we assume that there are finitely many firms and the number of firms is J.

In an economy, there are n kinds of commodities and the commodity space is denoted by n-dimensional Euclidian space R^n. We assume that all commodities can be used as consumption goods as well as capital goods.

The consumption set of each consumer is the non-negative orthant of space R^n, which is denoted by R^n_+. The set of possible utility functions of consumers is denoted by $\mathscr{U}$. The utility function of each consumer is uncertain but is an element of $\mathscr{U}$. We assume that $\mathscr{U}$ is a set of some real-valued continuous functions defined on R^n_+ and is endowed with the topology of uniform convergence. The family of possible production sets of firms is denoted by $\mathscr{Y}$. The production set of each firm is also uncertain but is an element of $\mathscr{Y}$. We assume that $\mathscr{Y}$ is a set of some closed subsets of R^n and is endowed with the topology of closed convergence.

Uncertainty in the economy can be described by a stochastic process. We assume that time is discrete, and it is denoted by the set of non-negative integers, $T=\{0, 1, 2, \cdots\}$. Let $(\Omega, \mathscr{F}, P)$ be a probability space, i.e., Ω is the set of all the states of nature, $\mathscr{F}$ is the set of all possible events and is a σ-field consisting of some subsets of Ω, and P is a probability measure. Uncertainty in consumers' utility functions and firms' production sets is described by a stochastic process $\{\mathscr{E}_t \mid t \in T\}$ defined on $(\Omega, \mathscr{F}, P)$. For each t, $\mathscr{E}_t$ is a measurable mapping denoted by

$$\omega \in \Omega \rightarrow (\mathbf{U}, \mathbf{Y}) \in \mathscr{U}^A \times \mathscr{Y}^J,$$

where $\mathscr{U}^A$ is the set of all measurable mappings from A to $\mathscr{U}$ and $\mathscr{Y}^J$ is a J-time product of $\mathscr{Y}$, i.e.,

$$\mathscr{U}^A = \{\mathbf{U} \mid \mathbf{U} \text{ is a measurable mapping denoted by } a \in A \rightarrow U_a \in \mathscr{U}.\}$$

and

$$\mathscr{Y}^J = \mathscr{Y} \times \cdots \times \mathscr{Y}.$$

When state ω of nature occurs at period t, consumers' utility functions and firms' production sets are denoted by $\mathscr{E}_t(\omega)$, say $(\mathbf{U}, \mathbf{Y})$. Then, $\mathbf{U}$ is a mapping, $a \in A \rightarrow U_a \in \mathscr{U}$ and value U_a is the utility function of consumer $a \in A$. In addition, $\mathbf{Y}$ is an element of set $\mathscr{Y}^J$ and the j-th coordinate Y_j of $\mathbf{Y}$ is the production set of the j-th firm. We assume that consumers' utility functions and firms' production sets at each period will be known at the beginning of the period.

Suppose that a consumer has utility u_t at each period $t=0, 1, 2, \cdots$. Let δ be the discount rate of utility, where $0<\delta<1$. The sum of discounted utilities that the consumer would have is $\sum_{t=0}^{\infty} \delta^t u_t$. However, since his utility functions in future are uncertain, the consumer is not able to know the levels of utilities that he will obtain in the future. Therefore, consumers will guess future utilities and behave to maximize the sum of expected utilities. On the other hand, firms are able to know their production sets at the beginning of each period and production takes place within one period. Therefore, there is no uncertainty for firms and they simply maximize their profits at each period in time.

Process $\{\mathscr{E}_t \mid t \in T\}$ also describes a transition of uncertainty in the economy. We assume that it is a Markov process. Let $S=\mathscr{U}^A \times \mathscr{Y}^J$ and $\mathscr{B}(S)$ be the set of all Borel subsets of S. We denote by $\mathscr{M}(S)$ the set of all measures defined on $\mathscr{B}(S)$, which is endowed with weak topology.

Assumption 2.1: There is a continuous mapping from S to $\mathscr{M}(S)$,

$$s \in S \to \mu_s \in \mathscr{M}(S),$$

which has the following property: For each $s \in S$, μ_s is a transition probability on S, i.e., for each $t \in T$,

$$\mu_s(B)=\text{Prob.}\{\mathscr{E}_{t+1} \in B \mid \mathscr{E}_t = s\} \text{ for all } B \in \mathscr{B}(S).$$

More precisely, for each $t \in T$ and $s \in S$,

$$\int_C \mu_s(B) d(P \cdot \mathscr{E}_t^{-1})(s) = P(\mathscr{E}_{t+1}^{-1}(B) \cap \mathscr{E}_t^{-1}(C)) \text{ for all } B, C \in \mathscr{B}(S).$$

The existence of such a transition probability means that the uncertainty at each period does not depend on time, but only on the state at the previous period. Therefore, if $s=(\mathbf{U}, \mathbf{Y}) \in S$ is realized at period t, then the uncertainty in the economy after period t depends only on $s=(\mathbf{U}, \mathbf{Y})$. In addition, the transition of uncertainty is the same at all periods, and in this sense, the economy is stationary. Because of this stationarity, when we describe the state of the economy at each period, we do not have to show index "t" of time in the arguments.

III. *The Definition of Equilibrium*

Let $\mathscr{L}^n_{\infty+}$ be the set of all essentially bounded measurable functions from A to R^n_+. We use a function in $\mathscr{L}^n_{\infty+}$ to denote the initial holdings of commodities by consumers at each period. Namely, for function $\kappa \in \mathscr{L}^n_{\infty+}$, we denote by $\kappa(a)$ the amounts of commodities held by consumer a. Let $\mathscr{L}^J_{1+}$ be the set of all integrable functions from A to R^J_+. We use a function in $\mathscr{L}^J_{1+}$ to denote the shares in firms owned by consumers at each period. Namely, for function

$\theta=(\theta_1,\cdots,\theta_J)\in\mathscr{L}^J_{1+}$, we denote by $\theta_j(a)$ the share of the j-th firm's equity owned by consumer a.

Let $\mathscr{L}_1$ be the set of all integrable functions from A to R. We use a function in $\mathscr{L}_1$ to denote the numbers of bonds owned by consumers. Namely, for function $\beta\in\mathscr{L}_1$, $\beta(a)$ denotes the amount of bonds owned by consumer a. We denote numbers of bonds that firms issue by vector $D=(D_1,\cdots,D_J)\in R^J$, where D_j is the amount of bonds that are issued by firm j. All bonds are measured in terms of money, and the value of a unit of bond is therefore equal to a unit of money.

The equilibrium of the economy is defined by pair $\{\psi, V\}$, where ψ *is* a correspondence from $\mathscr{L}^n_{\infty+}\times\mathscr{L}^J_{1+}\times\mathscr{L}_1\times R^J\times S$ to $R^n\times R^J\times R$ and V is a function from $A\times R^n_+\times R^J_+\times R\times\mathscr{L}^n_{\infty+}\times\mathscr{L}^J_{1+}\times\mathscr{L}_1\times R^J\times S$ to R.

Correspondence ψ: $\mathscr{L}^n_{\infty+}\times\mathscr{L}^J_{1+}\times\mathscr{L}_1\times R^J\times S\rightarrow R^n\times R^J\times R$ is called *a price correspondence* and it shows consumers' expectations on equilibrium prices and interest rates. Correspondence ψ is depicted in the following notation:

$$(\kappa,\theta,\beta,D;s)\in\mathscr{L}^n_{\infty+}\times\mathscr{L}^J_{1+}\times\mathscr{L}_1\times R^J\times S\rightarrow\psi(\kappa,\theta,\beta,D;s)\subset R^n\times R^J\times R.$$

Element $(\kappa,\theta,\beta,D;s)$ in $\mathscr{L}^n_{\infty+}\times\mathscr{L}^J_{1+}\times\mathscr{L}_1\times R^J\times S$ describes a situation of the whole economy at a period in time. Element (p,q,r) of set $\psi(\kappa,\theta,\beta,D;s)$ is a vector in $R^n_+\times R^J_+\times R$, where p is a vector of commodity prices, q is a vector of equity prices, and r is the interest rate of a bond. Price correspondence ψ describes how equilibrium prices and interest rates depend on the situation of the economy.

Function V: $A\times R^n_+\times R^J_+\times R\times\mathscr{L}^n_{\infty+}\times\mathscr{L}^J_{1+}\times\mathscr{L}_1\times R^J\times S\rightarrow R$ is called *a value function* and it shows consumers' expectations on utilities. Function V is depicted in the following notation:

$$(a,z,e,b;\kappa,\theta,\beta,D;s)\in A\times R^n_+\times R^J_+\times R\times\mathscr{L}^n_{\infty+}\times\mathscr{L}^J_{1+}\times\mathscr{L}_1\times R^J\times S$$
$$\rightarrow V_a(z,e,b;\kappa,\theta,\beta,D;s)\in R.$$

Element (z,e,b) in $R^n_+\times R^J_+\times R$ describes the state of consumer a. Value $V_a(z,e,b;\kappa,\theta,\beta,D;s)$ is the expected value of utilities that consumer a can have. Value function V describes how the expected utility of each consumer depends on his state (z,e,b) as well as the situation $(\kappa,\theta,\beta,D;s)$ of the whole economy. Since the measure space of consumers is atomless, each consumer is negligible in the whole economy and he can therefore choose a state (z,e,b) independently of the situation $(\kappa,\theta,\beta,D;s)$ of the economy.

In order for $\{\psi, V\}$ to be an equilibrium of the economy, it is required in the following definition that (p,q,r) in $\psi(\kappa,\theta,\beta,D;s)$ is a vector of prices and an interest rate which equilibrate all markets and that $V_a(z,e,b;\kappa,\theta,\beta,D;s)$ is the maximum expected utility that consumer a can have when the situation of the economy is $(\kappa,\theta,\beta,D;s)$.

Definition 3.1: Pair $\{\psi, V\}$ of a price correspondence and a value function is *an equilibrium of the economy*, if $\{\psi, V\}$ satisfies the following:

Let $(\kappa,\theta,\beta,D,s)\in\mathscr{L}^n_{\infty+}\times\mathscr{L}^J_{1+}\times\mathscr{L}_1\times R^J\times S$ and $(p,q,r)\in\psi(\kappa,\theta,\beta,D;s)$ with

$$\int_A\beta dv=\sum_{j=1}^J D_j \text{ and } \int_A\theta dv=\underline{\mathbf{1}},$$

where $s=(\mathbf{U},\mathbf{Y})\in S$ and $\underline{\mathbf{1}}=(1,\cdots,1)\in R^J$. Then, there exist $\hat{c}\in\mathscr{L}^n_{\infty+}$, $(\hat{\kappa},\hat{\theta},\hat{\beta},\hat{D})\in\mathscr{L}^n_{\infty+}\times$

$\mathscr{L}^J_{1+} \times \mathscr{L}_1 \times R^J \times S$, and $\hat{y}_j \in Y_j$ ($j=1, \cdots, J$), which satisfy the following conditions:

(1) Firms maximize their profits, i.e., for each $j=1, \cdots, J$,

$$p \cdot \hat{y}_j \geqq p \cdot y \text{ for all } y \in Y_j.$$

(2) Consumers maximize their expected utilities subject to their budget constraints, i.e., for almost all $a \in A$,

$$p \cdot (\hat{c}(a) + \hat{\kappa}(a)) + q \cdot \hat{\theta}(a) + \hat{\beta}(a) \leqq p \cdot \kappa(a) + q \cdot \theta(a) + (1+r)\beta(a) + \sum_{j=1}^{J} \theta_j(a)\ (p \cdot \hat{y}_j - rD_j)$$

and

$$\begin{aligned} V_a(\kappa(a), \theta(a), \beta(a); \kappa, \theta, \beta, D; s) &= U_a(\hat{c}(a)) + \delta \int_S V_a(\hat{\kappa}(a), \hat{\theta}(a), \hat{\beta}(a); \hat{\kappa}, \hat{\theta}, \hat{\beta}, \hat{D}; \cdot) d\mu_s \\ &\geqq U_a(x) + \delta \int_S V_a(z, e, b; \hat{\kappa}, \hat{\theta}, \hat{\beta}, \hat{D}; \cdot) d\mu_s \end{aligned}$$

for all $(x, z, e, b) \in R^n_+ \times R^n_+ \times R^J_+ \times R$ with

$$p \cdot (x+z) + q \cdot e + b \leqq p \cdot \kappa(a) + q \cdot \theta(a) + (1+r)\beta(a) + \sum_{j=1}^{J} \theta_j(a)(p \cdot \hat{y}_j - rD_j).$$

(3) All markets are in equilibrium, i.e.,

$$\int_A \hat{c} d\nu + \int_A \hat{\kappa} d\nu = \int_A \kappa d\nu + \sum_{j=1}^{J} \hat{y}_j, \int_A \hat{\beta} d\nu = \sum_{j=1}^{J} \hat{D}_j, \text{ and } \int_A \hat{\theta} d\nu = \underline{\mathbf{1}}.$$

IV. *The Modigliani-Miller Theorem*

In this section, we show a condition for value function V in the definition of equilibrium under which the Modigliani-Miller theorem holds.

Condition 4. 1: Let $(\kappa, \theta, \beta, D, s) \in \mathscr{L}^n_{\infty+} \times \mathscr{L}^J_{1+} \times \mathscr{L}_1 \times R^J \times S$ with $\int_A \beta d\nu = \sum_{j=1}^{J} D_j$ and $\int_A \theta d\nu = \underline{\mathbf{1}}$. Then, for each $a \in A$,

$$V_a(z, e, b + e \cdot \Delta D; \kappa, \theta, \beta + \theta \cdot \Delta D, D + \Delta D; s) = V_a(z, e, b; \kappa, \theta, \beta, D; s)$$

for all $(z, e, b) \in R^n_+ \times R^J_+ \times R$ and $\Delta D \in R^J$.

The above condition means that consumers' expected utilities are independent of any change ΔD of amounts of bonds issued by firms, as long as each consumer changes the amount of bonds by $e \cdot \Delta D$ proportionally to amount e of equities he holds. The condition is the essence of the Modigliani-Miller theorem asserting that the value of a firm is independent of the amount of the firm's debts. In fact, the following proposition shows how the equilibrium prices of equities change but the prices of commodities are unchanged.

Proposition 4.1: Let $\{\psi, V\}$ be an equilibrium of the economy and let us assume that value function V satisfies Condition 4.1. If $(p, q, r) \in \psi(\kappa, \theta, \beta, D; s)$, then

$$(p, q-\Delta D, r) \in \psi(\kappa, \theta, \beta+\theta\cdot\Delta D, D+\Delta D; s) \text{ for any } \Delta D \in R^J_+.$$

Proof: By Condition 4.1, (2) of Definition 3.1 can be rewritten in the following fashion. For almost all $a \in A$,

$$\begin{aligned} & p\cdot(\hat{c}(a)+\hat{\kappa}(a))+(q-\Delta D)\cdot\hat{\theta}(a)+\hat{\beta}(a)+\hat{\theta}(a)\cdot\Delta D \\ & \quad \leqq p\cdot\kappa(a)+(q-\Delta D)\cdot\theta(a)+(1+r)(\beta(a)+\theta(a)\cdot\Delta D)+\sum_{j=1}^{J}\theta_j(a)(p\cdot y_j-r(D_j+\Delta D_j)) \end{aligned}$$

and

$$\begin{aligned} & V_a(\kappa(a), \theta(a), \beta(a)+\theta(a)\cdot\Delta D; \kappa, \theta, \beta+\theta\cdot\Delta D, D+\Delta D; s) \\ & = V_a(\kappa(a), \theta(a), \beta(a); \kappa, \theta, \beta, D; s) \\ & = U_a(\hat{c}(a))+\delta\int_S V_a(\hat{\kappa}(a), \hat{\theta}(a), \hat{\beta}(a); \hat{\kappa}, \hat{\theta}, \hat{\beta}, \hat{D}; \cdot)d\mu_s \\ & \geqq U_a(x)+\delta\int_S V_a(z, e, b-e\cdot\Delta D; \hat{\kappa}, \hat{\theta}, \hat{\beta}, \hat{D}; \cdot)d\mu_s \\ & = U_a(x)+\delta\int_S V_a(z, e, b; \hat{\kappa}, \hat{\theta}, \hat{\beta}+\hat{\theta}\cdot\Delta D, \hat{D}+\Delta D; \cdot)d\mu_s \end{aligned}$$

for all $(x, z, e, b) \in R^n_+ \times R^n_+ \times R^J_+ \times R$ with

$$\begin{aligned} & p\cdot(x+z)+(q-\Delta D)\cdot e+b \\ & \quad \leqq p\cdot\kappa(a)+(q-\Delta D)\cdot\theta(a)+(1+r)(\beta(a)+\theta(a)\cdot\Delta D))+\sum_{j=1}^{J}\theta_j(a)(p\cdot y_j-r(D_j+\Delta D)). \end{aligned}$$

Moreover, we obviously have

$$\int_A(\hat{\beta}+\theta\cdot\Delta D)\, d\nu=\sum_{j=1}^{J}(\hat{D}_j+\Delta D_j).$$

This implies that $(p, q-\Delta D, r) \in \psi(\kappa, \theta, \beta+\theta\cdot\Delta D, D+\Delta D; s)$. Q.E.D.

Let $\{\psi, V\}$ be an equilibrium of the economy. If $(p, q, r) \in \psi(k, \theta, \beta, D; s)$, the values of firms are defined by $q+D$. Therefore, Proposition 4.1 implies that the prices of firms' equities become $q-\Delta D$ if the amounts of firms' debts change by ΔD. After D changes, the values of firms are $(q-\Delta D)+(D+\Delta D)=q+D$. Thus, the values of firms are unchanged and independent of the amounts of firms' debts.

In addition, price p of commodities and interest rate r remain constant. Moreover, since $V_a(\kappa(a), \theta(a), \beta(a)+\theta(a)\cdot\Delta D; \kappa, \theta, \beta+\theta\cdot\Delta D, D+\Delta D; s)=V_a(\kappa(a), \theta(a), \beta(a); \kappa, \theta, \beta, D; s)$ for each $a \in A$, all consumers can attain the same level of expected utility after D changes. Hence, Proposition 4.1 implies that the equilibrium of the economy is not affected by change of D, which is a theorem originally proved by Modigliani and Miller (1958) and extended to the framework of general equilibrium by Stiglitz (1989).

V. *An Aggregate Economy*

In this section we consider a simplified economy where there are many, but identical consumers and prove the existence of an equilibrium for the economy. In what follows, since we assume that the consumers in the economy are all identical, we have only to consider the behavior of a representative consumer. Such an aggregate model of the economy is useful particularly for macroeconomic analyses.

The utility functions of consumers are denoted by a mapping $\mathbf{U}: A \to \mathscr{U}$, which is an element of set $\mathscr{U}^A$. We assume that the utility functions of all consumers are the same, and that mapping $\mathbf{U}$ is constant, i.e., for some $u \in \mathscr{U}$, $\mathbf{U}(a) = u$ for all $a \in A$. Therefore, we can regard $\mathscr{U}^A$ as $\mathscr{U}$. Thus, in this section we assume that $S = \mathscr{U} \times \mathscr{Y}^J$, and Assumption 2.1 holds for set S in this case.

Moreover, we assume that consumers are all in the same situation, and that their holdings of commodities, equities, and bonds are the same. The amounts of commodities held by consumers are described by function $\kappa: A \to R^n_+$ which is an element of set $\mathscr{L}^n_{\infty+}$. When consumers have the same amounts of commodities, then function κ is constant, i.e., for some $k \in R^n_+$, $\kappa(a) = k$ for all $a \in A$. Therefore, we can regard $\mathscr{L}^n_{\infty+}$ as R^n_+.

Equity holdings by consumers are denoted by a function $\theta: A \to R^J_+$ which is an element of set $\mathscr{L}^J_{1+}$. Since the total equity of each firm is assumed to be unity, when all consumers have the same amounts of equities, $\theta(a) = \underline{\mathbf{1}}$ for all $a \in A$. Thus, function θ can be regarded as vector $\underline{\mathbf{1}} \in R^J_+$, and we can omit showing it.

The numbers of bonds held by consumers are described by a function $\beta: A \to R$, which is an element of set $\mathscr{L}_1$. When all consumers have the same amounts of bonds, then function β is constant, i.e., for some $B \in R$, $\beta(a) = B$ for all $a \in A$. Therefore, we can regard $\mathscr{L}_1$ as R.

By the above simplification, a macro-state $(\kappa, \theta, \beta, D; \mathbf{U}, \mathbf{Y})$ of the economy can be depicted in the aggregate economy by an element $(k, B, D; u, \mathbf{Y}) \in R^n_+ \times R \times R^J \times \mathscr{U} \times \mathscr{Y}^J$. By this procedure, we can define a price correspondence and a value function for a representative consumer in the following fashion.

Define price correspondence ψ by

$$(k, B, D; s) \in R^n_+ \times R \times R^J \times S \to \psi(k, B, D; s) \subset R^n \times R^J \times R,$$

where $s = (u, \mathbf{Y})$. Also, define a value function V by

$$(z, e, b; k, B, D; s) \in R^n_+ \times R^J_+ \times R \times R^n_+ \times R \times R^J \times S \to V(z, e, b; k, B, D; s) \in R.$$

We can now define an equilibrium for the aggregate economy. Definition 3.1 is reduced to the following.

Definition 5. 1: Pair $\{\psi, V\}$ of a price correspondence and a value function is called *an equilibrium for the aggregate economy*, if $\{\psi, V\}$ has the following property:

Let $(k, B, D) \in R^n_+ \times R \times R^J$, $s = (u, \mathbf{Y}) \in S$, and $(p, q, r) \in \psi(k, B, D; s)$ with $B = \sum_{j=1}^{J} D_j$. Then, there exist $\hat{x} \in R^n_+$, $(\hat{k}, \hat{B}, \hat{D}) \in R^n_+ \times R \times R^J$, and $\hat{y}_j \in Y_j (j = 1, \cdots, J)$, which satisfy the following conditions:

(1) Firms maximize their profits, i.e., for each $j=1,\cdots, J$,

$$p\cdot\hat{y}_j\geqq p\cdot y \text{ for all } y\in Y_j.$$

(2) Consumers maximize their expected utilities subject to their budget constraints, i.e.,

$$p\cdot(\hat{x}+\hat{k})+q\cdot\underline{\mathbf{1}}+\hat{B}\leqq p\cdot k+q\cdot\underline{\mathbf{1}}+(1+r)B+\sum_{j=1}^{J}(p\cdot\hat{y}_j-rD_j),$$

and

$$V(k,\underline{\mathbf{1}},B;k,B,D;s)=u(\hat{x})+\delta\int_S V(\hat{k},\underline{\mathbf{1}},\hat{B};\hat{k},\hat{B},\hat{D};\cdot)d\mu_s$$

$$\geqq u(x)+\delta\int_S V(z,e,b;\hat{k},\hat{B},\hat{D};\cdot)d\mu_s$$

for all $(x, z, e, b)\in R^n_+\times R^n_+\times R^J_+\times R$ with

$$p\cdot(x+z)+q\cdot e+b\leqq p\cdot k+q\cdot\underline{\mathbf{1}}+(1+r)B+\sum_{j=1}^{J}(p\cdot\hat{y}_j-rD_j).$$

(3) All markets are in equilibrium, i.e.,

$$\hat{x}+\hat{k}=k+\sum_{j=1}^{J}\hat{y}_j \text{ and } \hat{B}=\sum_{j=1}^{J}\hat{D}_j.$$

From (2) in the above definition, since $B=\sum_{j=1}^{J}D_j$, we can easily see that interest rate r is indeterminate and can be any number. The indeterminacy of interest rate is a peculiar phenomenon of the aggregate economy, where each agent virtually borrows from himself.

In what follows, we state the assumptions that ensure the existence of an equilibrium for the aggregate economy. For set $\mathscr{U}$ of utility functions and family $\mathscr{Y}$ of production sets, we assume the following.

Assumption 5.1: Let $u\in\mathscr{U}$. Then, u has the following properties.
(1) u is a continuous and concave function.
(2) u is a monotone increasing function, i.e., if $x\geq x'$ and $x\neq x'$, then $u(x)>u(x')$.
(3) $u(\underline{\mathbf{0}})=0$.
(4) There exists a number $\varepsilon_0>0$ such that $x\in R^n_+$ implies $|u(x)|\leq\varepsilon_0$.

Assumption 5.2: Let $\mathbf{Y}=(Y_1,\cdots,Y_J)\in\mathscr{Y}^J$. Then, $\mathbf{Y}$ has the following properties.
(1) Y_j is a closed and convex subset of R^n.
(2) $Y_j\cap R^n_+=\{\underline{\mathbf{0}}\}$.
(3) There exists a number $\varepsilon_1>0$ such that $y\in Y_j$ implies $\|y\|\leq\varepsilon_1$.

Under the above assumptions, we have the following theorem on the existence of an equilibrium for the aggregate economy, which includes a condition corresponding to Condition 4.1.

Theorem 5.1: Under Assumptions 5.1 and 5.2, there exists an equilibrium $\{\psi, V\}$ for the aggregate economy that has the following properties.
(1) Value function V is continuous and bounded and $V(z, e, b; k, B, D; s)$ is monotone non-decreasing and concave in (z, e, b).
(2) Let $(k, B, D; s) \in R^n_+ \times R \times R^J \times S$ and $B = \sum_{j=1}^{J} D_j$. Then

$$V(z, e, b + e \cdot \Delta D; k, B + \underline{\mathbf{1}} \cdot \Delta D, D + \Delta D; s) = V(z, e, b; k, B, D; s)$$

for all $(z, e, b) \in R^n_+ \times R^J_+ \times R$ and $\Delta D \in R^J$.

In the above theorem, value function V satisfies condition (2), which corresponds to Condition 4.1, and the M-M theorem therefore holds in the aggregate model.

VI. *Proof of Theorem 5.1*

In this section, we will prove Theorem 5.1. The proof of the theorem is a modification of the arguments in Takekuma (1990).

Let $\mathscr{C}_*$ be the space of all bounded continuous functions defined on $R^n_+ \times R^J_+ \times S$. For each $W \in \mathscr{C}_*$, define function MW on $R^n_+ \times R^J_+ \times S$ by

$$MW(k, e; s) = \sup \{u(x) + \delta \int_S W(z, e; \cdot) d\mu_s | \; x \in R^n_+, \; z \in R^n_+, \; y_j \in Y_j \; (j = 1, \cdots, J), \; x + z = k + \sum_{j=1}^{J} e_j \, y_j\},$$

where $s = (u, \mathbf{Y}) \in S$ and $\mathbf{Y} = (Y_1, \cdots, Y_J)$.

The following two lemmas are the same as Lemmas 6.1 and 6.2 in Takekuma (1990), and we will omit their proofs.

Lemma 6.1: For any $W \in \mathscr{C}_*$, MW is a function that has the following properties.
(1) $MW \in \mathscr{C}_*$, i.e., MW is a continuous and bounded function.
(2) If $W(z, e; s)$ is monotone non-decreasing and concave in (z, e), then so is $MW(z, e; s)$ in (z, e).
(3) If $W(\underline{\mathbf{0}}, e; s) = 0$ for all $(e; s)$, then $MW(\underline{\mathbf{0}}, e; s) = 0$ for all $(e; s)$.

By (1) of the above lemma, we have a mapping,

$$W \in \mathscr{C}_* \to MW \in \mathscr{C}_*,$$

which is denoted by $M: \mathscr{C}_* \to \mathscr{C}_*$. This mapping has the following property.

Lemma 6.2: There exists a unique function $W_0 \in \mathscr{C}_*$ that has the following properties.
(1) W_0 is a fixed-point of mapping M, i.e., $W_0 = MW_0$.
(2) For each s, $W_0(z, e; s)$ is monotone non-decreasing and concave in (z, e).
(3) $W_0(\underline{\mathbf{0}}, e; s) = 0$ for all e and s.

Let $k \in R^n_+$ and $s=(u, \mathbf{Y}) \in S$ where $\mathbf{Y}=(Y_1, \cdots, Y_J)$. Since $W_0=MW_0$, by Assumptions 5.1 and 5.2, there exist $\hat{x} \in R^n_+$, $\hat{k} \in R^n_+$, and $\hat{y}_j \in Y_j$ ($j=1, \cdots, J$) such that

$$W_0(k, \underline{\mathbf{1}}; s)=u(\hat{x})+\delta \int_S W_0(\hat{k}, \underline{\mathbf{1}}; \cdot) d\mu_s \text{ and } \hat{x}+\hat{k}=k+\sum_{j=1}^{J} \hat{y}_j. \tag{6.1}$$

Next, let $(B, D) \in R \times R^J$ with $B=\sum_{j=1}^{J} D_j$ and define subset $\Phi(k, B, D; s)$ of $R^n \times R^J$ by

$$\begin{aligned} \Phi(k, B, D; s)=\{(p, q) \in R \times R^J \mid & \\ & W_0(k, \underline{\mathbf{1}}; s)+B-\underline{\mathbf{1}} \cdot D \geqq u(x)+\delta \int_S (W_0(z, e; \cdot)+b-e \cdot D)\, d\mu_s \\ & \text{for all } (x, z, e, b) \in R^n_+ \times R^n_+ \times R^J_+ \times R \text{ with} \\ & p \cdot (x+z)+q \cdot e+b-\underline{\mathbf{1}} \cdot D \leqq p \cdot k+q \cdot \underline{\mathbf{1}}+\sum_{j=1}^{J} \sup p \cdot Y_j\}. \end{aligned} \tag{6.2}$$

Lemma 6.3: $\Phi(k, B, D; s) \neq \phi$.

Proof: Define two subsets F and G of $R^n \times R^J \times R$ by

$$\begin{aligned} F=\{(w, e, m) \mid\ & w=x+z,\ m=b-e \cdot D, \\ & u(x)+\delta \int_S (W_0(z, e; \cdot)+b-e \cdot D)\, d\mu_s > W_0(k, \underline{\mathbf{1}}; s)+B-\underline{\mathbf{1}} \cdot D\} \end{aligned}$$

$$G=\{(w, e, m) \mid w=k+\sum_{j=1}^{J} y_j \text{ for some } y_j \in Y_j (j=1, \cdots, J),\ e=\underline{\mathbf{1}},\ m=0\}.$$

By (6.1) and Assumption 5.1 (2), we can show that $F \neq \phi$. Also, Assumption 5.2 (2) implies that $G \neq \phi$. The convexity of F and G follows from Lemma 6.2 (2) and Assumptions 5.1 (1) and 5.2 (1).

Suppose that $F \cap G \neq \phi$. Then, there exist x', z', b', and $y'_j \in Y_j (j=1, \cdots, J)$ such that

$$u(x')+\delta \int_S (W_0(z', \underline{\mathbf{1}}; \cdot)+b'-\underline{\mathbf{1}} \cdot D)\, d\mu_s > W_0(k, \underline{\mathbf{1}}; s)+B-\underline{\mathbf{1}} \cdot D,$$

$$x'+z'=k+\sum_{j=1}^{J} y'_j, \text{ and } b'-\underline{\mathbf{1}} \cdot D=0.$$

That is, $u(x')+\delta \int_S W_0(z', \underline{\mathbf{1}}; \cdot)\, d\mu_s > W_0(k, \underline{\mathbf{1}}; s)$ and $x'+z'=k+\sum_{j=1}^{J} y'_j$. Since $W_0=MW_0$, we have a contradiction to the definition of mapping W_0. Hence, $F \cap G \neq \phi$.

By a separation theorem, there exists a vector $(p, q) \in R^n \times R^J$ such that vector $(p, q, 1)$ separates sets F and G, that is,

$$p \cdot w+q \cdot e+m \geqq p \cdot (k+\sum_{j=1}^{J} y_j)+q \cdot \underline{\mathbf{1}} \text{ for all } (w, e, m) \in F \text{ and } y_j \in Y_j (j=1, \cdots, J).$$

Since u is monotone increasing and W_0 is monotone non-decreasing, $p>0$, and $q \geqq 0$. Also, the above inequality implies that

$$p\cdot w+q\cdot e+m\geqq p\cdot k+q\cdot \underline{\mathbf{1}}+\sum_{j=1}^{J}\sup p\cdot Y_j \text{ for all } (w, e, m)\in F. \tag{6.3}$$

Suppose that equality in (6.3) holds for some $(w, e, m)\in F$. Then, there exist x', z', e', and b' so that

$$u(x')+\delta\int_S (W_0(z', e'; \cdot)+b'-e'\cdot D)\, d\mu_s > W_0(k, \underline{\mathbf{1}}; s)+B-\underline{\mathbf{1}}\cdot D$$

and

$$p\cdot(x'+z')+q\cdot e'+b'-\underline{\mathbf{1}}\cdot D=p\cdot k+q\cdot\underline{\mathbf{1}}+\sum_{j=1}^{J}\sup p\cdot Y_j.$$

However, by decreasing b' slightly, we have a contradiction to (6.3). Therefore, we have proved that $p\cdot w+q\cdot e+m>p\cdot k+q\cdot\underline{\mathbf{1}}+\sum_{j=1}^{J}\sup p\cdot Y_j$ for all $(w, e, m)\in F$, that is,

$$W_0(k, \underline{\mathbf{1}}; s)+B-\underline{\mathbf{1}}\cdot D\geqq u(x)+\delta\int_S (W_0(z, e; \cdot)+b-e\cdot D)d\mu_s$$

for all $(x, z, e, b)\in R^n_+\times R^n_+\times R^J_+\times R$ with

$$p\cdot(x+z)+q\cdot e+b-\underline{\mathbf{1}}\cdot D\leqq p\cdot k+q\cdot\underline{\mathbf{1}}+\sum_{j=1}^{J}\sup p\cdot Y_j,$$

which implies that $(p, q)\in\Phi(k, B, D; s)$. Q.E.D.

By Lemma 6.3 we have a correspondence,

$$(k, B, D; s)\in R^n_+\times R\times R^J\times S \to \Phi(k, B, D; s)\subset R^n\times R^J,$$

where $B=\sum_{j=1}^{J}D_j$. Thus, we can define correspondence ψ: $R^n_+\times R\times R^J\times S\to R^n\times R^J\times R$ by

$$\psi(k, B, D; s)=\Phi(k, B, D; s)\times R,$$

where $B=\sum_{j=1}^{J}D_j$. Also, let us define function V: $R^n_+\times R^J_+\times R\times R^n_+\times R^J_+\times R\times R^J\times S\to R$ by

$$V(z, e, b; k, B, D; s)=W_0(z, e; s)+b-e\cdot D.$$

Then, obviously, V is continuous and bounded. Also, by Lemma 6.2 and the definition of V, we can easily check that function V has properties (1) and (2) in Theorem 5.1. It remains to be shown that $\{\psi, V\}$ is an equilibrium for the aggregate economy in the sense of Definition 5.1.

Lemma 6.4: If $(p, q, r)\in\psi(k, B, D; s)$, then

$$V(k, \underline{\mathbf{1}}, B; k, B, D; s)=u(\hat{x})+\delta\int_S V(\hat{k}, \underline{\mathbf{1}}, B; \hat{k}, B, D; \cdot)d\mu_s$$

$$\geqq u(x)+\delta\int_S V(z,\ e,\ b;\ \hat{k},\ B,\ D;\ \cdot)\ d\mu_s$$

for all $(x, z, e, b)\in R^n_+\times R^n_+\times R^J_+\times R$ with

$$p\cdot(x+z)+q\cdot e+b\leqq p\cdot k+q\cdot\underline{\mathbf{1}}+(1+r)B+\sum_{j=1}^{J}(\sup p\cdot Y_j-rD_j).$$

Proof: Since $(p, q)\in\Phi(k, B, D; s)$, from (6.2) it follows that

$$W_0(k,\ \underline{\mathbf{1}};\ s)+B-\underline{\mathbf{1}}\cdot D\geqq u(x)+\delta\int_S (W_0(z,\ e;\ \cdot)+b-e\cdot D)\ d\mu_s$$

for all $(x, z, e, b)\in R^n_+\times R^n_+\times R^J_+\times R$ with

$$p\cdot(x+z)+q\cdot e+b-\underline{\mathbf{1}}\cdot D\leqq p\cdot k+q\cdot\underline{\mathbf{1}}+\sum_{j=1}^{J}\sup p\cdot Y_j.$$

Since $B=\underline{\mathbf{1}}\cdot D$, the definition of function V and (6.1) and (6.2) imply that

$$\begin{aligned}V(k,\ \underline{\mathbf{1}},\ B;\ k,\ B,\ D;\ s)&=W_0(k,\ \underline{\mathbf{1}};\ s)+B-\underline{\mathbf{1}}\cdot D\\&=u(\hat{x})+\delta\int_S (W_0(\hat{k},\ \underline{\mathbf{1}};\ \cdot)+B-\underline{\mathbf{1}}\cdot D)\ d\mu_s\\&=u(\hat{x})+\delta\int_S V\ (\hat{k},\ \underline{\mathbf{1}},\ B;\ \hat{k},\ B,\ D;\ \cdot)\ d\mu_s\\&\geqq u(x)+\delta\int_S (W_0(z,\ e;\ \cdot)+b-e\cdot D)\ d\mu_s\\&=u(x)+\delta\int_S V\ (z,\ e,\ b;\ \hat{k},\ B,\ D;\ \cdot)d\mu_s\end{aligned}$$

for all $(x, z, e, b)\in R^n_+\times R^n_+\times R^J_+\times R$ with

$$p\cdot(x+z)+q\cdot e+b\leqq p\cdot k+q\cdot\underline{\mathbf{1}}+(1+r)B+\sum_{j=1}^{J}(\sup p\cdot Y_j-rD_j).\qquad \text{Q.E.D.}$$

Put $\hat{B}=B$ and $\hat{D}=D$. Then, by (6.1) we have

$$p\cdot(\hat{x}+\hat{z})+q\cdot\underline{\mathbf{1}}+\hat{B}-\underline{\mathbf{1}}\cdot D=p\cdot(k+\sum_{j=1}^{J}\hat{y}_j)+q\cdot\underline{\mathbf{1}}\leqq p\cdot k+q\cdot\underline{\mathbf{1}}+\sum_{j=1}^{J}\sup p\cdot Y_j.$$

Suppose that strict inequality holds in the above. Then, by increasing $\hat{x}$ slightly, Assumption 5.1 (2) immediately implies a contradiction to Lemma 6.4. Therefore, equality holds in the above, and we have

$$p\cdot\sum_{j=1}^{J}\hat{y}_j=\sum_{j=1}^{J}\sup p\cdot Y_j,$$

which implies (1) of Definition 5.1. Thus, Lemma 6.4 implies (2) of Definition 5.1. (3) of Definition 5.1 also follows from (6.1). This completes the proof of Theorem 5.1.

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