Centroaffine minimal surfaces whose centroaffine curvature and Pick function are constants

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Abstract

We classify centroaffine minimal surfaces whose centroaffine curvature and Pick function are constants locally, which also gives classification of centroaffine minimal surfaces whose centroaffine curvature and generalized Pick function are constants locally.

Key words: Centroaffine minimal surfaces; Pick function
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1. Introduction

The notion of centroaffine minimal hypersurfaces was introduced by Wang [13] as extremals for the area integral of the centroaffine metric. Such a class of hypersurfaces is a natural generalization of proper affine hypersurfaces centered at the origin. In the following, we shall discuss the case the ambient space is $\mathbf{R}^3$ and consider surfaces in $\mathbf{R}^3$ locally. In this case, it is also worthwhile to point out that centroaffine minimal surfaces are considered as an interesting class of surfaces from the viewpoint of not only centroaffine differential geometry or variational problems but also integrable systems [10]. Fundamental examples of centroaffine minimal surfaces are centroaffine surfaces with vanishing centroaffine Tchebychev operator, which were classified by Liu and Wang [7]:

Theorem 1.1. *(Liu and Wang [7]). Let $f : M \to \mathbf{R}^3$ be an immersion from a 2-dimensional domain which is a centroaffine surface with vanishing centroaffine Tchebychev operator and put $f = (X, Y, Z)$. Then up to centroaffine congruence, f is one of the following:*

Example 1*: A piece of a quadric.*
Example 2*: A proper affine sphere centered at the origin.*
Example 3*: $X^\alpha Y^\beta Z^\gamma = 1$, where $\alpha, \beta, \gamma \in \mathbf{R}$ such that $\alpha\beta\gamma(\alpha + \beta + \gamma) \neq 0$.*
Example 4*: $\left\{\exp\left(-\alpha \arctan \dfrac{X}{Y}\right)\right\} (X^2+Y^2)^\beta Z^\gamma = 1$, where $\alpha, \beta, \gamma \in \mathbf{R}$ such that $\gamma(2\beta+\gamma)(\alpha^2+\beta^2) \neq 0$.*
Example 5*: $Z = -X(\alpha \log X + \beta \log Y)$, where $\alpha, \beta, \gamma \in \mathbf{R}$ such that $\beta(\alpha + \beta) \neq 0$.*
Example 6*: $Z = \pm X \log X + \dfrac{Y^2}{X}$.*

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Example 7*:* $f(u,v) = (e^u, A_1(u)e^v, A_2(u)e^v)$*, where* A_1 *and* A_2 *are linearly independent solutions of the linear ordinary differential equation* $A'' - A' - a(u)A = 0$ *for any function* $a = a(u)$*. In this case,* f *is indefinite.*

Remark 1.1. *The classification result in [7] and [2, Proposition* 2.2*] dropped Example* 6*. See also [4].*

Remark 1.2. *If a centroaffine surface with vanishing centroaffine Tchebychev operator has constant centroaffine curvature* κ*, then it is one of Example* 1*, Example* 2 *with* $\kappa = 0, 1$ *or Examples* 3, 4, 5, 6 *and* 7*, and the Pick function of the surface is constant. In the case of Examples* 3, 4, 5, 6 *and* 7*, we have* $\kappa = 0$*. Example* 2 *with* $\kappa = 0$ *and* $\kappa = 1$ *were classified by Magid and Ryan [8] and Simon [11] respectively.*

In this paper, we classify centroaffine minimal surfaces whose centroaffine curvature and Pick function are constants locally. Our main theorem is the following:

Theorem 1.2. *Let* $f : M \to \mathbf{R}^3$ *be an immersion from a* 2*-dimensional domain which is a centroaffine minimal surface whose centroaffine curvature* κ *and Pick function* J *are constants. Then up to centroaffine congruence,* f *is one of the following:*

(i) Example 1*, Example* 2 *with* $\kappa = 0, 1$ *or Examples* 3, 4, 5, 6 *and* 7*.*
(ii) The surface given by

$$f(u,v) = \left(\sum_{n=0}^{\infty} A_{n,1}(v)u^n, \sum_{n=0}^{\infty} A_{n,2}(v)u^n, \sum_{n=0}^{\infty} A_{n,3}(v)u^n \right), \tag{1}$$

where the coordinates (u,v) *are defined around* $(0, v_0)$ *such that* $v_0 \neq 0$*, and* $A_{0,1}, A_{0,2}$ *and* $A_{0,3}$ *are linearly independent solutions of the linear ordinary differential equation:*

$$vA''' + A'' - A = 0, \tag{2}$$

and

$$A_{n+1,i} = \frac{v}{n+1} A''_{n,i} \quad (i = 1, 2, 3). \tag{3}$$

In this case, f *is indefinite, and* $\kappa = 0$ *and* $J = -1$*.*
(iii) A ruled surface:

$$f(u,v) = A' + vA \tag{4}$$

for any $\mathbf{R}^3$*-valued function* $A = A(u)$ *such that*

$$\det \begin{pmatrix} A \\ A' \\ A'' \end{pmatrix} \neq 0. \tag{5}$$

In this case, f *is indefinite, and* $\kappa = 1$ *and* $J = 0$*.*

The surface given by (4) such that the left hand side of (5) is a constant, coincides with Example 2 with $\kappa = 1$. The surfaces given by (1) and (4) can also be found in the author's recent paper [3]. See also [5] for recent results about centroaffine ruled surfaces. Our result also gives classification of centroaffine minimal surfaces whose centroaffine curvature and generalized Pick function are constants locally, which generalize one of the results due to Liu and

Jung [6] about indefinite centroaffine minimal surfaces with constant centroaffine curvature and vanishing generalized Pick function. In [6, Theorem 3.3], they showed that the centroaffine curvature of indefinite centroaffine minimal surfaces with vanishing generalized Pick function is equal to 0 or 1. However, the surfaces for the non-flat case were not given explicitely. Hence the surface given by (4) also answers the classification problem for the non-flat case [6, Remark 3.3].

2. Preliminaries

A centroaffine surface f in the real affine 3-space $\mathbf{R}^3$ is given locally by a smooth immersion from a 2-dimensional domain to $\mathbf{R}^3$ such that the position vector f is transversal to the tangent plane at each point. In the following, we assume that f is nondegenerate, i.e., the centroaffine metric h is nondegenerate. For simplicity, we consider the case that h is indefinite. Then as can be seen in [10, Theorem 1], we can take asymptotic line coordinates (u, v) and the Gauss equations for f are given by

$$f_{uu} = \left(\frac{\varphi_u}{\varphi} + \rho_u\right) f_u + \frac{a}{\varphi} f_v, \ f_{uv} = -\varphi f + \rho_v f_u + \rho_u f_v, \ f_{vv} = \left(\frac{\varphi_v}{\varphi} + \rho_v\right) f_v + \frac{b}{\varphi} f_u, \tag{6}$$

where $\varphi = h(\partial_u, \partial_v)$,

$$a = \varphi \det \begin{pmatrix} f \\ f_u \\ f_{uu} \end{pmatrix} \Bigg/ \det \begin{pmatrix} f \\ f_u \\ f_v \end{pmatrix}, \ b = \varphi \det \begin{pmatrix} f \\ f_v \\ f_{vv} \end{pmatrix} \Bigg/ \det \begin{pmatrix} f \\ f_v \\ f_u \end{pmatrix}. \tag{7}$$

It is obvious to see that the cubic differentials adu^3 and bdv^3 are centroaffine invariants. On the other hand, the function ρ is an equicentroaffine invariant. Indeed, it is known that $\pm e^\rho$ is equal to the equiaffine support function from the origin. In particular, centroaffine transformations preserve the property that ρ is a constant, which was discovered by Tzitzéica [12]. Moreover, ρ is a constant if and only if f is a proper affine sphere centered at the origin. See [9] for basic facts about affine hyperspheres. In particular, flat affine spheres were classified by Magid and Ryan [8]. Affine spheres with constant curvature metric were classified by Simon [11]. It is easy to see that the integrability conditions for (6) are given by

$$(\log|\varphi|)_{uv} = -\varphi - \frac{ab}{\varphi^2} + \rho_u \rho_v, \ a_v + \rho_u \varphi_u = \rho_{uu} \varphi, \ b_u + \rho_v \varphi_v = \rho_{vv} \varphi. \tag{8}$$

The surface f is called to be centroaffine minimal if it extremizes the area integral of h, which is known to be equivalent to the condition that the trace of the centroaffine Tchebychev operator vanishes. Let ∇ be the connection induced by the immersion f and $\tilde{\nabla}$ the Levi-Civita connection of h. It is easy to see that the Christoffel symbols $\tilde{\Gamma}^k_{ij}$ $(i, j, k = 1, 2)$ for $\tilde{\nabla}$ with respect to (u, v) vanish except

$$\tilde{\Gamma}^1_{11} = \frac{\varphi_u}{\varphi}, \ \tilde{\Gamma}^2_{22} = \frac{\varphi_v}{\varphi}. \tag{9}$$

We denote $\nabla - \tilde{\nabla}$ by C, which defines a $(1, 2)$-tensor field. From (6) and (9), it is obvious to see that

$$C(\partial_u, \partial_u) = \rho_u \partial_u + \frac{a}{\varphi} \partial_v, \ C(\partial_u, \partial_v) = \rho_v \partial_u + \rho_u \partial_v, \ C(\partial_v, \partial_v) = \frac{b}{\varphi} \partial_u + \rho_v \partial_v. \tag{10}$$

Then the centroaffine Tchebychev vector field T is computed as

$$T = \frac{1}{2}\mathrm{tr}_h C = \frac{\rho_v}{\varphi}\partial_u + \frac{\rho_u}{\varphi}\partial_v = \mathrm{grad}_h \rho. \tag{11}$$

From the second and the third equations of (8) and (9), the centroaffine Tchebychev operator $\tilde{\nabla}T$ is computed as

$$\tilde{\nabla}T(\partial_u) = \frac{\rho_{uv}}{\varphi}\partial_u + \frac{a_v}{\varphi^2}\partial_v,\ \tilde{\nabla}T(\partial_v) = \frac{b_u}{\varphi^2}\partial_u + \frac{\rho_{uv}}{\varphi}\partial_v. \tag{12}$$

Hence f is centroaffine minimal if and only if $\rho_{uv} = 0$. Centroaffine surfaces such that $\tilde{\nabla}T$ is proportional to the identity are called to be centroaffine Tchebychev. In particular, f is centroaffine minimal and centroaffine Tchebychev if and only if $\tilde{\nabla}T = 0$, i.e., $\rho_{uv} = a_v = b_u = 0$. Such surfaces were classified by Liu and Wang [7] as in Theorem 1.1.

The centroaffine curvature κ is given by

$$\kappa = -\frac{(\log|\varphi|)_{uv}}{\varphi}. \tag{13}$$

Centroaffine Tchebychev surfaces with constant κ were classified by Binder [1]. In a previous paper [2], the author classified centroaffine minimal surfaces with constant κ, $a = b$ and $\rho = c_1 u + c_2 v + c_3$ for $c_1, c_2, c_3 \in \mathbf{R}$.

The Pick function J is computed as

$$J = \frac{1}{2}\|C\|^2 = \frac{3\rho_u\rho_v}{\varphi} + \frac{ab}{\varphi^3}. \tag{14}$$

We denote the traceless part of C by $\tilde{C}$, which is defined by

$$\tilde{C}(X,Y) = C(X,Y) - \frac{1}{2}(h(T,X)Y + h(T,Y)X + h(X,Y)T) \tag{15}$$

for vector fields X and Y on f. From (10), (11) and (15), we have

$$\tilde{C}(\partial_u,\partial_u) = \frac{a}{\varphi}\partial_v,\ \tilde{C}(\partial_u,\partial_v) = 0,\ \tilde{C}(\partial_v,\partial_v) = \frac{b}{\varphi}\partial_u. \tag{16}$$

Then from (11), (14) and (16), the generalized Pick function $\tilde{J}$ in [6] is computed as

$$\tilde{J} = \frac{1}{2}\|\tilde{C}\|^2 = \frac{ab}{\varphi^3} = J - \frac{3\rho_u\rho_v}{\varphi} = J - \frac{3}{2}\|T\|^2. \tag{17}$$

3. Indefinite case

Assume that the indefinite centroaffine surface f is centroaffine minimal. Since $\rho_{uv} = 0$, changing the coordinates, if necessary, we may assume that $\rho = c_1 u + c_2 v + c_3$ for any $c_1, c_2, c_3 \in \mathbf{R}$, which reduces (6) to the following:

$$f_{uu} = \left(\frac{\varphi_u}{\varphi} + c_1\right)f_u + \frac{a}{\varphi}f_v,\ f_{uv} = -\varphi f + c_2 f_u + c_1 f_v,\ f_{vv} = \left(\frac{\varphi_v}{\varphi} + c_2\right)f_v + \frac{b}{\varphi}f_u. \tag{18}$$

Moreover, from (14) and (17), we have

$$J = \frac{3c_1c_2}{\varphi} + \frac{ab}{\varphi^3} = \frac{3c_1c_2}{\varphi} + \tilde{J}. \tag{19}$$

Hence from (8), (13) and (19), we have

$$(J + 1 - \kappa)\varphi = 4c_1c_2,\ a_v + c_1\varphi_u = 0,\ b_u + c_2\varphi_v = 0, \tag{20}$$

or the first equation of (20) is equivalent to

$$(\tilde{J} + 1 - \kappa)\varphi = c_1c_2. \tag{21}$$

The following is a key lemma to our classification result.

Lemma 3.1. *Let f be an indefinite centroaffine minimal surface. If both κ and J are constants, then changing the coordinates u and v, if necessary, we have one of the following:*

(i) $\kappa = 0$, $J = 3$, $\varphi = c_1c_2 \neq 0$, $a = a(u)$ *and* $b = 0$.
(ii) $\kappa = 0$, $J = \dfrac{4c_1c_2}{\varphi} - 1 \neq 3, -1$, *and* a, b *and* φ *are non-zero constants.*
(iii) $\kappa = 0$, $J = -1$, $c_2 = 0$ *and* $b = b(v) \neq 0$.
(iv) $\kappa = 1$, $J = 0$, $c_2 = 0$ *and* $b = 0$.

Proof. In case of $J + 1 \neq \kappa$, since $\varphi \neq 0$, from the first equation of (20), we have $c_1c_2 \neq 0$ and φ is a non-zero constant. Then from (13) we have $\kappa = 0$. Moreover, from the second and the third equations of (20), we have $a = a(u)$ and $b = b(v)$. Since $\kappa = 0$ and $c_1c_2 \neq 0$, from the first equation of (20), we have

$$J = \frac{4c_1c_2}{\varphi} - 1 \neq -1. \tag{22}$$

From (19) and (22), $ab = 0$ if and only if $\varphi = c_1c_2$. If $ab \neq 0$, from (19) a and b are non-zero constants. Hence we have (i) and (ii).

In case of $J + 1 = \kappa = 0$, from the first equation of (20), we have $c_1 = 0$ or $c_2 = 0$. If $c_2 = 0$, we have $b = b(v)$ as above. Moreover, since $J = -1$, from (19) we have $b \neq 0$. Hence we have (iii).

In case of $J + 1 = \kappa \neq 0$, we have $c_1 = 0$ or $c_2 = 0$ as above. If $c_2 = 0$, we have $b = b(v)$ as above. Note that (13) is the Liouville equation, whose solution is given by

$$\varphi = -\frac{2}{\kappa}\frac{p_uq_v}{(p(u) + q(v))^2} \tag{23}$$

for any functions $p = p(u)$, $q = q(v)$ such that $p_u, q_v \neq 0$. Then the second equation of (20) becomes

$$a_v - \frac{2c_1}{\kappa}\left\{\frac{p_{uu}q_v}{(p+q)^2} - \frac{2p_u^2q_v}{(p+q)^3}\right\} = 0, \tag{24}$$

which can be integrated as

$$a = \frac{2c_1}{\kappa}\left\{-\frac{p_{uu}}{p+q} + \frac{p_u^2}{(p+q)^2}\right\} + r(u) \tag{25}$$

for any function $r = r(u)$. If $b(v) \neq 0$, since $c_2 = 0$, from (19) and (25), we have $J = c_1 = r = 0$, so that $a = 0$. Hence we have (iv). □

Proposition 3.1. *Depending on the case of (i)~(iv) in Lemma 3.1, we have* $\tilde{J} = 0, \dfrac{c_1 c_2}{\varphi} - 1, -1$ *and* 0 *respectively.*

Proof. It is obvious from (19). □

Proposition 3.2. *Let f be an indefinite centroaffine minimal surface with constant κ. Then J is a constant if and only if $\tilde{J}$ is a constant. In particular, Theorem 1.2 also gives classification of centroaffine minimal surfaces with constant κ and $\tilde{J}$.*

Proof. This is a direct consequence of the centroaffine Theorema egregium. □

In the case of (i) in Lemma 3.1, up to centroaffine congruence, f is Example 7, which includes a piece of the hyperbolic paraboloid.

In the case of (ii) in Lemma 3.1, up to centroaffine congruence, f is one of Examples 3, 4, 5 or 6.

In the case of (iii) in Lemma 3.1, it is better to come back to (6) as follows.

Theorem 3.1. *In the case of (iii) in Lemma 3.1, f is one of Examples* 3, 4 *or* 5*, which include Example* 2 *with $\kappa = 0$, or the surface given by (1).*

Proof. Since $\kappa = 0$, from (13) we have $\varphi = p(u)q(v)$ for any functions $p = p(u)$, $q = q(v)$ such that $pq \neq 0$. Changing the coordinates, if necessary, we may assume that $\varphi = -1$. Since $c_2 = 0$, we have $\rho = \rho(u)$. Since $J = -1$, $c_2 = 0$ and $b = b(v)$, from (19) we have

$$ab = a(v)b(v) = 1. \tag{26}$$

Then from the second equation of (8), we have

$$-\frac{b_v}{b^2} = -\rho_{uu}. \tag{27}$$

Hence we have

$$\rho = \frac{1}{2}\hat{c}_1 u^2 + \hat{c}_2 u + \hat{c}_3,\ b = -\frac{1}{\hat{c}_1 v + \hat{c}_4} \tag{28}$$

for any $\hat{c}_1, \hat{c}_2, \hat{c}_3, \hat{c}_4 \in \mathbf{R}$. Then from (26) we have

$$a = -(\hat{c}_1 v + \hat{c}_4). \tag{29}$$

If $\hat{c}_1 = 0$, then f is one of Examples 3, 4 or 5, which include Example 2 with $\kappa = 0$.

If $\hat{c}_1 \neq 0$, changing the coordinates, if necessary, we may assume that the coordinates (u, v) are defined around $(0, v_0)$ such that $v_0 \neq 0$, and $\hat{c}_1 = 1$ and $\hat{c}_2 = \hat{c}_4 = 0$. Then (6) becomes

$$f_{uu} = uf_u + vf_v,\ f_{uv} = f + uf_v,\ f_{vv} = \frac{1}{v}f_u. \tag{30}$$

Moreover, if we put

$$f = \sum_{n=0}^{\infty} A_n(v)u^n \tag{31}$$

for some $\mathbf{R}^3$-valued functions $A_n = A_n(v)$ $(n = 0, 1, 2, \dots)$, a direct computation using (30) and (31) shows that

$$(n+1)(n+2)A_{n+2} = nA_n + vA_n', \tag{32}$$

$$A_1' = A_0,\ (n+2)A_{n+2}' = A_{n+1} + A_n', \tag{33}$$

$$A_n'' = \frac{n+1}{v} A_{n+1}, \tag{34}$$

where $n = 0, 1, 2, \ldots$. Note that the second equation of (33) can also be deduced from (32) and (34). Combining the first equation of (33) and (34) with $n = 0$, we have

$$vA_0''' + A_0'' - A_0 = 0. \tag{35}$$

Moreover, if we define A_n $(n = 0, 1, 2, \ldots)$ by (34) and (35), it is straightforward to see by induction on n that (32) is satisfied. Hence f is the surface given by (1). □

In order to complete the proof of Theorem 1.2 in the case that the surface is indefinite, it remains to show the following:

Theorem 3.2. *In the case of (iv) in Lemma 3.1, f is the surface given by (4).*

Proof. The equations (23) and (25) with $\kappa = 1$ become

$$\varphi = -\frac{2p_u q_v}{(p+q)^2},\ a = 2c_1 \left\{ -\frac{p_{uu}}{p+q} + \frac{p_u^2}{(p+q)^2} \right\} + r(u). \tag{36}$$

If $c_1 = 0$, then f is Example 2 with $\kappa = 1$, which includes a piece of the hyperboloid of one sheet.

If $c_1 \neq 0$, note that the centroaffine Tchebychev operator $\tilde{\nabla} T$ is not semisimple. Then by [3, Theorem 4.6], f is the surface given by (4). □

4. Definite case

In the following, we consider the case that f is a definite centroaffine surface whose centroaffine metric is h. We can carry out all computations in a similar manner to the indefinite case. The Gauss equations for f are given by

$$f_{zz} = \left(\frac{\varphi_z}{\varphi} + \rho_z \right) f_z + \frac{a}{\varphi} f_{\bar{z}},\ f_{z\bar{z}} = -\varphi f + \rho_{\bar{z}} f_z + \rho_z f_{\bar{z}} \tag{37}$$

for a holomorphic coordinate z, where $\varphi = h(\partial_z, \partial_{\bar{z}})$,

$$a = \varphi \det \begin{pmatrix} f \\ f_z \\ f_{zz} \end{pmatrix} \Bigg/ \det \begin{pmatrix} f \\ f_z \\ f_{\bar{z}} \end{pmatrix}. \tag{38}$$

Similar to the indefinite case, the cubic differential adz^3 is a centroaffine invariant, while the function ρ is an equicentroaffine invariant. The integrability conditions for (37) are given by

$$(\log|\varphi|)_{z\bar{z}} = -\varphi - \frac{|a|^2}{\varphi^2} + |\rho_z|^2,\ a_{\bar{z}} + \rho_z \varphi_z = \rho_{zz} \varphi. \tag{39}$$

Assume that f is centroaffine minimal, i.e., $\rho_{z\bar{z}} = 0$. Then changing the coordinate z, if necessary, we may assume that $\rho = c_1 z + \bar{c}_1 \bar{z} + c_2$ for any $c_1 \in \mathbf{C}$ and $c_2 \in \mathbf{R}$, which reduces (37) to the following:

$$f_{zz} = \left(\frac{\varphi_z}{\varphi} + c_1 \right) f_z + \frac{a}{\varphi} f_{\bar{z}},\ f_{z\bar{z}} = -\varphi f + \bar{c}_1 f_z + c_1 f_{\bar{z}}. \tag{40}$$

Note that the centroaffine curvature κ, the Pick function J and the generalized Pick function $\tilde{J}$ become

$$\kappa = -\frac{(\log|\varphi|)_{z\bar{z}}}{\varphi}, \ J = \frac{3|c_1|^2}{\varphi} + \frac{|a|^2}{\varphi^3}, \ \tilde{J} = \frac{|a|^2}{\varphi^3}. \tag{41}$$

Then (39) becomes

$$(J + 1 - \kappa)\varphi = 4|c_1|^2, \ a_{\bar{z}} + c_1\varphi_z = 0, \tag{42}$$

or the first equation of (42) is equivalent to

$$(\tilde{J} + 1 - \kappa)\varphi = |c_1|^2. \tag{43}$$

Lemma 4.1. *Let f be a definite centroaffine minimal surface. If both κ and J are constants, then we have one of the following:*

(i) $\kappa = 0$, $J = 3$, $\varphi = |c_1|^2 > 0$ and $a = 0$.
(ii) $\kappa = 0$, $J = \dfrac{4|c_1|^2}{\varphi} - 1 \neq 3, -1$, and a and φ are non-zero constants.
(iii) $\kappa = 0$, $J = -1$, $c_1 = 0$ and $a = a(z) \neq 0$.
(iv) $\kappa = 1$, $J = 0$, $c_1 = 0$ and $a = 0$.

Depending on the case of $(i) \sim (iv)$, we have $\tilde{J} = 0, \dfrac{|c_1|^2}{\varphi} - 1, -1$ and 0 respectively.

Proof. We can carry out a similar computation to the proof of Lemma 3.1 except the case that $J + 1 = \kappa \neq 0$ and obtain (i), (ii) and (iii).

In case of $J + 1 = \kappa \neq 0$, from the first equation of (42), we have $c_1 = 0$. Then from the second equation of (42), we have $a = a(z)$. Since $c_1 = 0$, changing the coordinate, if necessary, we may assume that $a = 0, 1$. Then from the first equation of (39) and the first equation of (41), we have

$$(\kappa - 1)\varphi^3 = 0, 1 \tag{44}$$

Note that φ is not a constant since $\kappa \neq 0$. Hence we have $\kappa = 1$ and $a = 0$, so that $J = 0$. Therefore we have (iv). □

Similar to Proposition 3.2, we have the following:

Proposition 4.1. *Let f be a definite centroaffine minimal surface with constant κ. Then J is a constant if and only if $\tilde{J}$ is a constant.*

Combining the following proposition with the result in §3, we complete the proof of Theorem 1.2.

Proposition 4.2. *Depending on the case of $(i) \sim (iv)$ in Lemma 4.1, up to centroaffine congruence, we have*

(i) f is a piece of the elliptic paraboloid,
(ii) f is one of Examples 3, 4, 5 or 6,
(iii) f is Example 2 with $\kappa = 0$,
(iv) f is a piece of the ellipsoid or the hyperboloid of two sheets,

respectively.

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