Centroaffine minimal surfaces with non-semisimple centroaffine Tchebychev operator

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Dedicated to the memory of Professor Katsumi Nomizu

Abstract. We study centroaffine minimal surfaces with non-semisimple centroaffine Tchebychev operator and classify such surfaces with constant centroaffine curvature. We also study the center map of such surfaces and show that it becomes a centroaffine surface if and only if the centroaffine curvature is not equal to 1.

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1. Introduction

Centroaffine minimal surfaces are an interesting class of surfaces in centroaffine differential geometry, which was originally defined for centroaffine hypersurfaces by Wang [16] as extremals for the area integral of the centroaffine metric. In particular, such surfaces include proper affine spheres centered at the origin. Centroaffine minimal surfaces are also characterized by centroaffine surfaces whose trace of the centroaffine Tchebychev operator vanishes. Liu and Wang [7] classified centroaffine surfaces with vanishing centroaffine Tchebychev operator, so that they gave fundamental examples of centroaffine minimal surfaces.

On the other hand, there seems to be only a few examples of centroaffine minimal surfaces with non-vanishing centroaffine Tchebychev operator. The first examples of such surfaces were obtained by Vrancken [15], who investigated centroaffine minimal surfaces whose centroaffine Tchebychev vector field is an eigenvector of the centroaffine Tchebychev operator. The author [2] classified centroaffine minimal

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surfaces with constant centroaffine curvature under some condition on cubic differentials and gave other examples. Liu and Jung [6] showed that the centroaffine curvature of indefinite centroaffine minimal surfaces with vanishing generalized Pick function is equal to 0 or 1. Following their result, the author [3] classified centroaffine minimal surfaces whose centroaffine curvature and Pick function are constants, which also gave classification of centroaffine minimal surfaces whose centroaffine curvature and generalized Pick function are constants. In particular, some of the surfaces obtained there gave also other examples.

In this paper, we study centroaffine minimal surfaces with non-semisimple centroaffine Tchebychev operator, called non-semisimple centroaffine minimal surfaces, and classify such surfaces with constant centroaffine curvature. In particular, we show that if the centroaffine curvature of non-semisimple centroaffine minimal surfaces is a constant then it is equal to 0 or 1. Note that the centroaffine Tchebychev operator is symmetric with respect to the centroaffine metric so that the condition that the surface is non-semisimple implies that the centroaffine metric is indefinite. As an application of our classification result, we also study the center map of non-semisimple centroaffine minimal surfaces, which was introduced for affine hypersurfaces by Furuhata and Vrancken [4] as a generalization of the center of proper affine hyperspheres. In particular, we show that the center map of such surfaces becomes a centroaffine surface if and only if the centroaffine curvature is not equal to 1.

In [4] they studied affine hypersurfaces whose center map is centroaffine congruent with the original hypersurfaces, called to be self congruent. In particular, they showed that the center map of a definite centroaffine surface in the Euclidean 3-space which is not a proper affine sphere centered at the origin is self congruent if and only if the centroaffine Tchebychev operator vanishes. In the Appendix we consider the indefinite case and determine indefinite centroaffine surfaces with vanishing centroaffine Tchebychev operator whose center map is self congruent. In contrast to the definite case, we have examples whose center map is not self congruent.

2. Affine surfaces and the center map

Any affine surface f is given locally by a smooth immersion from a 2-dimensional domain to the Euclidean 3-space $\mathbf{R}^3$ equipped with a transversal vector field ξ on f. We denote the standard inner product on $\mathbf{R}^3$ by $\langle \cdot, \cdot \rangle$. Then taking coordinates (x_1, x_2) on f, we have the following Gauss equations:

$$f_{x_i x_j} = \Gamma_{ij}^1 f_{x_1} + \Gamma_{ij}^2 f_{x_2} + \langle f_{x_i x_j}, n \rangle n \quad (i, j = 1, 2), \tag{2.1}$$

where n is the unit normal to f and Γ_{ij}^k $(i, j, k = 1, 2)$ are the Christoffel symbols defined by

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{2} g^{kl} (g_{il,j} + g_{jl,i} - g_{ij,l}), \; g_{ij} = \langle f_{x_i}, f_{x_j} \rangle, \; (g^{ij}) = (g_{ij})^{-1}. \tag{2.2}$$

On the other hand, $f_{x_ix_j}$ $(i,j=1,2)$ are also expressed by a linear combination of f_{x_1}, f_{x_2} and ξ:

$$f_{x_ix_j} = \hat{\Gamma}^1_{ij} f_{x_1} + \hat{\Gamma}^2_{ij} f_{x_2} + h(\partial_{x_i}, \partial_{x_j})\xi \quad (i,j=1,2). \tag{2.3}$$

Hence ξ induces the torsion free affine connection $\hat{\nabla}$, the symmetric $(0,2)$-tensor field h, and the volume form θ defined by

$$\theta(\partial_{x_1}, \partial_{x_2}) = \det \begin{pmatrix} f_{x_1} \\ f_{x_2} \\ \xi \end{pmatrix}. \tag{2.4}$$

Note that from (2.3) we have

$$\langle f_{x_ix_j}, n\rangle = h(\partial_{x_i}, \partial_{x_j})\langle \xi, n\rangle \quad (i,j=1,2) \tag{2.5}$$

and $\langle \xi, n\rangle$ does not vanish since ξ is transversal to f. Hence h becomes a definite or indefinite metric, called the affine metric, if and only if f has positive or negative Euclidean Gaussian curvature, respectively. Moreover, h induces the volume form ω defined by

$$\omega(X_1, X_2) = |\det(h(X_i, X_j))|^{1/2}, \tag{2.6}$$

where X_1 and X_2 are vector fields on f such that $\theta(X_1, X_2) = 1$.

From now on and throughout the rest of the paper, we assume that the surface has negative Euclidean Gaussian curvature, which is equivalent to saying that the centroaffine metric, and hence also the Blaschke metric are indefinite. Then we can take asymptotic line coordinates (u, v) for (x_1, x_2), so that (2.1) becomes

$$f_{uu} = \Gamma^1_{11} f_u + \Gamma^2_{11} f_v, \; f_{uv} = \Gamma^1_{12} f_u + \Gamma^2_{12} f_v + Mn, \; f_{vv} = \Gamma^1_{22} f_u + \Gamma^2_{22} f_v, \tag{2.7}$$

where $M = \langle f_{uv}, n\rangle$. Moreover we have the following Weingarten equations:

$$n_u = \frac{FM}{EG-F^2} f_u - \frac{EM}{EG-F^2} f_v, \; n_v = -\frac{GM}{EG-F^2} f_u + \frac{FM}{EG-F^2} f_v, \tag{2.8}$$

where $E = \langle f_u, f_u\rangle,\ F = \langle f_u, f_v\rangle$ and $G = \langle f_v, f_v\rangle$.

Proposition 2.1. *There exists a unique ξ up to sign such that $\hat{\nabla}\theta = 0$ and $\theta = \omega$. Choosing the orientation of f such that*

$$\det \begin{pmatrix} f_u \\ f_v \\ n \end{pmatrix} = \sqrt{EG - F^2}, \tag{2.9}$$

we have

$$\xi = -\frac{\lambda_v}{M} f_u - \frac{\lambda_u}{M} f_v + \lambda n, \; \lambda = \pm(-K)^{1/4}, \tag{2.10}$$

where K is the Euclidean Gaussian curvature of f.

Proof. We choose the orientation of f as above and put $\xi = \zeta f_u + \eta f_v + \lambda n$ for some functions ζ, η and λ on f. Assume that $\hat{\nabla}\theta = 0$ and $\theta = \omega$. Then from (2.4) and (2.9), we have

$$\theta(\partial_u, \partial_v) = \lambda \det \begin{pmatrix} f_u \\ f_v \\ n \end{pmatrix} = \lambda\sqrt{EG - F^2}. \tag{2.11}$$

Since (u, v) are asymptotic coordinates, from (2.5) we have

$$h(\partial_u, \partial_u) = h(\partial_v, \partial_v) = 0,\ h(\partial_u, \partial_v) = \frac{M}{\lambda}, \tag{2.12}$$

so that from (2.6) we have

$$\omega(\partial_u, \partial_v) = \varepsilon \left| \frac{M}{\lambda} \right|, \tag{2.13}$$

where $\varepsilon = 1$ or $\varepsilon = -1$ if $\theta(\partial_u, \partial_v) > 0$ or $\theta(\partial_u, \partial_v) < 0$ respectively. Since $\theta = \omega$, if we combine (2.11) and (2.13), λ is given by the second equation of (2.10).

On the other hand, from (2.3), (2.7), (2.8) and (2.11), we have

$$\begin{aligned} (\hat{\nabla}_{\partial u}\theta)(\partial_u, \partial_v) &= \frac{\partial}{\partial u}\theta(\partial_u, \partial_v) - \theta(\hat{\nabla}_{\partial_u}\partial_u, \partial_v) - \theta(\partial_u, \hat{\nabla}_{\partial_u}\partial_v) && (2.14) \\ &= \left\{ \lambda_u + \lambda(\Gamma^1_{11} + \Gamma^2_{12} - \hat{\Gamma}^1_{11} - \hat{\Gamma}^2_{12}) \right\} \det \begin{pmatrix} f_u \\ f_v \\ n \end{pmatrix}. && (2.15) \end{aligned}$$

Since $\hat{\nabla}\theta = 0$, we have

$$\lambda_u + \lambda(\Gamma^1_{11} + \Gamma^2_{12} - \hat{\Gamma}^1_{11} - \hat{\Gamma}^2_{12}) = 0. \tag{2.16}$$

Note that from (2.3), (2.7) and (2.12), we have

$$\hat{\Gamma}^1_{11} = \Gamma^1_{11},\ \hat{\Gamma}^2_{12} + \frac{M\eta}{\lambda} = \Gamma^2_{12}. \tag{2.17}$$

Hence from (2.16) and (2.17), η is given by the coefficient of f_v of the first equation of (2.10). We can carry out a similar computation for ζ. □

Let ξ be as in Proposition 2.1. The line through each point of f in the direction of ξ is called the equiaffine normal line, which is independent of the sign of ξ. From (2.7), (2.8) and (2.10), it is easy to see that ξ_u and ξ_v are expressed by a linear combination of f_u and f_v:

$$\begin{pmatrix} \xi_u \\ \xi_v \end{pmatrix} = -S \begin{pmatrix} f_u \\ f_v \end{pmatrix}, \tag{2.18}$$

where S is a 2×2 matrix-valued function on f, called the equiaffine shape operator. If S is a zero matrix, f is called an improper affine sphere. If $S = \mu I$ for some $\mu \in \mathbf{R} \setminus \{0\}$, where I is the identity matrix, f is called a proper affine sphere. In particular, affine spheres with flat Blaschke metric were classified by Magid and Ryan [8]. Affine spheres with constant curvature Blaschke metric were classified by Simon [11]. See [9] for more about affine spheres as well as affine differential geometry. It is obvious to see that f is an improper affine sphere if and only if ξ is a constant vector, i.e., the equiaffine normals are parallel. On the other hand, we have the following:

Proposition 2.2. *The affine surface f is a proper affine sphere if and only if the equiaffine normals meet at one point.*

Proof. Let f be a proper affine sphere, i.e., $S = \mu I$ for some $\mu \in \mathbf{R} \setminus \{0\}$. We define a map $\hat{Z}$ by

$$\hat{Z} = f + \frac{1}{\mu}\xi. \tag{2.19}$$

Then it is easy to see that from (2.18) we have $\hat{Z}_u = \hat{Z}_v = 0$. Hence $\hat{Z}$ is a constant vector and the equiaffine normals meet at $\hat{Z}$. We can also prove the converse easily. □

The point where the equiaffine normals of proper affine spheres meet is called the center, which can be generalized to a map for affine surfaces as follows. We decompose f as

$$f = sf_u + tf_v + r\xi = Z + r\xi, \tag{2.20}$$

where s, t and r are functions on f and Z is an $\mathbf{R}^3$-valued function tangent to f. In particular, r is called the equiaffine support function from the origin.

Proposition 2.3. *The affine surface f is a proper affine sphere centered at the origin if and only if r is a non-zero constant.*

Proof. From (2.3) and (2.12), differentiating (2.20) by u and v, and taking the coefficient of ξ, we have

$$\frac{tM}{\lambda} + r_u = 0, \ \frac{sM}{\lambda} + r_v = 0. \tag{2.21}$$

If r is a non-zero constant, from (2.21) we have $s = t = 0$. Hence from (2.20) we have

$$f - r\xi = Z = 0, \tag{2.22}$$

that is, f is a proper affine sphere centered at the origin. We can also prove the converse easily. □

After Furuhata and Vrancken [4], we call Z the center map. See also [5, 12, 13] for more about related topics.

3. The fundamental equations for centroaffine surfaces

We assume that the surface f as in Section 2 is a centroaffine surface, i.e., the position vector f is transversal to the tangent plane at each point. We take $-f$ as a transversal vector field on f. Then we have the centroaffine metric $\tilde{h}$, which satisfies

$$\tilde{h}(\partial_u, \partial_u) = \tilde{h}(\partial_v, \partial_v) = 0, \ \tilde{h}(\partial_u, \partial_v) = -\frac{M}{\langle f, n\rangle}. \tag{3.1}$$

The absolute value of $\langle f, n\rangle$ is the distance from the origin of $\mathbf{R}^3$ to the tangent plane, called the Euclidean support function, which we denote by d. In the following, we choose n such that $d = \langle f, n\rangle$. If we compute f_{uuv} and f_{uvu} using (2.7) and (2.8) and compare the coefficient of n, we have

$$\Gamma^1_{11} M = \Gamma^2_{12} M + M_u. \tag{3.2}$$

Then from (2.2) it is straightforward to see that

$$M_u = \frac{GE_u - 2FF_u + 2FE_v - EG_u}{2(EG - F^2)} M \tag{3.3}$$

and hence

$$\left(\frac{M}{\sqrt{EG - F^2}}\right)_u + \frac{2\Gamma^2_{12} M}{\sqrt{EG - F^2}} = 0, \tag{3.4}$$

which gives Γ^2_{12} in terms of the Gaussian curvature K. We can carry out a similar computation as above for Γ^1_{12} and obtain

$$\Gamma^2_{12} = -\frac{1}{4}\frac{K_u}{K},\ \Gamma^1_{12} = -\frac{1}{4}\frac{K_v}{K}. \tag{3.5}$$

Therefore if we put $\varphi = \tilde{h}(\partial_u, \partial_v)$, from the second equations of (2.7) and (3.1), and (3.5), we have

$$f_{uv} = -\frac{1}{4}\frac{K_v}{K} f_u - \frac{1}{4}\frac{K_u}{K} f_v - \varphi d n. \tag{3.6}$$

On the other hand, from (2.8) we have

$$\langle n_u, n_u\rangle = \frac{EM^2}{EG - F^2},\ \langle n_u, n_v\rangle = -\frac{FM^2}{EG - F^2},\ \langle n_v, n_v\rangle = \frac{GM^2}{EG - F^2} \tag{3.7}$$

and hence

$$\langle n_u, n_u\rangle\langle n_v, n_v\rangle - \langle n_u, n_v\rangle^2 = \frac{M^4}{EG - F^2}, \tag{3.8}$$

which shows that n_u, n_v and n are linearly independent. Then from (2.8), the second equation of (3.1), (3.7) and (3.8), it is straightforward to see that

$$f = \frac{Gd_u + Fd_v}{M^2} n_u + \frac{Fd_u + Ed_v}{M^2} n_v + dn = \frac{d_v}{\varphi d} f_u + \frac{d_u}{\varphi d} f_v + dn. \tag{3.9}$$

Note that we have the same equation as (3.9) even if we choose n such that $d = -\langle f, n\rangle$. Combining (3.6) and (3.9), we have one of the Gauss equations for the centroaffine surface f:

$$f_{uv} = -\varphi f + \rho_v f_u + \rho_u f_v, \tag{3.10}$$

where

$$\rho = -\frac{1}{4}\log\left(-\frac{K}{d^4}\right). \tag{3.11}$$

It is easy to see that

$$-\frac{K}{d^4} = \det\begin{pmatrix} f_u \\ f_v \\ f_{uv}\end{pmatrix}^2 \Bigg/ \det\begin{pmatrix} f \\ f_u \\ f_v\end{pmatrix}^4, \tag{3.12}$$

so that ρ is an equicentroaffine invariant, i.e., an invariant under equiaffine transformations fixing the origin. Moreover, centroaffine transformations, i.e., affine transformations fixing the origin, preserve the property that ρ is a constant, which was discovered by Tzitzéica [14].

Proposition 3.1. *The center map Z and the equiaffine support function r is given by*

$$Z = \frac{\rho_v}{\varphi} f_u + \frac{\rho_u}{\varphi} f_v, \ r = \pm e^{\rho}. \tag{3.13}$$

Proof. It is straightforward to see from (2.10), (2.20), the second equation of (3.1), (3.9) and (3.11). □

Corollary 3.2. *The centroaffine surface f is a proper affine sphere centered at the origin if and only if ρ is a constant.*

Proof. It is obvious by Propositions 2.3 and 3.1. □

In order to obtain the remaining Gauss equations, we write the first and the third equations of (2.7) as

$$f_{uu} = \Gamma_{11}^1 f_u + \frac{a}{\varphi} f_v, \ f_{vv} = \Gamma_{22}^2 f_v + \frac{b}{\varphi} f_u, \tag{3.14}$$

where

$$a = \varphi \det \begin{pmatrix} f \\ f_u \\ f_{uu} \end{pmatrix} \Big/ \det \begin{pmatrix} f \\ f_u \\ f_v \end{pmatrix}, \ b = \varphi \det \begin{pmatrix} f \\ f_v \\ f_{vv} \end{pmatrix} \Big/ \det \begin{pmatrix} f \\ f_v \\ f_u \end{pmatrix}. \tag{3.15}$$

Then the cubic differentials adu^3 and bdv^3 are centroaffine invariants. If we compute f_{uvu}, f_{uuv} or f_{uvv}, f_{vvu} using (3.10) and (3.14) and compare the coefficient of f, we have

$$\Gamma_{11}^1 = \frac{\varphi_u}{\varphi} + \rho_u, \ \Gamma_{22}^2 = \frac{\varphi_v}{\varphi} + \rho_v. \tag{3.16}$$

Now the integrability conditions for (3.10) and (3.14) with (3.16) are easy to compute. Therefore as can be seen in [10], we can summarize as follows.

Proposition 3.3. *The Gauss equations for the centroaffine surface f are*

$$f_{uu} = \left(\frac{\varphi_u}{\varphi} + \rho_u\right) f_u + \frac{a}{\varphi} f_v, \ f_{uv} = -\varphi f + \rho_v f_u + \rho_u f_v, \ f_{vv} = \left(\frac{\varphi_v}{\varphi} + \rho_v\right) f_v + \frac{b}{\varphi} f_u \tag{3.17}$$

with the integrability conditions:

$$(\log|\varphi|)_{uv} = -\varphi - \frac{ab}{\varphi^2} + \rho_u \rho_v, \ a_v + \rho_u \varphi_u = \rho_{uu} \varphi, \ b_u + \rho_v \varphi_v = \rho_{vv} \varphi. \tag{3.18}$$

4. Non-semisimple centroaffine minimal surfaces with constant centroaffine curvature

The surface f is called to be centroaffine minimal if it extremizes the area integral of $\tilde{h}$, which is known to be equivalent to the condition that the trace of the centroaffine Tchebychev operator vanishes. Let ∇ be the connection induced by the immersion f and $\tilde{\nabla}$ the Levi-Civita connection of $\tilde{h}$. It is easy to see that the Christoffel symbols $\tilde{\Gamma}_{ij}^k$ $(i, j, k = 1, 2)$ for $\tilde{\nabla}$ with respect to (u, v) vanish except

$$\tilde{\Gamma}_{11}^1 = \frac{\varphi_u}{\varphi},\ \tilde{\Gamma}_{22}^2 = \frac{\varphi_v}{\varphi}. \tag{4.1}$$

We denote $\nabla - \tilde{\nabla}$ by C, which defines a $(1, 2)$-tensor field. From (3.17) and (4.1), it is obvious to see that

$$C(\partial_u, \partial_u) = \rho_u\partial_u + \frac{a}{\varphi}\partial_v,\ C(\partial_u, \partial_v) = \rho_v\partial_u + \rho_u\partial_v,\ C(\partial_v, \partial_v) = \frac{b}{\varphi}\partial_u + \rho_v\partial_v. \tag{4.2}$$

Then the centroaffine Tchebychev vector field T is computed as

$$T = \frac{1}{2}\mathrm{tr}_h C = \frac{\rho_v}{\varphi}\partial_u + \frac{\rho_u}{\varphi}\partial_v = \mathrm{grad}_h \rho. \tag{4.3}$$

From the second and the third equations of (3.18) and (4.1), the centroaffine Tchebychev operator $\tilde{\nabla}T$ is computed as

$$\tilde{\nabla}T(\partial_u) = \frac{\rho_{uv}}{\varphi}\partial_u + \frac{a_v}{\varphi^2}\partial_v,\ \tilde{\nabla}T(\partial_v) = \frac{b_u}{\varphi^2}\partial_u + \frac{\rho_{uv}}{\varphi}\partial_v. \tag{4.4}$$

Hence f is centroaffine minimal if and only if $\rho_{uv} = 0$. Centroaffine surfaces such that $\tilde{\nabla}T$ is proportional to the identity are called to be centroaffine Tchebychev. In particular, f is centroaffine minimal and centroaffine Tchebychev if and only if $\tilde{\nabla}T = 0$, i.e., $\rho_{uv} = a_v = b_u = 0$. Such surfaces were classified by Liu and Wang [7], and include proper affine spheres centered at the origin by Corollary 3.2 and the second and the third equations of (3.18). The centroaffine scalar curvature κ is given by

$$\kappa = -\frac{(\log|\varphi|)_{uv}}{\varphi}. \tag{4.5}$$

Centroaffine Tchebychev surfaces with constant κ were classified by Binder [1]. In the previous paper, the author [2] classified centroaffine minimal surfaces with constant κ, $a = b$ and $\rho = c_1 u + c_2 v + c_3$ for $c_1, c_2, c_3 \in \mathbf{R}$.

Definition 4.1. A centroaffine surface is called to be semisimple if and only if the centroaffine Tchebychev operator is semisimple.

Assume that f is a non-semisimple centroaffine minimal surface. Then changing the asymptotic line coordinates (u, v), if necessary, we may assume that $\rho = c_1 u + c_2 v + c_3$ for some $c_1, c_2, c_3 \in \mathbf{R}$, $a_v \neq 0$ and $b_u = 0$. In particular, $b = b(v)$. Then from (3.18) and (4.5), we have

$$(\kappa - 1)\varphi - \frac{ab}{\varphi^2} + c_1 c_2 = 0,\ a_v + c_1\varphi_u = 0,\ c_2\varphi_v = 0. \tag{4.6}$$

Lemma 4.2. *If κ is a constant, then $\kappa = 0, 1$; for $\kappa = 0$ we say that f is centroaffine flat.*

Proof. From the third equation of (4.6), we have $\varphi_v = 0$ or $c_2 = 0$.

In the case of $\varphi_v = 0$, from (4.5) we have $\kappa = 0$.

In the case of $c_2 = 0$, if $\kappa \neq 0$, (4.5) becomes the Liouville equation, whose solution is given by

$$\varphi = -\frac{2}{\kappa}\frac{p_u q_v}{(p(u)+q(v))^2} \tag{4.7}$$

such that $p_u, q_v \neq 0$. Then the second equation of (4.6) becomes

$$a_v - \frac{2c_1}{\kappa}\left\{\frac{p_{uu}q_v}{(p+q)^2} - \frac{2p_u^2 q_v}{(p+q)^3}\right\} = 0, \tag{4.8}$$

which can be integrated as

$$a = \frac{2c_1}{\kappa}\left\{-\frac{p_{uu}}{p+q} + \frac{p_u^2}{(p+q)^2}\right\} + w(u). \tag{4.9}$$

Assume that $\kappa \neq 1$. Since $c_2 = 0$, from the first equation of (4.6), we have $b \neq 0$. Then from the first equation of (4.6), (4.7) and (4.9), we have

$$\frac{8(1-\kappa)}{\kappa^3}\frac{p_u^3 q_v^3}{b} = \frac{2c_1}{\kappa}\left\{-p_{uu}(p+q)^5 + p_u^2(p+q)^4\right\} + w(p+q)^6. \tag{4.10}$$

Since $b = b(v)$, $p = p(u)$, $q = q(v)$ and $w = w(u)$, we have

$$\frac{8(1-\kappa)}{\kappa^3}\frac{q_v^3}{b} = \sum_{i=0}^{6}\alpha_i q^i \tag{4.11}$$

for some $\alpha_0, \ldots, \alpha_6 \in \mathbf{R}$. Substituting (4.11) into (4.10) and comparing the coefficients of q^6, q^5 and q^4, we have

$$\alpha_6 p_u^3 = w,\ \alpha_5 p_u^3 = -\frac{2c_1}{\kappa}p_{uu} + 6pw,\ \alpha_4 p_u^3 = -\frac{10c_1}{\kappa}pp_{uu} + \frac{2c_1}{\kappa}p_u^2 + 15p^2 w. \tag{4.12}$$

Then we have

$$(\alpha_4 - 5\alpha_5 p + 15\alpha_6 p^2)p_u = \frac{2c_1}{\kappa}. \tag{4.13}$$

Since $a_v \neq 0$, from the second equation of (4.6), we have $c_1 \neq 0$. Hence we have $\alpha_4 - 5\alpha_5 p + 15\alpha_6 p^2 \neq 0$. Then from (4.13) it is easy to see that

$$p_{uu} = -\frac{20c_1^2}{\kappa^2}\frac{-\alpha_5 + 6\alpha_6 p}{(\alpha_4 - 5\alpha_5 p + 15\alpha_6 p^2)^3}. \tag{4.14}$$

From the second equation of (4.12), (4.13) and (4.14), it is straightforward to see that

$$288\alpha_6 p - 48\alpha_5 = 0. \tag{4.15}$$

Since p is not a constant, we have $\alpha_5 = \alpha_6 = 0$. Hence from (4.10), the first equation of (4.12), (4.13) and (4.14), we have

$$\frac{8(1-\kappa)}{\alpha_4\kappa^3}\frac{q_v^3}{b} = (p+q)^4, \tag{4.16}$$

which is a contradiction. Therefore we have $\kappa = 1$. $\square$

Theorem 4.3. *Let f be a flat non-semisimple centroaffine minimal surface. Then changing the coordinates, if necessary, we have $\varphi = 1$, $b = \alpha/v$ and*

$$(a, \rho) = \left(-\frac{v}{\alpha}, -\frac{1}{2\alpha}u^2 + c_2\right), \left(\frac{c_1\beta^2}{\alpha^2}ve^{\frac{\beta}{\alpha}u}, c_1e^{\frac{\beta}{\alpha}u} + \frac{1}{\beta}u + \beta v + c_2\right), \tag{4.17}$$

where $\alpha, \beta, c_1 \in \mathbf{R} \setminus \{0\}$ and $c_2 \in \mathbf{R}$.

Proof. Since $\kappa = 0$ and f is non-semisimple, changing the coordinates, if necessary, we may assume that $\varphi = 1$, $a_v \neq 0$ and $b = b(v)$. Moreover, since $\rho_{uv} = 0$, we have $\rho = p(u) + q(v)$. Then (3.18) becomes

$$-1 - ab + p_u q_v = 0, \ a_v = p_{uu}, \ q_{vv} = 0. \tag{4.18}$$

From the third equation of (4.18), we have $q_v = \beta$ for some $\beta \in \mathbf{R}$. Then the first equation of (4.18) becomes

$$-1 - ab + \beta p_u = 0. \tag{4.19}$$

If $b = 0$, from (4.19) we have $p_u = 1/\beta \neq 0$, which contradicts the second equation of (4.18) since $a_v \neq 0$. Hence we have $b \neq 0$.

From the second equation of (4.18), we have

$$a = p_{uu}v + w(u), \tag{4.20}$$

so that (4.19) becomes

$$-1 - p_{uu}bv - wb + \beta p_u = 0. \tag{4.21}$$

Since $b = b(v)$, differentiating (4.21) by u, we have

$$-p_{uuu}bv - w_u b + \beta p_{uu} = 0. \tag{4.22}$$

Note that (4.21) and (4.22) are equivalent to

$$\begin{pmatrix} p_{uu} & w \\ p_{uuu} & w_u \end{pmatrix} \begin{pmatrix} bv \\ b \end{pmatrix} = \begin{pmatrix} \beta p_u - 1 \\ \beta p_{uu} \end{pmatrix}. \tag{4.23}$$

If $p_{uu}w_u \neq p_{uuu}w$, from (4.23) both bv and b are constants. Hence we have $b = 0$, which is a contradiction. Therefore we have $p_{uu}w_u = p_{uuu}w$, which can be solved as $w = \hat{c}p_{uu}$ for some $\hat{c} \in \mathbf{R}$. Then (4.20) becomes

$$a = p_{uu}(v + \hat{c}). \tag{4.24}$$

Using translation of the coordinate v, if necessary, we may assume that $\hat{c} = 0$. Then (4.21) becomes

$$-1 - p_{uu}bv + \beta p_u = 0. \tag{4.25}$$

Note that $p_{uu} \neq 0$ since $a_v \neq 0$. Moreover since $b \neq 0$, from (4.25) we have $b = \alpha/v$ for some $\alpha \in \mathbf{R} \setminus \{0\}$. Hence (4.25) becomes

$$\alpha p_{uu} - \beta p_u + 1 = 0, \tag{4.26}$$

which can be solved as

$$p = \begin{cases} \hat{c}_1 + \hat{c}_2 u - \dfrac{1}{2\alpha} u^2 & \text{if } \beta = 0, \\ \hat{c}_1 + \hat{c}_2 e^{\frac{\beta}{\alpha} u} + \dfrac{1}{\beta} u & \text{if } \beta \neq 0 \end{cases} \tag{4.27}$$

for some $\hat{c}_1, \hat{c}_2 \in \mathbf{R}$. Note that $\hat{c}_2 \neq 0$ if $\beta \neq 0$, since $p_{uu} \neq 0$. Using translation of the coordinate u, if necessary, we have (4.17). □

Remark 4.4. The author [3] classified centroaffine minimal surfaces whose centroaffine curvature and Pick function are constants. The Pick function J is computed as

$$J = \frac{1}{2}\|C\|^2 = \frac{3\rho_u \rho_v}{\varphi} + \frac{ab}{\varphi^3}. \tag{4.28}$$

In the case of Theorem 4.3, we have

$$(a, \rho, J) = \begin{cases} \left(-\dfrac{v}{\alpha}, -\dfrac{1}{2\alpha} u^2 + c_2, -1 \right), \\ \left(\dfrac{c_1 \beta^2}{\alpha^2} v e^{\frac{\beta}{\alpha} u}, c_1 e^{\frac{\beta}{\alpha} u} + \dfrac{1}{\beta} u + \beta v + c_2, \dfrac{4 c_1 \beta^2}{\alpha} e^{\frac{\beta}{\alpha} u} + 3 \right). \end{cases} \tag{4.29}$$

In particular, the former case was also obtained in [3].

Remark 4.5. Liu and Jung [6] studied indefinite centroaffine minimal surfaces with constant centroaffine curvature and vanishing generalized Pick function. If we denote the traceless part of the $(1, 2)$-tensor field C by $\tilde{C}$, we have

$$\tilde{C}(X, Y) = C(X, Y) - \frac{1}{2}(\tilde{h}(T, X)Y + \tilde{h}(T, Y)X + \tilde{h}(X, Y)T) \tag{4.30}$$

for vector fields X and Y on f. From (4.2) and (4.3), we have

$$\tilde{C}(\partial_u, \partial_u) = \frac{a}{\varphi}\partial_v, \ \tilde{C}(\partial_u, \partial_v) = 0, \ \tilde{C}(\partial_v, \partial_v) = \frac{b}{\varphi}\partial_u. \tag{4.31}$$

Then from (4.3), (4.29) and (4.31), the generalized Pick function $\tilde{J}$ in [6] is computed as

$$\tilde{J} = \frac{1}{2}\|\tilde{C}\|^2 = \frac{ab}{\varphi^3} = J - \frac{3\rho_u \rho_v}{\varphi} = J - \frac{3}{2}\|T\|^2. \tag{4.32}$$

In the case of Theorem 4.3, we have

$$(a, \rho, \tilde{J}) = \begin{cases} \left(-\dfrac{v}{\alpha}, -\dfrac{1}{2\alpha} u^2 + c_2, -1 \right), \\ \left(\dfrac{c_1 \beta^2}{\alpha^2} v e^{\frac{\beta}{\alpha} u}, c_1 e^{\frac{\beta}{\alpha} u} + \dfrac{1}{\beta} u + \beta v + c_2, \dfrac{c_1 \beta^2}{\alpha} e^{\frac{\beta}{\alpha} u} \right). \end{cases} \tag{4.33}$$

It was proved in [3] that if f is a centroaffine minimal surfaces with constant κ, then J is a constant if and only if $\tilde{J}$ is a constant.

Theorem 4.6. *Let f be a non-semisimple centroaffine minimal surface with $\kappa = 1$. Then f is given by a ruled surface:*

$$f = A_x + yA, \tag{4.34}$$

where A is an $\mathbf{R}^3$-valued function of x such that

$$\det \begin{pmatrix} A \\ A_x \\ A_{xx} \end{pmatrix} \neq 0, \ \frac{d}{dx} \det \begin{pmatrix} A \\ A_x \\ A_{xx} \end{pmatrix} \neq 0. \tag{4.35}$$

Proof. As in the proof of Lemma 4.2, we may assume that $\rho = c_1 u + c_3$ for some $c_1 \in \mathbf{R} \setminus \{0\}$ and $c_3 \in \mathbf{R}$. Moreover, by a similar argument as in the proof of Lemma 4.2, we have $b = 0$. Then (3.17) becomes

$$f_{uu} = \left(\frac{\varphi_u}{\varphi} + c_1 \right) f_u + \frac{a}{\varphi} f_v, \ f_{uv} = -\varphi f + c_1 f_v, \ f_{vv} = \frac{\varphi_v}{\varphi} f_v. \tag{4.36}$$

Moreover, from (4.7) and (4.9), we have

$$\varphi = -\frac{2p_u q_v}{(p+q)^2}, \ a = 2c_1 \left\{ -\frac{p_{uu}}{p+q} + \frac{p_u^2}{(p+q)^2} \right\} + w(u). \tag{4.37}$$

From the third equation of (4.36) and the first equation of (4.37), we have

$$f_v = \hat{A}(u)\varphi, \ f = \hat{A}(u)\frac{2p_u}{p+q} + \hat{B}(u), \tag{4.38}$$

where $\hat{A}$ and $\hat{B}$ are $\mathbf{R}^3$-valued functions of u. Then the second equation of (4.36) becomes

$$\hat{A}_u \varphi + \hat{A}\varphi_u = -\varphi \left(\hat{A}\frac{2p_u}{p+q} + \hat{B} \right) + c_1 \hat{A}\varphi \tag{4.39}$$

and hence

$$\hat{B} = -\hat{A}_u + \left(-\frac{p_{uu}}{p_u} + c_1 \right) \hat{A}. \tag{4.40}$$

From the second equation of (4.38) and (4.40), we have

$$f = -\hat{A}_u + \left(\frac{2p_u}{p+q} - \frac{p_{uu}}{p_u} + c_1 \right) \hat{A}, \tag{4.41}$$

which implies that f is given in the form of (4.34).

Conversely, if f is given by (4.34), we have

$$f_y = A, \ f_x = A_{xx} + yA_x. \tag{4.42}$$

Hence if the first equation of (4.35) is satisified, f becomes a centroaffine surface. Then from (4.34) and (4.42), we have

$$f_{yy} = 0, \ f_{xy} = -yf_y + f \tag{4.43}$$

and hence

$$\tilde{h}(\partial_y, \partial_y) = 0, \ \tilde{h}(\partial_x, \partial_y) = -1, \tag{4.44}$$

which implies that f is indefinite. Moreover, from (4.34) and (4.42), we have

$$\tilde{h}(\partial_x,\partial_x) = -\det\begin{pmatrix} f_{xx} \\ f_x \\ f_y \end{pmatrix} \Big/ \det\begin{pmatrix} f \\ f_x \\ f_y \end{pmatrix} = \det\begin{pmatrix} A \\ A_{xx} \\ A_{xxx} \end{pmatrix} \Big/ \det\begin{pmatrix} A \\ A_x \\ A_{xx} \end{pmatrix} + y\alpha + y^2, \tag{4.45}$$

where

$$\alpha = \det\begin{pmatrix} A \\ A_x \\ A_{xxx} \end{pmatrix} \Big/ \det\begin{pmatrix} A \\ A_x \\ A_{xx} \end{pmatrix} = \frac{d}{dx}\log\left|\det\begin{pmatrix} A \\ A_x \\ A_{xx} \end{pmatrix}\right|. \tag{4.46}$$

Note that if the second equation of (4.35) is satisfied, $\alpha \neq 0$. From (4.44) and (4.45), we have

$$\tilde{h}(\tilde{\nabla}_{\partial_y}\partial_y,\partial_y) = \tilde{h}(\tilde{\nabla}_{\partial_x}\partial_y,\partial_y) = \tilde{h}(\tilde{\nabla}_{\partial_y}\partial_y,\partial_x) = 0,\ \tilde{h}(\tilde{\nabla}_{\partial_y}\partial_x,\partial_x) = \frac{1}{2}\alpha + y \tag{4.47}$$

and hence

$$\tilde{\nabla}_{\partial_y}\partial_y = 0,\ \tilde{\nabla}_{\partial_x}\partial_y = -\left(\frac{1}{2}\alpha + y\right)\partial_y. \tag{4.48}$$

If we denote the curvature operator of $\tilde{\nabla}$ by $\tilde{R}$, from (4.44) and (4.48), we have

$$\tilde{h}(\tilde{R}(\partial_x,\partial_y)\partial_y,\partial_x) = \tilde{h}(\partial_x,\partial_x)\tilde{h}(\partial_y,\partial_y) - \tilde{h}(\partial_x,\partial_y)^2 = -1 \tag{4.49}$$

which implies that $\kappa = 1$. Moreover, from (4.43) and (4.48), we have

$$C(\partial_y,\partial_y) = 0,\ C(\partial_x,\partial_y) = \frac{1}{2}\alpha\partial_y. \tag{4.50}$$

Hence from (4.44) and (4.50), we have

$$T = -\frac{1}{2}\alpha\partial_y. \tag{4.51}$$

Therefore from (4.48) and (4.51), we have

$$\tilde{\nabla}_{\partial_y}T = 0,\ \tilde{\nabla}_{\partial_x}T = \left\{-\frac{1}{2}\alpha_x + \frac{1}{2}\alpha\left(\frac{1}{2}\alpha + y\right)\right\}\partial_y, \tag{4.52}$$

which implies that f is a non-semisimple centroaffine minimal surface if $\alpha \neq 0$. $\square$

Remark 4.7. Any non-semisimple centroaffine minimal surface with $\kappa = 1$ satisfies $J = \tilde{J} = 0$.

If f is a centroaffine surface given by (4.43) with $\alpha = 0$, then f is a proper affine sphere centered at the origin with $\kappa = 1$.

5. The center map of non-semisimple centroaffine minimal surfaces

As an application of our classification result in Section 4, we study the center map of non-semisimple centroaffine minimal surfaces. Let f be an indefinite centroaffine surface. We use the same notations as in the previous sections.

Lemma 5.1. *The center map Z of f becomes a centroaffine surface if and only if*

$$2(\rho_u\rho_v\rho_{uv} + \rho_u^2\rho_v^2)\varphi \neq \rho_v^2(a_v + a\rho_v) + \rho_u^2(b_u + b\rho_u). \tag{5.1}$$

Proof. From the first equation of (3.13), (3.17) and the second and the third equations of (3.18), it is straightforward to see that

$$\begin{cases} Z_u = -\rho_u f + \dfrac{\rho_{uv} + 2\rho_u\rho_v}{\varphi} f_u + \left(\dfrac{a_v + a\rho_v}{\varphi^2} + \dfrac{\rho_u^2}{\varphi}\right) f_v, \\ Z_v = -\rho_v f + \dfrac{\rho_{uv} + 2\rho_u\rho_v}{\varphi} f_v + \left(\dfrac{b_u + b\rho_u}{\varphi^2} + \dfrac{\rho_v^2}{\varphi}\right) f_u. \end{cases} \tag{5.2}$$

Then from the first equation of (3.13) and (5.2), it is straightforward to see that

$$\det\begin{pmatrix} Z \\ Z_u \\ Z_v \end{pmatrix} = \left\{\frac{2\rho_u\rho_v\rho_{uv}}{\varphi^2} + \frac{2\rho_u^2\rho_v^2}{\varphi^2} - \frac{\rho_v^2(a_v + a\rho_v)}{\varphi^3} - \frac{\rho_u^2(b_u + b\rho_u)}{\varphi^3}\right\} \det\begin{pmatrix} f \\ f_u \\ f_v \end{pmatrix}. \tag{5.3}$$

□

Theorem 5.2. *Let f be a non-semisimple centroaffine minimal surface. Then Z becomes a centroaffine surface if and only if $\kappa \neq 1$.*

Proof. As in Section 4, changing the coordinates, if necessary, we may assume that $\rho = c_1 u + c_2 v + c_3$ for some $c_1, c_2, c_3 \in \mathbf{R}$, $a_v \neq 0$ and $b = b(v)$. Assume that Z does not become a centroaffine surface. Then from (5.1) we have

$$2c_1^2 c_2^2 \varphi = c_2^2(a_v + c_2 a) + c_1^3 b. \tag{5.4}$$

Note that from the third equation of (4.6), we have $\varphi_v = 0$ or $c_2 = 0$.

In the case of $\varphi_v = 0$, from (4.5) we have $\kappa = 0$. Then changing the coordinates, if necessary, we may assume that φ, a, b and ρ are given as in Theorem 4.3. In the former case in (4.17), the left hand side of (5.1) vanishes, while the right hand side is equal to $-u^3/(\alpha^2 v)$. Hence Z becomes a centroaffine surface, which contradicts the assumption. In the latter case in (4.17), we have also a contradiction since (5.1) becomes

$$2\left(\frac{c_1\beta^2}{\alpha} e^{\frac{\beta}{\alpha}u} + 1\right)^2 \neq \frac{c_1\beta^4}{\alpha^2}(\beta v + 1)e^{\frac{\beta}{\alpha}u} + \frac{\alpha}{v}\left(\frac{c_1\beta}{\alpha} e^{\frac{\beta}{\alpha}u} + \frac{1}{\beta}\right)^3. \tag{5.5}$$

In the case of $c_2 = 0$, from (5.4) we have $c_1^3 b = 0$. Note that $c_1 \neq 0$ since f is non-semisimple. Hence we have $b = 0$, which corresponds to the surface with $\kappa = 1$ given by Theorem 4.6. □

Remark 5.3. If f is a non-semisimple centroaffine minimal surface with $\kappa = 1$ given by Theorem 4.6, then from the first equation of (3.13) and (4.3), we have

$$Z = f_* T = -\frac{1}{2}\alpha A. \tag{5.6}$$

Appendix A. The center map of centroaffine surfaces with vanishing centroaffine Tchebychev operator

In this Appendix, we consider indefinite centroaffine surfaces and use the same notations as in the previous sections. The following fact can be seen in [7].

Lemma A.1. *Let f be a centroaffine minimal surface with $\tilde{\nabla} T = 0$ which is not a proper affine sphere centered at the origin. Then $\kappa = 0$.*

Proposition A.2. *Let f be a centroaffine surface with $\tilde{\nabla} T = 0$ which is not a proper affine sphere centered at the origin. Changing the coordinates, if necessary, we have $\rho = c_1 u + c_2 v + c_3$ for some $c_1, c_2, c_3 \in \mathbf{R}$ and one of the following (i) and (ii).*

(*i*) $\varphi = c_1 = c_2 = 1$, $a = a(u)$ *and* $b = 0$. (A.1)

(*ii*) φ *is a non-zero constant,* $c_1 \neq 0$ *or* $c_2 \neq 0$, *and* $a = b = 1$. (A.2)

Proof. Note that $a = a(u)$ and $b = b(v)$ since $a_v = b_u = 0$, and $\kappa = 0$ by Lemma A.1. From (4.5) changing the coordinates, if necessary, we may assume that $\varphi = 1$. Then from the second and the third equations of (3.18), we have $\rho_{uu} = \rho_{vv} = 0$ and hence

$$\rho = c_0 uv + c_1 u + c_2 v + c_3 \tag{A.3}$$

for some $c_0, c_1, c_2, c_3 \in \mathbf{R}$. Moreover, from the first equation of (3.18) and (A.3), we have

$$0 = -1 - ab + (c_0 v + c_1)(c_0 u + c_2). \tag{A.4}$$

In case of $ab = 0$, i.e., $a = 0$ or $b = 0$, from (A.4) we have $c_0 = 0$ and $c_1 c_2 = 1$. Changing the coordinates, if necessary, we have (i).

In case of $ab \neq 0$, since $a = a(u)$ and $b = b(v)$, from (A.4) we have

$$a = a_1 u + a_2, \ b = b_1 v + b_2 \tag{A.5}$$

for some $a_1, a_2, b_1, b_2 \in \mathbf{R}$, so that

$$c_0^2 = a_1 b_1, \ c_0 c_1 = a_1 b_2, \ c_0 c_2 = a_2 b_1, \ c_1 c_2 - a_2 b_2 = 1. \tag{A.6}$$

If $c_0 \neq 0$, from the first, the second and the third equations of (A.6), it is easy to see that $c_1 c_2 = a_2 b_2$, which contradicts the fourth equation of (A.6). Hence we have $c_0 = 0$. Then from the first equation of (A.6), changing the coordinate u and v, if necessary, we may assume that $b_1 = 0$. Since $ab \neq 0$, we have $b_2 \neq 0$. Then from the second equation of (A.6), we have $a_1 = 0$ and hence $a_2 \neq 0$ as above. Since f is not a proper affine sphere centered at the origin, rescaling the coordinates, if necessary, we have (ii). □

In the case of (i) in Proposition A.2, (3.17) becomes

$$f_{uu} = f_u + a f_v, \ f_{uv} = -f + f_u + f_v, \ f_{vv} = f_v, \tag{A.7}$$

which can be integrated explicitely and up to centroaffine congruence

$$f = (e^u, A_1(u) e^v, A_2(u) e^v), \tag{A.8}$$

where A_1 and A_2 are linearly independent solutions of the linear differential equation:

$$A_{uu} - A_u - aA = 0. \tag{A.9}$$

If a is a constant, we have up to centroaffine congruence

$$f = \begin{cases} \left(e^u, e^{\frac{1+\sqrt{1+4a}}{2}u+v}, e^{\frac{1-\sqrt{1+4a}}{2}u+v}\right) & \text{if } a > -\frac{1}{4}, \\ \left(e^u, e^{\frac{1}{2}u+v}, ue^{\frac{1}{2}u+v}\right) & \text{if } a = -\frac{1}{4}, \\ \left(e^u, e^{\frac{1}{2}u+v}\cos\frac{\sqrt{-1-4a}}{2}u, e^{\frac{1}{2}u+v}\sin\frac{\sqrt{-1-4a}}{2}u\right) & \text{if } a < -\frac{1}{4}. \end{cases} \tag{A.10}$$

In particular, if $a = 0$, we have a piece of the hyperbolic paraboloid, which is known as an improper affine sphere.

In the case of (ii) in Proposition A.2, (3.17) becomes

$$f_{uu} = c_1 f_u + \frac{1}{\varphi} f_v, \; f_{uv} = -\varphi f + c_2 f_u + c_1 f_v, \; f_{vv} = c_2 f_v + \frac{1}{\varphi} f_u. \tag{A.11}$$

From the first equation of (3.18), we have

$$\varphi^3 - c_1 c_2 \varphi^2 + 1 = 0. \tag{A.12}$$

We define $\hat{C} > 0$, $\alpha, \beta, \gamma, D \in \mathbf{R}$ by

$$\hat{C}^3 + c_2\varphi\hat{C}^2 - c_1\varphi\hat{C} - 1 = 0, \tag{A.13}$$

$$\alpha = \frac{2\hat{C}(\hat{C} + c_2\varphi)}{\varphi}, \; \beta = \frac{-\hat{C}^2 + c_1\varphi}{\varphi}, \; \gamma = c_1 - c_2\hat{C}, \; D = \gamma^2 - \beta^2 + \alpha\beta. \tag{A.14}$$

Note that $\alpha, \beta \neq 0$ since $\alpha\beta = 2\varphi\hat{C} \neq 0$. Then as can be seen in [2, Proposition 2.2], if we put $u = x + y$, $v = \hat{C}(x - y)$, we have

$$f = \begin{cases} \left(e^{\alpha x}, e^{\beta x + (\gamma + \sqrt{D})y}, e^{\beta x + (\gamma - \sqrt{D})y}\right) & \text{if } \alpha \neq \beta, \; D > 0, \\ \left(e^{\alpha x}, e^{\beta x + \gamma y}, ye^{\beta x + \gamma y}\right) & \text{if } \alpha \neq \beta, \; D = 0, \\ \left(e^{\alpha x}, e^{\beta x + \gamma y}\cos\sqrt{-D}y, e^{\beta x + \gamma y}\sin\sqrt{-D}y\right) & \text{if } \alpha \neq \beta, \; D < 0, \\ \left(\left(x + \frac{\alpha}{2\gamma}y\right)e^{\alpha x}, e^{\alpha x}, e^{\alpha x + 2\gamma y}\right) & \text{if } \alpha = \beta, \; \gamma \neq 0, \\ \left((x + y^2)e^{-2x}, e^{-2x}, ye^{-2x}\right) & \text{if } \alpha = \beta, \; \gamma = 0. \end{cases} \tag{A.15}$$

Remark A.3. The classification result due to Liu and Wang and [2, Proposition 2.2] dropped the case $\alpha = \beta$, $\gamma = 0$ in (A.15). See also [4].

We call a centroaffine surface whose center map is centroaffine congruent with the original surface to be self congruent. The following was obtained by Furuhata and Vrancken [4] in more general cases.

Proposition A.4. *Let f be a self congruent centroaffine surface. Then $\tilde{\nabla}T = 0$.*

Considering the converse of Proposition A.4, we have the following theorem.

Theorem A.5. *Let f be a centroaffine surface with $\tilde{\nabla} T = 0$ which is not a proper affine sphere centered at the origin. Then f is self congruent if and only if f is centroaffine congruent to one of the surfaces given by* (A.10) *with $a \neq 2$, and* (A.15) *with*

$$(c_1^3 + c_2^3)^3 \neq 2c_1^3 c_2^3 (c_1^3 - c_2^3)^2. \tag{A.16}$$

Proof. In the case of (i) in Proposition A.2, by Lemma 5.1, Z becomes a centroaffine surface if and only if $a \neq 2$. From the first equation of (3.13) and (A.8), we have up to centroaffine congruence

$$Z = (e^u, (A_{1,u} + A_1)e^v, (A_{2,u} + A_2)e^v). \tag{A.17}$$

Then from (A.9) we have

$$Z_u = (e^u, (2A_{1,u} + aA_1)e^v, (2A_{2,u} + aA_2)e^v), \; Z_v = (0, (A_{1,u} + A_1)e^v, (A_{2,u} + A_2)e^v), \tag{A.18}$$

$$\begin{cases} Z_{uu} = (e^u, \{(2+a)A_{1,u} + (a_u + 2a)A_1\}e^v, \{(2+a)A_{2,u} + (a_u + 2a)A_2\}e^v), \\ Z_{uv} = (0, (2A_{1,u} + aA_1)e^v, (2A_{2,u} + aA_2)e^v), \\ Z_{vv} = (0, (A_{1,u} + A_1)e^v, (A_{2,u} + A_2)e^v). \end{cases} \tag{A.19}$$

If f is self congruent, Z satisfies the following Gauss equations:

$$\begin{cases} Z_{uu} = \left(\dfrac{\varphi_u}{\varphi} + \rho_u\right) Z_u + \dfrac{a}{\varphi} Z_v, \\ Z_{uv} = -\varphi Z + \rho_v Z_u + \rho_u Z_v, \\ Z_{vv} = \left(\dfrac{\varphi_v}{\varphi} + \rho_v\right) Z_v + \dfrac{b}{\varphi} Z_u. \end{cases} \tag{A.20}$$

From (A.17) and (A.18), the right hand sides of (A.20) become

$$\begin{cases} Z_u + aZ_v = (e^u, \{(2+a)A_{1,u} + 2aA_1\}e^v, \{(2+a)A_{2,u} + 2aA_2\}e^v), \\ -Z + Z_u + Z_v = (0, (2A_{1,u} + aA_1)e^v, (2A_{2,u} + aA_2)e^v), \\ Z_v = (0, (A_{1,u} + A_1)e^v, (A_{2,u} + A_2)e^v) \end{cases} \tag{A.21}$$

respectively. If we compare (A.19) and (A.21), f is self congruent if and only if $a_u = 0$ and $a \neq 2$.

In the case of (ii) in Proposition A.2, by Lemma 5.1, Z becomes a centroaffine surface if and only if

$$2c_1^2 c_2^2 \varphi \neq c_1^3 + c_2^3, \tag{A.22}$$

which is equivalent to (A.16) from (A.12). From the first equation of (3.13), we have

$$Z = \frac{c_2}{\varphi} f_u + \frac{c_1}{\varphi} f_v. \tag{A.23}$$

Then from (A.11) and (A.12), we have

$$Z_u = -c_1 f + \frac{2c_1 c_2}{\varphi} f_u + \left(\frac{c_2}{\varphi^2} + \frac{c_1^2}{\varphi}\right) f_v, \; Z_v = -c_2 f + \frac{2c_1 c_2}{\varphi} f_v + \left(\frac{c_1}{\varphi^2} + \frac{c_2^2}{\varphi}\right) f_u, \tag{A.24}$$

$$\begin{cases} Z_{uu} = -\left(\dfrac{c_2}{\varphi} + c_1^2\right) f + \left(\dfrac{c_1}{\varphi^3} + \dfrac{c_2^2}{\varphi^2} + \dfrac{2c_1^2 c_2}{\varphi}\right) f_u + \left(\dfrac{3c_1 c_2}{\varphi^2} + \dfrac{c_1^3}{\varphi}\right) f_v, \\ Z_{uv} = -2c_1 c_2 f + \left(\dfrac{c_1^2}{\varphi^2} + \dfrac{3c_1 c_2^2}{\varphi} - c_2\right) f_u + \left(\dfrac{c_2^2}{\varphi^2} + \dfrac{3c_1^2 c_2}{\varphi} - c_1\right) f_v. \end{cases} \tag{A.25}$$

From (A.23), (A.24) and (A.25), it is straightforward to see that the first and the second equations of (A.20) are satisfied. We can carry out a similar computation for the third equation of (A.20). Therefore f is self congruent if and only if (A.16) is satisfied. □

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