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Investigating Finite Sample Properties of Estimators for Approximate Factor Models When N Is Small

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Investigating Finite Sample Properties of Estimators for Approximate Factor Models When N Is Small

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Abstract

This paper examines the finite sample properties of estimators for approximate factor models when N is small via simulation study. Although the "rule-of-thumb" for factor models does not support using approximate factor models when N is small, we find that the principal component analysis estimator and quasi-maximum likelihood estimator proposed by Doz et al. (2008) perform very well even in this case. Our findings provide an opportunity for applying approximate factor models to low-dimensional data, which was thought to have been inappropriate for a long time.

JEL Classification : C32, C33, C63

Keywords: Approximate factor model, Principal components, Quasi-maximum likelihood.

1. Introduction

Let us consider the following N dimensional r static factor model;

$$x_t = \Lambda F_t + \varepsilon_t, \qquad t = 1, 2, \ldots, T, \tag{1}$$

where ε_t is an $N \times 1$ idiosyncratic error vector with mean zero and variance Ω, Λ is an $N \times r$ factor loading matrix, and F_t is an unobservable $r \times 1$ factor respectively. It is well known that there are two types of models, depending on the assumptions on idiosyncratic

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errors: one is the "exact factor model," (hereafter, EFM) in which ε_t is cross-sectionally independent so that Ω is diagonal, the other is the "approximate factor model," (hereafter, AFM) in which the limited cross-sectional correlation on ε_t is allowed.

Since factor models are frequently used to extract the co-movements of macroeconomic time series in economic analysis, it appears that AFM is preferable because there exists cross-sectional dependence among the time series in general. However, as Stock and Watson (2002), Bai (2003) and others show, in theory, AFM requires both T and N to go to infinity whereas EFM requires only $T \to \infty$ for estimating Λ and F_t. Then, consider the following question: Is it not possible to employ AFM in practice when we only have low-dimensional data in hand?

In this paper, we examine the finite sample properties of estimators for AFM when N is small via Monte Carlo experiment. Specifically, we investigate the properties of principal component analysis (PCA) estimator, quasi-maximum likelihood (QML) estimator and state-space subspace (SSS) estimator. Our results suggest that PCA and QML estimators perform very well so that AFM is applicable even if N is small in practice. Furthermore, we see that the PCA estimator is robust to the degree of cross-sectional dependence while the QML estimator is suitable for weak-dependence AFM.

2. Monte Carlo Experiment

2.1. Data Genarating Process

We set the basic structure of our DGP to be based on Doz et al. (2008): for $i = 1, 2, \ldots, N,\ t = 1, 2, \ldots, T$,

$$x_{it} = \lambda_i' F_t + \varepsilon_{it}, \tag{2}$$

$$\Phi(L) F_t = u_t, \quad \text{with } u_t \sim i.i.d.\ N(0, D), \tag{3}$$

$$\Gamma(L) \varepsilon_t = v_t, \quad \text{with } v_t \sim i.i.d.\ N(0, \Sigma), \tag{4}$$

where $\lambda_i \sim i.i.d.\ N(0, I_r)$, $\Phi(L) = diag(1 - \phi L, \cdots, 1 - \phi L)$, $\Gamma(L) = diag(1 - \gamma L, \cdots, 1 - \gamma L)$, $D = diag(1 - \phi^2, \cdots, 1 - \phi^2)$ and $\Sigma = \{\sigma_{ij}\}$ $(i, j = 1, 2, \ldots, N)$. Then we assume AR(1) orthogonal r factors with mean zero and unit variance and AR(1) cross-correlated idiosyncratic errors with mean zero. Furthermore, to satisfy the cross-sectional dependence assumptions on ε_t[3], we specify Σ as follows:

$$\sigma_{ij} = \tau^{|i-j|}\left(1-\gamma^2\right)\sqrt{\alpha_i \alpha_j}, \quad 0 \le \tau < 1$$
$$\alpha_i = \frac{\kappa_i}{1-\kappa_i}\lambda_i' \lambda_i.$$

Notice that τ determines the degree of cross-sectional dependence of the idiosyncratic errors while κ_i is idiosyncratic to the total variance ratio of series i $[= var(\varepsilon_{it}) \,/\, var(x_{it})]$. Hence, the larger values of τ and κ produce higher cross-sectional dependent and "noisy" series.

In our experiment, we set $\phi = 0.5, 0.9$, $\gamma = 0, 0.5$, $\tau = 0, 0.2, 0.5, 0.8$, $r = 1$ and $\kappa_i \sim U[0.1, 0.5]$ where $U[a, b]$ signifies a uniform distribution ranging from a to b. Moreover, we divide the series by their sample standard deviation before estimation because the estimators of factors and factor loadings are not scale invariant[4]. It should be mentioned that we assume a much narrower range of κ_i than Doz et al. (2008) that set $\kappa_i \sim U[0.1, 0.9]$. Because their assumption of $E(\kappa_i) = 0.5$ means that we try to extract co-movement from the set of series that consists of 50% of noisy variations on average, it appears to be inappropriate for the small data sets.

2.2. Estimation Methods

We now compare the finite sample properties of the three types of estimators in this study: (i) principal component analysis estimator, (ii) quasi-maximum likelihood estimator, and (iii) state-space subspace estimator.

(i) Principal Component Analysis (PCA) Estimator

The PCA estimator is the most widely used for AFM and its asymptotic properties are investigated rigorously by Stock and Watson (2002), Bai (2003) and many others[5]. The PCA estimators of λ_i and F_t, denoted by $\widetilde{\lambda}_i, \widetilde{F}_t$, minimize the following objective function V subject to r factors being orthogonal to each other:

$$V = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(x_{it} - \lambda_i' F_t\right)^2. \tag{5}$$

[3]See, e.g., Stock and Watson (2002) or Bai(2003) for details.

[4]That is, we define $\widetilde{x}_{it} = x_{it}/s_i$ and use $\widetilde{x}_{it}$ for estimation where $s_i^2 = \frac{1}{T}\sum_t x_{it}^2$.

[5]Forni et.al. (2000) and others study the dynamic principal component (DPCA) estimator for dynamic approximate factor models. However, we consider the static AFM so that DPCA is not treated in this paper.

Obviously, if F_t is known, $\widetilde{\lambda}_i$ is obtained by:

$$\widetilde{\lambda}_i = \frac{1}{T} F' x_i, \tag{6}$$

where $x_i = [x_{i1}, x_{i2}, \ldots, x_{iT}]'$ and $F = [F_1, F_2, \ldots, F_T]'$. Furthermore, substituting (6) into (5), it can be shown that $\widetilde{F}$ is obtained by multiplying $\sqrt{T}$ by the eigenvectors corresponding to the r largest eigenvalues of XX' where $X = [x_1, x_2, \ldots, x_N]$. Then, the PCA estimator is much easier to compute. However, it requires N and T going to infinity for consistency in theory.

(ii) Quasi-Maximum Likelihood (QML) Estimator

Although maximum likelihood (ML) estimation with a Kalman smoother is well known as a standard tool for EFM with fixed N[6], it seems that ML estimation is not suitable for AFM because AFM requires to estimate $N(N-1)/2$ additional parameters compared to EFM. However, Doz et.al. (2008) proposes the QML estimators for factors and factor loadings that are still valid for AFM under similar assumptions to the PCA estimator with $N, T \to \infty$. The word "quasi" comes from the fact that we intentionally misspecify the model as EFM and construct a maximum likelihood function with unautocorrelated idiosyncratic errors.

The estimation procedure for QML is the same as standard ML estimation: we first estimate the parameters in the misspecified model, then we estimate the factors using a Kalman smoother. Since the selection of the initial parameter values plays a crucial role in ML estimation, we consider three types of initial values to achieve a global maximum: (i) the true parameter values, (ii) the PCA estimates of parameters, and (iii) the initial values of ϕ $(\equiv \phi^{(ini)})$ ranging from 0 to 0.90 in increments of 0.1, and we define:

$$\lambda_i^{(ini)} = \lambda_i + \phi^{(ini)},$$
$$\sigma_{ii}^{(ini)} = \sigma_{ii} + \phi^{(ini)}.$$

(iii) State-Space Subspace (SSS) Estimator

The SSS estimator is an alternative estimator for AFM proposed by Kapetanios and Marcellino (2009), which takes a different approach to the PCA estimator. Let $\underline{\mathrm{X}}^p = \left[X^p_{\ell+1}, X^p_{\ell+2}, \ldots, X^p_{T-(s-1)}\right]'$

[6]See Anderson (2003) for time-independent factors and idiosyncratic errors, and Kim and Nelson (1999) for auto-correlated ones, for example.

and $\underline{X}^f = \left[X^f_{\ell+1}, X^f_{\ell+2}, \dots, X^f_{T-(s-1)}\right]'$, where $X^p_t = [x_{t-1}, x_{t-2}, \dots, x_{t-\ell}]'$ and $X^f_t = [x_t, x_{t+1}, \dots, x_{t-s+1}]'$. Furthermore, we denote the singular value decomposition of $X^f X^p (X^{p'} X^p)^+$ by USV' where A^+ represents the Moore–Penrose generalized inverse matrix of A. The SSS estimator of F_t is obtained by:

$$F_t^{SSS} = \mathscr{K} X_t^p,$$

where $\mathscr{K} = S_k^{1/2} V_k'$, S_k is the leading $k \times k$ submatrix of S, V_k is the first k columns of V and k is the number of factors. We must specify the lead and lag truncation parameters s and ℓ in the estimation; therefore we employ $s = 1$ and $p = (\log(T))^{1.25}$ following Kapetanios and Marcellino(2009).

The SSS estimator is a clever tool for estimating the factors because the estimation procedure is straightforward and easy to implement. Moreover, Kapetanios and Marcellino (2009) prove the consistency of the SSS estimator assuming that N and T tend to infinity with $N = o(T^{1/6})$. Although this divergence rate of N is somewhat restrictive for standard AFM in which both N and T are large, it would be appropriate for our study in which N is much smaller than T.

2.3. Criteria for Goodness of Fit

To evaluate the finite sample properties of the estimators, we employ the following two criteria:

$$AMSE = \frac{1}{I} \sum_{q=1}^{I} MSE^{(q)}, \tag{7}$$

$$AR_F^2 = \frac{1}{I} \sum_{q=1}^{I} R_F^{2\,(q)}, \tag{8}$$

where I is the total number of replications, $MSE^{(q)}$ and $R_F^{2\,(q)}$ are qth realizations of $MSE = \dfrac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(\lambda_i' F_t - \widehat{\lambda}_i' \widehat{F}_t\right)^2$ and $R_F^2 = \frac{Tr\left[F'\widehat{F}\left(\widehat{F}'\widehat{F}\right)^{-1}\widehat{F}'F\right]}{Tr[F'F]}$, and $\widehat{\lambda}_i, \widehat{F}_t, \widehat{F}$ denote the estimators of λ_i, F_t, F, respectively. $AMSE$ measures a goodness of fit of the common component $(= \lambda_i' F_t)$ while AR_F^2 is a criterion for the fitness of the unobserved factor F_t[7].

[7] R_F^2 is analogous to the coefficient of determination in a standard linear regression model. Boivin and Ng (2006) and Doz et.al. (2008) also use R_F^2 for evaluation. It should be mentioned that λ_i and F_t are not separately identifiable in general unless we impose additional restrictions on the model such as those proposed by Bai and Ng (2010). Then, we cannot compare directly $\widehat{F}_t$ with F_t for evaluation.

Note that a smaller value of $AMSE$ means a better fit of the common component while a larger value of AR_F^2 indicates a better estimate of the unobserved factor.

2.4. Simulation Results

We investigate the finite sample performance of the PCA, QML and SSS estimators when $N = 5, 10$, $T = 100, 200$, with $I = 500$. All results are obtained by using GAUSS and CML routine for ML estimation. The results are given in Tables 1 and 2. The values in parentheses are relative values scaled to that of the PCA estimator.

To begin with, let us consider the case of $N = 5$ and $T = 100$. The simulation result for this case is summarized in (i) of Table 1. Here we obtain the following findings. First, the PCA and QML estimators dominate the SSS estimator as a whole and perform quite well, whereas their availabilities are justified when N is so large in theory. This implies that AFM would be applicable even when only low dimensional data are available. Second, the QML estimator dominates the PCA estimator when $\tau = 0$ and 0.2, while the PCA estimator outperforms the QML estimator when $\tau = 0.5$ and 0.8. These findings tell us that the PCA estimator is suitable for strongly dependent AFM while the QML estimator is appropriate for EFM or weakly dependent AFM. Then, we cannot conclude which is the better estimator from our results. However, PCA would be useful when we have no firm beliefs about the dependence among series in hand because our result provides evidence that the PCA estimator is robust to the value of τ.

Next, we consider the other cases. We can confirm that the findings obtained above hold in these cases. Note that although the efficiency gains with respect to N and T boost the performance of the estimators as we expected, it appears that there are no drastic improvements on performances especially in the factor estimates. As a whole, we conclude that the PCA and QML estimators perform very well even when N is very small.

3. Concluding Remarks

In this paper, we have investigated the finite sample properties of estimators for approximate factor models when N is small via Monte Carlo experiment. We obtained the following findings from our study. First, the PCA and QML estimators perform very well even when N is small, which overturns the “rule-of-thumb” for factor analysis. Second, the

PCA estimator is suitable for strongly dependent approximate factor models while the QML estimator is appropriate for exact or weakly dependent approximate factor models. Moreover, we also found that the PCA estimator is more robust to the degree of cross sectional dependence of the series than the QML estimator. We expect that our results will provide an opportunity for applying approximate factor models to low-dimensional data, which was previously thought to be inappropriate.

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Table 1: Finite sample performance of estimators when $T = 100$

$(i) \quad N = 5, \ \ T = 100$

$(\phi,\ \gamma)$		$\tau = 0$				$\tau = 0.2$				$\tau = 0.5$				$\tau = 0.8$			
		$AMSE$		AR_F^2		$AMSE$		AR_F^2		$AMSE$		AR_F^2		$AMSE$		AR_F^2	
	PCA	0.061	(1.000)	0.923	(1.000)	0.062	(1.000)	0.921	(1.000)	0.061	(1.000)	0.923	(1.000)	0.086	(1.000)	0.904	(1.000)
(0.5, 0)	QML	0.049	(0.803)	0.933	(1.012)	0.053	(0.860)	0.928	(1.007)	0.049	(0.803)	0.933	(1.012)	0.177	(2.047)	0.800	(0.885)
	SSS	0.391	(6.446)	0.446	(0.483)	0.391	(6.309)	0.446	(0.484)	0.391	(6.446)	0.446	(0.483)	0.406	(4.697)	0.434	(0.480)
	PCA	0.064	(1.000)	0.914	(1.000)	0.066	(1.000)	0.912	(1.000)	0.074	(1.000)	0.905	(1.000)	0.103	(1.000)	0.883	(1.000)
(0.9, 0)	QML	0.038	(0.595)	0.945	(1.034)	0.040	(0.612)	0.942	(1.033)	0.061	(0.819)	0.917	(1.013)	0.164	(1.582)	0.811	(0.918)
	SSS	0.148	(2.303)	0.779	(0.852)	0.147	(2.237)	0.778	(0.854)	0.163	(2.207)	0.758	(0.838)	0.243	(2.348)	0.659	(0.747)
	PCA	0.063	(1.000)	0.923	(1.000)	0.064	(1.000)	0.922	(1.000)	0.069	(1.000)	0.917	(1.000)	0.089	(1.000)	0.903	(1.000)
(0.5, 0.5)	QML	0.058	(0.917)	0.925	(1.003)	0.063	(0.984)	0.918	(0.997)	0.100	(1.447)	0.875	(0.954)	0.187	(2.098)	0.792	(0.877)
	SSS	0.398	(6.356)	0.445	(0.483)	0.399	(6.259)	0.445	(0.482)	0.402	(5.794)	0.441	(0.481)	0.419	(4.712)	0.428	(0.474)
	PCA	0.068	(1.000)	0.914	(1.000)	0.070	(1.000)	0.912	(1.000)	0.078	(1.000)	0.905	(1.000)	0.108	(1.000)	0.883	(1.000)
(0.9, 0.5)	QML	0.057	(0.839)	0.925	(1.012)	0.060	(0.865)	0.921	(1.010)	0.087	(1.119)	0.890	(0.983)	0.178	(1.655)	0.802	(0.909)
	SSS	0.163	(2.397)	0.768	(0.840)	0.166	(2.375)	0.764	(0.838)	0.190	(2.442)	0.735	(0.812)	0.282	(2.616)	0.629	(0.713)

$(ii) \quad N = 10, \ \ T = 100$

$(\phi,\ \gamma)$		$\tau = 0$				$\tau = 0.2$				$\tau = 0.5$				$\tau = 0.8$			
		$AMSE$		AR_F^2		$AMSE$		AR_F^2		$AMSE$		AR_F^2		$AMSE$		AR_F^2	
	PCA	0.032	(1.000)	0.960	(1.000)	0.032	(1.000)	0.960	(1.000)	0.034	(1.000)	0.959	(1.000)	0.041	(1.000)	0.953	(1.000)
(0.5, 0)	QML	0.027	(0.844)	0.966	(1.006)	0.028	(0.870)	0.965	(1.005)	0.037	(1.114)	0.953	(0.994)	0.106	(2.579)	0.876	(0.919)
	SSS	0.216	(6.831)	0.698	(0.727)	0.215	(6.749)	0.699	(0.728)	0.217	(6.443)	0.697	(0.727)	0.230	(5.591)	0.684	(0.717)
	PCA	0.033	(1.000)	0.956	(1.000)	0.033	(1.000)	0.956	(1.000)	0.035	(1.000)	0.955	(1.000)	0.045	(1.000)	0.946	(1.000)
(0.9, 0)	QML	0.023	(0.713)	0.969	(1.013)	0.024	(0.726)	0.968	(1.013)	0.030	(0.864)	0.960	(1.006)	0.093	(2.063)	0.887	(0.937)
	SSS	0.085	(2.579)	0.878	(0.918)	0.086	(2.597)	0.876	(0.917)	0.107	(3.055)	0.848	(0.888)	0.259	(5.731)	0.666	(0.704)
	PCA	0.033	(1.000)	0.960	(1.000)	0.034	(1.000)	0.960	(1.000)	0.035	(1.000)	0.959	(1.000)	0.045	(1.000)	0.951	(1.000)
(0.5, 0.5)	QML	0.030	(0.915)	0.964	(1.004)	0.032	(0.942)	0.963	(1.002)	0.042	(1.201)	0.949	(0.990)	0.116	(2.579)	0.868	(0.913)
	SSS	0.217	(6.502)	0.702	(0.731)	0.216	(6.440)	0.702	(0.731)	0.218	(6.159)	0.701	(0.731)	0.241	(5.353)	0.674	(0.709)
	PCA	0.037	(1.000)	0.957	(1.000)	0.037	(1.000)	0.957	(1.000)	0.039	(1.000)	0.955	(1.000)	0.066	(1.000)	0.929	(1.000)
(0.9, 0.5)	QML	0.034	(0.909)	0.962	(1.005)	0.035	(0.927)	0.961	(1.004)	0.043	(1.098)	0.950	(0.995)	0.128	(1.941)	0.856	(0.922)
	SSS	0.090	(2.437)	0.878	(0.918)	0.091	(2.435)	0.878	(0.917)	0.106	(2.698)	0.857	(0.897)	0.245	(3.727)	0.687	(0.739)

Table 2: Finite sample performance of estimators when $T = 200$

(i) $N = 5, \quad T = 200$

$(\phi,\ \gamma)$		$\tau = 0$				$\tau = 0.2$				$\tau = 0.5$				$\tau = 0.8$			
		$AMSE$		AR_F^2		$AMSE$		AR_F^2		$AMSE$		AR_F^2		$AMSE$		AR_F^2	
(0.5, 0)	PCA	0.060	(1.000)	0.923	(1.000)	0.061	(1.000)	0.921	(1.000)	0.060	(1.000)	0.923	(1.000)	0.084	(1.000)	0.905	(1.000)
	QML	0.046	(0.775)	0.935	(1.014)	0.051	(0.825)	0.930	(1.009)	0.046	(0.775)	0.935	(1.014)	0.176	(2.104)	0.799	(0.882)
	SSS	0.443	(7.423)	0.373	(0.404)	0.442	(7.224)	0.373	(0.405)	0.443	(7.423)	0.373	(0.404)	0.447	(5.354)	0.371	(0.410)
(0.9, 0)	PCA	0.061	(1.000)	0.919	(1.000)	0.062	(1.000)	0.918	(1.000)	0.069	(1.000)	0.913	(1.000)	0.089	(1.000)	0.898	(1.000)
	QML	0.035	(0.569)	0.951	(1.034)	0.036	(0.583)	0.948	(1.033)	0.053	(0.777)	0.928	(1.016)	0.150	(1.679)	0.826	(0.920)
	SSS	0.149	(2.443)	0.783	(0.852)	0.149	(2.381)	0.784	(0.854)	0.150	(2.189)	0.782	(0.857)	0.205	(2.297)	0.709	(0.790)
(0.5, 0.5)	PCA	0.060	(1.000)	0.923	(1.000)	0.061	(1.000)	0.922	(1.000)	0.067	(1.000)	0.917	(1.000)	0.085	(1.000)	0.905	(1.000)
	QML	0.052	(0.864)	0.929	(1.006)	0.057	(0.926)	0.922	(1.001)	0.093	(1.393)	0.879	(0.959)	0.182	(2.156)	0.792	(0.875)
	SSS	0.448	(7.474)	0.371	(0.402)	0.448	(7.316)	0.371	(0.402)	0.451	(6.748)	0.369	(0.402)	0.463	(5.467)	0.359	(0.397)
(0.9, 0.5)	PCA	0.062	(1.000)	0.920	(1.000)	0.064	(1.000)	0.918	(1.000)	0.070	(1.000)	0.913	(1.000)	0.092	(1.000)	0.897	(1.000)
	QML	0.049	(0.786)	0.934	(1.015)	0.052	(0.811)	0.930	(1.013)	0.074	(1.057)	0.904	(0.989)	0.165	(1.807)	0.811	(0.904)
	SSS	0.160	(2.569)	0.774	(0.841)	0.160	(2.506)	0.774	(0.843)	0.169	(2.412)	0.764	(0.836)	0.257	(2.810)	0.655	(0.730)

(ii) $N = 10, \quad T = 200$

$(\phi,\ \gamma)$		$\tau = 0$				$\tau = 0.2$				$\tau = 0.5$				$\tau = 0.8$			
		$AMSE$		AR_F^2		$AMSE$		AR_F^2		$AMSE$		AR_F^2		$AMSE$		AR_F^2	
(0.5, 0)	PCA	0.030	(1.000)	0.961	(1.000)	0.030	(1.000)	0.960	(1.000)	0.032	(1.000)	0.958	(1.000)	0.040	(1.000)	0.951	(1.000)
	QML	0.024	(0.814)	0.967	(1.007)	0.025	(0.838)	0.966	(1.006)	0.034	(1.066)	0.954	(0.996)	0.103	(2.547)	0.875	(0.920)
	SSS	0.324	(10.850)	0.541	(0.563)	0.324	(10.698)	0.541	(0.563)	0.324	(10.042)	0.541	(0.564)	0.329	(8.146)	0.536	(0.563)
(0.9, 0)	PCA	0.031	(1.000)	0.959	(1.000)	0.031	(1.000)	0.958	(1.000)	0.033	(1.000)	0.956	(1.000)	0.043	(1.000)	0.947	(1.000)
	QML	0.021	(0.682)	0.971	(1.013)	0.021	(0.693)	0.970	(1.013)	0.027	(0.816)	0.963	(1.008)	0.087	(2.013)	0.893	(0.942)
	SSS	0.105	(3.441)	0.848	(0.884)	0.105	(3.400)	0.847	(0.884)	0.109	(3.295)	0.842	(0.881)	0.217	(4.993)	0.704	(0.743)
(0.5, 0.5)	PCA	0.031	(1.000)	0.961	(1.000)	0.031	(1.000)	0.960	(1.000)	0.033	(1.000)	0.958	(1.000)	0.041	(1.000)	0.952	(1.000)
	QML	0.027	(0.868)	0.965	(1.005)	0.028	(0.894)	0.964	(1.004)	0.038	(1.136)	0.952	(0.993)	0.108	(2.633)	0.871	(0.915)
	SSS	0.327	(10.601)	0.540	(0.562)	0.327	(10.459)	0.540	(0.562)	0.328	(9.892)	0.539	(0.562)	0.334	(8.148)	0.533	(0.560)
(0.9, 0.5)	PCA	0.033	(1.000)	0.959	(1.000)	0.033	(1.000)	0.958	(1.000)	0.035	(1.000)	0.956	(1.000)	0.046	(1.000)	0.947	(1.000)
	QML	0.028	(0.857)	0.965	(1.006)	0.029	(0.872)	0.963	(1.005)	0.036	(1.028)	0.954	(0.998)	0.101	(2.201)	0.880	(0.930)
	SSS	0.110	(3.350)	0.846	(0.882)	0.110	(3.314)	0.845	(0.882)	0.116	(3.273)	0.838	(0.876)	0.198	(4.326)	0.737	(0.778)