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We study theory of curves in the complex hyperbola and show that special motions of curves are linked with the Burgers hierarchy, which also leads to a Hamiltonian formulation of the hierarchy and to the difference Burgers equation via their discretization.
2. Curves in the complex hyperbola

The complex hyperbola $C$ is the subset of the complex 2-plane $\mathbb{C}^2$ given by

$$C = \{ (z, w) \in \mathbb{C}^2 \mid zw = 1 \}.$$ 

In this paper we call an immersion from an interval $I$ to $C$ a curve in $C$. We put $\gamma = (z, w)$ for a curve in $C$. Since

$$\frac{z'}{z} = -\frac{w'}{w},$$

we have $z', w' \neq 0$. Since

$$\det \begin{pmatrix} \gamma \\ \gamma' \end{pmatrix} = -2\frac{z'}{z} \neq 0,$$

$\gamma$ and $\gamma'$ are linearly independent over $\mathbb{C}$.

By a direct calculation, we have

$$(2.1) \quad \gamma'' = \tau^2 \gamma + \sqrt{-1} \kappa \gamma',$$

where

$$\tau = \frac{z'}{z}, \quad \kappa = -\sqrt{-1} \frac{\tau'}{\tau}.$$ 

Changing $\gamma$ to $\tilde{\gamma} = (\alpha z^\beta, \alpha^{-1} w^\beta)$ with $\alpha, \beta \in \mathbb{C}^\times$, we have

$$\tilde{\gamma}'' = (\beta \tau)^2 \tilde{\gamma} + \sqrt{-1} \kappa \tilde{\gamma}'.$$

Hence $\kappa$ is invariant under the change. We call $\kappa$ the curvature of $\gamma$.

REMARK 2.1. When a curve $\gamma$ is given from an arclength parametrized curve $\hat{\gamma}$ in the Euclidean 2-space $\mathbb{R}^2$ identified with the complex plane $\mathbb{C}$ by $\gamma = (e^{\hat{\gamma}}, e^{-\hat{\gamma}})$, then $\kappa$ coincides with the curvature of $\hat{\gamma}$, i.e., $\hat{\gamma}'' = \sqrt{-1} \kappa \hat{\gamma}'$ holds.

The fundamental theorem for curves in $C$ is stated as follows:

**Proposition 2.2.** For any function $\kappa: I \to \mathbb{C}$, there exists a curve $\gamma = (z, w)$ in $C$ with curvature $\kappa$. If $\tilde{\gamma} = (\tilde{z}, \tilde{w}): I \to C$ is another curve with curvature $\kappa$, there exist constants $\alpha, \beta \in \mathbb{C}^\times$ such that $\tilde{z} = \alpha z^\beta$. 
Remark 2.3. As in the theory of curves in the Euclidean 2-space, a curve $\gamma$ in $C$ with curvature $\kappa$ can be given explicitly:

$$\gamma = (z, z^{-1}), \quad z = \exp \left( \int \exp \left( -\sqrt{-1} \int \kappa \, ds \right) \, ds \right).$$

A curve $\gamma$ in $C$ with zero curvature is given by $\gamma = (\alpha e^{\beta s}, \alpha^{-1} e^{-\beta s})$ with $\alpha, \beta \in \mathbb{C}^\times$.

3. Motions of curves and the Burgers hierarchy

A motion of a curve in $C$ is given by a map $\gamma = \gamma(s, t)$ from the product of intervals $I$ and $J$ to $C$, where $t \in J$ is considered to be a time parameter and $s \in I$ is a parameter of a curve with fixed time. The time evolution of $\gamma$ is given by

$$\gamma_t = \lambda \gamma + \mu \gamma_s,$$

where $\lambda, \mu : I \times J \to \mathbb{C}$. Since $(zw)_t = 0$, we have $\lambda = 0$ from (3.1). Hence we have

$$\gamma_t = \mu \gamma_s.$$ (3.2)

Differentiating (3.2) by $s$ and using (2.1), we have

$$\gamma_{ss} = \tau^2 \mu \gamma + (\mu_s + \sqrt{-1} \kappa \mu) \gamma_s.$$ (3.3)

Proposition 3.1. The above time evolution exists if and only if $\kappa \neq 0$ and

$$\tau_s = \tau \mu_s + \tau_s \mu.$$ (3.4)

Proof. We have only to calculate the integrability condition:

$$\gamma_{sss} = \gamma_{tss}.$$ (3.5)

From (2.1) and (3.3), we have

$$\gamma_{ss} = (2 \tau \tau_t + \sqrt{-1} \kappa \tau^2 \mu) \gamma + (\tau^2 \mu + \sqrt{-1} \kappa \mu_s + \kappa^2 \mu) \gamma_s,$$

$$\gamma_{sss} = (2 \tau \tau_s \mu + 2 \tau^2 \mu_s + \sqrt{-1} \kappa \tau^2 \mu) \gamma + (\tau^2 \mu + \mu_{ss} + \sqrt{-1} \kappa \mu_s + 2 \sqrt{-1} \kappa \mu_s + \kappa^2 \mu) \gamma_s.$$ (3.5)

Comparing the coefficients of $\gamma$ and $\gamma_s$ in the right-hand sides, we have (3.4) and

$$\sqrt{-1} \kappa_t = \mu_{ss} + \sqrt{-1} \kappa \mu_s + \sqrt{-1} \kappa_s \mu.$$ (3.5)

Since

$$\kappa = -\sqrt{-1} (\log \tau)_s,$$ (3.6)

(3.5) is also derived from (3.4).
Note that (3.5) can be written as

$$\kappa_t = \Omega \mu_s,$$

where $\Omega = (-\sqrt{-1}D_s + \kappa + \kappa_s D_s^{-1})$ is the recursion operator of the Burgers equation:

$$\kappa_t = -\sqrt{-1}\kappa_{ss} + 2\kappa \kappa_s.$$

Moreover, it is not so hard to see by induction on $n \in \mathbb{N}$ that

$$(3.7) \quad D_s^{-1} \Omega^{n-1} \kappa_s = (-\sqrt{-1})^n D^n \tau.$$

Then we have the following:

**Theorem 3.2.** The curvature of a curve in $C$ associated to the time evolution:

$$\gamma_t = (D_s^{-1} \Omega^{n-1} \kappa_s) \gamma_s \quad (n \in \mathbb{N})$$

evolves according to the Burgers hierarchy:

$$(3.8) \quad \kappa_t = \Omega^n \kappa_s.$$

In particular, $\kappa$ and $\tau$ are related by the Cole-Hopf transformation (3.6) and $\tau$ evolves according to the equation:

$$(3.9) \quad \tau_t = (-\sqrt{-1})^n D^{n+1} \tau.$$

**Remark 3.3.** From (3.4), $\log \tau$ evolves according to the equation:

$$(3.10) \quad (\log \tau)_t = \mu_s + \mu (\log \tau)_s.$$

When $\mu = \sqrt{-1} \kappa$, (3.10) becomes the potential Burgers equation:

$$(\log \tau)_t = (\log \tau)_{ss} + (\log \tau)^2_s.$$

When $\mu = \log \tau$, (3.10) becomes the equation:

$$(\log \tau)_t = (\log \tau)_s + (\log \tau)(\log \tau)_s,$$

whose solutions are given explicitly:

$$\log \tau + 1 = \varphi(s + t (\log \tau + 1)),$$

where $\varphi$ is an arbitrary function.
Remark 3.4. We can reduce a curve \( \gamma \) in \( C \) to a curve \( p \) in \( C^\times \) by putting \( p = z \). Then \( \tau \) plays a role of the curvature of \( p \). If \( \tilde{p}: I \rightarrow C^\times \) is another curve with curvature \( \tau \), there exists a constant \( \alpha \in C^\times \) such that \( \tilde{p} = \alpha p \). The time evolution of \( \gamma \) reduces to that of \( p \):

\[
p_t = (\tau \mu)p,
\]

which satisfies (3.4).

4. A formulation as a Hamiltonian system

In this section, we give a formal Hamiltonian system describing the motion of closed curves in the complex hyperbola \( C \) given in Theorem 3.2.

We denote by \( M \) the space of all closed curves in \( C \), that is, the set of all curves \( \gamma: S^1 \rightarrow C \), where

\[
S^1 = \mathbb{R}/2\pi\mathbb{Z} \cong \{z \in \mathbb{C} \mid |z| = 1\}.
\]

From (3.2), the tangent space of \( M \) at \( \gamma \) can be identified with the set \( \{\mu \gamma' \mid \mu: S^1 \rightarrow C\} \). We note that any tangent vector \( \mu \gamma' \) with \( \mu: S^1 \rightarrow \mathbb{R} \) comes from an action of diffeomorphism group of the unit circle Diff(\( S^1 \)) by

\[
\varphi \cdot \gamma = \gamma \circ \varphi \quad (\varphi \in \text{Diff}(S^1), \gamma \in M).
\]

Indeed, if \( \varphi_t \) is a curve in Diff(\( S^1 \)) such that

\[
\varphi_0 = \text{identity}, \quad \frac{d}{dt} \bigg|_{t=0} \varphi_t = \mu \frac{d}{ds},
\]

we have

\[
\frac{d}{dt} \bigg|_{t=0} (\varphi_t \cdot \gamma) = \mu \gamma'.
\]

We define a 2-form \( \omega \) on \( M \) as follows:

\[
\omega(\mu_1 \gamma', \mu_2 \gamma') = \text{Im} \int_{S^1} \mu_1 \mu_2 |\tau|^2 \, ds \quad (\mu_1, \mu_2: S^1 \rightarrow \mathbb{C}).
\]

By a direct calculation, we can verify that \( \omega \) is closed, hence it defines a ‘symplectic structure’ on \( M \). For \( n \in \mathbb{N} \) we define a function on \( M \) by

\[
H_n(\gamma) = \frac{(-\sqrt{-1})^{n-1}}{2} \int_{S^1} \tau^{(n-1)} \bar{\tau} \, ds \quad (\gamma \in M).
\]

Then we have the following:
Theorem 4.1. The Hamiltonian vector field $X_n$ for $H_n$ with respect to $\omega$ is given by

$$(X_n)_\gamma = (D_s^{-1}\Omega^{n-1}\kappa')\gamma' \quad (\gamma \in \mathcal{M}).$$

Hence $H_n$ is the Hamiltonian of the motion of curves in $C$ associated with (3.8).

Proof. For a one-parameter family $\gamma(\cdot, t) \in \mathcal{M}$ such that $\gamma_t = \mu \gamma_s$, we have

$$\frac{d}{dt} H_n(\gamma) = \frac{(-\sqrt{-1})^{n-1}}{2} \int_{S^1} \frac{d}{dt} [(D_s^{-1}\tau)\bar{\tau}] \, ds$$

$$= \frac{(-\sqrt{-1})^{n-1}}{2} \int_{S^1} [(D_s^{-1}\tau_1)\bar{\tau} + (D_s^{-1}\tau)\bar{\tau}_1] \, ds$$

$$= \frac{(-\sqrt{-1})^{n-1}}{2} \int_{S^1} [(D_s^n(\tau\mu)]\bar{\tau} + (D_s^{-1}\tau)D_s(\tau\bar{\mu})] \, ds$$

(by Proposition 3.1)

$$= \frac{(-\sqrt{-1})^{n-1}}{2} \int_{S^1} [(-1)^n(D_s^n\tau)|\tau\mu - (D_s^n\tau)|\tau\bar{\mu}] \, ds$$

$$= \text{Im} \int_{S^1} (-\sqrt{-1})^n(D_s^n\tau)|\tau\bar{\mu} \, ds.$$

On the other hand,

$$\omega((D_s^{-1}\Omega^{n-1}\kappa_s)\gamma_s, \mu \gamma_s) = \text{Im} \int_{S^1} (D_s^{-1}\Omega^{n-1}\kappa_s)|\tau| \, ds$$

$$= \text{Im} \int_{S^1} (-\sqrt{-1})^n(D_s^n\tau)|\tau\bar{\mu} \, ds$$

by (3.7).

Remark 4.2. The symplectic structure $\omega$ on $\mathcal{M}$ comes from a flat Kähler structure. In fact, $\mathcal{M}$ has a natural complex structure $T\mathcal{M} \ni \mu \gamma' \mapsto \sqrt{-1}\mu \gamma' \in T\mathcal{M}$, and $\omega$ is the fundamental 2-form of a Hermitian metric $h$ defined by

$$h(\mu_1 \gamma', \mu_2 \gamma') = 2 \int_{S^1} \mu_1 \bar{\mu}_2 |\tau| \, ds.$$

For a vector field $\tilde{\mu} \gamma_s$ along a path $\gamma(\cdot, t)$ of $\mathcal{M}$ with the velocity vector $\gamma_t = \mu \gamma_s$, we set

$$\nabla_{\mu \gamma_s}(\tilde{\mu} \gamma_s) = (\tilde{\mu}_t + (\log \tau)_t \tilde{\mu}) \gamma_s.$$

Then, one can verify that $\nabla$ is the Levi-Civita connection for $h$ and $(\mathcal{M}, h)$ is flat as a Riemannian manifold.
5. Discretization

The Burgers equation can be discretized in terms of soliton theory ([8]). For a function \( f(x) \), the advanced and the central difference operators \( \Delta_{+x} \) and \( \Delta_x \) are defined by

\[
\Delta_{+x} f(x) = \frac{f(x + \Delta x) - f(x)}{\Delta x}, \quad \Delta_x f(x) = \frac{f(x + \Delta x/2) - f(x - \Delta x/2)}{\Delta x}.
\]

Using these operators, we discretize (3.9) as

\[
\Delta_x \tau(s, t) = \begin{cases} 
(\sqrt{-1})^n \Delta_x^n \tau(s, t) & (n = 2m - 1, \ m \in \mathbb{N}), \\
(\sqrt{-1})^n \Delta_x \Delta_x^n \tau(s, t) & (n = 2m, \ m \in \mathbb{N}).
\end{cases}
\]

By setting \( \tau_{i,j} = \tau(i \Delta x, j \Delta t) \ (i \in \mathbb{Z}, \ j \in \mathbb{N}) \), we have a more explicit equation:

\[
\tau_{i,j+1} - \tau_{i,j} = \begin{cases} 
\left(\sqrt{-1}\right)^n \Delta_t \sum_{k=-m}^m (-1)^{m+k} \binom{n + 1}{m+k} \tau_{i+k,j} & (n = 2m - 1), \\
\left(\sqrt{-1}\right)^n \Delta_t \sum_{k=-m}^{m+1} (-1)^{m+k+1} \binom{n + 1}{m+k+1} \tau_{i+k,j} & (n = 2m).
\end{cases}
\]

If we put

\[
\alpha = (-1)^n \left(\sqrt{-1}\right)^n \Delta t \left(\Delta s\right)^{n+1}, \quad c_k = (-1)^{|k|} \binom{n + 1}{m + k},
\]

we have

\[
\tau_{i,j+1} = \begin{cases} 
(1 + c_0 \alpha) \tau_{i,j} + \alpha \sum_{k=-m}^{-1} c_k \tau_{i+k,j} + \alpha \sum_{k=1}^m c_k \tau_{i+k,j} & (n = 2m - 1), \\
(1 - c_0 \alpha) \tau_{i,j} - \alpha \sum_{k=-m}^{m+1} c_k \tau_{i+k,j} - \alpha \sum_{k=1} c_k \tau_{i+k,j} & (n = 2m).
\end{cases}
\]

The difference Cole-Hopf transformation given by

\[
\kappa_{i,j} = -\sqrt{-1} \frac{\tau_{i+1,j}}{\tau_{i,j}}
\]
leads to the difference Burgers hierarchy:

\[(5.2) \quad \kappa_{i,j+1} = \kappa_{i,j} A_{i,j},\]

where

\[
A_{i,j} = \frac{1 + c_0 \alpha + \alpha \sum_{k=1}^{m} c_k \prod_{l=1}^{k} (\sqrt{-1} \kappa_{i+l,j}) + \alpha \sum_{k=-m}^{1} c_k \prod_{l=-k}^{0} (\sqrt{-1} \kappa_{i+l,j})^{-1}}{1 + c_0 \alpha + \alpha \sum_{k=1}^{m} c_k \prod_{l=1}^{k} (\sqrt{-1} \kappa_{i+l,j}) - \alpha \sum_{k=-m}^{1} c_k \prod_{l=-k}^{0} (\sqrt{-1} \kappa_{i+l,j})^{-1}}
\]

when \(n = 2m - 1\), and

\[
A_{i,j} = \frac{1 - c_0 \alpha - \alpha \sum_{k=1}^{m+1} c_k \prod_{l=k}^{1} (\sqrt{-1} \kappa_{i+l,j}) - \alpha \sum_{k=-m}^{1} c_k \prod_{l=-k}^{0} (\sqrt{-1} \kappa_{i+l,j})^{-1}}{1 - c_0 \alpha - \alpha \sum_{k=1}^{m+1} c_k \prod_{l=k}^{1} (\sqrt{-1} \kappa_{i+l,j}) + \alpha \sum_{k=-m}^{1} c_k \prod_{l=-k}^{0} (\sqrt{-1} \kappa_{i+l,j})^{-1}}
\]

when \(n = 2m\).

A discrete curve in \(C\) is a map \(\gamma_i = (z_i, u_i) (i \in \mathbb{Z})\) from integers \(\mathbb{Z}\) to \(C\) such that \(\gamma_i \neq \gamma_{i+1}\). The discrete curvature \(\kappa_i\) is given by

\[
\kappa_i = -\sqrt{-1} \frac{\log(z_{i+2}/z_{i+1})}{\log(z_{i+1}/z_i)} = -\sqrt{-1} \frac{\log(w_{i+2}/w_{i+1})}{\log(w_{i+1}/w_i)}.
\]

We define the discrete time evolution of a discrete curve in \(C\) by

\[(5.3) \quad z_{i,j+1} = \begin{cases} 
z_{i,j} \prod_{k=m}^{m+1} (z_{i+k,j})^{c_k \alpha} & (n = 2m - 1), \\
z_{i,j} \prod_{k=-m}^{m} (z_{i+k,j})^{-c_k \alpha} & (n = 2m).
\end{cases}\]

The right-hand side is determined independently of a choice of branch of \(\log\) because the sum of the exponents \(c_k \alpha\) is equal to zero. We can easily verify that \(\tau_{i,j} = \log(z_{i+1,j}/z_{i,j})\) of a curve evolving according to (5.3) satisfies (5.1), thereby we have the following:

**Theorem 5.1.** The discrete curvature of a discrete curve in \(C\) associated to the discrete time evolution (5.3) evolves according to the difference Burgers hierarchy (5.2), while \(\tau_{i,j}\) evolves according to the difference equation (5.1).

**Remark 5.2.** It is obvious to see that the discrete time evolution (5.3) keeps the periodicity of curves in \(C\).

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