BONNET SURFACES WITH NON-FLAT NORMAL BUNDLE IN THE HYPERBOLIC FOUR-SPACE

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Abstract

We study surfaces in the hyperbolic four-space admitting isometric deformations preserving the length of the mean curvature vector and especially focus on the case that surfaces are non-minimal and have non-flat normal bundle.

1. Introduction

There has been a long history of study of surfaces in 3-dimensional space forms admitting isometric deformations preserving the mean curvature (see [1, 5, 6, 7] and references therein), which can be traced back to the following result due to Bonnet [2].

Proposition 1.1. If a surface in a 3-dimensional space form has constant mean curvature and is not totally umbilic, then it admits isometric deformations preserving the mean curvature.

In the previous paper [4] the author studied surfaces in 4-dimensional space forms admitting isometric deformations preserving the length of

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the mean curvature vector, called Bonnet surfaces after Bonnet’s work, and obtained a generalization of Chen-Yau’s reduction theorem for surfaces with parallel mean curvature vector [3].

In this paper we study Bonnet surfaces in the hyperbolic 4-space and obtain an example of non-minimal surfaces with non-flat normal bundle.

2. Preliminaries

We denote the hyperbolic 4-space of curvature \( c < 0 \) by \( H^4(c) \), which is described as follows:

\[
H^4(c) = \text{a connected component of } \left\{ \mathbf{x} \in \mathbb{R}^5 \mid \langle \mathbf{x}, \mathbf{x} \rangle = \frac{1}{c} \right\},
\]

where \( \langle \cdot, \cdot \rangle \) is the Lorentzian inner product on \( \mathbb{R}^5 \) with signature \((1,4)\).

We assume that surfaces are sufficiently smooth. Any surface in \( H^4(c) \) is given by a conformal immersion \( F \) from a Riemann surface \( M \) to \( H^4(c) \). Using a local holomorphic coordinate \( z \), we write the induced metric on \( M \) as \( e^{\omega}dzd\bar{z} \).

Let \( N_1 \) and \( N_2 \) be orthogonal unit normals to \( F \). Then the Gauss-Weingarten equations are

\[
\begin{align*}
F_{zz} &= \omega_2 F_z + Q_1 N_1 + Q_2 N_2, \\
F_{z\bar{z}} &= -\frac{1}{2} ce^{\omega}F + \frac{1}{2} H_1 e^{\omega}N_1 + \frac{1}{2} H_2 e^{\omega}N_2, \\
\langle N_1 \rangle_z &= -H_1 F_z - 2Q_1 e^{-\omega}F_{\bar{z}} + AN_2, \\
\langle N_2 \rangle_z &= -H_2 F_z - 2Q_2 e^{-\omega}F_{\bar{z}} - AN_1,
\end{align*}
\]

where

\[
\langle F_{zz}, N_i \rangle = Q_i, \quad \langle F_{z\bar{z}}, N_i \rangle = \frac{1}{2} H_i e^{\omega} \quad (i = 1, 2), \quad \langle (N_1)_z, N_2 \rangle = A.
\]

The quartic differential \((Q_1^2 + Q_2^2)dz^4\) is independent of the choice of \( z \), \( N_1 \) and \( N_2 \) as well as the function \( H_1^2 + H_2^2 \), which is the length of the...
mean curvature vector. The compatibility conditions for (2.1) give the Gauss-Codazzi-Ricci equations:

\[
\begin{align*}
\omega_z + \frac{1}{2} (H_1^2 + H_2^2 + c)e^{\omega_0} - 2(|Q_1|^2 + |Q_2|^2)e^{-\omega_0} &= 0, \\
(Q_1)_z &= \frac{1}{2} (H_1)_z e^{\omega_0} + \overline{A} Q_2 - \frac{1}{2} \Delta H_2 e^{\omega_0}, \\
(Q_2)_z &= \frac{1}{2} (H_2)_z e^{\omega_0} - \overline{A} Q_1 + \frac{1}{2} \Delta H_1 e^{\omega_0}, \\
A_z - \overline{A}_z &= 2(Q_1 \overline{Q}_2 - \overline{Q}_1 Q_2)e^{-\omega_0},
\end{align*}
\]

which show that minimal surfaces or surfaces with parallel mean curvature vector are Bonnet surfaces (cf. [4]). Note that the normal bundle of minimal surfaces is non-flat in general. On the other hand, non-minimal surfaces with parallel mean curvature vector have flat normal bundle and are contained in some totally geodesic or umbilic 3-dimensional space form as surfaces with constant mean curvature, which is known as Chen-Yau’s reduction theorem [3]. We remark that the same equations as (2.1) and (2.2) hold for surfaces in the simply connected, complete, 4-dimensional space form of curvature \(c \geq 0\).

3. Bonnet Surfaces with Non-flat Normal Bundle

We consider a surface \(F : M \to H^4(c)\) such that \(Q_2 = \alpha Q_1\) for some \(\alpha \in \mathbb{C}\). Then the second and third equations of (2.2) give a linear equation for \(\alpha H_1 - H_2\) if and only if \(\alpha = \pm \sqrt{-1}\):

\[
(\alpha H_1 - H_2)_z = -\alpha \Lambda (\alpha H_1 - H_2).
\]

In the following we put \(\alpha = \pm \sqrt{-1}\). Exchanging the orthogonal unit normals, if necessary, we may assume that \(\alpha = \sqrt{-1}\). Let \(B : M \to \mathbb{C}\) be a function such that \(A = (\log B)_z\). Then (3.1) can be solved explicitly:

\[
\sqrt{-1} H_1 - H_2 = fB^{-\sqrt{-1}},
\]

where \(f : M \to \mathbb{C}\) is a holomorphic function. We consider the case that the mean curvature vector never vanishes. Then changing \(B\), if necessary, we may assume that \(f = 1\). Since \(H_1\) and \(H_2\) are real-valued, (3.2) is
equivalent to

\[ H_1 = -\frac{\sqrt{-1}}{2} (B^{-\sqrt{-1}} - \overline{B}^{\sqrt{-1}}), \quad H_2 = -\frac{1}{2} (B^{-\sqrt{-1}} + \overline{B}^{\sqrt{-1}}). \]

From the second equation of (2.2), we have the linear equation for \( Q_1 \):

\[ (Q_1)_z = \sqrt{-1} \frac{\overline{B}}{B} \, Q_1 + \frac{1}{4} \frac{\overline{B}^{\sqrt{-1}}}{B} \left( \log \frac{B}{\overline{B}} \right)_z e^{\omega}. \]  \hspace{1cm} (3.3)

Solutions of (3.3) are given by \( Q_1 = P \overline{B}^{\sqrt{-1}} \), where \( P : M \to \mathbb{C} \) is a function such that

\[ P_z = \frac{1}{4} \left( \log \frac{B}{\overline{B}} \right)_z e^{\omega}. \]

Let \( r \) and \( \theta \) be a positive or real valued functions on \( M \) respectively such that

\[ B = re^{\sqrt{-1} \theta}, \quad |B^{-\sqrt{-1}}| = e^\theta. \]

Then a direct computation leads to the following:

**Proposition 3.1.** The equations (2.2) are equivalent to

\[
\begin{cases}
\omega_z + \frac{1}{2} (e^{2\theta} + c) e^{\omega} - 4 |P|^2 e^{2\theta - \omega} = 0, \\
P_z = \sqrt{-1} \frac{1}{2} \theta e^{\omega}, \\
0_z = -2 |P|^2 e^{2\theta - \omega}.
\end{cases}
\]  \hspace{1cm} (3.4)

**Remark 3.2.** Note that the normal bundle of \( F \) is flat if and only if \( \theta_z = 0 \). In this case it is easy to see that \( F \) is a totally umbilic surface in some totally geodesic or umbilic 3-dimensional space form.

Now we assume that \( F \) is a Bonnet surface without umbilic points. Then \( H_1 \) and \( H_2 \) are deformed under the deformations as

\[ H_1 \to H_1 \cos \lambda - H_2 \sin \lambda, \quad H_2 \to H_1 \sin \lambda + H_2 \cos \lambda, \]

where \( \lambda \) is a deformation parameter. However transforming \( N_1 \) and \( N_2 \)
as
\[ N_1 \rightarrow N_1 \cos \lambda + N_2 \sin \lambda, \quad N_2 \rightarrow -N_1 \sin \lambda + N_2 \cos \lambda, \]
we may assume that \( H_1 \) and \( H_2 \) are invariants under the deformations.

In the following we consider the case that \( A \) is invariant under the deformations and \( F \) is simple, i.e., the deformations are given by the transformations:
\[ Q_1 \rightarrow \mu Q_1, \quad Q_2 \rightarrow \mu Q_2 \]  
for some function \( \mu : M \rightarrow \mathbb{C} \) with \( |\mu| = 1 \) (cf. [4]). Then from the second equation of (3.4), we have
\[ \{ (\mu - 1)P \} = 0. \]
Since \( F \) contains no umbilic points, we have \( Q_1 \neq 0 \) and hence \( P \neq 0 \). In particular from the third equation of (3.4), the normal bundle of \( F \) is non-flat. Then we have the following analog of the result due to Graustein [7] for Bonnet surfaces in 3-dimensional space forms.

**Proposition 3.3.** Changing the holomorphic coordinate, if necessary, we may assume that
\[ P = \frac{\sqrt{-1}}{g + \bar{g}} \]  
for some holomorphic function \( g : M \rightarrow \mathbb{C} \).

From the second equation of (3.4) and (3.6), we have
\[ -\frac{\overline{P_\tau}}{(g + \bar{g})^2} = \frac{1}{2} \partial_z e^\alpha. \]  
Hence if we assume \( g_z \neq 0 \) and put
\[ w = \int \frac{dz}{g_z}, \quad z = w + \overline{w}, \quad t = w - \overline{w}, \]  
it is easy to verify that \( \theta \) is a function of \( s \) only.

**Theorem 3.4.** Let \( r \) be a positive valued function on \( M \) and \( g \) be a holomorphic function on \( M \) such that \( g_z \neq 0 \) and \( g + \overline{g} \neq 0 \). Then a
surface $F : M \to H^4(c)$ given by

$$e^{\theta_0} = -\frac{2g_z}{(g + \bar{g})^2\theta_s}, \quad H_1 = -e^0 \sin r, \quad H_2 = -e^0 \cos r,$$

$$Q_1 = \sqrt{-1}e^{\theta_0 + \sqrt{-1}\log r}, \quad Q_2 = \sqrt{-1}Q_1, \quad A = \frac{r_z}{r} + \sqrt{-1}\frac{\theta_s}{g_z}$$

is a Bonnet surface with non-flat normal bundle, where $s$ is a real parameter given by (3.8) and

$$\theta = -\frac{1}{2} \log \left( \beta e^{-cs} - \frac{1}{c} \right), \quad \beta > 0. \quad (3.9)$$

Moreover the deformations of $F$ are given by (3.5) with

$$\mu = \frac{1 - 2\sqrt{-1}u\bar{u}}{1 + 2\sqrt{-1}u\bar{u}}, \quad u \in \mathbb{R}.$$

**Proof.** We continue the above argument to solve (3.4). From (3.6), (3.7), (3.8) and the third equation of (3.4), we have

$$e^{\theta_0} = -\frac{2g_z^2}{(g + \bar{g})^2\theta_s} = -\frac{2g_z}{(g + \bar{g})^2\theta_s}e^{2\theta}.$$ \hspace{1cm} (3.10)

Hence we have $\theta_s < 0$ and

$$\theta_{ss} = \theta_s e^{2\theta}. \quad (3.11)$$

From (3.6), (3.8), (3.10) and the first equation of (3.4), we have

$$2 - \frac{e^{2\theta} + c}{\theta_s} = \frac{(g + \bar{g})^2}{g_z^4} \left( \frac{\theta_{ss}}{\theta_s} - 2\theta_s e^{2\theta} \right). \quad (3.12)$$

Since the right-hand side of (3.12) vanishes from (3.11), we can integrate (3.11) as

$$\theta_s = \frac{1}{2} (e^{2\theta} + c). \quad (3.13)$$

Since $c < 0$, we can obtain solutions of (3.13) with $\theta_s < 0$ as (3.9). \hfill \square
References


