Far East Journal of Mathematical Sciences (FJMS)
Volume 30, Issue 2, 2008, Pages 381-387
Published online: September 16, 2008
This paper is available online at http://www.pphmj.com

BONNET SURFACES WITH NON-FLAT NORMAL BUNDLE IN THE HYPERBOLIC FOUR-SPACE

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Abstract

We study surfaces in the hyperbolic four-space admitting isometric deformations preserving the length of the mean curvature vector and especially focus on the case that surfaces are non-minimal and have non-flat normal bundle.

1. Introduction

There has been a long history of study of surfaces in 3-dimensional space forms admitting isometric deformations preserving the mean curvature (see [1, 5, 6, 7] and references therein), which can be traced back to the following result due to Bonnet [2].

Proposition 1.1. *If a surface in a* 3*-dimensional space form has constant mean curvature and is not totally umbilic*, *then it admits isometric deformations preserving the mean curvature*.

In the previous paper [4] the author studied surfaces in 4-dimensional space forms admitting isometric deformations preserving the length of

2000 Mathematics Subject Classification: 53A07, 53A10.

Keywords and phrases: Bonnet surfaces, mean curvature vector, non-flat normal bundle.

Partially supported by Grant-in-Aid for Scientific Research No. 18540074, Japan Society for the Promotion of Science.

Received June 19, 2008

the mean curvature vector, called *Bonnet surfaces* after Bonnet's work, and obtained a generalization of Chen-Yau's reduction theorem for surfaces with parallel mean curvature vector [3].

In this paper we study Bonnet surfaces in the hyperbolic 4-space and obtain an example of non-minimal surfaces with non-flat normal bundle.

2. Preliminaries

We denote the hyperbolic 4-space of curvature $c < 0$ by $H^4(c)$, which is described as follows:

$$H^4(c) = \text{ a connected component of } \left\{x \in \mathbf{R}^5 \,|\, \langle x, x\rangle = \frac{1}{c}\right\},$$

where $\langle\, ,\rangle$ is the Lorentzian inner product on $\mathbf{R}^5$ with signature (1, 4).

We assume that surfaces are sufficiently smooth. Any surface in $H^4(c)$ is given by a conformal immersion F from a Riemann surface M to $H^4(c)$. Using a local holomorphic coordinate z, we write the induced metric on M as $e^{\omega}dzd\bar{z}$.

Let N_1 and N_2 be orthogonal unit normals to F. Then the Gauss-Weingarten equations are

$$\begin{cases} F_{zz} = \omega_z F_z + Q_1 N_1 + Q_2 N_2, \\ F_{z\bar{z}} = -\dfrac{1}{2} ce^{\omega} F + \dfrac{1}{2} H_1 e^{\omega} N_1 + \dfrac{1}{2} H_2 e^{\omega} N_2, \\ (N_1)_z = -H_1 F_z - 2Q_1 e^{-\omega} F_{\bar{z}} + AN_2, \\ (N_2)_z = -H_2 F_z - 2Q_2 e^{-\omega} F_{\bar{z}} - AN_1, \end{cases} \tag{2.1}$$

where

$$\langle F_{zz}, N_i\rangle = Q_i, \ \ \langle F_{z\bar{z}}, N_i\rangle = \frac{1}{2} H_i e^{\omega} \ (i = 1, 2), \ \ \langle (N_1)_z, N_2\rangle = A.$$

The quartic differential $(Q_1^2 + Q_2^2)dz^4$ is independent of the choice of z, N_1 and N_2 as well as the function $H_1^2 + H_2^2$, which is the length of the mean curvature vector. The compatibility conditions for (2.1) give the Gauss-Codazzi-Ricci equations:

$$\begin{cases} \omega_{z\bar{z}} + \dfrac{1}{2}(H_1^2 + H_2^2 + c)e^{\omega} - 2(|Q_1|^2 + |Q_2|^2)e^{-\omega} = 0, \\ (Q_1)_{\bar{z}} = \dfrac{1}{2}(H_1)_z e^{\omega} + \overline{A}Q_2 - \dfrac{1}{2}AH_2e^{\omega}, \\ (Q_2)_{\bar{z}} = \dfrac{1}{2}(H_2)_z e^{\omega} - \overline{A}Q_1 + \dfrac{1}{2}AH_1e^{\omega}, \\ A_{\bar{z}} - \overline{A}_z = 2(Q_1\overline{Q}_2 - \overline{Q}_1Q_2)e^{-\omega}, \end{cases} \tag{2.2}$$

which show that minimal surfaces or surfaces with parallel mean curvature vector are Bonnet surfaces (cf. [4]). Note that the normal bundle of minimal surfaces is non-flat in general. On the other hand, non-minimal surfaces with parallel mean curvature vector have flat normal bundle and are contained in some totally geodesic or umbilic 3-dimensional space form as surfaces with constant mean curvature, which is known as Chen-Yau's reduction theorem [3]. We remark that the same equations as (2.1) and (2.2) hold for surfaces in the simply connected, complete, 4-dimensional space form of curvature $c \geq 0$.

3. Bonnet Surfaces with Non-flat Normal Bundle

We consider a surface $F : M \to H^4(c)$ such that $Q_2 = \alpha Q_1$ for some $\alpha \in \mathbf{C}$. Then the second and third equations of (2.2) give a linear equation for $\alpha H_1 - H_2$ if and only if $\alpha = \pm\sqrt{-1}$:

$$(\alpha H_1 - H_2)_z = -\alpha A(\alpha H_1 - H_2). \tag{3.1}$$

In the following we put $\alpha = \pm\sqrt{-1}$. Exchanging the orthogonal unit normals, if necessary, we may assume that $\alpha = \sqrt{-1}$. Let $B : M \to \mathbf{C}$ be a function such that $A = (\log B)_z$. Then (3.1) can be solved explicitly:

$$\sqrt{-1}H_1 - H_2 = \bar{f}B^{-\sqrt{-1}}, \tag{3.2}$$

where $f : M \to \mathbf{C}$ is a holomorphic function. We consider the case that the mean curvature vector never vanishes. Then changing B, if necessary, we may assume that $f = 1$. Since H_1 and H_2 are real-valued, (3.2) is equivalent to

$$H_1 = -\frac{\sqrt{-1}}{2}(B^{-\sqrt{-1}} - \overline{B}^{\sqrt{-1}}), \quad H_2 = -\frac{1}{2}(B^{-\sqrt{-1}} + \overline{B}^{\sqrt{-1}}).$$

From the second equation of (2.2), we have the linear equation for Q_1:

$$(Q_1)_{\bar{z}} = \sqrt{-1}\frac{\overline{B}_{\bar{z}}}{\overline{B}}Q_1 + \frac{1}{4}\overline{B}^{\sqrt{-1}}\left(\log\frac{B}{\overline{B}}\right)_z e^{\omega}. \tag{3.3}$$

Solutions of (3.3) are given by $Q_1 = P\overline{B}^{\sqrt{-1}}$, where $P : M \to \mathbf{C}$ is a function such that

$$P_{\bar{z}} = \frac{1}{4}\left(\log\frac{B}{\overline{B}}\right)_z e^{\omega}.$$

Let r and θ be a positive or real valued functions on M respectively such that

$$B = re^{\sqrt{-1}\theta}, \quad |B^{-\sqrt{-1}}| = e^{\theta}.$$

Then a direct computation leads to the following:

Proposition 3.1. *The equations* (2.2) *are equivalent to*

$$\begin{cases} \omega_{z\bar{z}} + \dfrac{1}{2}(e^{2\theta} + c)e^{\omega} - 4|P|^2 e^{2\theta-\omega} = 0, \\ P_{\bar{z}} = \dfrac{\sqrt{-1}}{2}\theta_z e^{\omega}, \\ \theta_{z\bar{z}} = -2|P|^2 e^{2\theta-\omega}. \end{cases} \tag{3.4}$$

Remark 3.2. Note that the normal bundle of F is flat if and only if $\theta_{z\bar{z}} = 0$. In this case it is easy to see that F is a totally umbilic surface in some totally geodesic or umbilic 3-dimensional space form.

Now we assume that F is a Bonnet surface without umbilic points. Then H_1 and H_2 are deformed under the deformations as

$$H_1 \to H_1\cos\lambda - H_2\sin\lambda, \quad H_2 \to H_1\sin\lambda + H_2\cos\lambda,$$

where λ is a deformation parameter. However transforming N_1 and N_2

as

$$N_1 \to N_1 \cos\lambda + N_2 \sin\lambda, \quad N_2 \to -N_1 \sin\lambda + N_2 \cos\lambda,$$

we may assume that H_1 and H_2 are invariants under the deformations. In the following we consider the case that A is invariant under the deformations and F is simple, i.e., the deformations are given by the transformations:

$$Q_1 \to \mu Q_1, \quad Q_2 \to \mu Q_2 \tag{3.5}$$

for some function $\mu : M \to \mathbf{C}$ with $|\mu| = 1$ (cf. [4]). Then from the second equation of (3.4), we have

$$\{(\mu - 1)P\}_{\bar{z}} = 0.$$

Since F contains no umbilic points, we have $Q_1 \neq 0$ and hence $P \neq 0$. In particular from the third equation of (3.4), the normal bundle of F is non-flat. Then we have the following analog of the result due to Graustein [7] for Bonnet surfaces in 3-dimensional space forms.

Proposition 3.3. *Changing the holomorphic coordinate, if necessary, we may assume that*

$$P = \frac{\sqrt{-1}}{g + \bar{g}} \tag{3.6}$$

for some holomorphic function $g : M \to \mathbf{C}$.

From the second equation of (3.4) and (3.6), we have

$$-\frac{\bar{g}_{\bar{z}}}{(g + \bar{g})^2} = \frac{1}{2}\theta_z e^{\omega}. \tag{3.7}$$

Hence if we assume $g_z \neq 0$ and put

$$w = \int \frac{dz}{g_z}, \; s = w + \bar{w}, \; t = w - \bar{w}, \tag{3.8}$$

it is easy to verify that θ is a function of s only.

Theorem 3.4. *Let r be a positive valued function on M and g be a holomorphic function on M such that $g_z \neq 0$ and $g + \bar{g} \neq 0$. Then a* *surface* $F : M \to H^4(c)$ *given by*

$$e^{\omega} = -\frac{2|g_z|^2}{(g+\bar{g})^2\theta_s}, \quad H_1 = -e^{\theta}\sin\log r, \quad H_2 = -e^{\theta}\cos\log r,$$

$$Q_1 = \frac{\sqrt{-1}e^{\theta+\sqrt{-1}\log r}}{g+\bar{g}}, \quad Q_2 = \sqrt{-1}Q_1, \quad A = \frac{r_z}{r} + \sqrt{-1}\,\frac{\theta_s}{g_z}$$

is a Bonnet surface with non-flat normal bundle, where s is a real parameter given by (3.8) *and*

$$\theta = -\frac{1}{2}\log\left(\beta e^{-cs} - \frac{1}{c}\right), \quad \beta > 0. \tag{3.9}$$

Moreover the deformations of F are given by (3.5) *with*

$$\mu = \frac{1 - 2\sqrt{-1}u\bar{g}}{1 + 2\sqrt{-1}ug}, \quad u \in \mathbf{R}.$$

Proof. We continue the above argument to solve (3.4). From (3.6), (3.7), (3.8) and the third equation of (3.4), we have

$$e^{\omega} = -\frac{2|g_z|^2}{(g+\bar{g})^2\theta_s} = -\frac{2|g_z|^2 e^{2\theta}}{(g+\bar{g})^2\theta_{ss}}. \tag{3.10}$$

Hence we have $\theta_s < 0$ and

$$\theta_{ss} = \theta_s e^{2\theta}. \tag{3.11}$$

From (3.6), (3.8), (3.10) and the first equation of (3.4), we have

$$2 - \frac{e^{2\theta} + c}{\theta_s} = \frac{(g+\bar{g})^2}{|g_z|^4}\left\{\left(\frac{\theta_{ss}}{\theta_s}\right)_s - 2\theta_s e^{2\theta}\right\}. \tag{3.12}$$

Since the right-hand side of (3.12) vanishes from (3.11), we can integrate (3.11) as

$$\theta_s = \frac{1}{2}(e^{2\theta} + c). \tag{3.13}$$

Since $c < 0$, we can obtain solutions of (3.13) with $\theta_s < 0$ as (3.9). □

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