Discussion Paper Series A No.551

Rationality and Solutions to Nonconvex Bargaining Problems: Rationalizable, Asymmetric and Nash Solutions

Yongsheng Xu
(Department of Economics, Andrew Young School of Policy Studies, Georgia State University)
and
Naoki Yoshihara
(Institute of Economic Research, Hitotsubashi University)

May 2011

Institute of Economic Research
Hitotsubashi University
Kunitachi, Tokyo, 186-8603 Japan

Rationality and solutions to nonconvex bargaining problems: rationalizable, asymmetric and Nash solutions[1]

Yongsheng Xu
Department of Economics
Andrew Young School of Policy Studies
Georgia State University, Atlanta, GA 30302

Naoki Yoshihara
Institute of Economic Research
Hitotsubashi University, Kunitachi, Tokyo

This Version: April 2011

[1] We are grateful to Michele Lombardi for his comments and suggestions.

Abstract. Conditions α and β are two well-known rationality conditions in the theory of rational choice. This paper examines the implications of weaker versions of these two rationality conditions in the context of solutions to non-convex bargaining problems. It is shown that, together with the standard axioms of efficiency and strict individual rationality, they imply rationalizability of solutions to nonconvex bargaining problems. We then characterize asymmetric Nash solutions by imposing a continuity and the scale invariance requirements. We also give a characterization of the Nash solution by using the two rationality conditions. These results make a further connection between solutions to non-convex bargaining problems and rationalizability of choice function in the theory of rational choice.
J.E.L. Classification Numbers: C71, C78, D63, D71

1 Introduction

In this paper, we study solutions to non-convex bargaining problem by examining their connections to two well-known rationality conditions, namely conditions α and β, in the theory of rational choice (see, for example, Sen (1971)). Condition α says that, when a set A contracts to another set B, and if an option x chosen from A continues to be available in B, then x must be chosen from B. Condition β, on the other hand, says that, when two options x and y are chosen from a set A and when A expands to another set B, then, *either* both x and y are chosen from B *or* neither x nor y are chosen from B.

In the literature on non-convex bargaining problems, a stronger version of condition α, often called *contraction independence*, has been used for characterizing the Nash solution (see, for example, Mariotti (1998, 1999), Xu and Yoshihara (2006)). Contraction independence requires that, when a bargaining problem A shrinks to another bargaining problem B and if B contains some options of the solution to A, then the solution to B coincides with the intersection of B and the solution to A. This version of contraction independence can also be regarded as a natural generalization of Nash's independence of irrelevant alternatives (IIA) (Nash (1950)) introduced for convex bargaining problems where a solution picks a single option from a bargaining problem.[2]

Building on the intuitions of conditions α and β, in this paper, we consider weaker versions of conditions α and β, to be called *binary condition* α and *binary condition* β, respectively. Binary condition α requires that, for any two options x and y, if either x or y is part of the solution to a bargaining problem A, then the solution to the bargaining problem given by the comprehensive hull (see Section 2 for a formal definition) of x and y must contain the intersection of $\{x, y\}$ and the solution to A. Binary condition β requires that, if two options x and y are the only chosen alternatives from the problem of the comprehensive hull of x and y, then when the problem is enlarged, either both belong to the solution to the enlarged problem or neither do not belong to the solution to the enlarged problem.

We will then use these two weaker rationality conditions to first study

[2] There are other variations of the generalised Nash's IIA. Mariotti (1998a) introduces a weaker variant of the standard Nash IIA which is solely applicable to single-valued solutions, whereas Thomson (1981) introduces a weaker variant of the contraction independence discussed in Mariotti (1998, 1999) and Xu and Yoshihara (2006).

rationalizability of solutions to nonconvex bargaining problems (see Section 3 for a formal definition). The interest of studying rationalizability of bargaining solutions in the literature is two-fold. In the first place, a solution can be interpreted as a fair arbitration scheme ratifiable by a committee (see Mariotti (1999)), and as a consequence, it represents the majority preferences of the committee. Secondly, as argued by Peters and Wakker (1991), a solution to bargaining problems may be thought to reveal the preferences of the players involved as a group, and thus the behavior of a solution may be linked to 'revealed group preference.' In the literature on bargaining problems, the rationalizability of solutions to convex bargaining problems has been fruitfully studied (see, among others, Peters and Wakker (1991), Bossert (1994), and Sanchez (2000)), and there is little research on the rationalizability of solutions to nonconvex bargaining problems (see, however, Kaneko (1980), Denicolo and Mariotti (2000), for some exceptions). We show that if we restrict a solution to be efficient and strict individually rational, then the rationalizability of a solution is equivalent to the combination of our two weaker rationality conditions. We also point out that neither efficiency nor strict individual rationality is necessary for a solution to be rationalizable and that there are non-rationalizable solutions satisfying efficiency and strict individual rationality.

Next, we examine the consequences of imposing scale invariance on solutions to nonconvex bargaining problems. We observe that the imposition of scale invariance on a rationalizable solution to nonconvex bargaining problems in our context is not sufficient to characterize asymmetric Nash solutions. This is in sharp contrast with the results obtained for convex problems (see, for example, Roth (1977)) and for nonconvex problems where solutions are restricted to be single-valued (see, for example, Zhou (1997)). To characterize asymmetric Nash solutions in our context, a continuity property on part of a solution is needed. The continuity axiom we use is fairly weak as it only restricts to behaviors of solutions to problems given by comprehensive hulls of two alternatives.

Finally, we use our two rationality conditions to study the Nash solution to nonconvex bargaining problems. In particular, we show that, together with the standard axioms of efficiency, scale invariance and anonymity, our two rationality conditions characterize the Nash solution. This result therefore improves the existing characterization of the Nash solution to non-convex bargaining problems (see Mariotti (1998, 1999), Xu and Yoshihara (2006)) and makes a close connection between solutions to non-convex bargaining problems and rationality conditions in the theory of rational choice. Together with our studies of rationalizable and asymmetric Nash solutions, our result on the characterization of the Nash solution also sheds new light on the axioms of anonymity and scale invariance: they together play a role of continuity and ensure the required representation of the binary relation rationalizing the underlying solution.

The remainder of the paper is organized as follows. In the following section, Section 2, we present notation and definitions. Section 3 studies the rationalizability of solutions to nonconvex bargaining problems, while Section 4 is devoted to the study of asymmetric Nash solutions. The characterization of the Nash solution is given in Section 5. We conclude in Section 6.

2 Notation and definitions

Let $N = \{1, 2, \ldots, n\}$ be the set of all individuals in the society. Let $\mathbf{R}_+$ be the set of all non-negative real numbers, and $\mathbf{R}_{++}$ be the set of all positive numbers. Let $\mathbf{R}^n_+$ (resp. $\mathbf{R}^n_{++}$) be the n-fold Cartesian product of $\mathbf{R}_+$ (resp. $\mathbf{R}_{++}$). For any $x, y \in \mathbf{R}^n_+$, we write $x \geq y$ to mean $[x_i \geq y_i$ for all $i \in N]$, $x > y$ to mean $[x_i \geq y_i$ for all $i \in N$ and $x \neq y]$, and $x \gg y$ to mean $[x_i > y_i$ for all $i \in N]$. For any $x \in \mathbf{R}^n_+$ and any non-negative number q, we write $z = (q; \mathbf{x}_{-i}) \in \mathbf{R}^n_+$ to mean that $z_i = q$ and $z_j = x_j$ for all $j \in N \setminus \{i\}$.

For any subset $A \subseteq \mathbf{R}^n_+$, A is said to be (i) *non-trivial* if there exists $a \in A$ such that $a \gg 0$, and (ii) *comprehensive* if for all $x, y \in \mathbf{R}^n_+$, $[x \geq y$ and $x \in A]$ implies $y \in A$. For all $A \subseteq \mathbf{R}^n_+$, define the *comprehensive hull of* A, to be denoted by *compA*, as follows:

$$compA \equiv \left\{ z \in \mathbf{R}^n_+ \mid z \leq x \text{ for some } x \in A \right\}.$$

Let Σ be the set of all non-trivial, compact and comprehensive subsets of $\mathbf{R}^n_+$. Elements in Σ are interpreted as (normalized) bargaining problems. A *bargaining solution* F assigns a nonempty subset $F(A)$ of A for every bargaining problem $A \in \Sigma$.

Let π be a permutation of N. The set of all permutations of N is denoted by Π. For all $x = (x_i)_{i \in N} \in \mathbf{R}^n_+$, let $\pi(x) = (x_{\pi(i)})_{i \in N}$. For all $A \in \Sigma$ and any permutation $\pi \in \Pi$, let $\pi(A) = \{\pi(a) \mid a \in A\}$. For any $A \in \Sigma$, we say that A is *symmetric* if $A = \pi(A)$ for all $\pi \in \Pi$.

Definition 1: A bargaining solution F over Σ is the *Nash solution* if for all $A \in \Sigma$, $F(A) = \{a \in A \mid \prod_{i \in N} a_i \geq \prod_{i \in N} x_i$ for all $x \in A\}$.

Denote the Nash solution by F^N. Note that, for nonconvex bargaining problems, the Nash solution is typically multi-valued.[3]

3 Rationalizable solutions

In this section, we study the problem of rationalizability of solutions to nonconvex problems. First, we define the notion of rationalizable solutions in our context.

Definition 2: A bargaining solution F over Σ is *rationalizable* if there exists a reflexive, complete and transitive binary relation R over $\mathbb{R}^n_+$ such that, for all $A \in \Sigma$, $F(A) = \{x \in A \mid xRy \text{ for all } y \in A\}$.

Under what condition is a solution F over Σ rationalizable? To answer this question, we begin by introducing some axioms to be imposed on a solution to nonconvex bargaining problems. The first two, Efficiency and Strict Individual Rationality, are well-known in the literature.

Efficiency (E): For any $A \in \Sigma$ and any $a \in F(A)$, there is no $x \in A$ such that $x > a$.

Strict Individual Rationality (SIR): For all $A \in \Sigma$, $x \in F(A) \Rightarrow x \gg 0$.

In the literature on Nash bargaining problems and on rational choice theory, various *contraction independence* properties have been proposed. The idea behind a contraction independence property is the following: given two bargaining problems, A and B, in which A is a subset of B, and suppose that a point x chosen from B as a solution to B continues to be available in A, then x should continue to be a solution to A provided certain restrictions are satisfied. The following axiom, to be called (BCα), is a weaker version of the contraction independence used in nonconvex bargaining problems (see, for example, Thomson (1981), Mariotti (1998, 1999), and Xu and Yoshihara (2006)). It requires that, for any two points x and y in a bargaining problem A, if either x or y is part of the solution to A, then the common points in

[3] The Nash solution to non-convex problems discussed in this paper was firstly proposed by Kaneko (1980). There are other extensions of the Nash solution into non-convex problems such as Herrero (1989) and Conley and Wilkie (1996).

$\{x, y\}$ and the solution to A must be contained in the solution to the problem given by the comprehensive hull of x and y. It may be noted that the origin of (BCα) goes back to Herzberger (1973) (see also Sen (1977)) where a similar condition is introduced for finite choice problems: if an option x is chosen from a set A then x must be chosen from any two-element set $\{x, y\}$ as long as y is contained in A as well. Clearly, (BCα) is also weaker than condition α in the literature on rational choice theory (also known as the Chernoff condition, see Chernoff (1954) and Sen (1971)). Formally, (BCα) is stated as follows.

Binary Condition α (BCα): For all $A \in \Sigma$ and all $x, y \in A$, if $\{x, y\} \cap F(A) \neq \varnothing$ then $F(A) \cap \{x, y\} \subseteq F(comp\{x, y\})$.

It may be noted that (BCα) is specific to non-convex bargaining problems and is not applicable to convex bargaining problems. We next introduce a weaker version of condition β (see Sen (1971)).

Binary Condition β (BCβ): For all $A \in \Sigma$ and all $x, y \in A$, if $\{x, y\} = F(comp\{x, y\})$, then $[x \in F(A) \Leftrightarrow y \in F(A)]$.

Thus, (BCβ) requires that, whenever the solution to the problem $comp\{x, y\}$ is consisted of both x and y, then, for any problem A containing both x and y, either $[x$ and y are both chosen as solutions to $A]$ or neither x nor y is chosen as a solution to A. In a way, (BCβ) stipulates that whenever two alternatives, x and y, are "informationally equivalent" in a pairwise comparison, then they must be treated "equally".

With the help of the above axioms, we now state and prove our first main result.

Theorem 1. Let a solution F over Σ satisfy (E) and (SIR). Then, F satisfies (BCα) and (BCβ) if and only if F is *rationalizable*.

Proof. We note that if a solution F over Σ is rationalizable, then F satisfies both (BCα) and (BCβ). Therefore, we need only to show that if a solution F over Σ satisfies (E), (SIR), (BCα) and (BCβ), then it must be rationalizable.

Let a solution F over Σ satisfy (E), (SIR), (BCα) and (BCβ). Define a binary relation R over $\mathbb{R}^n_+$ as follows: for all $x, y \in \mathbb{R}^n_+$,

if $x = y$, then xRx;

if $x \neq y$, then $xRy \Leftrightarrow [x \in F(comp\{x,y\})]$ or $[y \notin F(A)$ for all $A \in \Sigma$ with $x, y \in A]$.

We first note that, for any $x, y \in \mathbb{R}^n_+$, if $y_i = 0$ for some $i \in N$, then by (SIR), $y \notin F(A)$ for all $A \in \Sigma$ with $x, y \in A$. Therefore, for any $x, y \in \mathbb{R}^n_+$, $[y_i = 0$ for some $i \in N] \Rightarrow xRy$. Further, if $x \gg 0$ and $y_i = 0$ for some $i \in N$, then, by (E) and (SIR), $\{x\} = F(comp\{x,y\})$ implying that xRy and not(yRx). Therefore, the binary relation R is well-defined.

Note that R thus defined is reflexive and complete. We now show that R is transitive. To see that R is transitive, consider $x, y, z \in \mathbb{R}^n_+$ such that xRy and yRz. If $x_i = 0$ for some $i \in N$, then xRy implies that $y_j = 0$ for some $j \in N$, and yRz together with $[y_j = 0$ for some $j \in N]$ implies $z_k = 0$ for some $k \in N$. Therefore, xRz follows from the definition of R. If $y_i = 0$ for some $i \in N$, then yRz implies that $z_j = 0$ for some $j \in N$. In this case, xRz follows from the definition of R. If $z_i = 0$ for some $i \in N$, then xRz follows again from the definition of R. Consider therefore that $x \gg 0, y \gg 0$ and $z \gg 0$. Given that xRy and yRz, it must be the case that $x \in F(comp\{x,y\})$ and $y \in F(comp\{y,z\})$. We need to show that $x \in F(comp\{x,z\})$. Suppose to the contrary that $x \notin F(comp\{x,z\})$. By (E), $\{z\} = F(comp\{x,z\})$. Consider the problem $comp\{x,y,z\}$. Note that $comp\{x,y,z\} \in \Sigma$. Consider $F(comp\{x,y,z\})$. By (E), $F(comp\{x,y,z\}) \subseteq \{x,y,z\}$. If $x \in F(comp\{x,y,z\})$, noting that $\{x,z\} \cap F(comp\{x,y,z\}) \neq \varnothing$, it follows from (BC$\alpha$) that $F(comp\{x,y,z\}) \cap \{x,z\} \subseteq F(comp\{x,z\})$, that is, $x \in F(comp\{x,z\})$, a contradiction. Therefore, $x \notin F(comp\{x,y,z\})$. If $y \in F(comp\{x,y,z\})$, noting that $x \in F(comp\{x,y\})$ by xRy and (BCα), we must have $F(comp\{x,y\}) = \{x,y\}$ by (BCα). It then follows from (BCβ) that $x \in F(comp\{x,y,z\})$, which, from the above, leads to a contradiction. Therefore, $y \notin F(comp\{x,y,z\})$. If $z \in F(comp\{x,y,z\})$, noting that $y \in F(comp\{y,z\})$ by yRz and (BCα), it follows from (BCα) that $F(comp\{y,z\}) = \{y,z\}$. It then follows from (BCβ) that $y \in F(comp\{x,y,z\})$, which, from the above, leads to a contradiction. Therefore, $z \notin F(comp\{x,y,z\})$. Consequently, $F(comp\{x,y,z\}) \cap \{x,y,z\} = \varnothing$, a contradiction. Thus, it must be true that $x \in F(comp\{x,z\})$ implying xRz. Therefore, R is transitive.

To complete the proof of Theorem 1, we need to show that, for all $A \in \Sigma$, $F(A) = \{x \in A \mid xRy$ for all $y \in A\}$. Consider $A \in \Sigma$. Let $x \in F(A)$. By (SIR), for any $y \in A$ with $y_i = 0$ for some $i \in N$, we must have xRy. For any $y \in A$ with $y \gg 0$, since $x \in F(A)$, by (BCα), it follows that

$x \in F(comp\{x, y\})$ implying that xRy. Therefore, $F(A) \subseteq \{x \in A \mid xRy$ for all $y \in A\}$. We next show that $\{x \in A \mid xRy$ for all $y \in A\} \subseteq F(A)$. Suppose, to the contrary, that it is not true that $\{x \in A \mid xRy$ for all $y \in A\} \subseteq F(A)$. Then, there must exist $x \in A$ such that xRy for all $y \in A$, but $x \notin F(A)$. Note that it must be true that $x \gg 0$. Consider $z \in F(A)$. (SIR) implies that $z \gg 0$. By (BCα) and from $z \in F(A)$ and $x \in A$, we must have $z \in F(comp\{x, z\})$. Note that xRz, that is, $x \in F(comp\{x, z\})$. Therefore, $\{x, z\} = F(comp\{x, z\})$. By (BC$\beta$) and noting that $z \in F(A)$, it then follows that $x \in F(A)$, a contradiction. Therefore, $\{x \in A \mid xRy$ for all $y \in A\} \subseteq F(A)$. Hence, $F(A) = \{x \in A \mid xRy$ for all $y \in A\} \subseteq F(A)$. $\diamond$

Remark 1. It may be noted that (BCα) and (BCβ) are not sufficient for a solution F to be rationalizable. In particular, there exists non-rationalizable solution F over Σ that satisfies (BCα) and (BCβ) but violates (E) and (SIR). For example, for any $A \in \Sigma$, let $F^{WP}(A) = \{x \in A \mid$ there exists no $y \in A$ such that $y \gg x\}$ (that is, the solution is given by all weakly Pareto efficient utility vectors in A). Clearly, this solution satisfies both (BCα) and (BCβ), but violates (E) and (SIR). The solution is not rationalizable by any binary relation R over $\mathbb{R}^n_+$: consider $n = 2$, $x = (1, 2)$, $y = (2, 1)$, $A = comp\{(3, 2)\}$ and $B = comp\{(2, 3)\}$. Then, $x \in F(A), y \notin F(A), y \in F(B)$ and $x \notin F(B)$ implying no binary relation R can be defined over $\mathbb{R}^2_+$ that would rationalize the solution.

Remark 2. It may be noted that (E) is not necessary for a solution to be rationalizable: there exists a rationalizable solution that satisfies (SIR), (BCα) and (BCβ) but violates (E). To see this, consider the following solution: for all $A \in \Sigma$, let $F^E(A) = \{x \in A \mid x_1 = \cdots = x_n$, and there exists no $y \in A$ such that $y \gg x\}$ (the Egalitarian solution).

Remark 3. It is also interesting to note that (SIR) is not necessary for a solution to be rationalizable: there exists a rationalizable solution that satisfies (E), (BCα) and (BCβ) but violates (SIR). To see this, let $\geq_{lex}$ be a standard lexicographic binary relation defined over $\mathbb{R}^n_+$. Define the solution, F^{lex} as follows: for all $A \in \Sigma$, $F^{lex}(A) = \{x \in A : x \geq_{lex} y$ for all $y \in A\}$.

Remark 4. It may be noted that (E) is indispensable in Theorem 1: there are non-rationalizable solutions that satisfy (SIR), (BCα) and (BCβ), but violates (E). To show this, let $\#N = 2$ and for each $A \in \Sigma$, let $E(A) \equiv \{x \in A \mid x_1 = x_2 > 0\}$. Then, define F^1 as follows: for any $A \in \Sigma$,

$F^1(A) = F^N(A) \cup E(A)$. Note that F^1 satisfies (SIR) but violates (E). Note that, for all $\{x,y\} \in \Sigma$, $\{x,y\} = F^1(comp\{x,y\})$ does not hold since $F^1(comp\{x,y\})$ contains $E(comp\{x,y\})$. Thus, (BCβ) is vacuously satisfied by F^1. F^1 also satisfies (BCα). To see this, consider $A \in \Sigma$, $x, y \in A$ such that $\{x,y\} \cap F^1(A) \neq \varnothing$. Let $x, y \in F^1(A)$. If $x, y \in E(A)$, then $x, y \in F^1(comp\{x,y\})$. If $x \in F^N(A) \setminus E(A)$ and $y \in E(A)$, then $x, y \in F^1(comp\{x,y\})$. If $x, y \in F^N(A) \setminus E(A)$, then $x, y \in F^1(comp\{x,y\})$. Let $x \in F^1(A)$ and $y \notin F^1(A)$. If $x \in E(A)$, then $x \in F^1(comp\{x,y\})$. If $x \in F^N(A) \setminus E(A)$, then $x \in F^1(comp\{x,y\})$. Therefore, F^1 satisfies (BCα). Finally, F^1 is not rationalizable. Consider $A = comp\{(4,2)\}$ and $B = comp\{(4,1),(2,2)\}$. Then, $(4,2),(2,2) \in F^1(A)$ and $(4,1) \notin F^1(A)$, while $(4,1),(2,2) \in F^1(B)$. Suppose F^1 is rationalizable by a binary relation R. Then, by considering the problem A, we must have $(2,2)R(4,1)$ and not$(4,1)R(2,2)$, and by considering the problem B, we have $(2,2)\,R\,(4,1)$ and $(4,1)R(2,2)$, which is a contradiction. Therefore, F^1 is not rationalizable.

Remark 5. We note that (SIR) is indispensable in Theorem 1: there are non-rationalizable solutions satisfying (E), (BCα), (BCβ) and violating (SIR). To see this, let $m_i(A) = \max\{a_i \mid (a_1, \cdots, a_i, \cdots, a_n) \in A\}$ for all $A \in \Sigma$ and all $i \in N$. Therefore, $m(A) \equiv (m_i(A))_{i \in N}$ is the *ideal point* of A. For each $i \in N$, let $m^i(A) \equiv (m_i(A); \mathbf{0}_{-i})$. Let $P(A)$ be the set of Pareto efficient alternatives in $A \in \Sigma$. Again, let $\#N = 2$. Given $A \in \Sigma$, let $\widetilde{A} \subseteq A$ be defined as follows:

$$\widetilde{A} = \{x \in A \mid \forall \epsilon > 0, \exists y \in \mathbf{R}^2_{++} \text{ with } ||y - x|| < \epsilon, y \in A\}$$

(For example, when $A = comp\{(1,1),(2,0)\}$, then $\widetilde{A} = comp\{(1,1)\}$.) Let $F^U(A) \equiv \{x \in A \mid \forall y \in A : x_1 + x_2 \geq y_1 + y_2\}$ for any $A \in \Sigma$. Then, define F^2 as follows: for any $A \in \Sigma$,
1) if $\min\{m_1(A), m_2(A)\} > x_1 + x_2$ for any $x \in \widetilde{A}$, then $F^2(A) = \{m^1(A), m^2(A)\}$;
2) if $\min\{m_1(A), m_2(A)\} \leq x_1 + x_2$ for some $x \in \widetilde{A}$, but $F^U(A) \cap \{m^1(A), m^2(A)\} \neq \varnothing$, then $F^2(A) = F^U(A) \cap \{m^1(A), m^2(A)\}$; and
3) if $F^U(A) \cap \{m^1(A), m^2(A)\} = \varnothing$, then $F^2(A) = F^U(A)$.

This solution satisfies (E), but violates (SIR). It also satifies (BCα) and (BCβ). Consider (BCα) first. Take any $A \in \Sigma$ and any $x, y \in A$, and suppose $F^2(A) \cap \{x,y\} \neq \varnothing$. If case 3) above is applicable, then (BCα) is obviously satisfied. If case 2) above is applicable, then $F^2(A) \subseteq \{m^1(A), m^2(A)\}$. Without loss of generality, let $x \in \{m^1(A), m^2(A)\}$. If $y \in \{m^1(A), m^2(A)\}$, then $comp\{x,y\} \notin \Sigma$, which implies that (BC$\alpha$) is trivially satisfied. If $y \notin \{m^1(A), m^2(A)\}$ and $y \in \mathbf{R}^2_+ \backslash \mathbf{R}^2_{++}$, again $comp\{x,y\} \notin \Sigma$. If $y \notin \{m^1(A), m^2(A)\}$ and $y \in \mathbf{R}^n_{++}$, then $comp\{x,y\} \in \Sigma$ and it corresponds to case 2), so that $x \in F^2(comp\{x,y\})$. If case 1) above is applicable, then $F^2(A) = \{m^1(A), m^2(A)\}$. Let $x \in \{m^1(A), m^2(A)\}$ and $y \notin \{m^1(A), m^2(A)\}$. Then, if $y \in \widetilde{A} \cap \mathbf{R}^2_{++}$, $comp\{x,y\} \in \Sigma$ and we are back to case 2). Thus, $x \in F^2(comp\{x,y\})$. Therefore, F^2 satisfies (BCα).

We now consider (BCβ). Note that for any $A \in \Sigma$, either $F^2(A) \subseteq \{m^1(A), m^2(A)\}$ or $F^2(A) \subseteq \mathbf{R}^2_{++}$. Take any $A \in \Sigma$ and any $x, y \in A$, and suppose $F^2(comp\{x,y\}) = \{x,y\}$. First of all, $comp\{m^1(A), m^2(A)\} \notin \Sigma$. Moreover, if $x \in \{m^1(A), m^2(A)\}$ and $y \in \mathbf{R}^2_{++}$, then $F^2(comp\{x,y\}) = \{x,y\}$ does not hold. Thus, that $F^2(comp\{x,y\}) = \{x,y\}$ implies case 3) is applicable. Consequently, $F^U(comp\{x,y\}) = \{x,y\}$. Therefore, F^2 satisfies (BCβ).

To see that F^2 is not rationalizable, let $\triangle$ be the unit simplex, and consider $x = (0,2)$, $y = (2,0)$, $A = comp(\{x, (3,0)\} \cup \triangle)$, and $B = comp(\{y, (0,3)\} \cup \triangle)$. Note that $x \in F^2(A), y \notin F^2(A), y \in F^2(B)$ and $x \notin F^2(B)$ implying no binary relation R can be defined over $\mathbb{R}^2_+$ that will rationalize F^2.

Remark 6. (BCα) is indispensable in Theorem 1: there are non-rationalizable solutions satisfying (E), (SIR) and (BCβ) but violating (BCα). To see this, consider the lexicographic Kalai-Smorodinsky solution F^{lexKS} which is defined as usual. This solution satisfies all the axioms in Theorem 1 except (BCα) and is not rationalizable.

Remark 7. (BCβ) is indispensable in Theorem 1: there are non-rationalizable solutions satisfying (E), (SIR) and (BCα) but violating (BCβ). To see this, for any $A \in \Sigma$, let $F^{PSIR}(A) = \{x \in A \cap \mathbf{R}^n_{++} \mid \text{there exists no } y \in A \text{ such that } y > x\}$. This solution satisfies all the axioms in Theorem 1 except (BCβ) and is not rationalizable.

Remark 8. It is interesting to note that several works (for example, Roth (1977) and Zhou (1997)) have used (SIR) without (E) to derive the (asymmetric) Nash solution. In our domain, however, there is even a non-rationalizable solution that satisfies (SIR), (BCα) and (BCβ) as shown earlier in Remark 4.

4 Asymmetric Nash solutions

In this section, we show that under (E), (SIR), (BCα) and (BCβ), if we impose Nash's scale invariance axiom and require a solution to be continuous (see the formal definition below), then we obtain an asymmetric Nash solution defined below.

Definition 3. A bargaining solution F over Σ is an asymmetric Nash solution if there exist $t_1 \geq 0, \cdots, t_n \geq 0$ and $t_1 + \cdots + t_n > 0$ such that, for all $A \in \Sigma$, $F(A) = \{x \in A \mid \prod_{i=1}^n x_i^{t_i} \geq \prod_{i=1}^n y_i^{t_i}$ for all $y \in A\}$

To study asymmetric Nash solutions, we first introduce the axioms of scale invariance and continuity.

Scale Invariance (SI): For all $A \in \Sigma$ and all $t \in \mathbf{R}^n_{++}$, if $tA = \{(t_i a_i)_{i \in N} \mid a \in A\}$ then $F(tA) = \{(t_i a_i)_{i \in N} \mid a \in F(A)\}$.

Continuity (CON): For any $x, y \in \mathbb{R}^n_+$ with $x \neq y$, if $\{x\} = F(comp\{x, y\})$ then there exists $\epsilon > 0$ such that for all $z \gg 0$ and all $z' \in \mathbb{R}^n_+$,

$$[||z - x|| < \epsilon \Rightarrow \{z\} = F(comp\{y, z\})] \text{ and } [||z' - y|| < \epsilon \Rightarrow \{x\} = F(comp\{x, z'\})].$$

It may be noted that (CON) introduced above is very different from various continuity properties discussed in the literature on bargaining problems (see, for example,). To a certain degree, (CON) is a weaker requirement as it restricts its applicability to a class of problems each consisting of the comprehensive hull of two points.

With the help of (CON), we present and prove the second main result of our paper.

Theorem 2. A solution F over Σ satisfies (E), (SIR), (BCα), (BCβ), (SI) and (CON) if and only if it is an asymmetric Nash.

Proof. It can be checked that an asymmetric Nash solution satisfies (E), (SIR), (BCα), (BCβ), (SI) and (CON). We now show that if a solution satisfies (E), (SIR), (BCα), (BCβ), (SI) and (CON), then it must be an asymmetric Nash solution.

Let F satisfy (E), (SIR), (BCα), (BCβ), (SI) and (CON). From Theorem 1, F is rationalizable by a reflexive, transitive and complete binary relation R over $\mathbb{R}^n_+$. Define the binary relation R as in the proof of Theorem 1. Note that F satisfies (CON). Then, R must be continuous over $\mathbb{R}^n_{++}$. Since F satisfies (SI), R satisfies the following property: for all $x, y \in \mathbb{R}^n_{++}$ and all $\lambda \in \mathbb{R}^n_{++}$, $xRy \Leftrightarrow (\lambda_1 x_1, \cdots, \lambda_n x_n) R (\lambda_1 y_1, \cdots, \lambda_n y_n)$. Then, following Tsui and Weymark (1997) (see also Xu (2002)), there exist $t_1, \cdots, t_n$ such that, for all $x, y \in \mathbb{R}^n_{++}$, $xRy \Leftrightarrow \prod_{i=1}^n x_i^{t_i} \geq \prod_{i=1}^n y_i^{t_i}$. By (E), it follows that $t_1 \geq 0, \cdots, t_n \geq 0$ and $t_1 + \cdots + t_n > 0$. Note that if $y_i = 0$ for some $i \in N$, then xRy. Therefore, R can be represented by a Cobb-Douglas function. Hence, F is an asymmetric Nash solution. $\diamond$

Remark 9. Note that (CON) is indispensable in Theorem 2. Indeed, there exists a solution satisfying (E), (SIR), (BCα), (BCβ) and (SI) while violating (CON). For instance, consider the following solution: for all $A \in \Sigma$, $F^3(A) = \{x \in F^N(A) \mid x \geq_{lex} y \text{ for all } y \in F^N(A)\}$, where $\geq_{lex}$ is a usual lexicographic relation. This solution satisfies (E), (SIR), (BCα), (BCβ), and (SI), but violates (CON). By the definition, F^3 is neither an asymmetric Nash nor the (symmetric) Nash solution.

Remark 10. Roth (1977) uses (SIR) to derive an asymmetric Nash solution for convex bargaining problems. Zhou (1997) uses (SIR) together with Nash's Independence of Irrelevant Alternatives (which is stronger than our (BCα) and (SI) to derive an asymmetric Nash solution for nonconvex bargaining problems. In Zhou's approach, a solution is assumed to be single-valued. In our context with a multi-valued solution, if we drop (E), the solution F^{WP-} which is defined as, for any $A \in \Sigma$, $F^{WP-}(A) = \{x \in A \cap \mathbf{R}^n_{++} \mid \text{there exists no } y \in A \text{ such that } y \gg x\}$, satisfies (SIR), (BC$\alpha$), (BC$\beta$), and (SI). Thus, there are solutions other than asymmetric Nash solutions satisfying (SIR), (BCα), (BCβ), and (SI). Note that F^{WP-} satisfies (CON).

Remark 11. It is interesting to note that, there are single-valued solutions satisfying (E), (BCα), (BCβ) and (SI), which are different from asymmetric Nash solutions. Let $\geq_{lex}$ be a standard lexicographic binary relation defined over $\mathbb{R}^n_+$. Define the solution, F^{lex}, as follows: for all $A \in \Sigma$, $F^{lex}(A) = \{x \in A \mid x \geq_{lex} y \text{ for all } y \in A\}$. This solution satisfies (E), (BCα) (because the solution is rationalizable), (BCβ) (because the solution is again rationalizable) and (SI). F^{lex} is single-valued. Note that F^{lex} violates (SIR) and (CON). This suggests that (E), (BCα), (BCβ), (SI) and the single-valuedness of a solution are not sufficient to derive asymmetric Nash solutions.

5 Nash solution

We now turn to the Nash solution. We first note that the Nash solution satisfies the following property:

Anonymity (A): For any $A \in \Sigma$, if A is symmetric, then $[a \in F(A) \Rightarrow \pi(a) \in F(A)$ for all $\pi \in \Pi]$.

Theorem 3. A solution F over Σ satisfies (E), (BCα), (BCβ), (SI) and (A) if and only if $F = F^N$.

Proof. It can be checked that the Nash solution satisfies (E), (BCα), (BCβ), (SI) and (A). We need only to show that, if a solution satisfies (E), (BCα), (BCβ), (SI) and (A), then it must be the Nash solution.

Let F be a solution over Σ satisfying (E), (BCα), (BCβ), (SI) and (A). From Theorem 1, F is rationalizable by the binary relation R over $\mathbb{R}^n_+$ defined below: $xRy \Leftrightarrow x \in F(comp\{x, y\})$. We next show that R is actually representable by the Cobb-Douglas function over $\mathbb{R}^n_+$ with equal weights. Given that R is complete and transitive, we need only to show that, for all $x, y \in \mathbb{R}^n_+$, $\prod_{i \in N} x_i = \prod_{i \in N} y_i > 0 \Rightarrow F(comp(x, y)) = \{x, y\}$, and $\prod_{i \in N} x_i > \prod_{i \in N} y_i \geq 0 \Rightarrow F(comp(x, y)) = \{x\}$.

Let $\prod_{i \in N} x_i = \prod_{i \in N} y_i > 0$. Consider an appropraite $t \in \mathbf{R}^n_{++}$ such that $tx = (t_1 x_1, \cdots, t_i x_i, \cdots t_n x_n)$ and $ty = (t_1 y_1, \cdots, t_i y_i, \cdots, t_n y_n)$ are permutations of each other (this is always possible due to the fact that x and y have the same Nash product). Let $S \equiv comp\,\{tx, ty\}$. Then, let $T \equiv \cup_{\pi \in \Pi} \pi(S)$. By construction, T is symmetric, and $\{\pi(tx), \pi(ty) \mid \pi \in \Pi\} \subseteq T$ is the set of all efficient outcomes in T. Thus, $F(T) \subseteq \{\pi(tx), \pi(ty) \mid \pi \in \Pi\}$, and let $tx \in F(T)$. Then, by (A), $\{\pi(tx) \mid \pi \in \Pi\} \subseteq F(T)$. Also, since tx and ty are permutations of each other, $ty \in F(T)$ by (A). Then, again by (A), $\{\pi(ty) \mid \pi \in \Pi\} \subseteq F(T)$. Thus, $F(T) = \{\pi(tx), \pi(ty) \mid \pi \in \Pi\}$. Thus, $tx, ty \in F(T)$. Then, by (E) and (BCα), $\{tx, ty\} = F(S)$. Thus, by (SI), $F(comp\,\{x, y\}) = \{x, y\}$.

Next, let $\prod_{i \in N} x_i > \prod_{i \in N} y_i \geq 0$. Then, by choosing an appropriate $\varepsilon \in \mathbf{R}^n_+$ with $\varepsilon_j > 0$ for some j, we can have $\prod_{i \in N} x_i = \prod_{i \in N} z_i$ for $z \equiv y + \varepsilon$. Then, from the last paragraph, $F(comp\,\{x, z\}) = \{x, z\}$. Note that, by the construction, $y \in comp\{x, z\}$, and $F(comp\{x, z\}) \cap \{x, y\} = \{x\}$. Therefore, $x \in F(comp\{x, y\})$ follows from (BCα). If $y \in F(comp\{x, y\})$, then (BCβ) would imply that $y \in F(comp\{x, z\})$, a contradiction. Therefore, $y \notin F(comp\{x, y\})$.

Therefore, for all $x, y \in \mathbb{R}^n_+$, $xRy \Leftrightarrow \prod_{i \in N} x_i \geq \prod_{i \in N} y_i$. Hence, F is the Nash solution. $\diamond$

Theorem 3 thus gives an alternative characterization of the Nash solution to non-convex bargaining problems. From the characterization result of the Nash solution to nonconvex bargaining problems in Xu and Yoshihara (2006), it is clear that, in the presence of (E), (A) and (SI), Contraction Independence is equivalent to (BCα) and (BCβ). As a matter of fact, it can be easily checked that, under (E), (BCβ) is implied by Contraction Independence, which states that, for all $A, B \in \Sigma$ with $B \subseteq A$ and all $x \in A \cap B$, if $x \in F(A)$ then $x \in F(B)$. Therefore, the result of Theorem 3 strengthens the characterization result given in Xu and Yoshihara (2006).

Note that in Theorem 3, (CON) is not needed, though is implied by the axioms figured in the theorem. The reason why (CON) is not needed here is that, from Theorem 1 and in the presence of (E), (BCα) and (BCβ), the solution is rationalizable; then, (A) and (SI) ensure the construction of "indifference surfaces" and such indifference surfaces are based on Cobb-Douglas function with equal weights.

Given the remarks following Theorem 1 and from Theorem 2, the independence of the axioms figured in Theorem 3 can be readily checked.

6 Conclusion

In this paper, we have examined the implications of two weaker versions of conditions α and β in the context of solutions to non-convex bargaining problems. In particular, we have shown that, (i) under efficiency and strict individual rationality, they are equivalent to rationalizable solutions, (ii) together with efficiency, strict individual rationality, scale invariance and a weak continuity requirement, they characterize asymmetric Nash solutions, and (iii) together with efficiency, anonymity and scale invariance, they characterize the Nash solution. Conditions α and β, together, characterize rationalizability of a choice function defined over the set of all non-empty subsets of a finite universal set in terms of an ordering. It is therefore interesting to note that, in non-convex bargainging problems, (BCα) and (BCβ) are associated with "rationalizability" of a solution to bargaining problems. Our results clarify several issues relating to rationalizable solutions to nonconvex bargaining problems and to asymmetric Nash bargaining solutions, make further connections between two widely used rationality conditions in rational choice theory and solutions to non-convex bargaining problems, and improve characterizations of asymmetric Nash and the Nash solutions to nonconvex bargaining problems.

References

Bossert, W. (1994), Rational choice and two-person bargaining solutions, *Journal of Mathematical Economics* 23: 549-563.

Chernoff, H. (1954), Rational selection of decision functions, *Econometrica* 22: 422-443.

Conley, J. and S. Wilkie (1996), An extension of the Nash bargaining solution to nonconvex problems, *Games and Economic Behavior* 13, 26-38.

Herrero, M. J. (1989), The Nash program: non-convex bargaining problems, *Journal of Economic Theory* 49, 266-277.

Herzberger, H.G. (1973), Ordinal preference and rational choice, *Econometrica* 41, 187-237.

Kaneko, M. (1980), An extension of the Nash bargaining problem and the Nash social welfare function, *Theory and Decision* 12, 135-148.

Mariotti, M. (1998), Nash bargaining theory when the number of alternatives can be finite, *Social Choice and Welfare* 15, 413-421.

Mariotti, M. (1998a), "Extending Nash's axioms to nonconvex problems," *Games and Economic Behavior* 22, 377-383.

Mariotti, M. (1999), Fair bargains: distributive justice and Nash bargaining theory, *Review of Economic Studies* 66, 733-741.

Nash, J. (1950), The bargaining problem, *Econometrica* 18: 155-162.

Peters, H. and P. Wakker (1991), Independence of irrelevant alternatives and revealed groups preferences, *Econometrica* 59: 1787-1801.

Roth, A. (1977), Individual rationality and Nash's solution to the bargaining problem, *Mathematics of Operations Research* 2, 64-65.

Sánchez, M. C. (2000), Rationality of bargaining solutions, *Journal of Mathematical Economics* 33: 389-399.

Sen, A.K. (1971), Choice functions and revealed preference, *Review of Economic Studies* 38, 307-317.

Sen, A.K. (1977), Social choice theory: a re-examination, *Econometrica* 45, 53-88.

Tsui, K. and J. Weymark (1997), Social welfare orderings for ratio-scale measurable utilities, *Economic Theory* 10, 241-256.

Thomson, W. (1981), A class of solutions to bargaining problems, *Journal of Economic Theory* 25, 431-441.

Xu, Y. (2002), Functioning, capability and the standard of living an axiomatic approach, *Economic Theory* 20, 387-399.

Xu, Y. and N. Yoshihara (2006), Alternative characterizations of three bargaining solutions for nonconvex problems, *Games and Economic Behavior* 57, 86-92.

Zhou, L. (1997), The Nash bargaining theory with non-convex problems, *Econometrica* 65, 681-685.