Impossibilities of Paretian Social Welfare Functions for Infinite Utility Streams with Distributive Equity

Norihito Sakamoto*

Hitotsubashi University

June, 2007

This version: August, 2011

Abstract

This paper examines the logical relationship between distributive equity and efficiency in aggregating infinite utility streams. Our main results show that there exist social welfare functions which satisfy the axioms of Pigou-Dalton Transfer Principle and a weak version of efficiency, but there exists no social welfare function which satisfies all of the distributive equity requirements and Weak Pareto Principle at the same time. Thus, we can prove that no Paretian

*Department of Economics, Hitotsubashi University, 2-1 Naka, Kunitachi, Tokyo, Japan, 186-8601. E-mail: b111126u@r.hit-u.ac.jp; Tel: +81-42-580-8192; Fax: +81-42-580-8195. This paper is part of my Ph. D. thesis at Hitotsubashi University. The main results of this paper (Propositions 3-4 and Corollaries 1-2) overlap with Alcantud (2010), Alcantud (2011), and Alcantud and Garcia-Sanz (2010). These works independently proved the propositions around the same time. I would like to thank Kotaro Suzumura, Motohiro Sato, Peter Lambert, Kiwako Sakamoto, Reiko Gotoh and Toyotaka Sakai for helpful comments and suggestions. I am also grateful to the anonymous referees whose helpful comments contributed to the improvement of this paper. Financial support from the 21st century Centers of Excellence (COE) project on "Normative Evaluation and Social Choice of Contemporary Economic Systems" is gratefully acknowledged. Any remaining errors are my responsibility.

ranking can satisfy the numerical representability and all of the distributive equity properties in the setting of intertemporal social choice.

1 Introduction

This paper examines the logical relationship between intergenerational equity and efficiency in terms of the existence of a social welfare function. In analyzing the intertemporal social choice problem, two classes of equity axioms have considered in the previous literature. The first one is a class of anonymity axioms that demands each generation to be treated equally in the evaluation of infinite utility streams (Diamond 1965; Svensson 1980; Campbell 1985; Epstein 1986; Lauwers 1997; Shinotsuka 1998; Basu and Mitra 2003; 2007b; Mitra and Basu 2007). The second one is a class of distributive equity properties that prefers a more equitable distribution of utilities (Fleurbaey and Michel 2001; 2003; Sakai 2003; 2006; Asheim and Tungodden 2004; Bossert, et al. 2007; Hara, et al. 2007).

Basu and Mitra's pioneering study reveals some impossibility/possibility results between Finite Anonymity requirement and variations of Pareto efficiency (Basu and Mitra 2007b). In this paper, we will focus on the distributive equity properties and analyze the compatibility of these properties with efficiency requirements. One of the most fundamental equity properties is Pigou-Dalton Transfer Principle adapted to the infinite-horizon framework. Our main results show that there exist social welfare functions which satisfy Pigou-Dalton Transfer Principle and a weak version of efficiency (called Weak Dominance in the previous literature), but there exists no social welfare function which satisfies all of the distributive equity requirements and Weak Pareto

Principle at the same time. Thus, we can prove impossibility results that show no Paretian social welfare function can satisfy all of the distributive equity properties. These impossibility results form a contrast to a social ranking approach in which many possibility results can be gained by sacrificing the continuity requirement (Basu and Mitra 2007a; Bossert, Sprumont, and Suzumura 2007).

2 Basic Notations and Definitions

Let $\mathbb{R}$ denote the set of real numbers. Let $\mathbb{Z}_+$, $\mathbb{Z}_{++}$ stand, respectively, for the set of non-negative and positive integers. We also denote the set of all possible utility levels for each generation by Y, so the set of infinite utility streams is $X = Y^{\mathbb{Z}_{++}}$. For simplicity of notation, we denote the K times repetition of the finite vector $(x_1, ..., x_T)$ by $(x_1, ..., x_T)_{K\text{-rep}}$, where $T, K \in \mathbb{Z}_{++}$. If $K = \infty$, then we write $(x_1, ..., x_T)_{K\text{-rep}}$ as $(x_1, ..., x_T)_{\text{rep}}$. Throughout this paper, we interpret the utilities as either income levels or level-plus comparable ones following Blackorby et al. (1984).

There are at least two ways to evaluate infinite utility streams defined as above. One way is to use a *primitive binary relation* on X, and the other way is to consider a *social welfare function*, *i.e.*, a real-valued function defined on X. Here we use the latter approach and examine the possibility/impossibility results of social welfare functions satisfying some appealing properties.

As Basu and Mitra (2003; 2007b) show, Finite Anonymity contradicts Pareto principle in the context of social welfare function approach. The anonymity axiom, however, treats each generation equally in their utility levels and ignores a distributional aspect of utility levels.

Hence our paper focuses on some concepts of distributional equity which are proposed by the previous studies (Fleurbaey and Michel 2001; 2003; Sakai 2003; 2006; Asheim and Tungodden 2004; Hara, et al. 2007) as a solution to the problem concerning the aggregating infinite utility streams.

First, we introduce *Strict Equity Preference* axiom which is used by the axiomatic characterization of leximin rankings in Bossert, et al. (2007).

Strict Equity Preference (SPE): $\forall \mathbf{x}, \mathbf{y} \in X, \forall i, j \in \mathbb{Z}_{++}$;

$$[y_i > x_i \geq x_j > y_j \;\; \& \;\; \forall k \neq i, j, x_k = y_k] \Rightarrow W(\mathbf{x}) > W(\mathbf{y}).$$

In this axiom, social welfare is improved whenever a income transfer from the better-off generation to the worse-off one is exercised, which may increase or decrease total surplus between the loss of the rich and the gain of the poor.

This axiom can be divided into the following three variations of equity requirements according to the degree of inequality aversion over an income transfer.

The first axiom is the well-known condition called *Pigou-Dalton Transfer Principle*, which is a standard axiom in the study of income inequality measures.

Pigou-Dalton Transfer Principle (PDT): $\forall \mathbf{x}, \mathbf{y} \in X, \forall \epsilon > 0, \forall i, j \in \mathbb{Z}_{++}$;

$$[x_i = y_i - \epsilon \geq y_j + \epsilon = x_j \;\; \& \;\; \forall k \neq i, j, x_k = y_k] \Rightarrow W(\mathbf{x}) > W(\mathbf{y}).$$

Intuitively, this axiom says that transferring income from the rich to the poor is always good as long as the relative position between them will not be reversed.

The definitions of the following two variations of SEP depend on differences between the welfare loss of the rich and the welfare gain of the poor.

Altruistic Equity-1 (AE-1): $\forall \mathbf{x}, \mathbf{y} \in X, \forall \epsilon > \delta > 0, \forall i, j \in \mathbb{Z}_{++}$;

$$[x_i = y_i - \delta \geq y_j + \epsilon = x_j \ \ \& \ \ \forall k \neq i, j, x_k = y_k] \Rightarrow W(\mathbf{x}) > W(\mathbf{y}).$$

Altruistic Equity-2 (AE-2): $\forall \mathbf{x}, \mathbf{y} \in X, \forall \epsilon > \delta > 0, \forall i, j \in \mathbb{Z}_{++}$;

$$[x_i = y_i - \epsilon \geq y_j + \delta = x_j \ \ \& \ \ \forall k \neq i, j, x_k = y_k] \Rightarrow W(\mathbf{x}) > W(\mathbf{y}).$$

Altruistic Equity-1, the axiom proposed by Hara, et al. (2007), states that a transfer from the rich to the poor should be done if the utility gain of the poor is greater than the utility loss of the rich through this transfer.

On the contrary, *Altruistic Equity-2* requires that society should accept a transfer from the rich to the poor even if the utility gain of the poor is smaller than the utility loss of the rich.

The last of the equity axioms we use is the one investigated by Asheim, Mitra and Tungodden (2007) and Banerjee (2006).

Hammond Equity for Future Generations (HEF): $\forall \mathbf{x}, \mathbf{y} \in X$;

$$[y_1 > x_1 > \bar{x} > \bar{y} \ \ \& \ \ \forall i \geq 2, x_i = \bar{x} \ \& \ y_i = \bar{y}] \Rightarrow W(\mathbf{x}) > W(\mathbf{y}).$$

This axiom states that social welfare must increase if inequity between generation 1 and all future generations is wholly improved. Note that this axiom is logically independent of Strict Equity Preference.

Now, we continue by introducing the efficiency requirement in the framework of social welfare function. The first axiom is the standard Pareto criterion.

Pareto Principle (P): $\forall \mathbf{x}, \mathbf{y} \in X$;

$$[\forall i \in \mathbb{Z}_{++},\, x_i \geq y_i \ \ \& \ \ \exists j \in \mathbb{Z}_{++}\, x_j > y_j] \Rightarrow W(\mathbf{x}) > W(\mathbf{y}).$$

In the previous studies, this efficiency condition is often shown to be too strong to gain the possibility results. Therefore, this paper considers two weak forms of Pareto Principle as in Basu and Mitra (2007b). The first version can be written as follows;

Weak Pareto Principle (WP): $\forall \mathbf{x}, \mathbf{y} \in X$; $\forall i \in \mathbb{Z}_{++}, x_i > y_i \Rightarrow W(\mathbf{x}) > W(\mathbf{y})$.

Weak Pareto Principle requires that social welfare must be improved whenever all generations strictly increase their utilities. The second condition of efficiency is defined as follows;

Weak Dominance (WD): $\forall \mathbf{x}, \mathbf{y} \in X$;

$$[\exists j \in \mathbb{Z}_{++},\, x_j > y_j \ \ \& \ \ \forall i \neq j,\, x_i = y_i] \Rightarrow W(\mathbf{x}) > W(\mathbf{y}).$$

Weak Dominance axiom states that an improvement of exactly one generation's utility increases social welfare. Under Weak Dominance, we can easily show that AE-2 implies PDT and PDT implies AE-1[1].

[1]Note that the logical relationships among three equity axioms are all independent under Weak Pareto Principle.

3 Consistency of the Dominance Principle

This section examines the logical compatibility between Weak Dominance –one of the Pareto requirements– and three variations of SEP.

First, we show an impossibility result for AE-2 which is the strongest requirement among our three axioms.

Proposition 1: *Let the range of utility function Y include a closed interval $[0,1]$. Then, there exists no social welfare function satisfying AE-2 and WD.*

[Proof] Suppose to the contrary. Let a social choice function W satisfy AE-2 and WD. Define the two utility streams $\mathbf{x}(\epsilon), \mathbf{y}(\epsilon)$ as follows;

For $\epsilon \in (3/10, 1/2)$,

$$\mathbf{x}(\epsilon) = \left(2\epsilon,\ \frac{1}{2}\epsilon,\ (\delta)_{\text{rep}}\right),$$

$$\mathbf{y}(\epsilon) = \left(\epsilon,\ \frac{1}{2}\epsilon,\ (\delta)_{\text{rep}}\right),$$

where $\delta \in [0,1]$. It is easy to show that both $\mathbf{x}(\epsilon)$ and $\mathbf{y}(\epsilon)$ are in $(0,1)^{\mathbb{Z}_{++}}$ for any ϵ. Now, for each $\epsilon \in (3/10, 1/2)$, the condition WD implies $W(\mathbf{x}(\epsilon)) > W(\mathbf{y}(\epsilon))$. Then, for each $\epsilon, \epsilon' \in (3/10, 1/2)$, $\epsilon > \epsilon'$ implies $2\epsilon' > \epsilon > \epsilon/2 > \epsilon'/2$ and $2\epsilon' - \epsilon > \frac{1}{2}(\epsilon - \epsilon')$. Hence, we have $W(\mathbf{y}(\epsilon)) > W(\mathbf{x}(\epsilon'))$ because of AE-2. By the above argument, $W(\mathbf{x}(\epsilon')) > W(\mathbf{y}(\epsilon'))$ holds. Therefore, for any $\epsilon, \epsilon' \in (3/10, 1/2)$ with $\epsilon > \epsilon'$, both of the two closed intervals $[W(\mathbf{y}(\epsilon)), W(\mathbf{x}(\epsilon))]$ and $[W(\mathbf{y}(\epsilon')), W(\mathbf{x}(\epsilon'))]$ are non-degenerate and the intersection of them must be empty. Thus, for any real number $\epsilon \in (3/10, 1/2)$, we can choose a distinct rational number $r(\epsilon)$ from the closed interval $[W(\mathbf{y}(\epsilon)), W(\mathbf{x}(\epsilon))]$. This implies, however, that the cardinality of continuum is a countable cardinality: a contradiction. $\|$

Since the axiom SEP implies AE-2, the following corollary can be immediately established by the above proposition.

Corollary 1[2]: *Let* $Y \supseteq [0,1]$. *Then, there is no social welfare function satisfying SEP and WD.*

Hammond Equity for Future Generations is logically independent of Altruistic Equity-2, but Banerjee (2006) proves that this equity requirement contradicts Weak Dominance.

Proposition 2 (Banerjee 2006): *Let* $Y \supseteq [0,1]$. *Then, there is no social welfare function satisfying HEF and WD.*

By replacing WD by WP, we can, however, immediately obtain a possibility result for HEF. For example, $W(\mathbf{x}) = \min\{x_1, x_2\}$ is a social welfare function satisfying both HEF and WP (Asheim, Mitra and Tungodden 2007, Example 1). But this function is not particularly useful because it does not impose value on procedural and distributive equity. The condition HEF, in this sense, is so weak as a concept of distributive equity that a class of social welfare functions satisfying this axiom is not desirable.

The next proposition says that Pigou-Dalton Transfer Principle can be consistent with Weak Dominance[3].

Proposition 3: *There exist social welfare functions satisfying PDT and WD.*

[2] Under Weak Dominance, the axiom SEP is equivalent to the requirement of Hammond Equity (Alcantud 2011, Lemma 1). Hence, this corollary means that there is no SWF satisfying the conditions of Weak Dominance and Hammond Equity.

[3] Weak Dominance is a kind of weakened efficiency conditions. However, there is another way to relax Pareto Principle. For example, Monotonicity is a weaker version of Pareto Principle, which requires that weakly improvement of all generations' utilities never decreases social welfare. But the proofs of propositions 4-6 in the next section show the non-existence of social welfare functions satisfying Monotonicity and our three axioms of distributive equity.

[Proof] The following functions are a variation of SWFs which are proposed by Basu and Mitra (2007b). For any infinite utility streams $\mathbf{x} \in X$, define a set $E(\mathbf{x})$ as follows;

$$E(\mathbf{x}) = \{\mathbf{y} \in X | \exists T \in \mathbb{Z}_{++}, \forall t \geq T, x_t = y_t\}.$$

That is, $E(\mathbf{x})$ is the set of the same utility streams for all generations beyond some generation T. The set E clearly forms an equivalence class, and the universal set X is partitioned by E. We denote the set of partitions of X as $\mathcal{E}$. Then, by the axiom of choice, there exists a function which assigns each $E \in \mathcal{E}$ into $\mathbf{x} \in X$ such that $g(E) \in E$ for all $E \in \mathcal{E}$. Using this function g, define a social welfare function as follows;

$$W(\mathbf{x}) = \lim_{n \to \infty} \sum_{t=1}^{n} \left[U(x_t) - U(g_t(E(\mathbf{x}))) \right],$$

where $U(\bullet)$ is a strictly increasing, real-valued, and strictly concave function. The function $W(\mathbf{x})$ must have a limit for all $\mathbf{x} \in X$ since both $\mathbf{x}$ and $g(E((x)))$ are in the $E(\mathbf{x})$. Therefore, this social welfare function is well-defined and satisfies WD by definition. Because of the strict concavity of $U(\bullet)$, this function satisfies PDT. $\|$

Since combining WD with PDT implies AE-1, the social welfare function above readily satisfies AE-1.

The results of this section show that the non-negativeness of total surplus by income transfer guarantee the existence of social welfare functions satisfying Weak Dominance. That is, if society accepts only income transfers from a richer generation to a poorer one, which keep total surplus of the aggregate welfare of the two generations to be non-negative, then we have a class of social welfare functions satisfying Weak Dominance.

The next section, however, proves that Weak Pareto Principle is inconsistent with any axiom of distributive equity regardless of whether an aggregate welfare between the rich and the poor after a transfer is negative or non-negative.

4 Impossibilities of the Weak Pareto Principle

This section investigates the logical relationships between three variations of distributive equity and Weak Pareto Principle. Impossibility results for Weak Pareto Principle are more fundamental than impossibility results for Weak Dominance because Weak Pareto Principle is considered as a basic concept of efficiency in that it requires social welfare to increase if utilities of all generations could be strictly improved. In the following discussion, we show that all concepts of distributive justice are inconsistent with Weak Pareto Principle. It is, therefore, proved that there is substantial collision between distributive equity and efficiency requirements in the social welfare function approach for infinite generations setting.

First, we begin by an impossibility result about social welfare functions satisfying Weak Pareto Principle and Altruistic Equity-1 which is the weakest requirement among our three equity axioms under Weak Dominance.

Proposition 4: *Let $Y \supseteq [0,1]$. Then, there is no social welfare function satisfying AE-1 and WP.*

[Proof] Suppose to the contrary. Let a social welfare function W satisfy AE-1 and WP. Define the two utility streams $\mathbf{x}(\epsilon), \mathbf{y}^k(\epsilon)$ as follows;

For $\epsilon \in (3/10, 1/2)$,

$$\mathbf{x}(\epsilon) = ((\epsilon,\ 2\epsilon)_{\text{rep}}),$$

$$\mathbf{y}^k(\epsilon) = \left((\epsilon + (2/3)^{k-1}(2/10),\ 2\epsilon - (1/3)^{k-1}(1/10))_{2^{k-1}\text{-rep}},\ (\epsilon,\ 2\epsilon)_{\text{rep}} \right).$$

By definition, both $\mathbf{x}(\epsilon)$ and $\mathbf{y}^k(\epsilon)$ are in $(0,1)^{\mathbb{Z}_{++}}$ for all $\epsilon \in (3/10, 1/2)$ and all natural number k.

Now, we will show that AE-1 implies $W(\mathbf{x}(\epsilon)) < W(\mathbf{y}^1(\epsilon)) < W(\mathbf{y}^2(\epsilon)) < \dots$. It is easy to check that $\mathbf{y}^1(\epsilon)$ is derived from $\mathbf{x}(\epsilon)$ by an income transfer where generation 2 loses $(1/10)$ but generation 1 gains $(2/10)$. Hence, we have $W(\mathbf{x}(\epsilon)) < W(\mathbf{y}^1(\epsilon))$.

Next, we show that for all k, the stream $\mathbf{y}^{k+1}(\epsilon)$ is constructed from $\mathbf{y}^k(\epsilon)$ by the repeated applications of an appropriate income transfer.

In the stream $\mathbf{y}^k(\epsilon)$, odd-numbered generations are relatively poor but there are inequalities among them, that is, one has $\epsilon + (2/3)^{k-1}(2/10)$ but the other has only ϵ. By implementing the transfer where the relatively rich odd-numbered generations loses $(2/3)^k(1/10)$ but the relatively poor odd-numbered generations gains $(2/3)^k(2/10)$, each generation has the same utility $\epsilon + (2/3)^k(2/10)$. Specifically, for any $i = 1, 2, ..., 2^{k-1}$, we take $(2/3)^k(1/10)$ from $(2i-1)$-th generation and give $(2/3)^k(2/10)$ to $(2^k + 2i - 1)$-th generation. Similarly, among even-numbered generations which are relatively rich in $\mathbf{y}^k(\epsilon)$, there are also inequalities. Then, these generations can have the same utility level $2\epsilon - (1/3)^k(1/10)$ through a transfer in which the rich who have 2ϵ loses $(1/3)^k(1/10)$ but the poor who only have $(2\epsilon - (1/3)^{k-1}(1/10))$ gain $(1/3)^k(2/10)$. That is, for any $i = 1, ..., 2^{k-1}$, we take $(1/3)^k(1/10)$ from $2i$-th generation and give $(1/3)^k(2/10)$ to $(2^k + 2i)$-th generation. Therefore, for all k, $W(\mathbf{y}^k(\epsilon)) < W(\mathbf{y}^{k+1}(\epsilon))$.

Then, for all distinct $\epsilon, \epsilon' \in (3/10, 1/2)$ with $\epsilon > \epsilon'$, there exists a natural number K^*

such that $\epsilon > \epsilon' + (2/3)^{K^*-1}(2/10)$. Obviously, $2\epsilon > 2\epsilon'$ and $2\epsilon > 2\epsilon' - (1/3)^{K^*-1}(1/10)$, so WP implies $W(\mathbf{x}(\epsilon)) > W(\mathbf{y}^{K^*}(\epsilon'))$. Because $W(\mathbf{y}^{K^*}(\epsilon')) > W(\mathbf{y}^1(\epsilon'))$ by the above argument, for any $\epsilon, \epsilon' \in (3/10, 1/2)$ with $\epsilon > \epsilon'$, the two closed intervals $[W(\mathbf{x}(\epsilon)), W(\mathbf{y}^1(\epsilon))]$ and $[W(\mathbf{x}(\epsilon')), W(\mathbf{y}^1(\epsilon'))]$ are non-degenerate, and the intersection between them is empty.

Thus, for any real number $\epsilon \in (3/10, 1/2)$, we can choose a distinct rational number $r(\epsilon)$ from the closed interval $[W(\mathbf{x}(\epsilon)), W(\mathbf{y}^1(\epsilon))]$. This implies, however, the cardinality of continuum equals the countable cardinality: a contradiction. ||

Since it holds true that AE-2 $\rightarrow$ PDT $\rightarrow$ AE-1 under Weak Dominance, Proposition 4 is interpreted as a fundamental result for Weak Pareto Principle. In fact, the remaining two equity axioms are incompatible with Weak Pareto Principle in the social welfare function approach. The following two impossibility results can be established by proofs which are similar to that of proposition 4. That is, whatever an aggregate welfare resulted from an income transfer between the rich and the poor is, the requirements of distributive equity and Weak Pareto Principle always leads to an impossibility result.

Proposition 5: *Let $Y \supseteq [0,1]$. Then, there is no social welfare function satisfying PDT and WP.*

[Proof] Since the proof of this proposition is similar to that of Prop.4, we omit some details of the proof. Consider the two utility streams $\mathbf{x}(\epsilon), \mathbf{y}^k(\epsilon)$ as follows;
For $\epsilon \in (1/5, 1/2)$,

$$\mathbf{x}(\epsilon) = ((\epsilon,\ 2\epsilon)_{\text{rep}}) ,$$

$$\mathbf{y}^k(\epsilon) = \left(\left(\epsilon + (1/2)^{k-1}(1/10),\ 2\epsilon - (1/2)^{k-1}(1/10)\right)_{2^{k-1}\text{-rep}},\ (\epsilon,\ 2\epsilon)_{\text{rep}} \right).$$

The stream $\mathbf{y}^1(\epsilon)$ is constructed from $\mathbf{x}(\epsilon)$ by the income transfer $1/10$ from generation 2 to generation 1. Hence, PDT implies $W(\mathbf{x}(\epsilon)) < W(\mathbf{y}^1(\epsilon))$. Next for all k, we show that the stream $\mathbf{y}^{k+1}(\epsilon)$ is induced from $\mathbf{y}^k(\epsilon)$ through the following transfer.

First, we transfer $(1/2)^k \times (1/10)$ from the better-off odd-numbered generations who have $\epsilon + (1/2)^{k-1}(1/10)$ to the worse-off odd-numbered generations who have only ϵ in the stream $\mathbf{y}^k(\epsilon)$. Then, each generation has the same utility $\epsilon + (1/2)^k(1/10)$. Similarly, even-numbered generations can have the same utility level $2\epsilon - (1/2)^k(1/10)$ by the transfer $(1/2)^k(1/10)$ from the rich who have 2ϵ to the poor who have only $(2\epsilon - (1/2)^{k-1}(1/10))$. Therefore, we show $W(\mathbf{y}^k(\epsilon)) < W(\mathbf{y}^{k+1}(\epsilon))$ for all k.

The remainder of this proof can be shown by the same argument used in the proof of proposition 4. ||

Proposition 6: *Let $Y \supseteq [0,1]$. Then, there is no social welfare function satisfying AE-2 and WP.*

[Proof] Define two utility streams $\mathbf{x}(\epsilon), \mathbf{y}^k(\epsilon)$ as follows;

For $\epsilon \in (3/10, 1/2)$,

$$\mathbf{x}(\epsilon) = ((\epsilon,\ 2\epsilon)_{\text{rep}}),$$

$$\mathbf{y}^k(\epsilon) = \left(\left(\epsilon + (1/3)^{k-1}(1/10),\ 2\epsilon - (2/3)^{k-1}(2/10)\right)_{2^{k-1}\text{-rep}},\ (\epsilon,\ 2\epsilon)_{\text{rep}} \right).$$

The stream $\mathbf{y}^1(\epsilon)$ is gained from $\mathbf{x}(\epsilon)$ by the transfer in which generation 2 loses $(2/10)$ but generation 1 gains $(1/10)$. Next, we show that, for all k, the stream $\mathbf{y}^{k+1}(\epsilon)$ is constructed from $\mathbf{y}^k(\epsilon)$ by repeated applications of an appropriate transfer. In the stream $\mathbf{y}^k(\epsilon)$, if we implement the transfer in which the relatively rich odd-numbered generations loses $(1/3)^k(2/10)$ but the relatively poor odd-numbered generations gains $(1/3)^k(1/10)$, then each odd-numbered generation has the same utility $\epsilon+(1/3)^k(1/10)$. In addition, even-numbered generations can have the same utility level $2\epsilon-(2/3)^k(2/10)$ by means of a transfer in which the rich who have 2ϵ lose $(2/3)^k(2/10)$ but the poor who only have $(2\epsilon-(2/3)^{k-1}(2/10))$ gain $(2/3)^k(1/10)$. Therefore, for all k, $W(\mathbf{y}^k(\epsilon)) < W(\mathbf{y}^{k+1}(\epsilon))$.

The remainder of the proof can be shown by the same argument used in the proof of proposition 4. $\|$

Since SEP implies the axioms PDT, AE-1, and AE-2, the following corollary follows directly from our impossibility results (propositions 4-6).

Corollary 2: *Let $Y \supseteq [0,1]$. Then, there is no social welfare function satisfying SEP and WP.*

5 Concluding Remarks

In this paper, we have examined the positive and negative relationships among the distributive equity axioms and two weak forms of Pareto principle for problems involving the aggregating infinite utility streams. As a result, we show that there exists no Paretian social welfare function satisfying any variation of the distributive justice requirement in our framework. Table 1 summarizes the results obtained in this paper.

Table 1: Logical relationships between efficiency and distributive equity

	AE-1	PDT	AE-2	SEP	HEF
WD	◯	◯	×	×	×
WP	×	×	×	×	◯

The symbols "◯" and "×" indicate, respectively, "possibility" and "impossibility" results.

As we can see from table 1, we can construct social welfare functions satisfying Weak Dominance and the two variations of distributive equity. These possibility results, however, should not be construed essentially as "positive" because an attempt to aggregate infinite utility streams never satisfies any form of equity axioms and Weak Pareto Principle simultaneously –the latter axiom is a more basic efficiency requirement than Weak Dominance. Accordingly, our analysis reveals the substantial difficulties of the social welfare function approach in the context of intergenerational equity.

Now we comment on the further discussion.

First, the non-existence of a social welfare function does not necessarily mean impossibility results of social ranking for infinite utility streams. Indeed, Bossert, Suzumura and Sprumont (2007) characterize orderings satisfying the axioms of Strong Pareto Principle, Finite Anonymity and Pigou-Dalton Transfer Principle. Therefore, without a numerical representation of social evaluation for intergenerational equity, we can construct social orderings which satisfy both equity and efficiency properties.

Secondly, we can attain some possibility results by weakening the domain restriction of Y. Indeed, if $Y = Z_+$, then we can easily construct a social welfare function satisfying Pigou-Dalton Transfer Principle and Pareto Principle following Basu and Mitra (2007b).

Lastly, the requirement of a numerical representation for social welfare functions is logically independent of the continuity condition of social rankings. If a social ranking on X is continuous with some topology, then the numerical representation of this ranking must be continuous on X as well. But it does not imply the continuity of the social ranking on X in which the numerical representation of the ranking is possible in general. In their seminal paper, Sakai (2006) and Hara, et al. (2007) show that there is no continuous social ranking on X satisfying both Pigou-Dalton Transfer Principle and some axioms of collective consistency such as acyclicity. But their impossibility results are all independent of our results because the numerical representability of social rankings does not imply the continuity of these rankings. The results established in this paper suggest a trade-off between intergenerational equity and efficiency in the objective function in usual dynamic optimization problems.

References

[1] Alcantud, J.C.R. (2010) "The Compromise Efficiency vs. Egalitarianism among Generations with an Infinite Horizon," *MPRA Paper 22284*, University Library of Munich, Germany.

[2] Alcantud, J.C.R. (2011) "Inequality Averse Criteria for Evaluating Infinite Utility Streams: The Impossibility of Weak Pareto," forthcoming in *Journal of Economic Theory*.

[3] Alcantud, J.C.R. and M.D. Garcia-Sanz (2010) "Evaluations of Infinite Utility Streams: Pareto-efficient and egalitarian axiomatics," *MPRA Paper 20133*, University Library of Munich, Germany.

[4] Asheim, G.B. and B. Tungodden (2004) "Resolving Distributional Conflicts between Generations," *Economic Thoery*, Vol.24, pp.221-230.

[5] Asheim, G.B., T. Mitra, and B. Tungodden (2007) "A New Equity Condition for Infinite Utility Streams and the Possibility of Being Paretian," in J. Roemer and K. Suzumura ［eds.］, *Intergenerational Equity and Sustainability*, Palgrave Macmillan, London.

[6] Banerjee, K. (2006) "On the Equity-Efficiency Trade off in Aggregating Infinite Utility Streams," *Economics Letters*, Vol.93, pp.63-67.

[7] Basu, K. and T. Mitra (2003) "Aggregating Infinite Utility Streams with Intergenerational Equity: The Impossibility of Being Paretian," *Econometrica*, Vol.71, pp.1557-1563.

[8] Basu, K. and T. Mitra (2007a) "Utilitarianism for Infinite Utility Streams: A New Welfare Criterion and its Axiomatic Characterization," *Journal of Economic Theory*, Vol.133, pp.350-373.

[9] Basu, K. and T. Mitra (2007b) "Possibility Theorems for Equitably Aggregating Infinite Utility Streams," in J. Roemer and K. Suzumura [eds.], *Intergenerational Equity and Sustainability*, Palgrave Macmillan, London.

[10] Blackorby, C., D. Donaldson, and J.A Weymark (1984) "Social Choice with Interpersonal Utility Comparisons: A Diagrammatic Introduction," *International Economic Review*, Vol.25, pp.327-356.

[11] Bossert, W., Y. Sprumont and K. Suzumura (2007) "Ordering Infinite Utility Streams," *Journal of Economic Theory*, Vol.135, pp.579-589.

[12] Campbell, D. (1985) "Impossibility Theorems and Infinite Planning," *Social Choice and Welfare*, Vol.2, pp.283-293.

[13] Diamond, P.A. (1965) "The Evaluation of Infinite Utility Streams," *Econometrica*, Vol.33, pp.170-177.

[14] Epstein, L. (1986) "Intergenerational Preference Orderings," *Social Choice and Welfare*, Vol.3, pp.151-160.

[15] Fleurbaey, M. and P. Michel (2001) "Transfer Principles and Inequality Aversion, with an Application to Optimal Growth," *Mathematical Social Sciences*, Vol.42, pp.1-11.

[16] Fleurbaey, M. and P. Michel (2003) "Intertemporal Equity and the Extension of the Ramsey Principle," *Journal of Mathematical Economics*, Vol.39, pp.777-802.

[17] Hara, C., T. Shinotsuka, K. Suzumura, and Y. Xu (2007) "On the Possibility of Continuous, Paretian and Egalitarian Evaluation of Infinite Utility Streams," Working Paper, Project on Intergenerational Equity, Institute of Economic Research, Hitotsubashi University.

[18] Lauwers, L. (1997) "Continuity and Equity with Infinite Horizons," *Social Choice and Welfare*, Vol.14, pp.345-356.

[19] Mitra, T. and Basu, K. (2007) "On the Existence of Paretian Social Welfare Quasi Orderings for Infinite Utility Streams with Extended Anonymity," in J. Roemer and K. Suzumura ［eds.］, *Intergenerational Equity and Sustainability*, Palgrave Macmillan, London.

[20] Sakai, T. (2003) "An Axiomatic Approach to Intergenerational Equity," *Social Choice and Welfare*, Vol.20, pp.167-176.

[21] Sakai, T. (2006) "Equitable Intergenerational Preferences on Restricted Domains," *Social Choice and Welfare*, Vol.27, pp.41-54.

[22] Shinotsuka, T. (1998) "Equity, Continuity, and Myopia: A Generalization of Diamond's Impossibility Theorem," *Social Choice and Welfare*, Vol.15, pp.21-30.

[23] Suzumura, K. and K. Tadenuma (2007) "Normative Approaches to the Issues of Global Warming: Responsibility and Compensation," in J. Roemer and K. Suzumura ［eds.］, *Intergenerational Equity and Sustainability*, Palgrave Macmillan, London, forthcoming.

[24] Svensson, L.G. (1980) "Equity among Generations," *Econometrica*, Vol.48, pp.1251-1256.