A LOCALLY OPTIMAL TEST FOR NO UNIT ROOT IN CROSS-SECTIONALLY DEPENDENT PANEL DATA

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Abstract

This paper develops a simple test for the null hypothesis of no unit root for panel data with cross-sectional dependence in the form of a common factor in the disturbance. We do not estimate the common factor but mop-up its effect by employing the same method as the one proposed in Pesaran (2007) in the unit root testing context. We show that our test is asymptotically locally optimal, although the optimality is not guaranteed under a wide range of the alternative.

Key words: KPSS test, unit root, cross-sectional dependence, LM test, locally best test
JEL classification: C12, C33

I. Introduction

Since the beginning of the 1990s, much theoretical and empirical econometrics literature was devoted to testing unit root and stationarity in panel data with a large $T$ (time dimension) and a large $N$ (cross-section dimension). The main motive for applying unit root and stationarity tests to panel data is to improve the power of the tests relative to their univariate counterparts. This was supported by the ensuing applications and simulations. The early
theoretical contributions were made from the mid-1990s to the early 2000s under the assumption that the cross-sectional units are independent or at least not cross-sectionally correlated. Banerjee (1999), Baltagi and Kao (2000), and Baltagi (2001) provide comprehensive surveys on the first generation panel tests.

However, in most empirical applications, this assumption is erroneous. O’Connell (1998) was the first to show via simulation that the panel tests are considerably distorted when the independence assumption is violated. Banerjee, Marcellino and Osbat (2001, 2004) argued against the use of panel unit root tests due to this problem. Therefore, it became imperative to develop panel tests that take the possibility of cross-sectional dependence into account. This led, recently, to a flurry of papers accounting for cross-sectional dependence in different forms or to the arrival of second generation panel unit root tests. The most noticeable proposals in this area are by Chang (2004), Phillips and Sul (2003), Bai and Ng (2004), Moon and Perron (2004), Choi and Chue (2007), and Pesaran (2007) for unit root panel tests. For panel stationarity tests, the only contributions thus far are by Bai and Ng (2005) and Harris, Leybourne and McCabe (2005), both of which corrected for cross-sectional dependence by using the principal component analysis proposed by Bai and Ng (2004).

In this paper, we focus on a test for the null hypothesis that there is no unit root in cross-sectionally dependent panel data against the alternative of the existence of unit roots. To deal with cross-sectional dependence, we adapt the Pesaran (2007) approach to the panel stationarity test of Hadri (2000) due to its conceptual simplicity. Our test is basically the same as the Kwiatkowski et al. (1992) test (KPSS test), and therefore, we call it the augmented KPSS test. We also derive a Lagrange multiplier (LM) test, which is known to be locally optimal under the assumption of normality. We show that these two tests have the same asymptotic property under the null of no unit root and under the local alternative. This implies that the augmented KPSS test is asymptotically locally optimal. Since it is much easier to construct the augmented KPSS test statistic than the LM test statistic while both tests have the same asymptotic optimality, our test is useful in practical analysis.

The paper is organized as follows. Section 2 sets up the model and assumptions, and defines the augmented test statistic. We also develop the LM test allowing for cross-sectional dependence. Section 3 is devoted to the comparison of our augmented KPSS test under restrictive assumptions with the LM test under the null of no unit root, under the local alternative and under the fixed alternative. We show that the limiting null distribution of the augmented KPSS test is the same as that of Hadri’s (2000) test. In Section 4, we examine whether our theoretical result is valid in finite samples via simple Monte Carlo simulations. Section 5 gives concluding remarks. All the proofs are relegated to the Appendix.

We now give a summary on the notations. We define $M_A = I_T - A(A'A)^{-1}A'$ for a full column rank matrix $A$. The symbols $p_{(N,T)}$ and $\Rightarrow_{(N,T)}$ imply joint convergence in probability and joint weak convergence, respectively, when both $N$ and $T$ approach infinity simultaneously, while $\Rightarrow$ and $\Rightarrow$ imply weak convergence when only $T$ or $N$ approaches infinity.
II. Model and Test Statistics

1. Model and Assumptions

Let us consider the following model:

\[ y_{it} = \gamma_i t + r_{it} + u_{it}, \quad r_{it} = r_{it-1} + v_{it}, \quad u_{it} = f_t \gamma_i + \varepsilon_{it} \]  \hspace{1cm} (1)

for \( i = 1, \ldots, N \) and \( t = 1, \ldots, T \), where \( z_t \) is deterministic and \( r_{i0} = 0 \) for all \( i \). The commonly used specification of \( z_t \) in the literature is either \( z_t = z^i_t = 1 \) or \( z_t = z^i_{1,t} \). In this paper, we consider these two cases. Accordingly, we define \( \delta_i = \alpha_i \) when \( z_t = 1 \) and \( \delta_i = [\alpha_i, \beta_i]' \) when \( z_t = [1, t]' \). In model (1), \( z^i_0 \delta_i \) is the individual effect while \( f_t \) is the one-dimensional unobserved common factor, \( \gamma_i \) is the loading factor, and \( \varepsilon_{it} \) is the individual-specific (idiosyncratic) error.

By stacking \( y_{it} \) with respect to \( t \), model (1) can be expressed as

\[
\begin{bmatrix}
    y_{i1} \\
    y_{i2} \\
    \vdots \\
    y_{iT}
\end{bmatrix}
= Z \delta_t + \begin{bmatrix}
    r_{i1} \\
    r_{i2} \\
    \vdots \\
    r_{iT}
\end{bmatrix} + \begin{bmatrix}
    f_1 \\
    f_2 \\
    \vdots \\
    f_T
\end{bmatrix} \gamma_i + \begin{bmatrix}
    \varepsilon_{i1} \\
    \varepsilon_{i2} \\
    \vdots \\
    \varepsilon_{iT}
\end{bmatrix},
\]

or

\[ y_i = Z \delta_t + r_i + f_i \gamma_i + \varepsilon_i \]  \hspace{1cm} (2)

where \( Z = [\tau, d] \) with \( \tau = [1, 1, \ldots, 1]' \) and \( d = [1, 2, \ldots, T]' \) being \( T \times 1 \) vectors, \( L \) is a \( T \times T \) matrix with ones on the main diagonal and everywhere below it. Further, we have

\[
\begin{bmatrix}
    y_1 \\
    y_2 \\
    \vdots \\
    y_N
\end{bmatrix}
= \begin{bmatrix}
    Z \\
    Z \\
    \vdots \\
    Z
\end{bmatrix} + \begin{bmatrix}
    \delta_1 \\
    \delta_2 \\
    \vdots \\
    \delta_N
\end{bmatrix} + \begin{bmatrix}
    L \\
    L \\
    \vdots \\
    L
\end{bmatrix} \begin{bmatrix}
    v_1 \\
    v_2 \\
    \vdots \\
    v_N
\end{bmatrix}
+ \begin{bmatrix}
    f_1 \gamma_1 \\
    f_2 \gamma_2 \\
    \vdots \\
    f_N \gamma_N
\end{bmatrix} + \begin{bmatrix}
    \varepsilon_1 \\
    \varepsilon_2 \\
    \vdots \\
    \varepsilon_N
\end{bmatrix}
\]

or

\[ y = (I_N \otimes Z) \delta + r + (\gamma \otimes f) + \varepsilon \]  \hspace{1cm} (3)

In this paper, we make the following assumption.

**Assumption 1.** (i) The stochastic processes \( \{\varepsilon_{it}\}, \{f_t\}, \) and \( \{v_{it}\} \) are independent and \( \varepsilon_{it} \sim i.i.d. N(0, \sigma^2_i), \ f_t \sim i.i.d. N(0, \sigma^2_t), \ v_{it} \sim i.i.d. N(0, \sigma^2_t) \) with known variances.

(ii) There exist real numbers \( M_1, M_2, \) and \( \bar{M} \) such that \( |\gamma_i| < M_1 < \infty \) for all \( i \) and \( 0 < M_2 < |\bar{\gamma}| < \bar{M} < \infty \) for all \( N \), where \( \bar{\gamma} = N^{-1} \sum_{i=1}^N \gamma_i \).
The assumption of normality with homoskedasticity in (i) is required to derive the LM test and to discuss the optimal property of the tests. The variances $\sigma^2_e$, $\sigma^2_f$, and $\sigma^2_v$ are assumed to be known in order to make the theoretical investigation as simple as possible. The unknown case will be discussed later. (ii) implies that each individual is possibly affected by the common factor with the finite weight $\gamma_i$ and that the absolute value of the average of $\gamma_i$ is bounded away from 0 and above both in finite samples and in asymptotics. The latter property is important in order to eliminate the common factor effect from the regression. See also Pesaran (2007).

We consider a test for the null hypothesis of no unit root component against the alternative of the existence of unit roots for model (1). Since all the innovations are homoskedastic, the testing problem is given by

$$H_0 : \rho = 0 \quad \text{vs.} \quad H_1 : \rho > 0$$

where $\rho = \sigma^2_v / \sigma^2_e$ is the signal-to-noise ratio. Under $H_0$, all $r_i$s become equal to zero and thus do not have unit root components, unlike under $H_1$.

### 2. A Simple Stationarity Test

Panel stationarity tests have already been proposed by Hadri (2000) and Shin and Snell (2006) for cross-sectionally independent data, and we extend Hadri’s test to the cross-sectionally dependent case. Hadri (2000) showed that if there is no cross-sectional dependence in a model, we can construct the LM test using the regression residuals of $y_{it}$ on $z_t$ in the same way as KPSS (1992), and that the limiting distribution of the standardized LM test statistic is standard normal under the null hypothesis. However, it can be shown that Hadri’s (2000) test depends on nuisance parameters even asymptotically if there exits cross-sectional dependence; we then need to develop a stationarity test that takes into account cross-sectional dependence.

In order to eliminate the effect of the common factor from the test statistic, we make use of the simple method proposed by Pesaran (2007), which develops panel unit root tests with cross-sectional dependence. As in Pesaran (2007), we first take a cross-sectional average of the model:

$$\bar{y}_t = z_t \mathbf{\bar{d}} + \mathbf{\bar{r}}_t + f_t \gamma + \bar{e}_t,$$

where $\bar{y}_t = N^{-1} \sum_{i=1}^N y_{it}$, $\mathbf{\bar{d}} = N^{-1} \sum_{i=1}^N \mathbf{d}_i$, $\mathbf{\bar{r}}_t = N^{-1} \sum_{i=1}^N r_{it}$, $\gamma = N^{-1} \sum_{i=1}^N \gamma_i$, and $\bar{e}_t = N^{-1} \sum_{i=1}^N e_{it}$. Since $\gamma \neq 0$ by assumption, we can solve equation (5) with respect to $f_t$ as

$$f_t = \frac{1}{\gamma} (\bar{y}_t - z_t \mathbf{\bar{d}} - \mathbf{\bar{r}}_t - \bar{e}_t).$$

By inserting this solution of $f_t$ into model (1), we obtain the following augmented regression model:

$$y_{it} = z_t \mathbf{\bar{d}} + \bar{\gamma} \bar{y}_t + \epsilon_{it},$$

where $\mathbf{\bar{d}}_i = \mathbf{d}_i - \bar{\gamma} \bar{y}_t$, $\bar{\gamma}_i = \gamma_i / \bar{\gamma}$, and $\epsilon_{it} = r_{it} - \bar{\gamma} \bar{r}_t + e_{it} - \bar{\gamma} \bar{e}_t$. Based on (6), we propose to regress $y_{it}$ on $z_t$ and $\bar{y}_t$ for each $i$, and construct the test statistic in the same way as Hadri (2000). That is,
\[ Z_d = \frac{\sqrt{N}(ST - \xi)}{\zeta}, \]  

(7)

where \( \overline{ST} = N^{-1} \sum_{i=1}^{N} ST_i \) with \( ST_i = (\sigma_i^2 T)^{-1} y_i' M_i L'L M_i y_i \) and

\[
\begin{align*}
\xi &= \xi_\mu = \frac{1}{6}, \\
\zeta &= \zeta_\mu = \frac{1}{45} \quad \text{when } z_i = z_t^\mu = 1, \\
\xi &= \xi_t = \frac{1}{15}, \\
\zeta &= \zeta_t = \frac{11}{6300} \quad \text{when } z_i = z_t^\mu = [1, t].
\end{align*}
\]

Note that \( ST_i \) can also be expressed as

\[ ST_i = \frac{1}{\sigma_i^2 T} \sum_{t=1}^{T} (S_t^n)^2 \quad \text{where } S_t^n = \sum_{i=1}^{N} \hat{\varepsilon}_t, \]

with \( \hat{\varepsilon}_t \) obtained for each \( i \) by regressing \( y_{it} \) on \( w_t = [\hat{z}_t', \bar{y}_t]' \) for \( t = 1, \ldots, T. \)

From (7), we can see that \( \overline{ST} \) is the average of the KPSS test statistic across \( i \) and \( Z_d \) corresponds to its normalized version. We call \( Z_d \) the augmented KPSS test statistic.

### 3. An LM Test for Panel Stationarity

Although the augmented KPSS test is easy to implement, we do not know whether it has an optimal property. Since the LM test is known to be a locally best invariant test under the assumption of normality as shown by Tanaka (1996), we derive the LM test, and then, in the later section, compare it with the augmented KPSS test.

Under Assumption 1, the log-likelihood function of \( y \), denoted by \( l \), is expressed as

\[ l = \text{const} - \frac{1}{2} \log |\Omega| - \frac{1}{2} \{ y - (I_N \otimes Z) \bar{\delta} \}' \Omega^{-1} \{ y - (I_N \otimes Z) \bar{\delta} \}, \]

where \( \Omega = \text{Var}(y) = \rho(\sigma_i^2 I_N \otimes LL') + A \otimes I_T \) with \( A = \sigma_i^2 T' + \sigma_i^2 I_N \). The partial derivative of \( l \) with respect to \( \rho \) is given by

\[ \frac{\partial l}{\partial \rho} = \text{const} - \frac{1}{2} \text{tr} \left( \Omega^{-1} \frac{\partial \Omega}{\partial \rho} \right) + \frac{1}{2} \{ y - (I_N \otimes Z) \bar{\delta} \}' \Omega^{-1} \frac{\partial \Omega}{\partial \rho} \Omega^{-1} \{ y - (I_N \otimes Z) \bar{\delta} \}. \]  

(8)

Noting that

\[ \Omega \big|_{H_0} = A \otimes I_T \quad \text{and} \quad \frac{\partial \Omega}{\partial \rho} \big|_{H_0} = \sigma_i^2 I_N \otimes LL', \]  

(9)

the maximum likelihood estimator (MLE) of \( \bar{\delta} \) under \( H_0 \) is given by

\[ \hat{\bar{\delta}} = \left[ (I_N \otimes Z') \Omega^{-1} \big|_{H_0} (I_N \otimes Z) \right]^{-1} \left( I_N \otimes Z' \right) \Omega^{-1} \big|_{H_0} y \]

\[ = [I_N \otimes (Z'Z)^{-1}Z']y. \]  

(10)

Thus, the MLE of \( \bar{\delta} \) under \( H_0 \) is the same as the OLS estimator. By evaluating (8) under the null hypothesis using (9) and (10), the LM test statistic is given by
\[
LM = \frac{1}{NT^2} \{ y - (I_S \otimes Z) \hat{\delta} \}' (A^{-1} \otimes I_T) (\sigma_e^2 I_S \otimes LL') (A^{-1} \otimes I_T) \{ y - (I_S \otimes Z) \hat{\delta} \} = \frac{1}{NT^2} y'(\sigma_e^2 A^{-1} \otimes M_z LL'M_z)y.
\]

Then, the normalized version of the LM test statistic is given by
\[
Z_{LM} = \frac{\sqrt{N}(LM - \xi)}{\zeta}, \quad (11)
\]
where \( \xi \) and \( \zeta \) are the same as in \( Z_A \).

### III. Limiting Distributions of the Test Statistics

In this section, we compare the augmented KPSS test with the LM test. Note that the LM test is known to be a locally best invariant test under Assumption 1. Because there is no one-to-one transformation between \( Z_A \) and \( Z_{LM} \), the augmented KPSS test does not have local optimality in finite samples. As such, we now focus on whether the KPSS test is asymptotically locally optimal or not.

In order to investigate the asymptotic local optimality of the augmented KPSS test, we compare it with the LM test statistic under the null hypothesis, under the local alternative and under the fixed alternative. We first give the limiting distributions of the two test statistics under the null hypothesis.

**Theorem 1.** Suppose that Assumption 1 holds. Under \( H_0 \), as \( N \) and \( T \) approach infinity simultaneously with \( N/T \rightarrow 0 \), the augmented KPSS and LM test statistics have a limiting standard normal distribution for both cases of \( z_t = 1 \) and \( z_t = [1, t]' \). That is, \( Z_A, Z_{LM} \overset{(N,T)}{\rightarrow} N(0,1) \).

Note that the rejection regions of both \( Z_A \) and \( Z_{LM} \) are the right-hand tails as in Hadri’s (2000) test. Theorem 1 shows that Pesaran’s (2007) method works well to eliminate cross-sectional dependence for testing the null hypothesis of stationarity. We also note that the condition that \( N/T \rightarrow 0 \) as \( N \) and \( T \) approach infinity, means that the tests are suitable for panels where \( T \) is larger than \( N \).

We now investigate the asymptotic property of the test statistics under the local alternative, which is expressed as
\[
H_1 : \rho = -\frac{c^2}{\sqrt{NT^2}}, \quad \text{where } c \text{ is some constant.}
\]

Note that for a single time series analysis, the local alternative is given by \( \rho = c^2/T^2 \). Since the sum of \( ST_i \) is normalized by \( \sqrt{N} \) as in \( Z_A \), the local alternative for panel stationarity tests becomes \( \rho = c^2(\sqrt{N}/T^2) \).

**Theorem 2.** Suppose that Assumption 1 holds. Under \( H_1 \), as \( N \) and \( T \) approach infinity simultaneously with \( N/T \rightarrow 0 \), the augmented KPSS and LM test statistics have the same
limiting distribution given by
\[
Z_{A}, Z_{LM} \overset{(N,T)}{\longrightarrow} N(0, 1) + \frac{\sigma^2}{\kappa} E \left[ \int_0^T F_i(r)^2 dr \right],
\]
where \( F_i(r) = \int_0^r B_i(s) ds + \int_0^r z(s) B_i'(s) ds \) with \( B_i(r) \) being independent Brownian motions, \( z(r) = 1 \) and \( E[\int_0^T F_i(r)^2 dr] \kappa = \sqrt{1/180} \) when \( z_i = 1 \), and \( z(r) = [1, r]' \) and \( E[\int_0^T F_i(r)^2 dr] \kappa = \sqrt{1/25200} \) when \( z_i = [1, 1]' \).

This result implies that both the augmented KPSS and extended LM test statistics have the same asymptotic local distribution. Since the LM test is locally best invariant, we can see that the augmented KPSS test has the same asymptotic local optimality.

We can also deduce from Theorem 2 that both tests are more powerful when only a constant is included in the regression than in the trending case, much like the univariate KPSS test, because \( 1/90 > 1/12600 \).

We finally investigate the asymptotic property of the test statistics under the fixed alternative \( H_1 \). The following theorem gives the difference in the powers of the two tests when the alternative is not local but far away from \( \rho = 0 \).

**Theorem 3.** Suppose that Assumption 1 holds. Under \( H_1 \), as \( N \) and \( T \) approach infinity simultaneously with \( N/T \to 0 \),
\[
\frac{1}{\sqrt{N T}} Z_A \overset{(N,T)}{\longrightarrow} \frac{\rho}{\kappa} E_{\omega} \left[ \int_0^T G_i(r)^2 dr \right], \quad \text{and} \quad \frac{1}{\sqrt{N T}} Z_{LM} \overset{(N,T)}{\longrightarrow} \frac{\rho}{\kappa} E \left[ \int_0^T F_i(r)^2 dr \right],
\]
where \( G_i(r) = \int_0^r B_i(s) ds - \int_0^r z_2(s) \int_0^s z_2(s) ds \int_0^s z_2(s) ds \int_0^s B_i'(s) ds \) with \( z_2(r) = [z'(r), B_i'(r)] \), \( B_i'(r) \) is a standard Brownian motion independent of \( B_i'(r) \), and \( E_{\omega} \) denotes the expectation operator with respect to \( B_i'(r) \).

Note that since \( G_i'(r) \) depends on \( B_i'(r) \) and \( B_i'(r) \), which are independent, we can see that \( E_{\omega} \left[ \int_0^T G_i(r)^2 dr \right] \) still depends on \( B_i'(r) \) and is thus stochastic, while \( E \left[ \int_0^T F_i(r)^2 dr \right] \) is deterministic. This is an interesting result because when the asymptotic local powers are the same for the two tests, it is often the case that they also have the same limiting distribution under the fixed alternative. In our situation, the two tests have the same local asymptotic power from Theorem 2 but the powers are different under the fixed alternative from Theorem 3. This implies that although the two tests are locally optimal, they are not equivalent in a wide range under the alternative.

Finally, we discuss the case where the variances are unknown. In this case, we can estimate \( \sigma^2 \) consistently under \( H_0 \) by \( (NT)^{-1} \sum_{i=1}^N \sum_{t=1}^T \hat{\varepsilon}_i^2 \), where \( \hat{\varepsilon}_i^2 \) is the residual from the augmented regression. Then, we can still construct \( Z_A \) in practical analysis. However, the construction of the LM test requires the knowledge of not only \( \sigma^2 \) but also \( \sigma^2 \gamma^T \gamma \) as in the definition of \( A \), which can be obtained by the method in Bai (2003). However, since \( Z_A \) is much simpler than \( Z_{LM} \) and \( Z_A \) is asymptotically locally optimal, the augmented KPSS test would be convenient and useful in practical analysis.
IV. Finite Sample Property

In this section, we investigate how accurately does the asymptotic theory approximate the finite sample behavior of the augmented KPSS and LM tests. We consider the following data generating process for finite sample simulations:

\[ y_{it} = z_i \delta_i + r_{it} + \gamma_i t + \varepsilon_{it}, \quad \varepsilon_{it} \sim \text{i.i.d.} N(0, 1), \quad \delta_i \sim \text{i.i.d.} N(0, 1), \]

where \( \delta_i = \alpha_i \) for the constant case while \( \delta_i = [\alpha_i, \beta_i]' \) for the trend case with \( \alpha_i \) and \( \beta_i \) being drawn from independent \( U(0, 0.02) \), \( \gamma_i \) are drawn from \( -1 + U(0, 4) \) for the strong cross-sectional correlation case (SCC) and from \( U(0, 0.02) \) for the weak cross-sectional correlation case (WCC), and \( \alpha_i, \beta_i, \) and \( \gamma_i \) are fixed throughout the iterations. Since our purpose is to see if the asymptotic theory obtained in the previous section can approximate the finite sample behavior, we assume that the variances are known throughout the simulations. We consider all the pairs of \( N=10, 20, 30, 50, \) and \( 100, \) and \( T=50, 100, \) and \( 200. \) The level of significance is 0.05 and the number of replications is 10,000 in all experiments.

Table 1 shows the sizes of the tests. We can observe that the empirical size of the augmented KPSS test is close to the nominal one for any value of \( T \) for the SCC case while it is slightly undersized for the WCC case. On the other hand, the size of the LM test is close to the nominal one irrespective of \( N \) and \( T \) but it is slightly undersized for the SCC case while it is slightly oversized for the WCC case. Overall, the null distributions of the two tests seem to be well approximated by a standard normal distribution as suggested by Theorem 1 in view of the size of the tests.

Table 2 reports the powers of the tests. For given \( N \) and \( T \), the upper, middle, and lower
## Table 2. Power of the Tests

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<th>N</th>
<th>T</th>
<th>constant case</th>
<th>trend case</th>
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entries are the powers of the tests for \( \rho = 0.0001, 0.001, \) and 0.01, respectively. From the table, the powers of the tests become higher for larger \( \rho \) and \( T \), although the tests have low power when \( T \) is small. We can also observe that the powers become higher for larger \( N \). For example, the size of the augmented KPSS test for \( T=50 \), SCC, and the constant case is relatively close to 0.05 for all the values of \( N \) while the empirical power when \( \rho = 0.001 \) is 0.145, 0.202, 0.254, 0.342, and 0.539 for \( N=10, 20, 30, 50, \) and 100, respectively. Table 2 implies that the tests are consistent as proved by Theorem 3.

In order to see if the augmented KPSS test can be seen as the asymptotically locally best test indicated by Theorem 2, we calculated the size adjusted power of the tests. Figure 1 draws the power curves for selected cases. From the figure, we observe that the power of the augmented KPSS test is almost the same as that of the LM test for the constant case. When a linear trend is included, the augmented KPSS test is as powerful as the LM test when \( \rho \) is small while the former is slightly less powerful than the latter for the trend case.

As a whole, the finite sample behavior of the augmented KPSS and LM tests is well approximated by the asymptotic theory established in the previous section when \( N \) and \( T \) are of
moderate size.

V. Conclusion

In this paper we extended Hadri’s (2000) test to correct for cross-sectional dependence à la Pesaran (2007). We showed that the limiting null distribution of the augmented KPSS test is the same as that of Hadri’s test that assumes cross-sectional dependence. We also derived the LM test under the assumption of cross-sectional dependence. Then, we compared these two tests and found that the augmented KPSS test is asymptotically locally optimal but it is not asymptotically equivalent to the LM test under the fixed alternative.

Although the augmented KPSS test has a local optimal property, we do not know the theoretical and finite sample property of the test when the idiosyncratic errors are serially correlated. In addition, we assumed a one-dimensional common factor in this paper but it would be worth considering multi-dimensional common factors. The modification of our test to such a general case is our ongoing research.

APPENDIX

In this appendix, we denote some constants independent of $N$, $T$, and the subscripts $i$ and $t$ as $C$, $C_1$, $C_2$, $\cdots$. To save space, we give the outline of the proof of the theorems only for the case where $z_i=[1, t]$. Details are available upon request. The proof for the level case with $z_i=1$ proceeds in exactly the same way, and is thus omitted. We also assume that $\sigma^2=1$ in this appendix without loss of generality because we know $\sigma^2$ under Assumption 1(i).

We first express $\bar{y}$, in matrix form. Since $\bar{y}=z'\bar{\sigma}+\bar{r}+f'\bar{y}+\bar{\varepsilon}$, we have

$$\bar{y}=Z\bar{\sigma}+\bar{r}+f'\bar{y}+\bar{\varepsilon},$$

(12)

where, for example, $\bar{y}=[\bar{y}_1', \bar{y}_2', \cdots, \bar{y}_t']'$ and the other vectors and matrices are defined similarly. Since $\bar{\gamma} \neq 0$, we have $\bar{f}=(\bar{y}-Z\bar{\sigma}-\bar{r}-\bar{\varepsilon})/\bar{\gamma}$. By inserting this into (2), the model becomes

$$y_i=Z(\delta_i-\bar{\gamma}\bar{\delta})+\bar{z}\tilde{y}+(r_i-\bar{\gamma}\bar{r})+(\varepsilon_i-\bar{\gamma}\bar{\varepsilon}),$$

(13)

where $\tilde{y}_i=y_i/\bar{\gamma}$.

Let $W=[\tau, d, \bar{y}]=[Z, \bar{y}]$ and $W'=WQ=[Z, \bar{y}']$, where $\bar{y}'=\bar{y}-Z\bar{\sigma}+\bar{r}+f'\bar{y}+\bar{\varepsilon}$,

$$Q=\begin{bmatrix} I_2 & -\bar{\delta} \\ 0 & 1 \end{bmatrix}, \quad D=\begin{bmatrix} D_2 & 0 \\ 0 & \sqrt{T} \end{bmatrix} \quad \text{and} \quad D_2=\begin{bmatrix} \sqrt{T} & 0 \\ 0 & T/\sqrt{T} \end{bmatrix}.$$  

Because $M_w=M_w^\star$, $ST_i$ in the augmented KPSS test statistic can be expressed in matrix form as

$$ST_i=1/Ty_i'M_{w^\star}L'LM_wy_i.$$  

Before proceeding with the proof of the theorems, we state two lemmas, which will be used in the proof repeatedly.

**Lemma A.1.** Let $v_n \sim i.i.d. N(0, \sigma^2)$ for $i=1, \cdots, N$ and $t=1, \cdots, T$, $r_n=\sum_{i=1}^{N}v_{nt}$, and $\bar{r}_T=N^{-1}\sum_{t=1}^{T}r_{nt}$.  

$$
Then,
\[
E[r_u r_a] = \sigma_i^2 \min(s, t),
\]
\[
E \left[ \sum_{j=1}^{t} r_{u}^2 \right] = \frac{\sigma_i^2}{6} (t+1)(2t+1),
\]
\[
E \left[ \sum_{j=1}^{t} r_{u}^2 \right] = \frac{\sigma_i^2}{30} (t+1)(2t+1)(2t^2+2t+1),
\]
\[
E \left[ r_u r_a \right] = \frac{\sigma_i^2}{N} \min(s, t),
\]
\[
E \left[ \sum_{j=1}^{t} r_{u}^2 \right] = \frac{\sigma_i^2}{6N} (t+1)(2t+1),
\]
\[
E \left[ \sum_{j=1}^{t} \tilde{r}_u^2 \right] = \frac{\sigma_i^2}{30N} (t+1)(2t+1)(2t^2+2t+1),
\]
\[
E \left[ \sum_{j=1}^{t} r_{u} \right] \left( \sum_{j=1}^{t} r_{a} \right) = \frac{\sigma_i^2}{6} (t+1)(3T-t+1),
\]
\[
E \left[ \sum_{j=1}^{t} r_{u} \right] \left( \sum_{j=1}^{t} r_{a} \right) = \frac{\sigma_i^2}{24} (t+1)(6T^2+6T-t^2-t+2),
\]
\[
E \left[ \sum_{j=1}^{t} r_{u} \right] \left( \sum_{j=1}^{t} r_{a} \right) = \frac{\sigma_i^2}{24} T(T+1)(5T^2+5T+2),
\]
\[
E[r_u r_a r_u r_a] = \sigma_i^2 (2st+su) \text{ for } s \leq t \leq u \leq v.
\]

The next lemma gives the sufficient conditions on the equivalence of the sequential limit to the joint limit. Notice that when the statistic $S_{iT}$ weakly converges to $S_{m}$ as $T \to \infty$, we can construct the probability space on which both $S_{iT}$ and $S_{m}$ exist, as discussed in Phillips and Moon (1999).

**Lemma A.2.** Let $S_{iT}$ and $S_{m}$ be i.i.d. sequences across $i$ ($i = 1, \ldots, N$) on the same probability space. Assume that $S_{m}$ does not depend on $N$, $S_{iT}$ is independent of $S_{m}$ for $i \neq j$, and $S_{iT} \underset{T}{\to} S_{m}$ as $T \to \infty$.

(i) If (a) $E[S_{i\tau}] \to \mu_{i} = E[S_{m}] < \infty$ as both $N$ and $T$ approach infinity, and (b) $\sup_{\tau} E[S_{i\tau}] < \infty$, then,
\[
\frac{1}{N} \sum_{i=1}^{N} S_{i\tau} \overset{p}{\underset{(N, T)}{\to}} \mu_{i}.
\]

(ii) If (a) $N^{-1/2} \sum_{i=1}^{N} S_{i\tau} \overset{d}{\to} S$ as $N \to \infty$, (b) $S_{iT}$ does not depend on $N$ and $\sup_{\tau} E[S_{i\tau}^{2+\kappa_{1}}] < \infty$ for some $\kappa_{1}>0$ or $E[S_{i\tau}] \to \mu_{i} = E[S_{m}] < \infty$ as $T \to \infty$, (c) $\sup_{\tau} E[S_{i\tau}] < \infty$ and $E[S_{i\tau}^{2+\kappa_{1}}] < \infty$ for some $\kappa_{1}>0$, then,
\[
\frac{1}{N} \sum_{i=1}^{N} S_{i\tau} \overset{d}{\underset{(N, T)}{\to}} S.
\]

Proof of Lemma A.2: (i) Since $S_{iT}$ is an i.i.d. sequence, we have for any arbitrary $\varepsilon > 0$,
\[
P \left( \left| \frac{1}{N} \sum_{i=1}^{N} S_{i\tau} - E[S_{i\tau}] \right| \geq \varepsilon \right) \leq \frac{1}{\varepsilon^2 N} E \left( (S_{i\tau} - E[S_{i\tau}])^2 \right) \leq \frac{1}{\varepsilon^2 N} \sup_{N, T} E[S_{i\tau}^2] \to 0
\]
by condition (b) as both $N$ and $T$ approach infinity. Because $E[S_{ij}] \to \mu_1$ by condition (a), we can see that $N^{-1} \sum_{i=1}^{N} S_{ij} \overset{p}{\to} \mu_1$.

(ii) Since $S_{ij}$ and $S_{km}$ are i.i.d. sequences, we have for any arbitrary $\varepsilon > 0$,

$$
\Pr \left( \frac{1}{\sqrt{N}} \sum_{j=1}^{N} (S_{ij} - S_{km}) \geq \varepsilon \right) \leq \frac{1}{\varepsilon} (E[S_{ij}^2] + E[S_{km}^2] - 2E[S_{ij}S_{km}]).
$$

(24)

If $\sup_T E[S_{ij}^{2+k}] < \infty$, then we can replace the limit and the expectation by Theorem 4.5.2. of Chung (1974), and thus, $\lim_T E[S_{ij}] = E[S_{ij}]$ under condition (b). On the other hand, by Hölder’s inequality, we have for any arbitrary $0 < \delta < 1$,

$$
E \left[ |S_{ij}S_{km}|^{1+\delta} \right] \leq (E[S_{ij}^2])^{1+\delta/2} \left( E \left[ |S_{km}|^{2(1+\delta)/(\delta-1)} \right] \right)^{1-\delta/2}.
$$

The right-hand side of the above inequality with $\delta = \kappa/(4 + \kappa)$ is bounded above uniformly over $T$ by condition (c). This implies that $\sup_T E[S_{ij} S_{km}^{1+k/(4+k)]} < \infty$, and again, we can replace the limit and the expectation, so that $\lim_T E[S_{ij} S_{km}] = E[S_{ij}^m]$. As a result, the right-hand side of (24) approaches zero as both $N$ and $T$ approach infinity. Combining this result with condition (a), we obtain (ii). □

**Proof of Theorem 1**

Because $r_i$ and $\bar{r}_i$ disappear under the null hypothesis, $ST_i$ can be expressed in matrix form under $H_0$ as

$$
ST_i = \frac{1}{T} y' M_{ij} L' L M_{ij} y_i
$$

$$
= \frac{1}{T} \bar{y}' M_{ij} L' L M_{ij} \bar{y} - 2 \frac{\bar{y}' \bar{y}}{T} M_{ij} L' L M_{ij} \bar{y}_i + \frac{\bar{y}_i^2}{T} \bar{y}' M_{ij} L' L M_{ij} \bar{y}_i
$$

$$
= ST_i' - 2 \bar{y} ST_i + \bar{y}^2 ST_i, \text{ say.}
$$

Let $ST_i' = T^{-2} \bar{y}_i M_{ij} L' L M_{ij} \bar{y}_i$. Since Shin and Snell (2006) showed that

$$
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( ST_i' - \bar{y}_i \right) \overset{p}{\to} \mathcal{N}(0, I),
$$

it is sufficient for us to prove that

$$
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} (ST_i' - ST_i') \overset{p}{\to} 0,
$$

(25)

$$
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} ST_i' \overset{p}{\to} 0,
$$

(26)

$$
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} ST_i' \overset{p}{\to} 0.
$$

(27)

Let $J_0 = T^{-1} L \bar{y}_i, [J_1, J_2] = T^{-1} L W D^{-1} = [T^{-1} L D^{-1} T^{-1/2} L S_1, T^{-1/2} L S_1], [J_3'] = D^{-1} W' \bar{y}_i, [(D^{-1} Z' \bar{y}_i)']'$, $(T^{-1/2} \bar{y}' \bar{y})^2]$, $K_1 = [K_{ij}] = D^{-1} W' W D^{-1}$, and $K_2 = [K_{ij}]$ for $i, j = 1, 2$. Then, we have

$$
\frac{1}{T} L M_{ij} \bar{y}_i = \frac{1}{T} L \bar{y}_i - \frac{1}{T} L W' W D^{-1} (D^{-1} W' W D^{-1})^{-1} D^{-1} W' \bar{y}_i.
$$
Therefore, we obtained (25).

Similarly, by letting $J_0 = N^{-1} \sum_{i=1}^{N} J_{0i}$, $J_1 = N^{-1} \sum_{i=1}^{N} J_{1i}$, and $J_4 = N^{-1} \sum_{i=1}^{N} J_{4i}$, we can see that

$$
\frac{1}{T} \hat{LM}_{\alpha} \hat{e}_i = \bar{J}_0 - (J_1 K_{11}^{\dagger} + J_2 K_{21}^{\dagger}) \bar{J}_3 - (J_1 K_{11}^{\dagger} + J_2 K_{21}^{\dagger}) \bar{J}_4.
$$

(29)

We first prove (25). Using expression (28), $ST_{ii}$ can be decomposed into

$$
ST_{ii} = (J_0 - J_1 K_{11}^{\dagger} J_0)(J_0 - J_1 K_{11}^{\dagger} J_0)
$$

$$
-2(J_0 - J_1 K_{11}^{\dagger} J_0)'(J_0 - J_1 K_{11}^{\dagger} J_0)
$$

$$
+ \{J_2 K_{21}^{\dagger} J_0 + (J_1 K_{11}^{\dagger} + J_2 K_{21}^{\dagger}) J_0\}' \{J_2 K_{21}^{\dagger} J_0 + (J_1 K_{11}^{\dagger} + J_2 K_{21}^{\dagger}) J_0\}
$$

$$
= ST_{ii} + ST_{ii} + ST_{ii},
$$

(30)

In order to evaluate each term, we use the following lemma.

**Lemma A.3.** Under the null hypothesis, as both $N$ and $T$ approach infinity simultaneously, (i) $E\|J_0\|^2 \leq C$, $\frac{1}{N} \sum_{i=1}^{N} \|J_{0i}\|^2 = O_p(N^{-1})$, and $\|J_0\| = O_p\left(\frac{1}{\sqrt{T}}\right)$; (ii) $J_0 = O(1)$; (iii) $E\|J\|^2 \leq C$ and $\|J\| = O_p\left(\frac{1}{\sqrt{T}}\right)$; (iv) $K_1 = O(1)$, $K_2 = O(1)$, $K_3 = K_4 = O_p\left(\frac{1}{\sqrt{T}}\right)$, $K_5 = O_p\left(\frac{1}{T}\right)$, and $K_6 = O(1)$; (v) $E\|J\|^2 \leq C$, $\frac{1}{N} \sum_{i=1}^{N} \|J_{2i}\|^2 = O_p(N^{-1})$, and $\|J\| = O_p\left(\frac{1}{\sqrt{T}}\right)$; (vi) $E\|J\|^2 \leq C$, $\frac{1}{N} \sum_{i=1}^{N} \|J_{4i}\|^2 = O_p(N^{-1})$, and $\|J_4\| = O_p\left(\frac{\sqrt{T}}{T}\right)$; and (vii) $\frac{1}{N} \sum_{i=1}^{N} \|J_{4i}\|^2 = O_p(N^{-1})$ for $l, m = 3, 4$ and $\frac{1}{N} \sum_{i=1}^{N} \|J_{4i}\|^2 = O_p(N^{-1})$.

Since $ST_{ii} = (J_0 - J_1 K_{11}^{\dagger} J_0)(J_0 - J_1 K_{11}^{\dagger} J_0)$, we have using Lemma A.3,

$$
\left| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (ST_{ii} - ST_{ii}) \right| = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \|J_0 (K_{11}^{\dagger} - K_{11}^{\dagger}) J_0 \|^2
$$

$$
+ \frac{2}{\sqrt{N}} \sum_{i=1}^{N} \|J_0 - J_1 K_{11}^{\dagger} J_0\| \|J_0 - J_1 K_{11}^{\dagger} J_0\|^2
$$

$$
= O_p\left(\frac{\sqrt{T}}{T}\right) + O_p\left(\frac{\sqrt{T}}{T}\right),
$$

which converges to 0 in probability when both $N$ and $T$ approach infinity because $N/T \to 0$ by Assumption 1(iii).

In exactly the same manner, we have

$$
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \|ST_{ii}\| = O\left(\frac{\sqrt{T}}{T}\right)
$$

and

$$
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \|ST_{ii}\| = O\left(\frac{\sqrt{T}}{T}\right).
$$

Therefore, we obtained (25).

To prove (26) and (27), note that

$$
\left| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} ST_{ii} \right| \leq C \frac{\sqrt{T}}{T^2} \tilde{e}' M_{\alpha} L'L M_{\alpha} \tilde{e}
$$

and

$$
\left| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} ST_{ii} \right| \leq C \frac{\sqrt{T}}{T^2} \tilde{e}' M_{\alpha} L'L M_{\alpha} \tilde{e}.
$$

Then, it is sufficient to show that $\frac{\sqrt{N}}{T} \tilde{e}' M_{\alpha} L'L M_{\alpha} \tilde{e} \overset{D}{\to} 0$, which can be proved by noting that
\[ \frac{1}{T} L M_{0 \nu} \tilde{\varepsilon} = O \left( \frac{1}{\sqrt{N}} \right), \]

using expression (29) and Lemma A.3. We thus obtain the result for the augmented KPSS test statistic.

To derive the limiting distribution of the LM test statistic, we first note that under \( H_0 \),

\[
(IS \otimes M_L) y = (\gamma \otimes M_L) + (IS \otimes M_L) \varepsilon \\
\sim N(0, A \otimes M_L) = (A^{1/2} \otimes M_L) \eta,
\]

where \( \eta = [\eta_1', \cdots, \eta_N'] \sim N(0, I_N \otimes I_d) \). Then, \( \overline{LM} \) can be expressed as

\[
\overline{LM} = \frac{1}{NT} \eta' (IS \otimes M_L)(A^{-1} \otimes I_d) (IS \otimes L'M_L) \eta.
\]

(31)

We first investigate the matrix \( A \). Note that \( A^{-1} \) can be expressed as

\[
A^{-1} = (\sigma_{\gamma' \gamma'}^2 + 1)^{-1} = \left( I_N - \frac{1}{1 + \sigma_{\gamma' \gamma'}^2} \sigma_{\gamma' \gamma'}^2 \right).
\]

Since \( \text{rk}(\gamma' \gamma') = 1 \) and \( (\gamma' \gamma')^{-1} = (\gamma' \gamma) \), the \( (N - 1) \) eigenvalues of \( \gamma' \gamma \), for which the corresponding eigenvector is \( \gamma \). Then, there exists an \( N \times N \) orthonormal matrix \( P \) such that \( PP^* = I_N \) and \( P \gamma' \gamma P^* = \text{diag} \{ \gamma' \gamma, 0, \cdots, 0 \} = \Lambda_\gamma \). This implies that

\[
P^* A^{-1} P = I_N - \frac{1}{1 + \sigma_{\gamma' \gamma'}^2} \sigma_{\gamma' \gamma'}^2 \Lambda_\gamma = \text{diag} \left\{ \frac{1}{1 + \sigma_{\gamma' \gamma'}^2}, 1, \cdots, 1 \right\} = \Lambda_\gamma^{-1}.
\]

(32)

By inserting (32) into (31), we obtain

\[
\overline{LM} = \frac{1}{NT} \eta' (IS \otimes M_L)(PP^* \sigma_{\gamma'}^2 A^{-1} P P^* \otimes I_d) (IS \otimes L'M_L) \eta
\]

\[
= \frac{1}{NT} \eta' (IS \otimes M_L)(\Lambda_\gamma^{-1} \otimes I_d) (IS \otimes L'M_L) \eta
\]

\[
= \frac{1}{1 + \sigma_{\gamma' \gamma'}^2} \frac{1}{NT} \eta' M L'M \eta + \frac{1}{NT} \sum_{i=1}^{N} \eta_{i}' M L'L M \eta_{i}.
\]

where \( \eta_i' = [\eta_{i1}', \cdots, \eta_{iN}'] = (P \otimes I_d) \eta \sim N(0, I_N \otimes I_d) \). Note that the first term converges to zero in probability as both \( N \) and \( T \) approach infinity, while the second term has the same structure as \( ST^2 \). We then obtain the result for the LM test statistic.

**Proof of Theorem 2**

We first note that Lemma A.3 still holds under \( H_1 \) using the fact that \( \tilde{y}_i = \tilde{y}_i + \tilde{y}_i + \tilde{e}_i \). Let \( J_0 = T^{-1} L r_0, \quad [J_0', J_0'] = D^{-1} W' r_i, \quad [D^{-1} W' r_i, (T^{-1} y' r_i)'] \), \( J_0 = N^{-1} \sum_{i=1}^{N} J_0, \quad J_5 = N^{-1} \sum_{i=1}^{N} J_0 \), and \( J_5 = N^{-1} \sum_{i=1}^{N} J_0 \). Under \( H_1 \), we have the following lemma.

**Lemma A.4.** Under the local alternative \( H_1 \), as both \( N \) and \( T \) approach infinity simultaneously, (i) \( E \| J_0 \|^2 \leq \frac{C}{N^{1/4}}, \quad \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \| J_0 \|^2 = O_\delta(1), \) and \( \| J_5 \| = O \left( \frac{1}{\sqrt{N}} \right) \); (ii) \( E \| J_5 \|^2 \leq \frac{C}{N^{1/4}}, \quad \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \| J_5 \|^2 = O_\delta(1), \) and \( \| J_5 \| = O \left( \frac{1}{\sqrt{N}} \right) \); (iii) \( E \| J_5 \|^2 \leq \frac{C}{N^{1/4}}, \quad \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \| J_5 \|^2 = O_\delta(1), \) and \( \| J_5 \| = O \left( \frac{1}{\sqrt{N}} \right) \); (iv) \( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \| J_0 \| = O_\delta(1), \) \( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \| J_0 \| = O_\delta(1), \) and \( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \| J_3 \| \| J_3 \| = O \left( \frac{1}{\sqrt{N}} \right) \).
Note that, under the local alternative, 
\[ \frac{1}{T} LM_{\omega \nu} y = \frac{1}{T} LM_{\omega \nu} \varepsilon - \frac{1}{T} LM_{\omega \nu} \varepsilon + \frac{1}{T} LM_{\omega \nu} r - \frac{1}{T} LM_{\omega \nu} r. \]

Using Lemmas A.3 and A.4, it can be shown that 
\[ \left\| \frac{1}{T} LM_{\omega \nu} \varepsilon \right\| = O_P \left( \frac{1}{\sqrt{N}} \right) \quad \text{and} \quad \left\| \frac{1}{T} LM_{\omega \nu} r \right\| = O_P \left( \frac{1}{\sqrt{N}} \right), \]

and thus, since \( |\bar{\gamma}| \leq C \), we can see that both 
\[ \left\| \frac{1}{\sqrt{N} T^3} \sum_{i=1}^{N} \bar{\gamma}_i \varepsilon_i M_{\omega \nu} L M_{\omega \nu} \bar{r} \right\| \quad \text{and} \quad \left\| \frac{1}{\sqrt{N} T^2} \sum_{i=1}^{N} \bar{\gamma}_i r_i M_{\omega \nu} L M_{\omega \nu} \bar{r} \right\| \]

are \( O_P(N^{-3/4}) \). Therefore, the cross products between the terms related with \( \varepsilon \), \( \bar{r} \), \( r \), and \( \bar{\varepsilon} \), and \( \varepsilon \) and \( \bar{r} \) converge to zero in probability as both \( N \) and \( T \) approach infinity.

In addition, using expression (28) and Lemmas A.3 and A.4, it is observed that both 
\[ \left\| \frac{1}{T} LM_{\omega \nu} \varepsilon - (J_0 - J_1 K_{\nu}^{-1} J_3) \right\| \quad \text{and} \quad \left\| \frac{1}{T} LM_{\omega \nu} r - (J_0 - J_1 K_{\nu}^{-1} J_3) \right\| \]

are \( O_P(T^{-1/2}) \), which implies that we only have to consider \( (J_0 - J_1 K_{\nu}^{-1} J_3) \) and \( (J_0 - J_1 K_{\nu}^{-1} J_3) \) in the limit. Moreover, the cross product between these two terms can be seen to be negligible. Therefore, we have 
\[ Z_1 = \frac{1}{\xi \sqrt{N}} \sum_{i=1}^{N} \left\{ (J_0 - J_1 K_{\nu}^{-1} J_3)' (J_0 - J_1 K_{\nu}^{-1} J_3) - \bar{\xi}_i \right\} \]

\[ + \frac{1}{\xi N} \sum_{i=1}^{N} \sqrt{N} (J_0 - J_1 K_{\nu}^{-1} J_3)' (J_0 - J_1 K_{\nu}^{-1} J_3) + o_4(1). \]

The first term weakly converges to a standard normal distribution as proved in Theorem 1, whereas the probability limit of the second term is obtained by applying Lemma A.2 (i). To see this, we first note that, using Lemma A.1,
\[ E \left[ \sqrt{N} (J_0 - J_1 K_{\nu}^{-1} J_3)' (J_0 - J_1 K_{\nu}^{-1} J_3) \right] = \frac{11}{12600} \xi^2 + O \left( \frac{1}{T^1} \right), \]

while the second moment is bounded above uniformly over \( N \) and \( T \) using (23). On the other hand, since \( N^{1/4} T_{(\tau)} \to c \mathbb{B}(r) \), we can see that 
\[ \sqrt{N} (J_0 - J_1 K_{\nu}^{-1} J_3)' (J_0 - J_1 K_{\nu}^{-1} J_3) \Rightarrow c^3 \int_0^1 F(r)^2 dr, \]

whose moment is \( 11 c^2 / 12600 \) by direct calculation. Then, by Lemma A.2 (i), we have
\[ \frac{1}{\xi \sqrt{N}} \sum_{i=1}^{N} (J_0 - J_1 K_{\nu}^{-1} J_3)' (J_0 - J_1 K_{\nu}^{-1} J_3) \Rightarrow c^2 E \left[ \int_0^1 F(r)^2 dr \right] = \frac{11}{12600} \frac{\xi^2}{c^2}. \]

When \( z_1 = 1 \), the above probability limit can be shown to be \( c^2 (90 \xi) \) in exactly the same manner.

In order to derive the limiting distribution of the LM test statistic, note that \( (I_8 \otimes M_3) y = (I_8 \otimes M_3) r + (A^{1/2} \otimes M_3) \eta \). Then, the denominator of the LM test statistic can be expressed as
\[
\sqrt{N}(LM - \xi) = \frac{1}{\sqrt{NT}}(\eta' (A^{-2} \otimes M, LL'M_r)N - \xi) + \frac{1}{\sqrt{NT}} \eta' (A^{-2} \otimes M, LL'M_r)\varepsilon.
\]

The first term on the right-hand side of (33) converges in distribution to a standard normal distribution as proved in Theorem 1.

Since \( A^{-2} = P \Lambda_\varepsilon P' \), the second term on the right-hand side of (33) is expressed as
\[
\frac{1}{\sqrt{NT}} r' (A^{-2} \otimes M, LL'M_r) \mathbf{r} = \frac{1}{\sqrt{NT}} r' (P \otimes I_d)(\Lambda_\varepsilon \otimes M_r, LL'M_r)(P \otimes I_d)\mathbf{r}.
\]

Note that \((P \otimes I_d)\mathbf{r} = (P \otimes I_d)(I_0 \otimes L)v = (I_0 \otimes L)v' = \mathbf{r}' \), where \( v' = (P \otimes I_d)v \sim N(0, \rho(I_0 \otimes I_d)) \) and \( \mathbf{r}' = (I_0 \otimes L)v' \). Using this expression, we have
\[
\frac{1}{\sqrt{NT}} r' (A^{-2} \otimes M_r, LL'M_r) \mathbf{r}' = \frac{1}{(1 + \sigma^2 \gamma^2)} \frac{1}{\sqrt{NT}} r_1'M_r, LL'M_r r_1 + \frac{1}{\sqrt{NT}} \sum_{i=2}^N r_i'M_r, LL'M_r r_i.
\]

It is not difficult to see that the first term on the right-hand side converges to zero as both \( N \) and \( T \) approach infinity, while
\[
\frac{1}{\sqrt{NT}} \sum_{i=2}^N r_i'M_r, LL'M_r r_i = \frac{1}{\sqrt{N}} \sum_{i=2}^N (J_{i0} - J_i K_{1i} J_{10}') (J_{i0} - J_i K_{1i} J_{10}'),
\]

where \( J_{i0} \) and \( J_{10} \) are defined in the same way as \( J_0 \) and \( J_1 \) with \( r_i \) being replaced by \( r_i' \). Since \( \mathbf{r}' \) has the same distribution as \( \mathbf{r} \), (34) converges in probability to \( 11c^2/12600 \) as proved for the case of \( Z_\varepsilon \).

Similarly, we can see that the third term on the right-hand side of (33) converges to zero in probability as proved for the case of \( Z_\varepsilon \). ■

**Proof of Theorem 3**

**Lemma A.5.** Under the fixed alternative \( H_1 \), as both \( N \) and \( T \) approach infinity simultaneously, (i) \( E\| \mathbf{J}_0 \|^2 \leq C, \frac{1}{\sqrt{N}} \sum_{i=1}^N \| J_0 \|^2 = O_p(\sqrt{N}) \), and \( \| J_0 \| = O_p(1) \); (ii) \( E\| \mathbf{J}_1 \|^2 \leq C T \sqrt{N} \) and \( \| J_1 \| = O_p(T) \); (iv) \( K_{11} = O(1) \), \( K_{12} = K_{21} = O_p(\sqrt{N}) \), and \( K_{22} = O_p(T/N) \); (v) \( E\| J_0 \|^2 \leq C, \frac{1}{\sqrt{N}} \sum_{i=1}^N \| J_0 \|^2 = O_p(\sqrt{N}) \), and \( \| \mathbf{J}_0 \| = O_p\left(\frac{T}{\sqrt{N}}\right) \); (vi) \( E\| J_0 \|^2 \leq C T \sqrt{N}, \frac{1}{\sqrt{N}} \sum_{i=1}^N \| J_0 \|^2 = O_p(\sqrt{N}) \), and \( \| \mathbf{J}_0 \| = O_p\left(\frac{T}{\sqrt{N}}\right) \); and (vii) \( \frac{1}{\sqrt{N}} \sum_{i=1}^N \| J_0 \| = O_p(\sqrt{N}) \), \( \frac{1}{\sqrt{N}} \sum_{i=1}^N \| J_0 \| = O_p(T) \), and \( \| \mathbf{J}_0 \| = O_p(T) \).

**Lemma A.6.** Under the fixed alternative \( H_1 \), as both \( N \) and \( T \) approach infinity simultaneously, (i) \( E\| J_0 \|^2 \leq C T \sqrt{N} \), \( \frac{1}{\sqrt{N}} \sum_{i=1}^N \| J_0 \|^2 = O_p(\sqrt{N} T) \), and \( \| \mathbf{J}_0 \| = O_p\left(\frac{T}{\sqrt{N}}\right) \); (ii) \( E\| J_1 \|^2 \leq C T \sqrt{N}, \frac{1}{\sqrt{N}} \sum_{i=1}^N \| J_1 \|^2 = O_p(\sqrt{N} T) \), and \( \| \mathbf{J}_1 \| = O_p\left(\frac{T}{\sqrt{N}}\right) \); and (iii) \( E\| J_1 \|^2 \leq C T \sqrt{N}, \frac{1}{\sqrt{N}} \sum_{i=1}^N \| J_1 \|^2 = O_p(\sqrt{N} T) \), and \( \| \mathbf{J}_1 \| = O_p\left(\frac{T}{\sqrt{N}}\right) \).
(iv) \( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \| J_i \| J_i^* = O_\mathcal{N}(\sqrt{NT^3}) \), \( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \| J_i \| J_i^* = O_\mathcal{N}(T^2 \sqrt{T}) \) and \( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \| J_i \| J_i^* = O_\mathcal{N}(T^2 \sqrt{T}) \).

Using Lemmas A.5 and A.6, it can be shown that
\[
\left\| \frac{1}{T} \sum_{i=1}^{N} M \cdot e_i \right\| = O_\mathcal{N}(\frac{1}{\sqrt{T}}), \left\| \frac{1}{T} \sum_{i=1}^{N} M \cdot e_i \right\| = O_\mathcal{N}(\frac{1}{\sqrt{T}}), \left\| \frac{1}{T} \sum_{i=1}^{N} M \cdot e_i \right\| = O_\mathcal{N}(\frac{1}{\sqrt{T}}),
\]
which implies that
\[
\frac{1}{\sqrt{NT}} Z_i = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{1}{T} \sum_{i=1}^{N} M \cdot e_i + o_\mathcal{N}(1).
\]

We decompose the first term on the right-hand side such that
\[
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{1}{T} \sum_{i=1}^{N} M \cdot e_i = \frac{1}{T} J_{\theta} - \left[ J_{\theta}, \frac{\sqrt{N}}{T} J_{\theta} \right] \left[ \frac{\sqrt{N}}{T} K_{11}, \frac{\sqrt{N}}{T} K_{12} \right]^{-1} \left[ \frac{1}{T} J_{\theta} \right].
\]

Using this expression and letting \( K = D \tilde{Z} K D^{-1} \), where \( D = \text{diag} \{ 1, \frac{\sqrt{N}}{T} \} \), we have
\[
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{1}{T} \sum_{i=1}^{N} M \cdot e_i = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{1}{T} J_{\theta} - 2 \text{tr} \left( K^{-1} \left[ J_{\theta}, \frac{\sqrt{N}}{T} J_{\theta} \right] \left[ \frac{\sqrt{N}}{T} K_{11}, \frac{\sqrt{N}}{T} K_{12} \right]^{-1} \left[ \frac{1}{T} J_{\theta} \right] \right)
\]
\[
+ \text{tr} \left( K^{-1} \left[ J_{\theta}, \frac{\sqrt{N}}{T} J_{\theta} \right] \left[ \frac{\sqrt{N}}{T} K_{11}, \frac{\sqrt{N}}{T} K_{12} \right]^{-1} \left[ \frac{1}{T} J_{\theta} \right] \right).
\]

By applying Lemmas A.1 and A.2, it can be shown that the joint limits of the three terms on the right-hand side of (36) are the same as the sequential limits, which are given by
\[
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{1}{T} J_{\theta} - 2 \text{tr} \left( K^{-1} \left[ J_{\theta}, \frac{\sqrt{N}}{T} J_{\theta} \right] \left[ \frac{\sqrt{N}}{T} K_{11}, \frac{\sqrt{N}}{T} K_{12} \right]^{-1} \left[ \frac{1}{T} J_{\theta} \right] \right)
\]
\[
\overset{(N,D)}{\Rightarrow} 2 \sigma_{\mathcal{N}}^{2} E_{\mathcal{N}} \left[ \left( \int_{0}^{1} B_{\theta}(s) ds \right)^{2} d\theta \right],
\]
\[
2 \text{tr} \left( K^{-1} \left[ J_{\theta}, \frac{\sqrt{N}}{T} J_{\theta} \right] \left[ \frac{\sqrt{N}}{T} K_{11}, \frac{\sqrt{N}}{T} K_{12} \right]^{-1} \left[ \frac{1}{T} J_{\theta} \right] \right)
\]
\[
\overset{(N,D)}{\Rightarrow} 2 \sigma_{\mathcal{N}}^{2} E_{\mathcal{N}} \left[ \left( \int_{0}^{1} \left( \int_{0}^{1} B_{\theta}(s) ds \right)^{2} d\theta \right) \right],
\]
\[
\text{tr} \left( K^{-1} \left[ J_{\theta}, \frac{\sqrt{N}}{T} J_{\theta} \right] \left[ \frac{\sqrt{N}}{T} K_{11}, \frac{\sqrt{N}}{T} K_{12} \right]^{-1} \left[ \frac{1}{T} J_{\theta} \right] \right)
\]
\[
\overset{(N,D)}{\Rightarrow} 2 \sigma_{\mathcal{N}}^{2} E_{\mathcal{N}} \left[ \left( \int_{0}^{1} \left( \int_{0}^{1} B_{\theta}(s) ds \right)^{2} d\theta \right) \right].
\]
\[ (N, T) \rightarrow \sigma^2 E \left[ \int_0^1 z(t)'B'(t) dt \left( \int_0^1 z(t)z(t)' dt \right)^{-1} \right. \\
\left. \times \int_0^1 \int_0^1 z(s)ds \int_0^1 z(t)ds \int_0^1 z(t)'B'(t) dt \right] \]  

(38)

Using these results, we obtain the joint weak limit of \( Z_4 \) under \( H_1 \) because \( \sigma^2 = \sigma^2 \rho = \rho \).

For the LM test statistic, we can see using expression (33) and Lemmas A.5 and A.6 that

\[ \frac{1}{\sqrt{NT}} Z_{LM} = \frac{1}{\sqrt{N}} \left( A^{-2} \Theta M L' M_3 \right) r + o_p(1) \]

\[ = \frac{1}{\sqrt{N}} \sum_{i=1}^N \left( J_{ii}^0 - J_i K_i J_i \right) \left( J_{ii}^0 - J_i K_i J_i \right)' + o_p(1) \]

\[ + o_p \left( \frac{(N, T)}{\sqrt{N}} \right) \frac{\sigma^2}{\rho^2} E \left[ \int_0^1 F'_v(r)^2 dr \right]. \]

Because \( F'_v(r) \) has the same distribution as \( F'_v(r) \), we obtain the theorem.\( \blacksquare \)

REFERENCES


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