On the Verification Theorem of Dynamic Portfolio-Consumption Problems with Stochastic Market Price of Risk *

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Abstract

In this paper, we study a dynamic portfolio-consumption optimization problem when the market price of risk is driven by linear Gaussian processes. We show sufficient conditions to verify that an explicit solution derived from the Hamilton-Jacobi-Bellman equation is in fact an optimal solution to the portfolio selection problem.

Keywords

Optimal portfolios, Hamilton-Jacobi-Bellman equation, stochastic market price of risk, verification theorem.

1 Introduction

Since Merton's seminal work (Merton [14], [15]), many studies have been done on continuous-time portfolio optimization problems. In particular, there has been increasing interest in finding an optimal portfolio strategy when investment opportunities are stochastic, because many empirical works conclude that investment opportunities are time varying. In this paper, we study a continuous-time utility maximization problem when the market price of risk is driven by linear Gaussian processes in a complete market model. The investor allocates his wealth among traded risky and riskless assets so that he can maximize his utility from terminal wealth and intermediate consumption. The utility function is assumed to be a power-utility.

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In general, it is difficult to solve optimal portfolio problems when the investment opportunity is time varying. In the last ten years, however, many authors have derived explicit solutions to the optimal portfolio problems when the investment opportunity set is time varying; see, for example, Kim and Omberg [10], Wachter [18], Liu [13], and references therein. An explicit solution allows detailed study of investor's behavior. In most papers, the associated Hamilton-Jacobi-Bellman (HJB) equation is transformed into an ordinary differential equation (ODE) and an optimal solution is conjectured by guessing the solution to the ODE. It is necessary to verify that the conjectured solution is in fact a solution to the original problem. However, as Korn and Kraft [11] emphasized, verification is often skipped since it is mathematically demanding. Indeed, Kim and Omberg [10] and Liu [13] did not provide any verification conditions although the former examined the finiteness of the conjectured value function very carefully. Thus the purpose of this paper is to show sufficient conditions which verify that an explicit solution derived in their papers is in fact an optimal solution.

We concentrate on a model that is essentially similar to that of Wachter [18] and Liu [13]. Wachter [18], using the martingale approach, gave a verification condition for investors who are more risk averse than log-utility investor in one-factor model. Thus in this paper we will give verification conditions for both more risk averse than log-utility investor and more risk seeking than log-utility investor in multifactor model. We will adapt the dynamic programming approach to solve the problem and give different sufficient conditions from that of Wachter [18].

Our assumption of a complete market is restrictive. In an incomplete market model, it seems to be impossible to obtain an explicit solution to the associated HJB equation. However, if we consider utility from terminal wealth only, we can obtain an explicit solution; see Kim and Omberg [10] and Liu [13]. In such a case, our analysis can be applied similarly.

There are many other interesting studies related to our problem. Brennan et al. [6], Brandt [5], and Campbell and Viceira [7] computed an optimal portfolio strategy using various numerical and approximation methods. Bielecki and Pliska [3], Bielecki et al. [4], and Nagai [16] studied the problem in the context of risk-sensitive control. Zariphopoulou [19], Stoikov and Zariphopoulou [17], and Castañeda-Leyva and Hernández-Hernández [8] consider more general factor processes than ours, but they essentially assume that coefficients of price processes are bounded. The coefficients of our model are not bounded.

From our results, we will see that a relative risk aversion coefficient of a power-utility function and the definition of the admissible portfolio strategy are important elements. If the investor is more risk seeking than log-utility investor, the solution to the Riccati equation related to the HJB equation may blow up on a time horizon. A verification theorem, however, can be proved under the usual definition of admissibility as long as the solution to the related Riccati equation exists. If, on the other hand, the investor is more risk averse than log-utility investor, the solution to the related Riccati equation always exits, but we may have to choose a restrictive set of admissible portfolio strategies to prove a verification theorem.

The rest of this paper is organized as follows. In Section 2, we describe the model and formulate a portfolio optimization problem. In Section 3, we derive the HJB equation. In Section 4, we study the related Riccati equation. In Section 5, we give verification theorems. Section 6 concludes the paper.

2 Formulation of the problem

We fix a complete probability space $(\Omega, \mathcal{F}, P)$ on which a K-dimensional standard Brownian motion B is defined, and we also fix a time interval $[0, T]$. Let $\mathcal{F}(t)$ be the augmentation of the filtration $\sigma(B(s)\,;\, 0 < s < t)$, $0 < t < T$.

There are K factors $X = (X_1, \ldots, X_K)^\top$, which determine investment opportunity and satisfy

$$dX(t) = \mu^X(X(t))dt + \sigma^X dB(t),\ X(0) = x_0 \in \mathbb{R}^K, \tag{1}$$

where $\mu^X(x) = \kappa - Mx$, $\kappa \in \mathbb{R}^K$, $M \in \mathbb{R}^{K\times K}$, and $\sigma^X \in \mathbb{R}^{K\times K}$. There is one riskless asset and K risky assets. Suppose that the price S_0 of the riskless asset satisfies

$$dS_0(t) = r(X(t))S_0(t)dt, \quad S_0(0) = 1.$$

Here $r\colon \mathbb{R}^K \to \mathbb{R}$ is defined by $r(x) = r_0 + r_1^\top x + x^\top r_2 x/2$, where $r_0 \in \mathbb{R}$, $r_1 \in \mathbb{R}^K$, and $r_2 \in \mathbb{R}^{K\times K}$ is nonnegative definite. The risky asset price S satisfies the stochastic differential equation

$$\begin{aligned} &dS_i(t) = S_i(t)\mu_i^S(X(t))dt + S_i(t)\sum_{j=1}^{K}\sigma_{ij}^S(X(t))dB_j(t), \\ &S_i(0) = s_i > 0,\ i = 1, 2, \ldots, K, \end{aligned} \tag{2}$$

where $\sigma^S(x) := (\sigma_{ij}^S(x))_{1\le i,j\le K}$ is a function from $\mathbb{R}^K$ to $\mathbb{R}^{K\times K}$ such that $\Sigma^S(x) := \sigma^S(x)(\sigma^S(x))^\top$ is positive definite for all $x \in \mathbb{R}^K$. A coefficient $\mu^S(x) := (\mu_1^S(x), \ldots, \mu_K^S(x))^\top$ is defined by

$$\mu^S(x) = r(x)\mathbf{1} + \sigma^S(x)x,$$

where $\mathbf{1}$ denotes an appropriate dimension vector with every component equal to one. This means that the market price of risk $\theta := (\sigma^S)^{-1}(\mu^S - r\mathbf{1})$ satisfies $\theta(x) = x$. Since the number of risky assets is equal to the total number of risk sources, the market is complete in the sense that any random processes are replicated by the self-financing strategy.

We consider an investor who allocates his wealth between risky assets and a riskless asset and chooses a consumption rate to maximize expected utility from terminal wealth and intermediate consumption in a self-financing way. Suppose that the investor has power-utility. Let $\eta_0(t)$ and $\eta_i(t)$ be the unit of the riskless asset and the i-th risky assets which the investor holds at time t, respectively. The investor's wealth $W(t)$ at time t is then defined by

$$W(t) = \eta_0(t)S_0(t) + \sum_{i=1}^{K} \eta_i(t)S_i(t).$$

Let $c(t)$ be the consumption rate at time t. The self-financing hypothesis implies that, given a initial wealth $w_0 > 0$, the wealth dynamics is

$$dW(t) = \eta_0(t)dS_0(t) + \sum_{i=1}^{K} \eta_i(t)dS_i(t) - c(t)dt, \ W(0) = w_0.$$

Let $\phi_0(t)$ and $\phi_i(t)$ $(i = 1, 2, \dots, K)$ be the processes such that

$$\phi_i(t) := \begin{cases} \eta_i(t)S_i(t)/W(t), & W(t) \neq 0 \\ 0, & W(t) = 0, \end{cases} \quad \phi_0(t) := \begin{cases} 1 - \sum_{i=1}^{K} \phi_i(t), & W(t) \neq 0 \\ 0, & W(t) = 0. \end{cases}$$

Then $\phi_i(t)$ and $\phi_0(t)$ denote the fraction of the wealth invested in the i-th risky asset and riskless asset at time t, respectively.

We call (ϕ, c) a portfolio-consumption strategy on $[t_0, t_1]$ if $\phi \in \mathcal{L}^2(t_0, t_1)$, $c \in \mathcal{L}^1(t_0, t_1)$, $c(t) \geq 0$ and $W(t) \geq 0$ for all $t \in [t_0, t_1]$ a.s., where W is the wealth process corresponding to (ϕ, c),

$$\mathcal{L}^1(t_0, t_1) := \left\{ f \colon \Omega \times [t_0, t_1] \to \mathbb{R} \ \middle| \ P\left(\int_{t_0}^{t_1} |f(t)| dt < \infty \right) = 1 \right\},$$

and

$$\mathcal{L}^2(t_0, t_1) := \left\{ f \colon \Omega \times [t_0, t_1] \to \mathbb{R}^K \ \middle| \ P\left(\int_{t_0}^{t_1} \|f(t)\|^2 dt < \infty \right) = 1 \right\}.$$

The set of all portfolio-consumption strategies on $[t_0, t_1]$ will be denoted by $\mathcal{H}(t_0, t_1)$. The investor's wealth process W corresponding to $(\phi, c) \in \mathcal{H}(0, T)$ is then given by

$$\begin{aligned} dW(t) = W(t)[\phi(t)^\top (\mu^S(X(t)) - r(X(t))\mathbf{1}) + r(X(t))]dt \\ + W(t)\phi(t)^\top \sigma^S(X(t))dB(t) - c(t)dt. \end{aligned} \tag{3}$$

Let π be a stochastic process such that $\pi := (\sigma^S)^\top \phi$. The wealth process (3) then can be rewritten as

$$dW(t) = W(t)[\pi(t)^\top \theta(X(t)) + r(X(t))]dt + W(t)\pi(t)^\top dB(t) - c(t)dt. \tag{4}$$

For simplicity, we will regard (π, c) a portfolio-consumption strategy instead of (ϕ, c).

The investor's problem is

$$\max_{(\pi,c) \in \mathcal{A}_\gamma(0,T)} \quad E\left[\int_0^T \frac{c(u)^{1-\gamma}}{1-\gamma} du + \frac{W(T)^{1-\gamma}}{1-\gamma} \right]. \tag{5}$$

Here $\mathcal{A}_\gamma \subset \mathcal{H}$ denotes the set of admissible strategies which will be defined later. The set of all admissible strategies on $[t_0, t_1]$ is denoted by $\mathcal{A}_\gamma(t_0, t_1)$. Let

$$J(t, w, x; \pi, c) = E^{t,w,x}\left[\int_t^T \frac{c(u)^{1-\gamma}}{1-\gamma}du + \frac{W(T)^{1-\gamma}}{1-\gamma}\right],$$

where W is the wealth process corresponding to (π, c). Here and in the sequel, we use the notation $E^{t,w,x}[\,\cdot\,] := E[\,\cdot\,|\,W(t) = w, X(t) = x]$. We then define $V\colon [0, T) \times (0, \infty) \times \mathbb{R}^K \to \mathbb{R}$ by

$$V(t, w, x) := \sup_{(\pi,c)\in\mathcal{A}_\gamma(t,T)} J(t, w, x; \pi, c). \tag{6}$$

The function V is called a value function.

3 The HJB equation

Using the dynamic programming principle, we obtain the HJB equation related to the problem (5) as follows:

$$\sup_{\pi\in\mathbb{R}^K, c\geq 0}\left\{\frac{c^{1-\gamma}}{1-\gamma} + \mathcal{D}^{\pi,c}G(t, w, x)\right\} = 0 \tag{7}$$

with the boundary condition

$$G(T, w, x) = \frac{w^{1-\gamma}}{1-\gamma}, \tag{8}$$

where

$$\begin{aligned}\mathcal{D}^{\pi,c}G(t, w, x) = G_t &+ w(\pi^\top\theta(x) + r(x))G_w + \mu^X(x)^\top G_x \\ &+ \frac{1}{2}w^2\|\pi\|^2 G_{ww} + \frac{1}{2}\mathrm{tr}[\Sigma^X G_{xx}] + w(\sigma^X\pi)^\top G_{wx} - cG_w,\end{aligned}$$

where $\Sigma^X := \sigma^X(\sigma^X)^\top$.

It follows from the first order condition for (7) that the candidate optimal strategy (π^*, c^*) is given by

$$\pi^* = -\frac{G_w}{wG_{ww}}\theta(x) - \frac{(\sigma^X)^\top G_{wx}}{wG_{ww}}, \quad c^* = G_w^{-\frac{1}{\gamma}}. \tag{9}$$

The HJB equation (7) is then reduced to

$$\begin{aligned}G_t - \frac{1}{2}\frac{G_w^2}{G_{ww}}\|\theta(x)\|^2 - \frac{G_w}{G_{ww}}\theta(x)^\top(\sigma^X)^\top G_{wx} + wr(x)G_w + \mu^X(x)^\top G_x \\ - \frac{1}{2G_{ww}}\left\|(\sigma^X)^\top G_{wx}\right\|^2 + \frac{1}{2}\mathrm{tr}\left[\Sigma^X G_{xx}\right] + \frac{\gamma}{1-\gamma}G_w^{\frac{\gamma-1}{\gamma}} = 0.\end{aligned} \tag{10}$$

It is well known from Kim and Omberg [10], Liu [13], and others that the function G is separable and has the following form:

$$G(t, w, x) = \frac{w^{1-\gamma}}{1-\gamma} f(t, x)^{\gamma}. \tag{11}$$

Substituting (11) into (10), we obtain the linear differential equation for f:

$$\begin{aligned} f_t + \left(\frac{1-\gamma}{\gamma} \sigma^X \theta(x) + \mu^X(x) \right)^{\top} f_x + \frac{1}{2} \mathrm{tr}[\Sigma^X f_{xx}] \\ + \left(\frac{1-\gamma}{2\gamma^2} \|\theta(x)\|^2 + \frac{1-\gamma}{\gamma} r(x) \right) f + 1 = 0 \end{aligned} \tag{12}$$

with the boundary condition $f(T, x) = 1$.

Now we conjecture that

$$\begin{aligned} f(t, x) = \int_t^T \exp\left(\alpha(u) + \beta(u)^{\top} x + \frac{1}{2} x^{\top} \zeta(u) x \right) du \\ + \exp\left(\alpha(t) + \beta(t)^{\top} x + \frac{1}{2} x^{\top} \zeta(t) x \right) \end{aligned} \tag{13}$$

with the boundary conditions $\alpha(T) = 0$, $\beta(T) = 0$, and $\zeta(T) = 0$, where $\zeta(t)$ is a symmetric matrix. Substituting (13) into (12) and using $\mu^X(x) = \kappa - Mx$, $\theta(x) = x$, and $r(x) = r_0 + r_1^{\top} x + x^{\top} r_2 x/2$, we obtain following ODEs:

$$\dot{\zeta}(t) = -\zeta(t) Z_2 \zeta(t) - \zeta(t) Z_1 - Z_1^{\top} \zeta(t) - Z_0 \tag{14}$$

$$\dot{\beta}(t) = -\zeta(t) B_2 \beta(t) - \zeta(t) B_{11} - B_{12} \beta(t) - B_0 \tag{15}$$

$$\dot{\alpha}(t) = -\beta(t)^{\top} A_2 \beta(t) - \beta(t)^{\top} A_1 - \frac{1}{2} \mathrm{tr}[\Sigma^X \zeta(t)] - A_0, \tag{16}$$

where

$$\begin{aligned} & Z_2 = \Sigma^X, \ Z_1 = \frac{1-\gamma}{\gamma} \sigma^X - M, \ Z_0 = \frac{1-\gamma}{\gamma} \left(\frac{1}{\gamma} I_K + r_2 \right) \\ & B_2 = Z_2, \ B_{11} = \kappa, \ B_{12} = Z_1, \ B_0 = \frac{1-\gamma}{\gamma} r_1 \\ & A_2 = \frac{1}{2} Z_2, \ A_1 = B_{11}, \ A_0 = \frac{1-\gamma}{\gamma} r_0. \end{aligned}$$

Here I_n is a n-dimensional identity matrix. Note that if the solution to (14) exists, then the solutions to (15) and (16) also exist, because these are usual linear differential equations.

The ODE (14) is called the Riccati equation. If the solution to (14) exists on $[0, T]$, then the candidate value function and optimal strategy are given by

$$G(t, w, x) = \frac{w^{1-\gamma}}{1-\gamma} f(t, x)^{\gamma}, \tag{17}$$

$$\pi^*(t) := \frac{1}{\gamma} X(t) + \frac{(\sigma^X)^\top f_x(t, X(t))}{f(t, X(t))}, \tag{18}$$

$$c^*(t) := \frac{W^*(t)}{f(t, X(t))}, \tag{19}$$

where W^* is the wealth process corresponding to (π^*, c^*),

$$f(t,x) = \int_t^T \exp\left(\alpha(u) + \beta(u)^\top x + \frac{1}{2} x^\top \zeta(u) x\right) du \\ + \exp\left(\alpha(t) + \beta(t)^\top x + \frac{1}{2} x^\top \zeta(t) x\right),$$

and ζ, β, and α are solutions to (14), (15), (16), respectively.

The first term of (18) is the usual mean-variance portfolio in a continuous-time model. The second term is a so-called hedging portfolio, which is held by investors in order to hedge against an unfavorable shift in the state variables.

In order to complete the whole story, we need to investigate the existence of the solution to the Riccati equation (14) and verify that $G = V$ and the candidate optimal portfolio-consumption strategy (π^*, c^*) is indeed a solution to (5). In the next section, we study the Riccati equation (14). A verification theorem is given in Section 5.

4 The Riccati equation

In this section, we discuss the solution to the Riccati equation (14). The representation of the solution to the Riccati equation is well known. Set

$$H := \begin{pmatrix} Z_1 & Z_2 \\ -Z_0 & -Z_1^\top \end{pmatrix} \in \mathbb{R}^{2K \times 2K},$$

which is the so-called Hamiltonian matrix. Let $(Q, P)^\top$ be a solution of the linear system of differential equations

$$\frac{d}{dt} \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = H \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix}, \ Q(T) = I_K, \ P(T) = \zeta(T)(= 0). \tag{20}$$

Then

$$\zeta(t) = P(t) Q^{-1}(t)$$

is the solution to the Riccati equation (14) as long as $Q^{-1}(t)$ exists. Further, the linear system (20) can be solved as follows. We assume that H is diagonalizable, that is, there exists $2K$-dimensional basis of eigenvectors $v_1, \ldots, v_{2K}$. Suppose that $\lambda_1, \ldots, \lambda_{2K}$ are the eigenvalues corresponding to $v_1, \ldots, v_{2K}$, respectively. Let $V := (v_1, \ldots, v_{2K})$. Then the solution to the linear system (20) has the form

$$\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = V e^{-\Delta(T-t)} V^{-1} \begin{pmatrix} Q(T) \\ P(T) \end{pmatrix} = V e^{-\Delta(T-t)} V^{-1} \begin{pmatrix} I_K \\ 0 \end{pmatrix}, \tag{21}$$

where $\Delta := \mathrm{diag}(\lambda_1, \ldots, \lambda_{2K})$.

Since $Q(T) = I_K$, $\zeta(t) = P(t)Q^{-1}(t)$ is a solution to the Riccati equation (14) at least on a small neighborhood of T. However it is a delicate matter whether $\zeta(t) = P(t)Q^{-1}(t)$ is a global solution on $[0,T]$, that is, $\det Q(t) \neq 0$ on $[0,T]$. It is well known that if Z_2 is positive definite and Z_0 is negative definite, then the solution to (14) exists globally on $[0,T]$, that is, $\det Q(t) \neq 0$ on $[0,T]$; see, for example, Fleming and Rishel [9, Theorem 4.5.2] and Abou-Kandil et al. [1, Theorem 4.1.6]. Hence, if $\gamma > 1$, then the solution to (14) exists globally.

Proposition 1. *If $\gamma > 1$, then the solution to the Riccati equation* (14) *exists on* $[0,T]$.

On the other hand, if $0 < \gamma < 1$, that is, Z_0 is not negative definite, then $\det Q(t)$ may be zero for some $t \in [0,T)$. The proof of global existence results is rather complicated; see Abou-Kandil et al. [1, Chapter 3 and 4] for a general discussion. In this paper, we will see how $\det Q(t)$ becomes zero in the following example.

Example. Assume that

$$0 < \gamma < 1,\ \sigma^X = \begin{pmatrix} \sigma_1^X & 0 \\ 0 & \sigma_2^X \end{pmatrix},\ M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix},\ r_2 = 0,$$

where $\sigma_1^X, \sigma_2^X, m_1, m_2 > 0$ with $\sigma_1^X \neq \sigma_2^X$ and $m_1 \neq m_2$. That is, we consider two independent factor processes which follow the Ornstein-Uhlenbeck process. Then

$$H = \begin{pmatrix} \frac{1-\gamma}{\gamma}\sigma_1^X - m_1 & 0 & (\sigma_1^X)^2 & 0 \\ 0 & \frac{1-\gamma}{\gamma}\sigma_2^X - m_2 & 0 & (\sigma_2^X)^2 \\ -\frac{1-\gamma}{\gamma^2} & 0 & -\frac{1-\gamma}{\gamma}\sigma_1^X + m_1 & 0 \\ 0 & -\frac{1-\gamma}{\gamma^2} & 0 & -\frac{1-\gamma}{\gamma}\sigma_2^X + m_2 \end{pmatrix}.$$

Eigenvalues of H are given by

$$\lambda_1 = \sqrt{d_1},\ \lambda_2 = -\sqrt{d_1},\ \lambda_3 = \sqrt{d_2},\ \lambda_4 = -\sqrt{d_2},$$

where

$$d_i = \frac{\gamma - 1}{\gamma}(\sigma_i^X)^2 + \frac{2m_i(\gamma-1)}{\gamma}\sigma_i^X + m_i^2,\ i = 1, 2$$

Eigenvectors $v_i = (v_{i1}, \ldots, v_{i4})^\top$ corresponding to λ_i $(i = 1, \ldots, 4)$ are given by

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{(\sigma_1^X)^2}(k_1 - \sqrt{d_1}) \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{(\sigma_1^X)^2}(k_1 + \sqrt{d_1}) \\ 0 \end{pmatrix},$$

$$v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{1}{(\sigma_2^X)^2}(k_2 - \sqrt{d_2}) \end{pmatrix},\ v_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{1}{(\sigma_2^X)^2}(k_2 + \sqrt{d_2}) \end{pmatrix},$$

where $k_i = (1-\gamma)\sigma_i^X/\gamma - m_i$ $(i=1,2)$. Set $V^{-1} = (v'_1, \dots, v'_4)$, where $v'_i = (v'_{i1}, \dots, v'_{i4})^\top$ $(i = 1, \dots, 4)$. We then have

$$v'_1 = \begin{pmatrix} \frac{k_1+\sqrt{d_1}}{2\sqrt{d_1}} \\ \frac{-k_1+\sqrt{d_1}}{2\sqrt{d_1}} \\ 0 \\ 0 \end{pmatrix}, \quad v'_2 = \begin{pmatrix} 0 \\ 0 \\ \frac{k_2+\sqrt{d_2}}{2\sqrt{d_2}} \\ \frac{-k_2+\sqrt{d_2}}{2\sqrt{d_2}} \end{pmatrix}.$$

It follows from (21) that

$$Q(t) = \begin{pmatrix} q_1(t) & 0 \\ 0 & q_2(t) \end{pmatrix},$$

where

$$\begin{aligned} q_1(t) &= v_{11}v'_{11}e^{-\sqrt{d_1}(T-t)} + v_{21}v'_{12}e^{\sqrt{d_1}(T-t)}, \\ q_2(t) &= v_{32}v'_{23}e^{-\sqrt{d_2}(T-t)} + v_{42}v'_{24}e^{\sqrt{d_2}(T-t)}. \end{aligned}$$

We first assume that $d_1, d_2 > 0$. We then have $k_i < 0$ $(i = 1, 2)$. In fact, it follows from $0 < \gamma < 1$ and

$$d_i = \left(\frac{1-\gamma}{\gamma}\sigma_i^X - m_i\right)^2 + \frac{\gamma-1}{\gamma^2}(\sigma_i^X)^2 > 0$$

that

$$k_i = \frac{1-\gamma}{\gamma}\sigma_i^X - m_i < -\frac{\sqrt{1-\gamma}}{\gamma}\sigma_i^X < 0.$$

We then obtain

$$q_i(t) = \frac{k_i+\sqrt{d_i}}{2\sqrt{d_i}}e^{-\sqrt{d_i}(T-t)} + \frac{-k_i+\sqrt{d_i}}{2\sqrt{d_i}}e^{\sqrt{d_i}(T-t)} > 0, \;\; i = 1, 2.$$

Therefore the condition $d_1, d_2 > 0$ is sufficient for $\det Q(t) \neq 0$ on $[0, T]$. Note that if $\gamma > 1$, then $d_i > 0$ and $k_i < 0$ $(i = 1, 2)$ always hold.

On the other hand, if $d_i < 0$ $(i = 1, 2)$, then

$$q_i(t) = \cos\left(\sqrt{-d_i}(T-t)\right) - \frac{k_i}{\sqrt{-d_i}}\sin\left(\sqrt{-d_i}(T-t)\right).$$

Hence if, for $i = 1$ or 2,

$$0 < (T_i^* :=)\ \frac{1}{\sqrt{-d_i}}\tan^{-1}\left(\frac{\sqrt{-d_i}}{k_i}\right) < T,$$

then $\det Q(t) = 0$ for some $t \in [0, T)$.

From the above discussion, we can see that if one of the following conditions

(i) $d_1 > 0$ and $d_2 > 0$

(ii) $d_1 < 0$, $T < T_1^*$, and $d_2 > 0$

(iii) $d_1 > 0$, $d_2 < 0$, and $T < T_2^*$

(iv) $d_1 < 0$, $d_2 < 0$, and $T < \min\{T_1^*, T_2^*\}$

is satisfied, then $\det Q(t) \neq 0$ on $[0,T]$. Further we can also see that when $0 < \gamma < 1$ and $\|\sigma^X\|$ and T are too large, then $\det Q(t)$ may be zero on $[0,T)$. □

5 Verification theorem

The following lemma is crucial to the proof of the verification theorem. For a stochastic process g, define

$$\mathcal{E}(t,g) := \exp\left\{ \int_0^t g(u)^\top dB(u) - \frac{1}{2}\int_0^t \|g(u)\|^2 du \right\}.$$

Lemma 2. *Let $g(t) := \tilde{g}(t, X(t))$, where $\tilde{g}\colon [0,T] \times \mathbb{R}^K \to \mathbb{R}^K$ satisfies the linear growth condition* [1]*. Then*

$$E\left[\mathcal{E}(T,g)\right] = 1.$$

Proof. Given the process (1) of X, the result is derived as in Bensoussan [2, Lemma 4.1.1] or Liptser and Shiryaev [12, Section 6.2]. □

5.1 Case $0 < \gamma < 1$

In this case, we define the set of admissible portfolio-consumption strategies $\mathcal{A}_\gamma$ by

$$\mathcal{A}_\gamma := \mathcal{H}.$$

Theorem 3. *Assume that the solution to* (14) *exists on* $[0,T]$*. Then the function G defined by* (17) *satisfies $G = V$. Further, (π^*, c^*), defined by* (18) *and* (19)*, is an optimal portfolio-consumption strategy.*

Proof. We first show that (π^*, c^*) is admissible. We have

$$\begin{aligned} dW^*(t) &= W^*(t)\left[\pi^*(t)^\top \theta(X(t)) + r(X(t))\right] dt + W^*(t)\pi^*(t)^\top dB(t) - c^*(t)dt \\ &= W^*(t)\left[\pi^*(t)^\top \theta(X(t)) + r(X(t)) - \frac{1}{f(X(t))}\right] dt + W^*(t)\pi^*(t)^\top dB(t), \end{aligned}$$

where f is given by (13) and W^* is the wealth process corresponding to (π^*, c^*). It then follows that

$$\begin{aligned} W^*(t) = w_0 \exp\Bigg\{ \int_0^t \left(\pi^*(u)^\top \theta(X(u)) + r(X(u)) - \frac{1}{f(X(u))} - \frac{1}{2}\|\pi^*(u)\|^2 \right) du \\ + \int_0^t \pi^*(u)^\top dB(u) \Bigg\} > 0, \end{aligned}$$

[1] A function $h\colon [0,T] \times \mathbb{R}^K \to \mathbb{R}^K$ is said to satisfy the linear growth condition if $\|h(t,x)\| \leq k(1 + \|x\|)$ for some $k > 0$.

which implies that $(\pi^*, c^*) \in \mathcal{A}_\gamma(0,T)$.

Let $(t, w, x) \in [0,T] \times [0,\infty) \times \mathbb{R}$ be fixed. We define the value process

$$g^{\pi,c}(s) := \int_t^s \frac{c(u)^{1-\gamma}}{1-\gamma} du + G(s, W(s), X(s)), \;\; s \in [t,T]. \tag{22}$$

Using Itô's formula, we obtain

$$dg^{\pi,c}(s) = \left[\frac{c(s)^{1-\gamma}}{1-\gamma} + \mathcal{D}^{\pi,c} G\left(s, W(s), X(s)\right)\right] ds + g^{\pi,c}(s) h^{\pi,c}(s)^\top dB(s), \tag{23}$$

where

$$h^{\pi,c}(s) := \left[(1-\gamma)\pi(s) + \gamma \frac{(\sigma^X)^\top f_x(s, X(s))}{f(s, X(s))}\right] \frac{G(s, W(s), X(s))}{g^{\pi,c}(s)}$$

for all $s \in [t,T]$ and $(\pi, c) \in \mathcal{A}_\gamma(t,T)$. Since G is the solution to the HJB equation (7) and (π^*, c^*) is the maximizer in (7), it follows that

$$g^*(s') = g^*(s) \frac{\mathcal{E}(s', h^*)}{\mathcal{E}(s, h^*)} \tag{24}$$

for all $s, s' \in [t,T]$ with $s' \geq s$, where $g^* := g^{\pi^*,c^*}$ and $h^* := h^{\pi^*,c^*}$. Here it is easy to see that

$$\left\| \frac{f_x(s,x)}{f(s,x)} \right\| \leq k(1 + \|x\|) \tag{25}$$

for some $k > 0$ and

$$\left| \frac{G(s, W_s, X_s)}{g^{\pi,c}(s)} \right| \leq 1 \tag{26}$$

for all $s \in [0,T]$ and $(\pi, c) \in \mathcal{A}_\gamma(0,T)$. From (18), (25), (26), and Lemma 2, $\{\mathcal{E}(s, h^*)\}_{s \in [t,T]}$ is a martingale. Hence, from (24), we have

$$\begin{aligned} E^{t,w,x}\left[\int_t^T \frac{c^*(u)^{1-\gamma}}{1-\gamma} du + \frac{W^*(T)^{1-\gamma}}{1-\gamma}\right] &= E^{t,w,x}[g^*(T)] \\ &= E^{t,w,x}\left[g^*(t) \frac{\mathcal{E}(T, h^*)}{\mathcal{E}(t, h^*)}\right] \\ &= G(t, w, x). \end{aligned} \tag{27}$$

On the other hand, it follows from the HJB equation (7) and (23) that

$$g^{\pi,c}(s) \leq g^{\pi,c}(t) + \int_t^s g^{\pi,c}(u) h^{\pi,c}(u)^\top dB(u) \tag{28}$$

for all $(\pi, c) \in \mathcal{A}_\gamma(t,T)$ and $s \in [t,T]$. Set

$$\Phi(s) := \int_t^s \|g^{\pi,c}(u) h^{\pi,c}(u)\|^2 du \tag{29}$$

and $\tau_n := T \wedge \inf\{s \in [t,T] \mid \Phi(s) \geq n\}$, $n \in \mathbb{N}$. It follows from $E[\Phi(s)] < n$ for $s \in [t, \tau_n]$ that the stochastic integral in (28) is a martingale for $s \in [t, \tau_n]$. Thus

$$\begin{aligned} E^{t,w,x}[g^{\pi,c}(\tau_n)] &\leq E^{t,w,x}[g^{\pi,c}(t)] + E^{t,w,x}\left[\int_t^{\tau_n} g^{\pi,c}(u)h^{\pi,c}(u)^\top dB(u)\right] \\ &= G(t,w,x). \end{aligned}$$

From $\lim_{n\to\infty} \tau_n = T$ a.s., $g^{\pi,c}(t) \geq 0$, and Fatou's lemma, we have

$$\begin{aligned} E^{t,w,x}\left[\int_t^T \frac{c(u)^{1-\gamma}}{1-\gamma}du + \frac{W(T)^{1-\gamma}}{1-\gamma}\right] &= E^{t,w,x}[g^{\pi,c}(T)] \\ &\leq \liminf_{n\to\infty} E^{t,w,x}[g^{\pi,c}(\tau_n)] \\ &\leq G(t,w,x) \end{aligned} \tag{30}$$

for all $(\pi, c) \in \mathcal{A}_\gamma(t,T)$.

Combining (27) and (30), we see that $G = V$ and (π^*, c^*) is an optimal strategy. ☐

5.2 Case $\gamma > 1$

In this case, since a power-utility function is unbounded from below, so is the (candidate) value process $g^{\pi,c}$ defined by (22). Therefore, we cannot use Fatou's lemma in proving the inequality (30). To prove (30), we will restrict the set of admissible portfolio-consumption strategies as follows:

$$\mathcal{A}_\gamma(t_0,t_1) := \left\{ (\pi,c) \in \mathcal{H}(t_0,t_1) \;\middle|\; \begin{array}{l} \text{for some function } \tilde{\pi}\colon [t_0,t_1]\times\mathbb{R}\to\mathbb{R} \\ \text{satisfying the linear growth condition,} \\ \pi(t) = \tilde{\pi}(t, X(t)). \end{array} \right\}.$$

Under this definition, we can show the following result. Recall that when $\gamma > 1$, the solution to (14) always exists.

Theorem 4. *$G = V$ and (π^*, c^*) is an optimal portfolio-consumption strategy, where G and (π^*, c^*) are given by* (17), (18) *and* (19), *respectively.*

Proof. It follows from (25) that π^* satisfies the linear growth condition with respect to $X(t)$. This implies $(\pi^*, c^*) \in \mathcal{A}_\gamma(0,T)$. We can show the equation (27) in the same way as in the proof of Theorem 3. We will show the inequality (30). By (28), we have

$$g^{\pi,c}(T) \leq g^{\pi,c}(t)\frac{\mathcal{E}(T, h^{\pi,c})}{\mathcal{E}(t, h^{\pi,c})} \tag{31}$$

for all $(\pi, c) \in \mathcal{A}_\gamma(t,T)$. It follows from the definition of $\mathcal{A}_\gamma$ and Lemma 2,

$\{\mathcal{E}(s, h^{\pi,c})\}_{s\in[t,T]}$ is a martingale for all $(\pi, c) \in \mathcal{A}_\gamma(t, T)$. Hence

$$
\begin{aligned}
E^{t,w,x}\left[\int_t^T \frac{c(u)^{1-\gamma}}{1-\gamma}du + \frac{W(T)^{1-\gamma}}{1-\gamma}\right] &= E^{t,w,x}[g^{\pi,c}(T)] \\
&\leq E^{t,w,x}\left[g^{\pi,c}(t)\frac{\mathcal{E}(T, h^{\pi,c})}{\mathcal{E}(t, h^{\pi,c})}\right] \\
&= G(t, w, x)
\end{aligned}
$$

for all $(\pi, c) \in \mathcal{A}_\gamma(t, T)$, which completes the proof. $\square$

From the above proof, we can see that one general definition of admissibility is that

$$(\pi, c) \in \mathcal{H}(t_0, t_1) \text{ and } \{\mathcal{E}(t, h^{\pi,c})\} \text{ is a martingale.}$$

Wachter [18] restricted market parameters instead of the set of admissible strategies to prove a verification result.

One may think that the choice of our admissible set of portfolio-consumption strategies is too restrictive. As long as the stochastic integral in a wealth process is well defined and the doubling strategy is excluded, there is no trivial reason to restrict investor's possible choice set. In our model, the wealth process is assumed to be nonnegative, then the doubling strategy is excluded. Given the price processes, an investor may choose the portfolio strategy $(\pi, c) \notin \mathcal{A}_\gamma$ if he is better off. However, from the mathematical viewpoint, additional restrictions to the set of admissible strategies or market parameters seem to be essential although it is hard to motivate economically.

6 Conclusion

In this paper, we have considered a dynamic portfolio-consumption problem when the market price of risk is driven by linear Gaussian processes. We have basically shown that if the Riccati equation related to the HJB equation has a global solution, the conjectured explicit solution is in fact a solution to the original problem. We however find that the definition of admissible portfolio strategies should be carefully chosen. If the investor is more risk seeking than log-utility investors and the solution to the related Riccati equation exists, then the conjectured optimal strategy is in fact optimal. However, the solution to the Riccati equation may not exist globally for some parameter combinations. On the other hand, if the investor is more risk averse than log-utility investors, then the solution to the Riccati equation always exists. However, the conjectured optimal strategy is verified when portfolio-consumption strategies are chosen from a rather restricted set of stochastic processes.

One limitation of our analysis is the assumption of a complete market. In the case of an incomplete market model, the partial differential equation (PDE) corresponding to (12) becomes essentially a nonlinear equation. It seems to be impossible to obtain an explicit solution to such a nonlinear PDE. Hence we have to show the existence of the solution f to the nonlinear PDE and calculate f_x using another method. However, if we do not consider utility from intermediate consumption, the associated HJB equation can be reduced to a linear PDE as (12); see Zariphopoulou [19]. In such a case, an explicit solution can be expected and hence our analysis will be applied in almost the same way.

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