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SURFACES IN THE COMPLEXIFIED SPHERE PARAMETERIZED BY A COMPLEX ORTHOGONAL NET

ATSUSHI FUJIOKA

Graduate School of Economics
Hitotsubashi University
2-1, Naka, Kunitachi
Tokyo 186-8601, Japan
e-mail: fujioka@math.hit-u.ac.jp

Abstract

We study surfaces in the complexified sphere parameterized by a complex orthogonal net and show that the fundamental theorem of these surfaces is stated in terms of two functions called the curvatures. We also exhibit fundamental examples by assuming natural conditions on the curvatures, which include a portion of the sphere, the hyperboloid of one sheet or the hyperboloid of two sheets, and the surfaces whose integrability condition is given by the complexified Liouville's equation.

1. Introduction

Differential geometry of curves or surfaces has now been studied intensively especially from a viewpoint of integrable systems (see [1, 6, 7] and references therein). For example, the integrability condition for umbilic-free surfaces with non-zero constant mean curvature in the Euclidean three-space is given by the sinh-

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Gordon equation, and their deformation parameter as constant mean curvature surfaces describes the spectral parameter. Similar properties to the above example can be seen in case of Bonnet surfaces ([2, 4]), centroaffine minimal surfaces ([3]), harmonic inverse mean curvature surfaces ([4]), isothermic surfaces ([4]) and Willmore surfaces ([4]). On the other hand, in a paper joint with Kurose [5, Theorem 4.1], we showed that special motions of curves in the complex hyperbola are linked with the Burgers hierarchy, which can be formulated as a Hamiltonian system. As shown in Appendix, these motions of curves correspond to those in the complexified circle. Hence it is natural to expect that there are certain relationships between surfaces in the complexified sphere and integrable systems.

In this paper, we shall study surfaces in the complexified sphere parameterized by a complex orthogonal net. In Section 2, we shall introduce the notion of surfaces in the complexified sphere parameterized by a complex orthogonal net and prove the fundamental theorem, which is stated in terms of two functions called the *curvatures*. In Sections 3 and 4, we shall exhibit fundamental examples by assuming natural conditions on the curvatures. Examples in Section 3 include the case that one of the curvatures is a constant, especially a portion of the sphere, the hyperboloid of one sheet or the hyperboloid of two sheets, while the integrability condition for examples in Section 4 is the complexified Liouville's equation.

2. Surfaces in the Complexified Sphere

In the following, we shall study local differential geometry of surfaces in the complexified sphere $S^2_{\mathbf{C}}$, which is defined by

$$S^2_{\mathbf{C}} = \{(x, y, z) \in \mathbf{C}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

In this paper, we call an immersion from a real 2-dimensional domain to $S^2_{\mathbf{C}}$ a surface in $S^2_{\mathbf{C}}$.

Lemma 2.1. *Let* $F : D \to S^2_{\mathbf{C}}$ *be a surface in* $S^2_{\mathbf{C}}$ *and* (u, v) *be coordinates on D. Then* F, F_u *and* F_v *are linearly independent over* $\mathbf{C}$.

Proof. Putting $F = (x, y, z)$, we have

$$x^2 + y^2 + z^2 = 1. \tag{2.1}$$

Differentiating (2.1) by u or v, we have

$$\begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -z\begin{pmatrix} z_u \\ z_v \end{pmatrix}. \tag{2.2}$$

Since F is an immersion, it is enough to consider the case that $x_u y_v - y_u x_v \neq 0$. Then from (2.1) and (2.2), we have $z \neq 0$. Moreover, a direct computation shows that

$$\det\begin{pmatrix} F \\ F_u \\ F_v \end{pmatrix} = \frac{1}{z}(x_u y_v - y_u x_v) \neq 0. \qquad \square$$

Definition 2.2. Let $F : D \to S^2_{\mathbf{C}}$ be a surface in $S^2_{\mathbf{C}}$. Coordinates (u, v) on D are called a *complex orthogonal net*, if

$$(F_u, F_v) = 0, \quad (F_u, F_u) \neq 0, \quad (F_v, F_v) \neq 0,$$

where $(\ ,\ )$ denotes the complex linear extension of the standard inner product on $\mathbf{R}^3$:

$$((x_1, y_1, z_1), (x_2, y_2, z_2)) = x_1x_2 + y_1y_2 + z_1z_2 \quad (x_i, y_i, z_i \in \mathbf{C}, i = 1, 2).$$

The fundamental equations for surfaces in $S^2_{\mathbf{C}}$ parameterized by a complex orthogonal net are given as follows.

Proposition 2.3. *Let $F : D \to S^2_{\mathbf{C}}$ be a surface in $S^2_{\mathbf{C}}$ parameterized by a complex orthogonal net (u, v). Then we have*

$$\begin{cases} F_{uu} = -KF + \dfrac{1}{2}\dfrac{K_u}{K}F_u - \dfrac{1}{2}\dfrac{K_v}{L}F_v, \\ F_{uv} = \dfrac{1}{2}\dfrac{K_v}{K}F_u + \dfrac{1}{2}\dfrac{L_u}{L}F_v, \\ F_{vv} = -LF + \dfrac{1}{2}\dfrac{L_v}{L}F_v - \dfrac{1}{2}\dfrac{L_u}{K}F_u, \end{cases} \tag{2.3}$$

where $K = (F_u, F_u)$, $L = (F_v, F_v)$.

Proof. By Lemma 2.1, F_{uu}, F_{uv} and F_{vv} can be expressed by a linear combination of F, F_u and F_v. Then we have only to use

$$(F, F) = 1, \quad (F, F_u) = (F, F_v) = (F_u, F_v) = 0. \qquad \square$$

Definition 2.4. The functions K and L in Proposition 2.3 are called the *curvatures* for F.

The fundamental theorem of surfaces in $S^2_{\mathbf{C}}$ parameterized by a complex orthogonal net is stated as follows.

Theorem 2.5. *Let D be a real 2-dimensional domain with coordinates (u, v). Given functions K, $L : D \to \mathbf{C}\setminus\{0\}$, there exists a surface in $S^2_{\mathbf{C}}$ parameterized by a complex orthogonal net (u, v) with curvatures K, L uniquely up to right action of the complex orthogonal matrices of third order $O(3, \mathbf{C})$, if and only if*

$$K_{vv} + L_{uu} - \frac{1}{2}\frac{K_v^2}{K} - \frac{1}{2}\frac{L_u^2}{L} - \frac{1}{2}\frac{K_u L_u}{K} - \frac{1}{2}\frac{K_v L_v}{L} + 2KL = 0. \tag{2.4}$$

Proof. From (2.3) we have

$$\begin{pmatrix} F \\ F_u \\ F_v \end{pmatrix}_u = A \begin{pmatrix} F \\ F_u \\ F_v \end{pmatrix}, \qquad \begin{pmatrix} F \\ F_u \\ F_v \end{pmatrix}_v = B \begin{pmatrix} F \\ F_u \\ F_v \end{pmatrix}, \tag{2.5}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -K & \frac{1}{2}\frac{K_u}{K} & -\frac{1}{2}\frac{K_v}{L} \\ 0 & \frac{1}{2}\frac{K_v}{K} & \frac{1}{2}\frac{L_u}{L} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2}\frac{K_v}{K} & \frac{1}{2}\frac{L_u}{L} \\ -L & -\frac{1}{2}\frac{L_u}{K} & \frac{1}{2}\frac{L_v}{L} \end{pmatrix}.$$

Then it is straightforward to see that (2.4) is equivalent to the integrability condition for (2.5):

$$A_v - B_u + [A, B] = 0.$$

Since

$$\begin{pmatrix} F \\ \frac{1}{\sqrt{K}} F_u \\ \frac{1}{\sqrt{L}} F_v \end{pmatrix} \in O(3, \mathbf{C}),$$

the surface exists uniquely up to right action of $O(3, \mathbf{C})$. □

3. A Generalization of the Case that One of the Curvatures is a Constant

Note that the curvatures K and L for a surface in $S^2_{\mathbf{C}}$ parameterized by a complex orthogonal net cannot be constants simultaneously from (2.4) since $K, L \neq 0$. The simplest examples will be given by the case that K or L is a non-zero constant, which is generalized to the condition that K is a function of u only or L is a function of v only. For simplicity, we consider the latter condition.

Proposition 3.1. *Let $F : D \to S^2_{\mathbf{C}}$ be a surface in $S^2_{\mathbf{C}}$ parameterized by a complex orthogonal net (u, v) with curvatures K, L. If L is a function of v only, then*

$$K = \left(\alpha(u) \exp\left(i \int \sqrt{L} dv \right) + \beta(u) \exp\left(-i \int \sqrt{L} dv \right) \right)^2, \tag{3.1}$$

where α and β are arbitrary $\mathbf{C}$-valued functions of u only such that $(\alpha, \beta) \neq (0, 0)$.

Proof. If L is a function of v only, then (2.4) becomes

$$K_{vv} - \frac{1}{2} \frac{K_v^2}{K} - \frac{1}{2} \frac{K_v L_v}{L} + 2KL = 0,$$

which is equivalent to

$$M_{ww} - M = 0$$

if we put

$$M = \sqrt{K}, \qquad w = i \int \sqrt{L} dv.$$ □

In the following, we have only to consider the case that $\alpha \neq 0$ in Proposition 3.1.

Theorem 3.2. *Let* $F : D \to S^2_{\mathbf{C}}$ *be a surface in* $S^2_{\mathbf{C}}$ *parameterized by a complex orthogonal net* (u, v) *with curvatures K, L. If L is a function of v only and K is given by* (3.1) *with* $\alpha \neq 0$, *then*

$$F = G_1(u)\exp\left(i\int\sqrt{L}dv\right) + G_2(u)\exp\left(-i\int\sqrt{L}dv\right), \tag{3.2}$$

where G_1 *and* G_2 *are* $\mathbf{C}^3$*-valued functions of u only satisfying the ordinary differential equations*:

$$\begin{cases} \alpha G_1'' = \alpha' G_1' - 2\alpha^2\beta G_1 - 2\alpha^3 G_2, \\ \beta G_1' = \alpha G_2', \end{cases} \tag{3.3}$$

such that

$$(G_1, G_1) = (G_2, G_2) = 0, \qquad (G_1, G_2) = \frac{1}{2}. \tag{3.4}$$

Proof. Since L is a function of v only, (2.3) becomes

$$\begin{cases} F_{uu} = -KF + \dfrac{1}{2}\dfrac{K_u}{K}F_u - \dfrac{1}{2}\dfrac{K_v}{L}F_v, \\ F_{uv} = \dfrac{1}{2}\dfrac{K_v}{K}F_u, \\ F_{vv} = -LF + \dfrac{1}{2}\dfrac{L_v}{L}F_v. \end{cases} \tag{3.5}$$

From the third equation of (3.5), F is given by (3.2). From (3.1), (3.2) and the first equation of (3.5), we have the first equation of (3.3) and

$$\begin{cases} \beta G_2'' = \beta' G_2' - 2\alpha\beta^2 G_2 - 2\beta^3 G_1, \\ \beta G_1'' + \alpha G_2'' = \beta' G_1' + \alpha' G_2' - 4\alpha\beta^2 G_1 - 4\alpha^2\beta G_2. \end{cases} \tag{3.6}$$

From (3.1), (3.2) and the second equation of (3.5), we have the second equation of (3.3). Note that (3.6) can be deduced from (3.3). Since $(F, F) = 1$, we have

$$(G_1, G_1)\exp\left(2i\int\sqrt{L}dv\right) + 2(G_1, G_2) + (G_2, G_2)\exp\left(-2i\int\sqrt{L}dv\right) = 1,$$

which implies that (3.4).

Conversely, if F is given by (3.2) with (3.3) and (3.4), a direct computation shows that F becomes a surface in $S^2_{\mathbf{C}}$ parameterized by a complex orthogonal net (u, v) with curvatures K, L. □

Example 3.3. Let $\lambda, \mu \in \mathbf{R} \setminus \{0\}$. In the case that

$$\alpha = \frac{\lambda}{2}, \frac{\lambda}{2} i, \ \alpha = \pm\beta, \ L = \pm\mu^2$$

in Theorem 3.2, the surface F can be represented by a portion of the sphere, the hyperboloid of one sheet or the hyperboloid of two sheets. Indeed, up to right action of $O(3, \mathbf{C})$, F is given by the following:

If $L = \mu^2$, then

$$F = \begin{cases} (\cos\lambda u \cos\mu v, \sin\lambda u \cos\mu v, \sin\mu v) & (K = \lambda^2 \cos^2 \mu v), \\ (i \sinh\lambda u \cos\mu v, \cosh\lambda u \cos\mu v, \sin\mu v) & (K = -\lambda^2 \cos^2 \mu v), \\ (i \sinh\lambda u \sin\mu v, \cosh\lambda u \sin\mu v, \cos\mu v) & (K = -\lambda^2 \sin^2 \mu v), \\ (\cos\lambda u \sin\mu v, \sin\lambda u \sin\mu v, \cos\mu v) & (K = \lambda^2 \sin^2 \mu v). \end{cases}$$

If $L = -\mu^2$, then

$$F = \begin{cases} (\cos\lambda u \cosh\mu v, \sin\lambda u \cosh\mu v, i \sinh\mu v) & (K = \lambda^2 \cosh^2 \mu v), \\ (i \sinh\lambda u \cosh\mu v, \cosh\lambda u \cosh\mu v, i \sinh\mu v) & (K = -\lambda^2 \cosh^2 \mu v), \\ (\sinh\lambda u \sinh\mu v, i \cosh\lambda u \sinh\mu v, \cosh\mu v) & (K = \lambda^2 \sinh^2 \mu v), \\ (i \cos\lambda u \sinh\mu v, i \sin\lambda u \sinh\mu v, \cosh\mu v) & (K = -\lambda^2 \sinh^2 \mu v). \end{cases}$$

4. The Case that the Curvatures are Symmetric or Anti-symmetric

The next case which should be considered will be that the curvatures are symmetric or anti-symmetric.

Theorem 4.1. *Let $F : D \to S^2_{\mathbf{C}}$ be a surface in $S^2_{\mathbf{C}}$ parameterized by a complex orthogonal net (u, v) with curvatures K, L. If $K = \varepsilon L$ with $\varepsilon = \pm 1$, then*

$$F = \frac{C_1(f(s) - g(t)) + 2C_2 f(s) g(t) + C_3}{f(s) + g(t)}, \tag{4.1}$$

where $C_1, C_2, C_3 \in \mathbf{C}^3$ *such that*

$$(C_1, C_1) = (C_2, C_3) = 1, \ (C_1, C_2) = (C_1, C_3) = (C_2, C_2) = (C_3, C_3) = 0, \quad (4.2)$$

and f and g are arbitrary **C**-*valued functions of*

$$s = \begin{cases} u + iv & (\varepsilon = 1), \\ u + v & (\varepsilon = -1) \end{cases} \quad (4.3)$$

only and

$$t = \begin{cases} u - iv & (\varepsilon = 1), \\ u - v & (\varepsilon = -1) \end{cases} \quad (4.4)$$

only, respectively such that

$$f + g, \ f', \ g' \neq 0. \quad (4.5)$$

Proof. A direct computation shows that (2.3) is equivalent to

$$F_{ss} = \frac{K_s}{K} F_s, \quad F_{st} = -\frac{1}{2} KF, \quad F_{tt} = \frac{K_t}{K} F_t \quad (4.6)$$

under the transformations (4.3) and (4.4). Moreover, we have

$$(F_s, F_s) = (F_t, F_t) = 0, \ (F_s, F_t) = \frac{1}{2} K. \quad (4.7)$$

Note that (2.4) becomes the complexified Liouville's equation:

$$(\log K)_{uu} + \varepsilon(\log K)_{vv} + 2K = 0$$

or equivalently

$$(\log K)_{st} + \frac{1}{2} K = 0,$$

whose solutions are given by

$$K = -4 \frac{f'(s) g'(t)}{(f(s) + g(t))^2}$$

for arbitrary **C**-valued functions $f = f(s)$ and $g = g(t)$ with (4.5). Then it is straightforward to see that the surface F given by (4.1) with (4.2) satisfies $(F, F) = 1$, (4.6) and (4.7). □

Appendix. Motions of Curves in the Complexified Circle

The complexified circle $S^1_{\mathbf{C}}$ is defined by

$$S^1_{\mathbf{C}} = \{(x, y) \in \mathbf{C}^2 \mid x^2 + y^2 = 1\}.$$

We call an immersion from an interval to $S^1_{\mathbf{C}}$ a curve in $S^1_{\mathbf{C}}$. A direct computation shows that the following:

Proposition A.1. *Let* $\gamma = (x, y)$ *be a curve in* $S^1_{\mathbf{C}}$. *If* $x' \neq 0$, *then*

$$\det\begin{pmatrix} \gamma \\ \gamma' \end{pmatrix} = -\frac{x'}{y} = -\frac{(x')^2 + (y')^2}{x'} y.$$

In particular, γ *and* γ' *are linearly independent over* $\mathbf{C}$ *and* $(\gamma', \gamma') \neq 0$, *where* $(\ ,\ )$ *denotes the complex linear extension of the standard inner product on* $\mathbf{R}^2$. *Moreover, we have*

$$\gamma'' = \tau^2\gamma + i\kappa\gamma',$$

where

$$\tau = i\sqrt{(\gamma', \gamma')}, \quad \kappa = -i\frac{\tau'}{\tau}.$$

A motion of a curve in $S^1_{\mathbf{C}}$ is given by a map $\gamma = \gamma(s, t)$ from the product of intervals I and J, where $t \in J$ is considered to be a time parameter and $s \in I$ is a parameter of a curve in $S^1_{\mathbf{C}}$ with fixed time. Since $(\gamma, \gamma) = 1$, the time evolution of γ is given by

$$\gamma_t = \lambda\gamma_s,$$

where λ is a $\mathbf{C}$-valued function on $I \times J$. Hence as shown in [5, Theorem 3.2], special motions of curves in $S^1_{\mathbf{C}}$ are linked with the Burgers hierarchy. In fact, it is straightforward to see that our motions of curves in $S^1_{\mathbf{C}}$ correspond to those in the complex hyperbola:

$$\{(z, w) \in \mathbf{C}^2 \mid zw = 1\}$$

in [5] via a map defined by

$$(x, y) \mapsto (x + iy, x - iy) \quad ((x, y) \in S^1_{\mathbf{C}}).$$

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