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EXTENSIONS OF THE FUNDAMENTAL WELFARE THEOREMS IN A NON-WELFARISTIC FRAMEWORK*

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Abstract

In a non-welfaristic framework, Sen (1993) extends the first fundamental welfare theorem by demonstrating that the market mechanism also promotes individual freedom efficiently. This paper has a two-fold purpose. First, in order to investigate extensions of the first and the second welfare theorems, we present an analytical framework in which each agent is endowed with three types of preference relations: an allocation preference relation, an opportunity preference relation, and an overall preference relation. We demonstrate that under certain conditions, the two welfare theorems can be extended. Second, we describe the restrictive nature of the underlying conditions for these positive results.

Keywords: fundamental welfare theorems, market mechanism, freedom, opportunity set, Pareto optimality.
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I. *Introduction*

It is often argued that the market mechanism promotes individual freedom that permits to choose freely. However, in the traditional framework of economic theory, the market mechanism is evaluated exclusively depending on its allocation efficiency. The concept of

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allocation efficiency is typically a "welfaristic" concept in which final allocations resulting from the market mechanism are judged by the utility levels of the individuals involved. Nothing can be said about individuals' freedom.

To incorporate the freedom aspect of the market mechanism, Amartya Sen (1993) developed a framework and made an argument for the market mechanism to promote individual freedom and to make individuals free to choose. He identified two aspects of freedom: "the opportunity aspect" and "the process aspect." The opportunity aspect relates to the opportunities of achieving goals that each individual values, whereas the process aspect is concerned with each individual's free decisions. Then, Sen established that under certain types of assessments of opportunities, the market mechanism attains efficient states in opportunity freedom. Following Sen (1993), formal frameworks for "non-welfaristic" analyses have been developed by Suzumura and Xu (2001, 2003), and the framework has been used to examine Arrow's impossibility result in social choice theory (Suzumura and Xu (2004), and Iwata (2009)). In this study, we first develop an alternative formal framework for analyzing welfaristic and non-welfaristic concerns, and then examine the performance of the market mechanism from a non-welfaristic perspective by focusing on opportunity efficiency, the concept of efficiency in the distributions of opportunity sets, and on overall efficiency, the concept of efficiency in the distributions of pairs of an opportunity set and a consumption bundle.

In the traditional framework of welfare economics, the performance of the market mechanism is best summarized by the two fundamental theorems: (i) the first theorem claims that under certain conditions, the market mechanism generates an efficient allocation, and (ii) *the second theorem* asserts that under some more restrictive conditions, any efficient allocation can be achieved by the market mechanism with an appropriate redistribution of agents' initial endowments. Note that in both theorems, the concept of efficiency is based solely on individual preferences of final consumption bundles, and is welfaristic in nature.

To elucidate the freedom aspects of the market mechanism, we must expand our framework in general and go beyond the usual concept of efficiency in particular. For this purpose, we define a *configuration* as a pair of an allocation and a distribution of opportunity sets. Although the usual concept of efficiency is based on only the allocation part of a configuration, a configuration also contains information about the distribution of opportunity sets. An agent's opportunity set is viewed as reflecting the opportunity aspect of freedom (Sen (1988, 1993, 2002) and Pattanaik and Xu (1990)). Depending on how each agent ranks his opportunity sets as well as on how he assesses his pairs of consumption bundles and opportunity sets, we can go beyond the concept of allocation efficiency to that of efficiency with respect to distributions of opportunity sets reflecting concerns for freedom, and to the concept of efficiency with respect to configurations reflecting agents' overall attitudes toward consumption bundles and opportunities.

To formalize these ideas, we assume that each agent is endowed with three types of preference relations: an allocation preference relation that ranks consumption bundles, an opportunity preference relation that ranks opportunity sets, and an overall preference relation that ranks pairs of consumption bundles and the associated opportunity sets. Using these preference relations, we introduce three concepts of efficiency: (i) *allocation-Pareto optimality*, which is the usual concept of efficiency of allocations, (ii) *opportunity-Pareto optimality*, which reflects the situation in which it is impossible to improve one agent's opportunities without reducing another agent's opportunities, and (iii) *overall-Pareto optimality*, which concerns the possibility of improving one agent's overall situation without hurting another agent's overall situation.

With these concepts in hand, we examine the relationship between the market mechanism and various concepts of Pareto-optimality, with the specific objective of extending the two fundamental theorems of welfare economics. First, we note a difficulty in extending the welfare theorems to opportunity-Pareto optimality and overall-Pareto optimality. Then, we introduce weaker concepts of Pareto optimality. We demonstrate that, if the opportunity preference relation of every agent is that discussed by Sen (1993), then the two welfare theorems can be extended with these weaker concepts of Pareto optimality. Therefore, we clarify and extend Sen's (1993) arguments.

However, when we venture out of Sen's type of opportunity-preference relations, we may no longer be able to extend the two welfare theorems. Therefore, although the conviction that the market mechanism promotes individual opportunity freedom efficiently is valid in certain limited cases, it is problematic in general.

The remainder of the paper is as follows. Section 2 presents basic notation and definitions, and introduces three types of preference relations. Extensions of the first and second theorems are examined in Sections 3 and 4, respectively. Section 5 discusses the restrictive nature of our positive results. We offer concluding remarks in Section 6. Proofs are in the Appendix.

II. *Allocations, Opportunities, and Configurations*

We consider an economy with n agents and k infinitely divisible goods. Let $N=\{1,...,n\}$ be the set of agents. An *allocation* is a vector $x=(x_1,..., x_n)\in\mathbb{R}^{nk}_+$ where for each $i\in N$, $x_i=(x_{i1}, ..., x_{ik})\in\mathbb{R}^k_+$ is a *consumption bundle* of agent i.[1] There exist fixed amounts of social endowments of goods, which are represented by the vector $\omega\in\mathbb{R}^k_{++}$. An allocation $x\in\mathbb{R}^{nk}_+$ is *feasible* if $\sum_{i=1}^{n} x_i\leq\omega$.[2] Let Z be the set of all feasible allocations.

For each $i\in N$, an *opportunity set* for agent i is a compact and comprehensive set in $\mathbb{R}^k_+$. Recall that a set $A\subseteq\mathbb{R}^k_+$ is *comprehensive* if $a\in A$ and $0\leq b\leq a$ imply $b\in A$. Comprehensiveness is a reasonable assumption for opportunity sets in economic environments. It means that if a consumption bundle a is available for an agent, then any consumption bundle b containing a smaller amount of every good than a is also available for the agent. The set of all opportunity sets is denoted by $\mathcal{O}$. A *distribution of opportunity sets* is an n-tuple $S=(S_1,..., S_n)\in\mathcal{O}^n$.

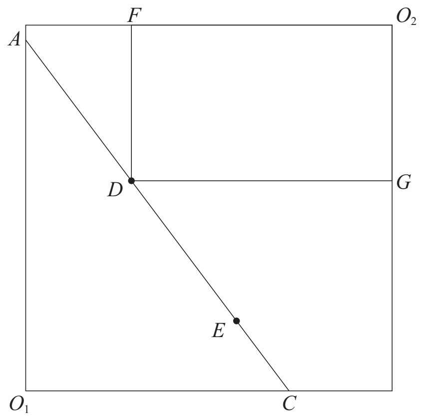
An opportunity set of an individual prescribes the range of alternatives from which she can choose. In social situations, however, choices made by individuals involved are often interdependent, as noticed in Basu (1987), Gravel, Laslier and Trannoy (1998), Pattanaik (1994), Pattanaik and Xu (2009), and Tadenuma and Xu (2010). Consider an example of a pure exchange economy with two agents and two goods as in Figure 1.

In Figure 1, the triangle O_1AC represents a typical budget set for agent 1. In this set, agent

[1] As usual, $\mathbb{R}_+$ is the set of all non-negative real numbers, and $\mathbb{R}_{++}$ is the set of all positive real numbers.

[2] Vector inequalities are defined as follows: for all x, $y\in\mathbb{R}^k_+$, $x\geq y \Leftrightarrow (x-y)\in\mathbb{R}^k_+$; $x>y \Leftrightarrow [x\geq y$ and $x\neq y]$; and $x\gg y \Leftrightarrow (x-y)\in\mathbb{R}^k_{++}$.

FIG. 1. INTERDEPENDENCE OF OPPORTUNITY SETS



1 can certainly choose $\overrightarrow{O_1D}$ if agent 2's choice is $\overrightarrow{O_2D}$ or in the region of O_2FDG. However, if agent 2 wanted to choose $\overrightarrow{O_2E}$, then at least one of the two agents' choices could not be attained. That is, in general, whether an agent can actually obtain a bundle in her budget set depends on the other agents' choices. Nevertheless, it is certain that any bundle in an agent's budget set is attainable given *certain* choices of the other agents.

Following the above observations, we propose the next definition of a feasible distribution of opportunity sets, which is a generalization of the concept of a distribution of budget sets.

For each $S\subseteq\mathbb{R}^k_+$, define

$$\partial^+ S=\{x\in S \mid \nexists y\in S \text{ such that } y>x\}.$$

That is, $\partial^+ S$ is the "undominated boundary of S." A distribution of opportunity sets $S=(S_1,\dots,S_n)\in\mathcal{O}^n$ is *feasible* if for every $i\in N$, and every $y_i\in S_i$, there exists $(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)\in\Pi_{j\neq i}\partial S_j$ such that $(y_1,\dots,y_n)\in Z$. That is, under a feasible distribution of opportunity sets, every consumption bundle in an agent's opportunity set is attainable given certain choices of the other agents from the boundaries of their opportunity sets.

As we have described above, a particular class of opportunity sets is the class of constrained budget sets. A *constrained budget set* for agent $i\in N$ at a price vector $p\in\mathbb{R}^k_+$ and a consumption bundle $x_i\in\mathbb{R}^k_+$ is defined by

$$B(p, x_i)=\{y_i\in\mathbb{R}^k_+ \mid p\cdot y_i\leq p\cdot x_i\}\cap\{a\in\mathbb{R}^k_+ \mid a\leq\omega\}.$$

Let $\mathcal{B}=\{B(p, x_i) \mid p\in\mathbb{R}^k_+, x_i\in\mathbb{R}^k_+\}$.

In our framework, an extended social state is described by a pair of a final allocationan dadistribution of opportunity sets. A *configuration* is a pair $(x, S)\in\mathbb{R}^{nk}_+\times\mathcal{O}^n$ such that $x_i\in S_i$ for every $i\in N$. Let $\mathbb{C}$ be the set of all configurations. For each $(x, S)\in\mathbb{C}$ and each $i\in N$, the *overall individual state of agent i at* (x, S) is the pair (x_i, S_i). A configuration $(x, S)\in\mathbb{C}$ is *feasible* if both x and S are feasible. Let $\mathbb{Z}$ be the set of all feasible configurations.

We assume that each agent is endowed with three preference relations. First, as in standard microeconomics, each $i\in N$ has a preference order over final consequences, specifically over consumption bundles in this economic environment.[3] We call it *allocation-preference order* and denote it by R_i^A.[4] We assume that R_i^A is continuous and strictly monotonic.

Second, as in the literature on opportunity set rankings, we assume that each $i \in N$ has an *opportunity-preference quasiorder* over opportunity sets, which we denote by R_i^O. The preference quasiorder R_i^O is *monotonic* in the following sense:

(i) if an opportunity set S_i is included in another opportunity set T_i, then T_i is at least as good as S_i;
(ii) if S_i is included in the interior of T_i, then T_i is strictly better than S_i.

Finally, we assume that each agent $i \in N$ has a preference quasiorder on overall individual states, particularly pairs of a consumption bundle and an opportunity set. We call it *overall-preference quasiorder* and denote it by $\overline{R}_i$. The relation $(x_i, S_i)\ \overline{R}_i\ (y_i, T_i)$ means the following: it is at least as good for agent i to have the consumption bundle x_i from the set S_i of attainable consumption bundles as to have the consumption bundle y_i from the set T_i. Each agent's evaluations on overall individual states should be linked naturally with his preferences about consumption bundles as well as his assessments of opportunity sets. Throughout this paper, we assume the following two conditions on the relationships between overall-preference quasiorders and the other two preference quasiorders: [5]

Condition A:
(i) An overall individual state (x_i, S_i) is at least as good as an overall individual state (y_i, T_i) in $\overline{R}_i$ *only if* the consumption bundle x_i is at least as good as the consumption bundle y_i in R_i^A or the opportunity set S_i is at least as good as the opportunity set T_i in R_i^O.
(ii) An overall individual state (x_i, S_i) is strictly better than an overall individual state (y_i, T_i) in $\overline{R}_i$ *only if* the consumption bundle x_i is strictly better than the consumption bundle y_i in R_i^A *or* the opportunity set S_i is strictly better than the opportunity set T_i in R_i^O.

Condition B:
(i) If a consumption bundle x_i is at least as good as a consumption bundle y_i in R_i^A and an opportunity set S_i is at least as good as an opportunity set T_i in R_i^O, then the overall individual state (x_i, S_i) is at least as good as (y_i, T_i).
(ii) If a consumption bundle x_i is strictly better than a consumption bundle y_i in R_i^A *and* an opportunity set S_i is strictly better than an opportunity set T_i in R_i^O, then the overall individual state (x_i, S_i) is strictly better than (y_i, T_i).[6]

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[3] A *preference quasiorder* is a reflexive and transitive binary relation, and A *preference order* is a complete preference quasiorder.

[4] As usual, the strict preference relation associated with R_i^A is denoted by P_i^A, and the indifference relation by I_i^A. Similar notation is used for other preference relations.

[5] In this paper, we regard allocation preferences and opportunity preferences as the bases, and introduce overall preferences with some natural relations to the foregoing two types of preferences. Alternatively, we might consider overall preferences as the primitive, and derive the other two kinds of preferences as projections of the overall preferences. Since our purpose here is to investigate extensions of the fundamental welfare theorems from the standard allocation preferences to opportunity preferences and overall preferences, we have taken the first approach.

[6] Note that in general, there is no logical relation between Conditions A and B. However, if the opportunity preference quasiorder is complete, then Condition B implies Condition A, whereas if the overall preference quasiorder

Opportunity preferences may be related to allocation preferences. Sen (1993, p.530) introduces the following conditions (Sen's Axiom O), which we assume in Sections 3 and 4 of this paper.

Sen's Condition:
(i) An opportunity set S_i is at least as good as T_i in R_i^O *only if* there exists a consumption bundle x_i in S_i that is at least as good as every consumption bundle in T_i.
(ii) An opportunity set S_i is strictly better than T_i in R_i^O *only if* there exists a consumption bundle x_i in S_i that is strictly better than every consumption bundle in T_i.

Note that (i) and (ii) in Sen's Condition (Sen's Axiom O) are *necessary conditions* for S_i R_i^O T_i and for S_i P_i^O T_i, respectively. Sen argues that the possible insufficiency distinguishes this condition from a purely instrumental view of freedom, as quoted below.

> The possible insufficiency of this condition is one important distinction between Axiom O and a purely instrumental view of freedom. ··· To be sure of an increase in freedom requires the presence of a more preferred alternative, but the presence of a more preferred alternative does not necessarily guarantee an enhancement of freedom. (Sen, 1993, p.530)

However, if if R_i^O is complete, then (i) and (ii) together are also *sufficient conditions* for S_i R_i^O T_i and S_i P_i^O T_i.

Furthermore, we consider certain "extreme" preferences. We say that agent $i \in N$ is a *consequentialist* if he evaluates his overall individual states based only on his preferences over consumptions bundles: formally, for all overall individual states (x_i, B_i) and (y_i, C_i), (x_i, B_i) $\overline{R}_i$ (y_i, C_i) if and only if x_i R_i^A y_i. In contrast, we say that agent $i \in N$ is a *non-consequentialist* if he assesses his overall individual states based solely on his preferences over opportunity sets: formally, for all overall individual states (x_i, B_i) and (y_i, C_i), (x_i, B_i) $\overline{R}_i$ (y_i, C_i) if and only if B_i R_i^O C_i.

III. *Extensions of the First Welfare Theorem*

We examine efficiency of social states specified by competitive equilibrium. Since we are interested not only in final allocations but also in opportunity distributions and configurations, we develop new concepts in Pareto optimality on distributions of opportunity sets and on configurations.

The traditional concept of Pareto optimality concerns only final allocations, and may be defined as follows. An allocation $y \in \mathbb{R}_+^{nk}$ *Pareto dominates* an allocation $x \in \mathbb{R}_+^{nk}$ if y_i R_i^A x_i for every $i \in N$ and y_i P_i^A x_i for some $i \in N$. A feasible configuration $(x, S) \in \mathbb{Z}$ is *allocation-Pareto* optimal if no feasible allocation Pareto-dominates x.

Symmetrically to allocation-Pareto optimality, we may define the Pareto optimality of distributions of opportunity sets, and of configurations as follows. A distribution of opportunity sets, $T=(T_1,...,T_n) \in \mathcal{O}^n$, *Pareto dominates* a distribution of opportunity sets, $S=(S_1,...,S_n) \in \mathcal{O}^n$,

is complete, then Condition A implies Condition B. (Hence, if both the opportunity preference quasiorder and the overall preference quasiorder are complete, then Conditions A and B are equivalent.)

if $T_i\ R_i^O\ S_i$ for every $i \in N$ and $T_i\ P_i^O\ S_i$ for some $i \in N$. A feasible configuration $(x, S) \in \mathbb{Z}$ is *opportunity-Pareto optimal* if no feasible distribution of opportunity sets Pareto-dominates S.

A configuration $(y, T) \in \mathbb{C}$ *Pareto dominates* a configuration $(x, S) \in \mathbb{C}$ if $(y_i, T_i)\ \overline{R}_i\ (x_i, S_i)$ for every $i \in N$ and $(y_i, T_i)\ \overline{P}_i\ (x_i, S_i)$ for some $i \in N$. A feasible configuration $(x, S) \in \mathbb{Z}$ is *overall-Pareto optimal* if no feasible configuration Pareto-dominates (x, S).

A configuration $(x, B) \in \mathbb{C}$ is a *Walrasian configuration* if it is feasible and there exists a price vector $p \in \mathbb{R}_+^k$ such that for every $i \in N$, $B_i = B(p, x_i)$ and $x_i\ R_i^A\ y_i$ for every $y_i \in B(p, x_i)$.

Theorem 1 (Extension of the First Fundamental Welfare Theorem: Part I)
(i) *Every Walrasian configuration is allocation-Pareto optimal.*
(ii) *If every agent is a consequentialist, then every Walrasian configuration is overall-Pareto optimal.*

Theorem 1 is essentially a restatement of the classical first welfare theorem in our framework. Note that part (ii) of Theorem 1 is applicable only to the case where every agent is a consequentialist.

As the following example illustrates, although opportunity-Pareto optimality and overall-Pareto optimality defined above are the exact counterparts of the standard definition of allocation-Pareto optimality, a Walrasian configuration may not be opportunity-Pareto optimal nor overall-Pareto optimal even if each individual's preferences satisfy Sen's Condition.

Example 1 Consider a standard two-person and two-good economy as described in Figure 2. We assume that $\omega = (2, 2)$, and that each agent has opportunity preferences and overall preferences such that:

(i) for all opportunity sets $S_i, T_i \in \mathcal{O}$, $S_i\ R_i^O\ T_i$ if and only if there exists $x_i \in S_i$ such that $x_i\ R_i^A\ y_i$ for every $y_i \in T_i$;

(ii) for all overall individual states (x_i, S_i), (y_i, T_i), if $x_i\ R_i^A\ y_i$ and $S_i\ P_i^O\ T_i$, then $(x_i, S_i)\ \overline{P}_i\ (y_i, T_i)$.

Clearly, the preferences of each agent satisfy Sen's Condition. For each agent i, let

$$B_i = \{z_i \in \mathbb{R}_+^2 \mid z_{i1} + z_{i2} \leq 2\}$$

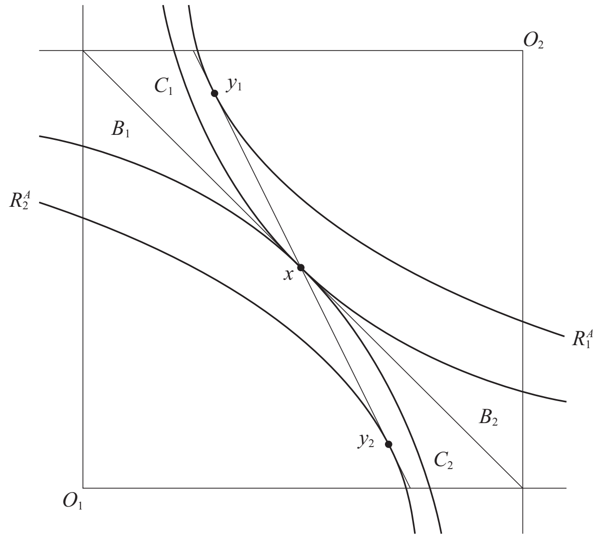
and $x_i = (1, 1)$. Then, with allocation preferences illustrated by the indifference curves in the figure, $(x, B) = ((x_1, x_2), (B_1, B_2)) \in \mathbb{Z}$ is a Walrasian configuration with the prices $(p_1, p_2) = (1, 1)$. Consider another feasible configuration $(x, C) \in \mathbb{Z}$ such that the opportunity set of each agent is

$$C_i = \{z_i \in \mathbb{R}_+^2 \mid 2z_{i1} + z_{i2} \leq 3 \text{ and } z_{i2} \leq 2\}.$$

Notice that for every agent i, $y_i \in C_i$ and $y_i\ P_i^A\ z_i$ for every $z_i \in B_i$, and by (i) above, $C_i\ P_i\ B_i$. By (ii), we also have $(y_i, C_i)\ \overline{P}_i\ (x_i, B_i)$. Therefore, the Walrasian configuration (x, B) is neither opportunity-Pareto optimal nor overall-Pareto optimal.

The message of this example sharply contrasts with Sen's (1993) claim that a Walrasian configuration is always opportunity-Pareto optimal. In fact, he established the result with a weaker concept of opportunity-Pareto optimality than ours, to which we will turn next. A major difference of our framework from Sen's (1993) is that we do not *a priori* assume that at each

FIG. 2. A WALRASIAN CONFIGURATION MAY NOT BE OPPORTUNITY-PARETO OPTIMAL



configuration, each agent attains the best consumption bundle in his opportunity set. We would rather consider this property *a notable feature of Walrasian configurations*. In fact, there are many games where agents do not attain their best alternatives in the set of attainable alternatives at equilibria.

Let us say that a configuration $(x, S) \in \mathbb{C}$ is *individually maximizing* if for every $i \in N$, and every $y_i \in S_i$, $x_i \ R_i^A \ y_i$. Observe that in Example 1, the configuration (x, C), which Pareto dominates the Walrasian configuration (x, B), is not individually maximizing. We now turn to the following weaker concepts of Pareto optimality.

Weak Allocation-Pareto Optimality: A feasible configuration $(x, S) \in \mathbb{Z}$ is *weakly allocation-Pareto optimal* if there is no feasible and individually maximizing configuration $(y, T) \in \mathbb{Z}$ such that y Pareto dominates x.

Weak Opportunity-Pareto Optimality: A feasible configuration $(x, S) \in \mathbb{Z}$ is *weakly opportunity-Pareto optimal* if there is no feasible and individually maximizing configuration $(y, T) \in \mathbb{Z}$ such that T Pareto dominates S.

Weak Overall-Pareto Optimality: A feasible configuration $(x, S) \in \mathbb{Z}$ is *weakly overall-Pareto optimal* if there is no feasible and individually maximizing configuration $(y, T) \in \mathbb{Z}$ that Pareto dominates (x, S).

Regarding Pareto optimality of allocations, it can be shown that weak allocation-Pareto optimality is equivalent to allocation-Pareto optimality.[7] Therefore, we use only the standard concept of allocation-Pareto optimality.

We demonstrate that under Sen's Condition of individual preferences, Walrasian

[7] See Lemma 1 in the Appendix.

configurations satisfy these weaker versions of Pareto optimality of opportunity distributions and of overall configurations.

Theorem 2 (Extension of the First Welfare Theorem: Part II)
Assume that the preferences of every agent satisfy Sen's Condition. Then:
(i) (Sen, 1993) *Every Walrasian configuration is weakly opportunity-Pareto optimal.*
(ii) *Every Walrasian configuration is weakly overall-Pareto optimal.*

Proof[1]: See the Appendix.

Therefore, with some weakenings of the conditions of Pareto optimality for opportunity distributions and configurations, we can extend the classical first welfare theorem to our framework whenever agents' preferences are of the type discussed by Sen (1993).

IV. *Extensions of the Second Welfare Theorem*

Throughout this section, we assume that for every $i \in N$, the allocation-preference order R_i^A is convex. When the opportunity-preference quasiorder of every agent is complete, we obtain the following result.

Theorem 3 (Extension of the Second Welfare Theorem: Part I)
Assume that the preferences of every agent $i \in N$ satisfy Sen's Condition, and that the opportunity preferences of every agent $i \in N$ are complete. Let $(x, S) \in \mathbb{Z}$ be an individually maximizing configuration. Suppose that one of the following three statements is true:

(a) (x, S) *is allocation-Pareto optimal.*
(b) (x, S) *is weakly opportunity-Pareto optimal.*
(c) (x, S) *is weakly overall-Pareto optimal.*

Then, there exists a Walrasian configuration (x, B) such that for every $i \in N$, B_i is indifferent to S_i in her opportunity-preference relation, and (x_i, B_i) is indifferent to (x_i, S_i) in her overall preference relation.

Proof: See the Appendix.

Theorem 3 extends the classical second theorem of welfare economics to our framework when every agent's opportunity-preference quasiorder is of the type discussed by Sen (1993), and is complete.

When the completeness of opportunity preference relations is not required, we have the following result.

Theorem 4 (Extension of the Second Welfare Theorem: Part II)
Assume that the preferences of every agent satisfy Sen's Condition. Let $(x, S) \in \mathbb{Z}$ be an individually maximizing configuration. Suppose that (x, S) is allocation-Pareto optimal. Then, there exists a Walrasian configuration (x, B) such that no agent $i \in N$ prefers S_i to B_i in his opportunity-preference relation, nor he prefers (x_i, S_i) to (x_i, B_i) in his overall preference relation.

Proof: See the Appendix.

Therefore, when every agent's preferences satisfy Sen's condition, we can associate to any allocation-Pareto optimal and individually maximizing configuration (x, S) a Walrasian configuration consisting of the same allocation x and a distribution of budget sets that gives all the agents *no worse opportunities* as well as *no worse individual overall states*. This can be regarded as an extension of the second welfare theorem to our framework when agents' opportunity preferences may be incomplete.

Therefore, our results in this section, together with the results obtained in the previous section, reinforce and extend Sen's insights for the operation of the market mechanism.

V. *Limitations*

Although our results obtained till now reinforce one's conviction that market mechanisms promote both allocation efficiency, opportunity efficiency, and overall efficiency, they rely on specific restrictions about agents' opportunity preferences. In particular, we have assumed that each agent's preferences satisfy Sen's Condition. However, the type of such preferences is only one of the possible types of preferences for opportunities. In this section, we discuss the possibility of extending the two welfare theorems for other classes of agents' opportunity-preference quasiorders.

Let us consider opportunity preference quasiorders satisfying the following "Superset Domination" condition. Let μ be the Lebesgue measure on $\mathbb{R}^k$.

Superset Domination: For all $S_i, T_i \in \mathcal{O}$,
(i) $S_i \supseteq T_i \Rightarrow S_i \; R_i^O \; T_i$, and
(ii) $S_i \supseteq T_i$ and $\mu(S_i) > \mu(T_i) \Rightarrow S_i \; P_i^O \; T_i$.

Although many opportunity-preference quasiorders discussed in the literature on ranking opportunity sets satisfy Superset Domination, this condition is not compatible with Sen's Condition. We now examine the possibility of extending the two welfare theorems when opportunity-preference quasiorders satisfy Superset Domination.

An interesting type of opportunity preferences satisfying Superset Domination is that of "additive" opportunity-preference orders. We say that an opportunity-preference order R_i^O is *additive* if there exists an integrable function $f_i : \mathbb{R}_+^k \to \mathbb{R}_{++}$ such that for all S_i, $T_i \in \mathcal{O}$,

$$S_i \; R_i^O \; T_i \Leftrightarrow \int_{S_i} f_i d\mu \geq \int_{T_i} f_i d\mu \qquad .$$

It is clear that any additive opportunity-preference order satisfies Superset Domination. Of particular interest is the case where the function f_i coincides with the utility function $u_i : \mathbb{R}_+^k \to \mathbb{R}_{++}$ representing the allocation-preference order. In this case, opportunity sets are ranked according to the sum of the utilities of all possible consumption bundles in each set.

The next example illustrates the possibility that a Walrasian configuration is not necessarily weakly opportunity-Pareto optimal, when every agent has an additive opportunity-preference order. Thus, the first fundamental welfare theorem cannot be extended to this type of opportunity-preference orders.

Example 2 Consider a standard two-person and two-good economy. Let $\omega = (10, 10)$. Each agent $i \in N$ has the following utility function $u_i : \mathbb{R}^2_+ \rightarrow \mathbb{R}_{++}$ defined by

$$u_i(x_{i1}, x_{i2}) = \begin{cases} \frac{5}{14}(x_{i1} + 10x_{i2}) & \text{if } x_{i2} \leq \frac{1}{4}x_{i1} \\ x_{i1} + x_{i2} & \text{if } \frac{1}{4}x_{i1} < x_{i2} < \frac{5}{4}x_{i1} \\ \frac{1}{5}(10x_{i1} + x_{i2}) & \text{if } x_{i2} \geq \frac{4}{5}x_{i1} \end{cases}$$

For each $i \in N$, and for all $A_i, B_i \in \mathcal{O}$,

$$A_i \, R_i^O \, B_i \Leftrightarrow \int_{A_i} u_i d\mu \geq \int_{B_i} u_i d\mu$$

Define $(x, B) \in \mathbb{Z}$ as follows: for each $i \in N$, $x_i = (5, 5)$ and $B_i = B(p, x_i)$ where $p = (1, 1)$. Then, (x, B) is a constrained Walrasian configuration. For each $i \in N$, define

$$C_i = \{x_i \in \mathbb{R}^2_+ \mid x_i \in B_i \text{ and } x_{i2} \leq 9\} \\ \cup \{x_i \in \mathbb{R}^2_+ \mid 0 \leq x_{i2} \leq 1 \text{ and } 0 \leq x_{i1} \leq 10\}.$$

Clearly, (x, C) is an individually maximizing configuration. It can be calculated that for every $i \in N$,

$$\int_{C_i} u_i d\mu \geq \int_{B_i} u_i d\mu$$

Therefore, for every $i \in N$, $C_i \, P_i^O \, B_i$. Thus, (x, B) is not weakly opportunity-Pareto optimal.

VI. *Concluding Remarks*

To a limited extent, we can extend the fundamental welfare theorems to situations involving not only weak allocation-Pareto optimality but also weak opportunity-Pareto optimality and weak overall-Pareto optimality. When agents' opportunity-preferences are of the types discussed in Sections 3 and 4, the market mechanism may be regarded as appealing because it generates configurations that are weakly allocation-Pareto optimal, weakly opportunity-Pareto optimal and weakly overall-Pareto optimal (extensions of the first fundamental theorem of welfare economics). Furthermore, for every configuration that is allocation-Pareto optimal, we can find a market equilibrium to support it, and the configuration achieved at the equilibrium is not worse than the given configuration in allocation, opportunity, and overall preferences (extensions of the second fundamental theorem of welfare economics).

We have also demonstrated, however, that the above results depend crucially on the particular classes of opportunity-preference relations. If we go beyond these classes, we may encounter difficulties in extending the two theorems of welfare economics. The difficulties point to the incompatibility of allocation-Pareto optimality, opportunity-Pareto optimality, and/or overall-Pareto optimality. We have used specific classes of opportunity-preference relations to illustrate those difficulties. Investigation into various types of opportunity-preference relations in the context of extending welfare theorems may deserve further exploration.

Appendix

Before we present the proofs for our main results, we first prove lemmata.

Lemma 1 A configuration $(x, S)\in\mathbb{Z}$ *is allocation-Pareto optimal if and only if it is weakly allocation-Pareto optimal.*

Proof: The "only if" part is obvious. To prove the "if" part, let $(x, S)\in\mathbb{Z}$ be weakly allocation-Pareto optimal. Suppose, to the contrary, that (x, S) is not allocation-Pareto optimal. Then, there exists $(y, C)\in\mathbb{Z}$ such that for every $i\in N$, $y_i\ R_i^A\ x_i$, and for some $j\in N$, $y_j\ P_i^A\ x_j$. Define a distribution of opportunity sets, $D=(D_1, \dots, D_n)$, by

$$D_i=\{v_i\in\mathbb{R}_+^k \mid v_i\leq y_i\} \text{ for each } i\in N.$$

Clearly, D is a feasible distribution of opportunity sets, and (y, D) is a feasible configuration. For every $i\in N$, by strict monotonicity of R_i^A, $y_i\ R_i^A\ v_i$ for every $v_i\in D_i$. Therefore, (y, D) is an individually maximizing configuration. This implies that (x, S) is not weakly allocation-Pareto optimal, which is a contradiction. Thus, (x, S) is allocation-Pareto optimal. ■

Lemma 2 *Assume that the preferences of every agent satisfy Sen's Condition.*

(i) *If* $(x, S)\in\mathbb{Z}$ *is allocation-Pareto optimal, then it is weakly opportunity-Pareto optimal and weakly overall-Pareto optimal.*

(ii) *Assume that for every* $i\in N$, R_i^O *is complete, and that* $(x, S)\in\mathbb{Z}$ *is an individually maximizing configuration. If* (a) (x, S) *is weakly opportunity-Pareto optimal or* (b) (x, S) *is weakly overall-Pareto optimal, then it is allocation-Pareto optimal.*

Proof:
(i) Let $(x, S)\in\mathbb{Z}$ be an allocation-Pareto optimal configuration. Suppose, to the contrary, that (x, S) is not weakly opportunity-Pareto optimal. Then, there exists an individually maximizing configuration $(y, C)\in\mathbb{Z}$ such that for every $i\in N$, $C_i\ R_i^O\ S_i$ and for some $j\in N$, $C_j\ P_j^O\ S_j$. By Sen's Condition, for every $i\in N$, there exists $z_i\in C_i$ such that $z_i\ R_i^A\ w_i$ for every $w_i\in S_i$. Because (y, C) is an individually maximizing configuration and $z_i\in C_i$, we have $y_i\ R_i^A\ z_i$. By the transitivity of R_i^A, $y_i\ R_i^A\ w_i$ for every $w_i\in S_i$. In particular, we have $y_i\ R_i^A\ x_i$. Similarly, we can prove that for agent j, $y_j\ P_j^A\ x_j$. Thus, (y, C) allocation-Pareto-dominates (x, S), which contradicts the allocation-Pareto optimality of (x, S). Therefore, (x, S) is weakly opportunity-Pareto optimal.

Next suppose, to the contrary, that (x, S) is not weakly overall-Pareto optimal. Then, there exists an individually maximizing configuration $(y, C)\in\mathbb{Z}$ such that for every $i\in N$, $(y_i, C_i)\ \overline{R}_i\ (x_i, S_i)$, and for some $j\in N$, $(y_j, C_j)\ \overline{P}_j(x_j, S_j)$. By Condition A, for every $i\in N$, $y_i\ R_i^A\ x_i$ or $C_i\ R_i^O\ S_i$, and for agent j, $y_j\ P_j^A\ x_j$ or $C_j\ P_j^O\ S_j$. Because (y, C) is an individually maximizing configuration, it follows that for every $i\in N$, $y_i\ R_i^A\ w_i$ for every $w_i\in C_i$. From this and by Sen's Condition, for every $i\in N$, if $C_i\ R_i^O\ S_i$, then $y_i\ R_i^A\ v_i$ for every $v_i\in S_i$, and in particular, $y_i\ R_i^A\ x_i$. Similarly, we can prove that for agent j, if $C_j\ P_j^O\ S_j$, then $y_j\ P_j^A\ x_j$. Thus, we conclude that $y_i\ R_i^A\ x_i$ for every $i\in N$ and $y_j\ P_j^A\ x_j$. This contradicts the allocation-Pareto optimality of (x, S). Therefore, (x, S) is weakly overall-Pareto optimal.
(ii) Assume that for every $i\in N$, R_i^O is complete, and that $(x, S)\in\mathbb{Z}$ is an individually maximizing configuration.

(a): Assume that (x, S) is weakly opportunity-Pareto optimal. Suppose, to the contrary, that (x, S) is not allocation-Pareto optimal. By Lemma 1, (x, S) is not weakly allocation-Pareto optimal. Then, there exists an individually maximizing $(y, C) \in \mathbb{Z}$ such that for every $i \in N$, $y_i\ R_i^A\ x_i$, and for some $j \in N$, $y_j\ P_j^A\ x_j$. Because (x, S) is an individually maximizing configuration, it follows that for every $i \in N$, and every $w_i \in S_i$, $x_i\ R_i^A\ w_i$. By transitivity of R_i^A, $y_i\ R_i^A\ w_i$ for every $w_i \in S_i$. By Sen's Condition, there cannot exist $i \in N$ such that $S_i\ P_i^O\ C_i$. Because R_i^O is complete, we have $C_i\ R_i^O\ S_i$ for every $i \in N$. By a similar argument, we can demonstrate that $C_j\ P_j^O\ S_j$. Therefore, (y, C) opportunity-Pareto dominates (x, S), which contradicts the assumption that (x, S) is weakly opportunity-Pareto optimal. Thus, (x, S) is allocation-Pareto optimal.
(b): Assume that $(x, S) \in \mathbb{Z}$ is weakly overall-Pareto optimal. Suppose, to the contrary, that (x, S) is not allocation-Pareto optimal. By Lemma 1, (x, S) is not weakly allocation-Pareto optimal. Then, there exists an individually maximizing configuration $(y, C) \in \mathbb{Z}$ such that for every $i \in N$, $y_i\ R_i^A\ x_i$, and for some $j \in N$, $y_j\ P_j^A\ x_j$. By the same argument as in part (a) above, we can show that $C_i\ R_i^O\ S_i$ for every $i \in N$, and $C_j\ P_j^O\ S_j$. By Condition B, it follows that $(y_i, C_i)\ \overline{R}_i\ (x_i, S_i)$ for every $i \in N$, and $(y_j, C_j)\ \overline{P}_j\ (x_j, S_j)$. Therefore, (y, C) weakly overall-Pareto dominates (x, S), which is a contradiction. Thus, (x, S) is allocation-Pareto optimal. ■

Proof of Theorem 2. Claims (i) and (ii) follows from Lemma 2(i) and Theorem 1. ■

Proof of Theorem 3. Let $(x, S) \in \mathbb{Z}$ be an individually maximizing configuration.
(a): Suppose that (x, S) is allocation-Pareto optimal. From the second fundamental theorem of welfare economics, there exists a Walrasian configuration (x, B). By the feasibility of (x, B), for every $i \in N$, $x_i \in B_i$. Because (x, S) is an individually maximizing configuration, $x_i\ R_i^A\ y_i$ for all $y_i \in S_i$. Therefore, by Sen's Condition, $B_i\ R_i^O\ S_i$. On the other hand, $x_i \in S_i$, and because (x, B) is a Walrasian configuration, $x_i\ R_i^A\ y_i$ for all $y_i \in B_i$. Again by Sen's Condition, $S_i\ R_i^O\ B_i$. Thus, we have $B_i\ I_i^O\ S_i$ for every $i \in N$. By the reflexivity of R_i^A, $x_i\ I_i^A\ x_i$ for every $i \in N$. It follows from Condition B that $(x_i, S_i)\ \overline{I}_i\ (x_i, B_i)$ for every $i \in N$.
(b), (c): Suppose that (x, S) is weakly opportunity-Pareto optimal or weakly overall-Pareto optimal. Then, by Lemma 2(ii), (x, S) is allocation-Pareto optimal. Therefore, the claim follows from the argument in (a). ■

Proof of Theorem 4. Let $(x, S) \in \mathbb{Z}$ be an individually maximizing and allocation-Pareto optimal configuration. From the second fundamental theorem of welfare economics, there exists a Walrasian configuration (x, B).

Suppose, to the contrary, that for some $i \in N$, $S_i\ P_i^O\ B_i$ holds true. By Sen's Condition, there exists $y_i \in S_i$ such that $y_i\ P_i^A\ w_i$ for every $w_i \in B_i$. In particular, $y_i\ P_i^A\ x_i$ (i). Because (x, S) is an individually maximizing configuration, $x_i\ R_i^A\ w_i$ for every $w_i \in S_i$, and hence $x_i\ R_i^A\ y_i$ (ii). (i) and (ii) are incompatible. Thus, $S_i\ P_i^O\ B_i$ cannot be true.

Next, suppose that for some $i \in N$,$(x_i, S_i)\ \overline{P}_i\ (x_i, B_i)$. From Condition A and $x_i\ I_i^A\ x_i$, we must have $S_i\ P_i^O\ B_i$. By the same argument as above, this leads to a contradiction. Therefore, $(x_i, B_i)\ \overline{P}_i\ (x_i, S_i)$ cannot be true. ■

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