Threshold Behavior and Aggregate Fluctuation

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Abstract

This paper concerns a propagation mechanism in an economy where many individuals follow a threshold rule and interact with a positive feedback. We derive an asymptotic distribution of the propagation size when the number of the agents tends to infinity. The propagation distribution exhibits a slower convergence to a deterministic value than it would if the agents followed a smooth adjustment policy. This gives rise to significant aggregate fluctuations in a finite lumpy-adjusting economy even when the agents are hit by small independent shocks.

Key words: Aggregation, propagation, cascade, law of large numbers, heavy-tailed distribution, (S, s) policy, lumpy investment, menu cost, strategic complementarity, best response dynamics
JEL classification: E1, E3

1 Introduction

Since the debate between Prescott and Summers [1,2] on aggregate technological shocks, it has been of continuous interest whether idiosyncratic shocks can cause aggregate fluctuations. Whereas idiosyncratic sectoral shocks have robust tendency to cancel out each other in the standard dynamic general

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equilibrium model (Dupor [3]), several models have been proposed to exhibit to the contrary (Jovanovic [4], Bak *et al.* [5]). This paper extends the latter literature by demonstrating that the aggregate fluctuations can occur when individual behavior follows a threshold rule, often called an (S, s) policy.

Threshold rules are widely observed in individual economic behavior. As their microeconomic foundation has been well established, researchers' interest has shifted to their aggregate consequences. Many macroeconomic studies attempted to establish the aggregate relevance of the individual threshold behavior. For example, the menu-cost pricing model (Mankiw [6]) claimed the aggregate effect of individual inertia in price settings. The model of lumpy investment (Cooper and Haltiwanger [7]) claimed that the micro-level lumpy adjustment generated aggregate fluctuations in production.

Yet few theoretical works have shown the relevance of the threshold behavior in aggregation when the shocks are independent across individual units, an environment in which the law of large numbers takes effect. Caplin and Spulber [8] and Caballero and Engel [9], for example, found that the mean aggregate behavior did not differ between the two economies, one with lumpy adjustments and the other with smooth adjustments. This neutrality result is due to the fact that adjustments in the extensive margin across agents work exactly like adjustments in the intensive margins within agents, when agents' positions in (S, s) bands are distributed uniformly.

Our model differs from these studies in that we have a finite number of agents, whereas the previous models use a continuum of agents. It turns out that the deviation from the continuum model can be quantitatively significant in a finite economy. Heuristically speaking, exogenous independent shocks across agents cause some agents to adjust. Their adjustments induce further adjustments of other agents through a feedback effect, which in turn induce even further adjustments, and so on. This chain reaction constitutes a propagation mechanism. If an economy is inhabited by a continuum of agents, then any positive shocks across agents cause a deterministic fraction of agents to adjust, due to the work of the law of large numbers at the limit. If the economy consists of countably many agents and the size of the shock is conditioned on the number of agents, then the outcome depends on how we condition on them. Suppose that the shock size is of order $1/N$. Then the *number* of agents that adjust due to the initial shocks asymptotically follows a non-degenerate Poisson distribution. The same mechanism applies to the subsequent propagation process when the feedback effect is of order $1/N$. The feedback effect is of order $1/N$ when the feedback is caused through an average behavior of all agents. In this case, the magnitude of the shocks is equal to the magnitude of the feedback effect, and the feedback effect plays a decisive role in determining the aggregate fluctuations. The *fraction* of agents that adjust converges to zero in probability, but we find that the convergence is slower than that in the

smoothly-adjusting economy. Thus, the finite agent model enables us to study the economic situations where individual shocks add up to a sizable aggregate shock, which the continuum of agents models could not account for.

This paper develops a mathematical method to evaluate the asymptotic distribution of aggregate variables in an (S, s) economy when the number of agents N tends to infinity. Our approach clarifies when and why the lumpy adjustment matters. We reproduce the neutrality theorem that the mean aggregate behaviors of lumpy and smooth economies coincide, but we find that the variance of the propagation size in the lumpy economy is much larger than that in the smooth economy. This raises the possibility that idiosyncratic shocks cause significantly large aggregate fluctuations for an economy with finite but many agents. Furthermore, the propagation size follows a heavy-tailed distribution when the idiosyncratic shocks are small relative to the size of the lumpiness, whereas it follows a normal distribution when the shock dominates the lumpiness. In the latter case, the aggregate behavior is the same as its smoothly-adjusting counterpart. The heavy-tailed distribution emerges from the propagation effect, which characterizes the aggregate fluctuations as long as the exogenous shocks do not overwhelm the feedback effect of individual lumpy adjustments.

It turns out that we can regard the smooth and lumpy economies as polar cases in terms of the ratio of lumpiness to shock size. When this ratio goes to zero, the propagation follows a normal distribution and the aggregate variance in the lumpy economy is as small as the smooth economy, whereas when the ratio diverges to infinity, it follows a heavy-tailed distribution and the variance in the lumpy economy is much bigger than the smooth economy. By numerically calculating the aggregate variances in the region between the polar cases, we determine that the transition between the two phases occurs at the point where the lumpiness is equal to the standard deviation of the shock. We also find that the heavy-tailed distribution is infinitely divisible. This property proves useful when we consider a dynamic extension of the model. Suppose that the exogenous shocks hit repeatedly over time. Then, the variance of the accumulated shocks increases linearly with the time horizon. Thus, the corresponding aggregate fluctuation follows the heavy-tailed distribution when the time horizon is short, whereas it follows the normal distribution when the time horizon is long. Using the infinite divisibility, we can derive an approximate stochastic process of the aggregate for a short time horizon, which progressively turns into a normal process for a long time horizon.

The paper is organized as follows. Section 2 presents a simple model of the (S, s) economy. Section 3 shows the distribution of the aggregate. Section 4 concludes. Proofs are deferred to the Appendix as well as to the working paper version of this paper [10].

2 General Framework

An individual's behavior is usually specified in such a way that the agent responds smoothly to a change in the environment. This behavioral specification may be expressed as follows:

$$x_i = Q_N(x) + e_i, \qquad i = 1, 2, \ldots, N. \tag{1}$$

Agent i's action is x_i and e_i is an agent specific factor. Define x and e as vectors with i-th coordinates x_i and e_i for $i = 1, 2, \ldots, N$ respectively. $Q_N(x)$ is an aggregator function of the action profile. We assume that $Q_N(x)$ is increasing in all the coordinates x_i, and $\partial Q_N(x)/\partial x_i = O(1/N)$. Thus an agent's action generates a positive feedback effect of order $1/N$ on other agents' actions through Q_N. This behavioral function is commonly derived from a first-order condition of the agent's utility maximization. In a Cournot competition, for example, x_i is the best reply in producer i's production level given the other producer's production levels. In many economic decisions, however, the adjustment in individual behavior exhibits inertia and occasional lumpy correction:

$$\begin{cases} x_i = Q_N(x) + e_i - \lambda_i s_i \\ s_i \equiv ((Q_N(x) + e_i)(\text{mod } \lambda_i))/\lambda_i \end{cases}, \qquad i = 1, 2, \ldots, N \tag{2}$$

where $x(\text{mod } \lambda)$ denotes the remainder of the division of x by λ. The domain of x_i is thus $\{0, \pm\lambda_i, \pm 2\lambda_i, \ldots\}$. By λ_i we denote the size of a lumpy adjustment in x_i. The variable s_i is normalized by the agent-specific bandwidth λ_i so that it always takes a value in $[0, 1)$. It represents the agent's position in the (S, s) band. The linear specification in e_i is standard as in [9]. The specification is versatile enough to allow, among others, the analysis of a standard sectoral business cycle model [10].

We construct a simple game to exemplify an economy that generates behavioral rules of the type (2). Consider a sequence of strategic games $G_N = \langle N, (X_i), (U_i^N) \rangle$, $N = N_0, N_0 + 1, \ldots$ for a large integer N_0. A game G_N is played by N players. The action set for player i is $X_i = \{0, \pm\lambda_i, \pm 2\lambda_i, \ldots\}$. Player i's payoff is given by a function $U_i^N(x) = -(Q_N(x) + e_i - \lambda_i/2 - x_i)^2$ which is a quadratic loss function utilized in Caplin and Leahy [11]. The payoff attains its maximum zero when $x_i = Q_N(x) + e_i - \lambda_i/2$. The optimal $x_i \in X_i$ lies between $\chi - \lambda_i$ and χ, where χ satisfies $U_i^N(\chi - \lambda_i, x_{-i}) = U_i^N(\chi, x_{-i})$, by the concavity of U_i^N and by $\partial Q_N(x)/\partial x_i = O(1/N)$. Hence $x_i = \chi - \chi(\text{mod } \lambda_i)$ is the global maximizer for large N. Solving χ, we obtain (2) as the best response of i. Thus a solution x of the system (2) is a Nash equilibrium of the game G_N. When $\lambda_i = 0$, we reset the action set as an entire real line: $X_i = R$.

Then the first-order condition of i's payoff maximization produces (1) as a behavioral rule for a smoothly adjusting case.

The existence of a solution x for the system of behavioral rules (2) is readily available. Let an underline and an overline denote a lower and an upper bound of the variable, respectively.

Lemma 1 (Existence of equilibrium) *Suppose that $Q_N(x)$ is increasing in $x \in R^N$ and that e_i and λ_i are bounded. Suppose that there exist scalars $\underline{x}$ and $\overline{x}$ that satisfy $\underline{x} = Q_N(\underline{x}, \underline{x}, \ldots, \underline{x}) + \underline{e} - \overline{\lambda}$ and $\overline{x} = Q_N(\overline{x}, \overline{x}, \ldots, \overline{x}) + \overline{e}$. Then the system (2) has a solution for any e and λ.*

Proof: If $Q_N(x)$ is increasing in x, then $Q_N(x) + e_i - (Q_N(x) + e_i)(\mathrm{mod}\ \lambda_i)$ is increasing in x as well. Let us construct a vector function by stacking N functions in (2). Since $0 \leq (Q_N(x)+e_i)(\mathrm{mod}\ \lambda_i) < \lambda_i$, the constructed $\underline{x}$ and $\overline{x}$ defines the lower and upper bound of x_i. The vector function thus has a fixed point by Tarski's theorem (Vives [12]). Any $x_i \in R$ that satisfies (2) belongs to X_i by construction. □

Lemma 1 implies that the equilibrium exists when Q_N is increasing and when the smooth counterpart (1) has an equilibrium for an extended space $[\underline{e} - \overline{\lambda}, \overline{e}]$ of the exogenous variable e_i. Since $Q_N(x)$ is increasing, players are situated in strategic complementarity. Thus, G_N is a particular case of supermodular games.

Suppose that x^1 and x^0 correspond to the equilibria that solve the system (2) given e^1 and e^0, respectively. We generate e^1 by perturbation as $e_i^1 = e_i^0 + \epsilon_i/N$ where ϵ_i is positive and i.i.d. across i. The best reply is a one-sided (S, s) policy due to the positivity of the perturbation. The perturbation shock is also normalized by N so that its impact matches the magnitude of the feedback effect $\partial Q_N(x)/\partial x_i$. This choice of normalization allows us a simple analysis of a polar case in which the lumpiness overwhelms the size of shocks. [2] Define a propagation size caused by the perturbation as $Q_N(x^1) - Q_N(x^0)$. This is the increment in the aggregate index Q_N. It is also the common factor of increments in individual actions x_i. Our goal is to asymptotically characterize the distribution of the propagation size for large N.

The system (2) allows multiple equilibria. To complete the definition of the perturbation, we need an equilibrium selection algorithm. We employ best response dynamics as such algorithm. Define u as the step of the dynamics $(x_{i,u}, s_{i,u})$. Set the initial value of the best response dynamics equal to the

[2] The case when the shock is mean-zero, or positive with the mean which is independent of N, can be analyzed similarly. The aggregate fluctuation becomes symmetric, but is still characterized by a heavy tail which is analogous to the result in this paper (see [13]).

initial equilibrium value: $x_{i,0} = x_i^0$ and $s_{i,0} = s_i^0$. For subsequent steps, we define the individual responses as follows:

$$x_{i,1} = \begin{cases} x_{i,0} + \lambda_i \text{ if } s_{i,0} + \epsilon_i/(N\lambda_i) \geq 1 \\ x_{i,0} \qquad \text{otherwise} \end{cases} \tag{3}$$

$$x_{i,u+1} = \begin{cases} x_{i,u} + \lambda_i \text{ if } s_{i,u} + (Q_N(x_u) - Q_N(x_{u-1}))/\lambda_i \geq 1 \\ x_{i,u} \qquad \text{otherwise} \end{cases}, \quad \text{for } u \geq 1 \tag{4}$$

$$s_{i,1} = s_{i,0} + (\epsilon_i/N - x_{i,1} + x_{i,0})/\lambda_i \tag{5}$$

$$s_{i,u+1} = s_{i,u} + (Q_N(x_u) - Q_N(x_{u-1}) - x_{i,u+1} + x_{i,u})/\lambda_i, \quad \text{for } u \geq 1. \tag{6}$$

Define T as a stopping time of the process $Q_N(x_u) - Q_N(x_{u-1})$. Namely, $T \equiv \min_{\{u|Q_N(x_u)-Q_N(x_{u-1})=0\}} u$. Then x_T satisfies the system (2) for e^1. Thus the best response dynamics constitutes an equilibrium selection algorithm that defines an equilibrium $x^1 = x_T$. The stopping time T is finite with probability one when $N \to \infty$, as we will prove when we show the propagation distribution. This equilibrium selection algorithm has been used by Vives [12] and Cooper [14]. The algorithm has a straightforward economic intuition. We start from an initial equilibrium that solves the system of best response functions, and add a disturbance to the system. Then we update the individual choice by applying the best response function iteratively until a new solution is reached. This particular equilibrium selection has immediate implications. The procedure imposes inertia on the individual actions. Agents do not adjust their actions unless they are strictly better off by adjusting. Also, it rules out an equilibrium far from an initial equilibrium that would require some kind of informational coordination. The preclusion of big jumps in equilibrium based on informational coordination suits our aim to focus on strategic complementarity alone as a propagation mechanism.

3 Main Results

In this section we derive the distribution of propagation size under a simplifying assumption (Assumption 3). Let us first define a perturbation experiment.

Assumption 2 (Perturbation) *We have an equilibrium x^0. In a perturbation $e_i^1 = e_i^0 + \epsilon_i/N$, ϵ_i is positive and bounded. An agent's position s_i^0 is a random variable with support $[0, 1)$ and i.i.d. across i. The cumulative distribution function of s_i^0 satisfies $\lim_{h\to 0}(F_s(1) - F_s(1-h))/h = \psi < \infty$.*

We let s_i^0 follow any distribution that has a density in the vicinity of the border $s_i^0 = 1$. Our initial condition of the perturbation is an equilibrium

x^0 and random variables e_i^0, such that $s_i^0 = (e_i^0 - x_i + Q_N(x))/\lambda_i$ obeys the distribution function F_s where λ_i, $i = 1, 2, \ldots, N$, is a prefixed sequence. [3]

As a simplifying assumption, we assume that the aggregator $Q_N(x)$ is symmetric and linear in x_i, and the lumpiness of adjustments is the same for all agents.

Assumption 3 (Linearity and homogeneity) *$Q_N(x) = \phi \sum_{i=1}^N x_i/N$ where $\phi\psi \leq 1$. Also $\lambda_i = \lambda$ for all i.*

The existence of solution x^0 is confirmed by Lemma 1 under Assumption 3 when $\phi < 1$, for $[(\underline{e}-\lambda)/(1-\phi), \overline{e}/(1-\phi)]^N$ is the compact domain of x that is mapped into itself. With this assumption, the response system with continuous adjustment (1) is analogous to that of Jovanovic [4]. His paper shows that the idiosyncratic shock is amplified to a fluctuation of average of x_i if ϕ approaches to 1 as $1 - 1/\sqrt{N}$. Revisiting his paper, Gabaix [4] argues that it essentially assumes an implausibly high magnitude of multiplication effect ($1/(1-\phi) = \sqrt{N}$). The present model extends Jovanovic's by incorporating a nonlinear effect due to (S, s) policy. The (S, s) policy transforms the adjustment at the intensive margin to that at the extensive margin. This gives rise to a new source of aggregate fluctuations even when ϕ is strictly below 1, and a heavy-tail distribution of the aggregate fluctuations when ϕ approaches to 1 as we see below.

Define $\mu = \mathrm{E}[\epsilon_i]\psi/\lambda$. We then obtain the distribution of the propagation size.

Proposition 4 (Propagation distribution) *Under Assumptions 2 and 3, the normalized propagation size $N(Q_N(x^1) - Q_N(x^0))$ converges in distribution to $w\phi\lambda$ when $N \to \infty$, where w is a random variable that follows a probability distribution:*

$$\Pr(w) = (\phi\psi w + \mu)^{w-1}\mu e^{-\phi\psi w - \mu}/w! \tag{7}$$

for $w = 0, 1, 2, \ldots$. The moment generating function of w is $e^{\mu(G(s)-1)}$ where $G(s)$ is a moment generating function which satisfies a functional equation $G(s) = e^{s+\phi\psi(G(s)-1)}$. The distribution of w is infinitely divisible. [5] *Its tail is*

[3] The assumption that s_i is identically distributed holds for any distribution of e_i when the equilibrium is symmetric: $x_i = x$ and $\lambda_i = \lambda$ for all i. The assumption imposes a restriction on e_i when the equilibrium is asymmetric. A particular distribution of e_i that satisfies the assumption is a uniform distribution over $[0, \lambda_i z_i)$ where z_i is an arbitrary integer. Lemmas 5 and 6 offer an alternative justification that the assumption holds at the stationary state when e_i follows a random walk.

[4] "Power laws and the origins of the aggregate fluctuations," 2004, mimeo.

[5] A distribution F is called *infinitely divisible* when for any integer n there exists a distribution F_n such that F is the n-fold convolution of F_n.

approximated by:

$$\Pr(w) \approx (\mu e^{-\mu}/(\phi\psi\sqrt{2\pi}))(\phi\psi e^{1-\phi\psi})^w w^{-1.5} \tag{8}$$

for large integer w.

Proof: See Appendix A.1.

The propagation size $Q_N(x^1)-Q_N(x^0)$ has an asymptotic mean $(\mathrm{E}[\epsilon_i]/N)\phi\psi/(1-\phi\psi)$ and variance $(\mathrm{E}[\epsilon_i]/N^2)\psi\lambda\phi^2/(1-\phi\psi)^3$. The distribution starts out as a power-law distribution $w^{-1.5}$ at $w = 0$ and exhibits an exponential truncation $(\phi\psi e^{1-\phi\psi})^w$ at the tail. The distribution becomes a pure power-law when $\phi\psi \to 1$ and its variance tends to infinity.

Proposition 4 marks a clear departure from an economy without inertia. Let us set up a model of a smoothly adjusting counterpart of our economy. Suppose that an agent's optimal decision follows a smooth adjustment rule (1). Then, under Assumption 3, we directly obtain that $Q_N(x^1) - Q_N(x^0) = (\phi/(1-\phi))\sum_i \epsilon_i/N^2$ when $\phi < 1$. By the central limit theorem, a normalized propagation size $N^{1.5}(Q_N(x^1) - Q_N(x^0) - (\mathrm{E}[\epsilon_i]/N)\phi/(1-\phi))$ asymptotically follows a normal distribution with mean zero and variance $\mathrm{Var}[\epsilon_i]\phi^2/(1-\phi)^2$. The normalization factor $N^{1.5}$ consists of two factors: the contribution from the normalized shock (N) and the central limit theorem ($N^{0.5}$). The propagation size in Proposition 4 has the normalization factor N, that is equal to the normalization factor for the smooth case less the contribution of the central limit theorem. An immediate implication of the comparison between the two economies is that the aggregate in the lumpy economy is much more volatile than its smooth counterpart. The asymptotic variance of the propagation size $Q_N(x^1)-Q_N(x^0)$ is scaled to N^{-3} in the smooth economy, whereas it is scaled to N^{-2} in the lumpy economy. This leads to the possibility that the threshold model may quantitatively explain the large magnitude of aggregate fluctuations when a smooth model cannot.

Let us note that the variance of perturbation shocks does not appear in our formula of the aggregate variance, whereas it does in its smoothly adjusting counterpart. In fact, the aggregate variance in the lumpy economy remains unchanged when the shock ϵ_i is replaced with its mean. The randomness of the shock is irrelevant in determining the fluctuation magnitude. To the contrary, the distribution of s_i^0 directly affects the variance through ψ. This implies that the configuration of agents' positions in the inaction intervals is the crucial variable in determining the variability of propagation sizes. The nonlinearity of (S, s) policy alone can generate a complex aggregate fluctuation, provided that the deterministic evolution of the configuration of s_i is ergodic.

The propagation distribution also differs in its shape between the lumpy and smooth economies. It is skewed and heavy-tailed in the lumpy economy, whereas it follows a normal distribution in the smooth economy. The heavy-tail accounts for an extra $1-\phi\psi$ in the denominator in the variance. The tail is stretched because of the propagation effect. To see this, let us consider an economy with lumpiness but without spillover, say, $Q_N(x)=0$. Then the number of agents who adjust follows a Poisson distribution asymptotically, and the variance of the average of x_i is $(\mathrm{E}[\epsilon_i]/N^2)\psi\lambda$. This corresponds to the initial adjustments caused directly by the shocks in our economy. The aggregate variance in our economy with spillover is larger than this by the factor of $\phi^2/(1-\phi\psi)^3$, which corresponds to the effect of propagation that follows the initial adjustments.

It is useful to compare our finite agent model with continuum of agents models. Let us set up the continuum of agents model as follows. Suppose that $Q(x)=\phi\int x_i di$. Consider a perturbation ϵ_i/N on e_i for some fixed N. Under Assumption 3, the best response is $x_i^1=x_i^0+\lambda$ if $s_i+(\epsilon_i/N+\phi\int x_i^1-x_i^0 di)/\lambda\geq 1$ and $x_i^1=x_i^0$ otherwise, if N is large enough so that no agent adjusts more than λ. Suppose that s_i^0 is uniformly distributed. Then the fraction of agents who adjust is a deterministic value $(\mathrm{E}[\epsilon_i]/N+\phi\int x_i^1-x_i^0 di)/\lambda$. Thus, we obtain that $Q(x^1)-Q(x^0)=(\mathrm{E}[\epsilon_i]/N)\phi/(1-\phi)$. This is a deterministic value that is equal to the mean of the propagation size in our finite model. The continuum of agents model does capture the mean impact of the propagation, but fails to capture its stochastic nature. The propagation size in the continuum of agents model is also equal to the mean propagation size in the smooth economy, thus we have reproduced the neutrality theorem.

Both the continuum of agents model and the smooth economy model eliminate the randomness of the propagation effect. In the continuum of agent model, the feedback effect of agents' actions on the aggregate is always deterministic by construction. In the smooth economy, the propagation effect cancels out across agents quickly due to the law of large numbers. Even in the lumpy economy, the lumpiness λ should be washed away when the shock ϵ_i/N overwhelms the lumpiness, and the aggregate fluctuation should resemble that of the smooth economy. A simulation verifies this intuition. Figure 1 plots the standard deviation of the propagation size for various sizes of λ and $\sigma_e\equiv\mathrm{Std}[\epsilon_i/N]$.[6] It is seen that the standard deviation of the propagation

[6] The simulation is executed as follows. Draw an initial value of e_i^0 for $i=1,2,\ldots,N$ from a distribution uniform over $[0,\lambda)$. Then $x_i^0=0$ for all i is an equilibrium. Draw a perturbation ϵ_i from a normal distribution with mean zero and various standard deviations σ_e. The new equilibrium x^1 is calculated by the best response dynamics (3,4). Then we obtain a propagation size $\phi\sum_{i=1}^N(x_i^1-x_i^0)/N$. We set $N=500$ and $\phi=0.5$. We compute the propagation size for 10^4 times and the standard deviation of the propagation size for each value of λ.

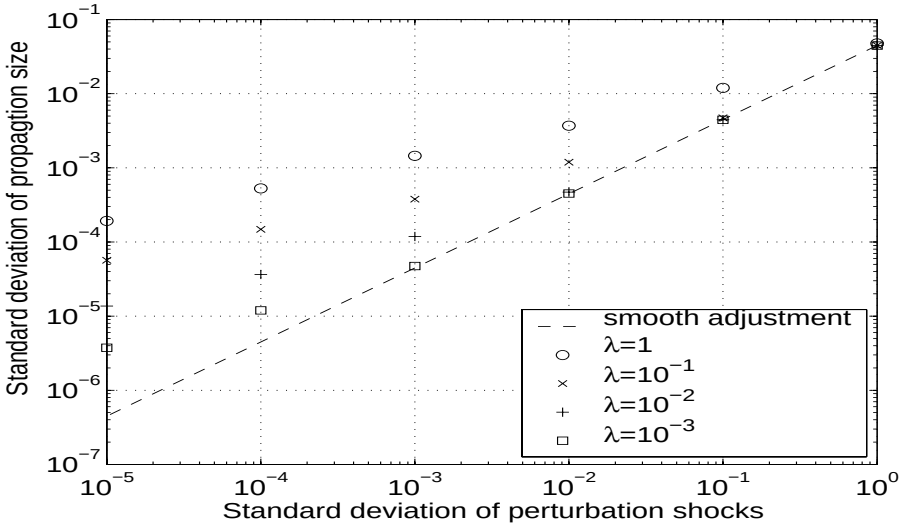

Fig. 1. Simulation for cross-over points of lumpy and smooth economies

increases proportionally with σ_e when $\sigma_e > \lambda$ so that it behaves as in the smooth economy, whereas it significantly deviates upward when $\sigma_e < \lambda$.

Using above arguments, we can deduce a time-series property of the aggregate $Q(x)$ when e_i evolves over time with variance increasing in time horizon. Intuitively speaking, the economy behaves as a lumpy economy for the time horizon in which the innovation of e_i is dominated by the lumpiness λ, whereas it behaves as a smooth economy for the time horizon in which the innovation washes out λ. The infinite divisibility of distribution (7) plays an important role technically. In general, a random variable having an infinitely divisible distribution is equivalent to the random variable being an increment of a stochastic process with independent increments [15, page 177]. We can see this point for our case. Suppose that e_i^t evolves as a stochastic process with independent and positive increments. We define an equilibrium path as a sequence of static equilibria. Then we can define the aggregate growth, $Q_N(x^t) - Q_N(x^0)$, for every time horizon t. The variance of a cumulative innovation $e_i^t - e_i^0$ grows linearly with the time horizon t. For a fixed large N, suppose that ϵ_i/N is equivalent to $e_i^1 - e_i^0$ which is an increment of e_i for a unit time horizon. Then, the normalized growth $N(Q_N(x^1) - Q_N(x^0))$ follows the moment generating function $e^{\mu(G(s)-1)}$, which is a compound Poisson distribution with Poisson mean μ and a random variable that follows $G(s)$. Now consider a growth in a shorter time horizon $N(Q_N(x^{1/n}) - Q_N(x^0))$ for any integer n. Then this growth follows a moment generating function $e^{(\mu/n)(G(s)-1)}$, which is a compound Poisson with the same random variable following $G(s)$ and with a Poisson mean μ/n which is linearly scaled by the time horizon $1/n$. Therefore, the sequence of the static equilibria of an economy with a fixed N is approximated by a compound Poisson process with hazard rate μ and a random variable that follows $G(s)$ for a time horizon shorter than the unit time. By a similar argument, a smoothly adjusting counterpart is approximated by a Brownian motion.

The compound Poisson process exactly obtains only for a vanishingly short time scale. This fact corresponds to that Proposition 4 holds only when the shock is small relative to the lumpiness. Our simulation in Figure 1 shows, however, that the deviation from the smooth economy is observable for the range $\sigma_e < \lambda$. This implies in the time-series context that the aggregate process is in transition from a compound Poisson to a Brownian up to the time scale for which the size of the accumulated shock matches the lumpiness. The cross-over time scale is characterized by λ/σ_e where σ_e is the size of shocks for a unit time.

Proposition 4 holds under various distributions of s_i^0. It is necessary, however, to know the stationary distribution of s_i, when we consider a sequence of static equilibria. We can indeed show that the uniform distribution is a stationary distribution of s_i with respect to the perturbation. The following Lemma is a reexpression of the result of Caplin and Spulber [8].

Lemma 5 (Stationary distribution of s_i) *Consider a perturbation $e_i^1 = e_i^0 + \epsilon_i$ where ϵ_i is bounded. Suppose that s_i^0 is independently and uniformly distributed over $[0, 1)$. Assume that $Q_N(x^1) - Q_N(x^0)$ is asymptotically independent of s_i^0 when $N \to \infty$. Then, s_i^1 is independently and uniformly distributed asymptotically when $N \to \infty$.*

Proof is provided in [10]. Lemma 5 shows that s_i stays uniformly distributed even after a sizable shock ϵ_i hits. The assumption of the asymptotic independence between $Q_N(x^1) - Q_N(x^0)$ and s_i^0 is natural, for we consider the situation in which the effect of a single agent's action is of order $1/N$ in aggregation.

The uniform distribution is also a convergent point of s_i if e_i evolves as a random walk. The following Lemma is analogous to Caballero and Engel [9].

Lemma 6 (Convergence of s_i) *Consider that e_i evolves as a random walk $e_{i,t+1} = e_{i,t} + \epsilon_{i,t}/N$ for $t = 0, 1, \ldots$ where $\epsilon_{i,t}$ is i.i.d. across both i and t. Pick $p > 2$ arbitrarily. Assume that $Q_N(x_{t=N^p}) - Q_N(x_0)$ and $\sum_{t=0}^{N^p-1} \epsilon_t/(N\lambda_i)$ are asymptotically independent. Then $s_{i,t=N^p}$ converges in distribution to a distribution uniform over a unit interval and independent across i when $N \to \infty$.*

Proof is provided in [10]. Above Lemmas imply that the uniform distribution of s_i serves as a steady state of the economy with respect to its aggregate variability when the environment e_i is diverging. Thus $\psi = 1$ is the stationary value of ψ. The condition $p > 2$ in Lemma 6 implies that the convergence takes no less than N^2 periods for which the accumulated shock achieves an observable size of standard deviation $\mathrm{Std}[\epsilon_i]/\lambda_i$.

The Lemmas are proved for nonlinear $Q_N(x)$ and heterogeneous λ_i. In fact, Proposition 4 can also be shown in the general environment. We allow λ_i to be

heterogeneous with a finite number of types K that take values $\lambda(1), \lambda(2), \ldots, \lambda(K)$. We call an agent with $\lambda(k)$ "type-k." Let σ_k denote the limit fraction of type-k agents among all the agents when $N \to \infty$. Consider two sequences of real numbers a_i and λ_i, $i = 1, 2, \ldots$. We assume that a_i and λ_i are mutually independent when i is drawn randomly. Define a function $b(\lambda_i)$ and let $b(k)$ denote $b(\lambda(k))$. We write the limit of the averages of $1/\lambda_i$ and $b(\lambda_i)/\lambda_i$ for $i = 1, 2, \ldots, N$ when $N \to \infty$ by using an expectation operator as $\mathrm{E}[1/\lambda] \equiv \sum_{k=1}^{K} \sigma_k/\lambda(k)$ and $\mathrm{E}[b/\lambda] \equiv \sum_{k=1}^{K} \sigma_k b(\lambda(k))/\lambda(k)$, respectively. The relaxed assumption is following.

Assumption 7 (Generalization) *For any finite set H, a sequence of bounded functions Q_N satisfies $N(Q_N(\{x_i + \lambda_i\}_{i \in H}, x_{-H}) - Q_N(x)) \to \sum_{i \in H} a_i(x) b(\lambda_i)$ as $N \to \infty$. A sequence λ_i, $i = 1, 2, \ldots$, takes a finite number of values. For each x, a sequence $a_i(x)$, $i = 1, 2, \ldots$, is bounded and satisfies $\sum_{i=1}^{N} a_i(x)/N \to \phi(x)$ where $\phi(x)\psi\mathrm{E}[b/\lambda] \leq 1$. The pair $(a_i(x), \lambda_i)$ is mutually independent for each x when i is randomly drawn.*

Assumption 7 allows a heterogeneous effect, $a_i(x)b(\lambda_i)$, of an adjustment of x_i on $Q(x)$. A simple example of such a $Q_N(x)$ is $\sum_i a_i x_i/N$, which includes the homogeneous case as a special case when $a_i = \phi$. Assumption 7 also allows the effect to depend on λ_i and x. Dependence on x is permissible since an asymptotic distribution of perturbation is determined only by the local characteristics of $Q_N(x)$ at x^0. An example of such a nonlinear aggregator $Q_N(x)$ is a CES-type function.

Define $\mu = \psi\mathrm{E}[\epsilon_i]\mathrm{E}[1/\lambda]$. Define $J_x(\cdot)$ as a moment generating function of $a_i(x)$ when i is randomly drawn. Under the relaxed assumption, we obtain the following distribution of the propagation size:

Proposition 8 (Propagation distribution in a general case) *Under Assumptions 1 and 7, the normalized propagation size $N(Q_N(x^1) - Q_N(x^0))$ asymptotically follows a moment generating function $e^{\mu(G(s)-1)}$ where $G(s)$ satisfies a functional equation:*

$$G(s) = \sum_{k=1}^{K} J_{x^0}((s + \psi\mathrm{E}[1/\lambda](G(s) - 1))b(k))\sigma_k/(\lambda(k)\mathrm{E}[1/\lambda]). \tag{9}$$

The propagation distribution is infinitely divisible.

Proof is provided in [10]. By the formula above, one can calculate any moment of the propagation size $Q_N(x^1) - Q_N(x^0)$ asymptotically. For example, its mean is $(\mathrm{E}[\epsilon_i]/N)\mathrm{E}[b/\lambda]\phi\psi/(1-\phi\psi\mathrm{E}[b/\lambda])$ and its variance is $(\mathrm{E}[\epsilon_i]/N^2)\mathrm{E}[b^2/\lambda]\mathrm{E}[a_i^2(x^0)]\psi/(1-\phi\psi\mathrm{E}[b/\lambda])^3$. The mean and variance in the homogeneous model can be reproduced by substituting $a_i = \phi$ and $b(\lambda_i) = \lambda$.

The aggregate behavior of the economy's smooth counterpart is analogous to the homogeneous case. Consider a linear case $Q_N(x) = \sum_i a_i x_i/N$. If agents adjust smoothly, then $x_i^1 - x_i^0 = Q_N(x^1) - Q_N(x^0) + \epsilon_i/N$ holds. Then we obtain: $\sum_{i=1}^N a_i(x_i^1 - x_i^0) = (\sum_{i=1}^N a_i/N)N(Q_N(x^1) - Q_N(x^0)) + \sum_{i=1}^N a_i\epsilon_i/N$. Thus, the normalized propagation size is $N(Q_N(x^1) - Q_N(x^0)) = (\sum_{i=1}^N a_i\epsilon_i/N)/(1 - \sum_{i=1}^N a_i/N)$. Hence, $N^{1.5}(Q_N(x^1) - Q_N(x^0) - \mathrm{E}[\epsilon_i]\phi/(1-\phi)N)$ follows a normal distribution with mean zero and variance $\mathrm{Var}[a_i\epsilon_i]/(1-\phi)^2$. The variance of the propagation size converges to zero as fast as N^{-3}. The homogeneous case is again a particular case of this result.

By the form of the moment generating function, the normalized propagation size follows a compound Poisson distribution with Poisson mean μ and a random variable that follows a moment generating function $G(s)$. Thus, the distribution function is infinitely divisible. Therefore, the time series implication derived for the homogeneous case applies to the heterogeneous case. Suppose that e_i^t evolves as a stochastic process with independent and positive increments. Then we can define a sequence of static equilibria. For a fixed large N, suppose that ϵ_i/N is equivalent to an increment of e_i for a unit time horizon $e_i^1 - e_i^0$. Then the sequence of static equilibria is approximated by a compound Poisson process with hazard rate μ and a random variable that follows $G(s)$ for a time horizon less than the unit time.

4 Conclusion

In this paper, we analyze a generic model of an (S, s) economy with finite agents where each agent follows a threshold adjustment policy. We derive an asymptotic distribution of propagation caused by a positive feedback effect across agents' policies. With homogeneous agents, we derive the closed-form distribution of the propagation. The distribution shows a slower convergence to a deterministic value than its counterpart in a smoothly-adjusting economy. Moreover, the distribution is skewed and heavy-tailed. The variance of the propagation is significantly larger than its smooth counterpart due to the slow convergence and the heavy tail, and hence contrasts the neutrality theorems on the (S, s) economy in which the threshold behavior does not cause significant aggregate fluctuations.

The distribution exhibits a phase transition depending on the size of the lumpiness relative to the size of an exogenous shock. The distribution is skewed and heavy-tailed, and slowly converging to a deterministic value when the lumpiness is larger than the shock, whereas it follows a normal distribution and converges as fast as the central limit theorem predicts when the lumpiness is overwhelmed by the shock. Applying this idea to the case when the shocks accumulate over time, we show that short-run fluctuations are characterized

by the skewed and heavy-tailed distribution, whereas long-run fluctuations are characterized by the normal distribution. Furthermore, by utilizing the infinite divisibility of the propagation distribution, the equilibrium path can be approximated by a compound Poisson process for a short time horizon, whereas the process progressively converges to a normal process as the time horizon becomes longer.

It is left as an open question what dynamics the (S, s) economy generates. The compound Poisson process is only an approximate when s_i is held at the stationary distribution. In a finite economy, ψ varies over time due to the ergodic evolution of s_i, which can generate rich structure in time series. In particular, the clustered adjustments tend to cause another clustering when those agents approach to the threshold as a mass again, generating so-called echo effects. It seems promising to explore the fluctuations in (S, s) economies in relation to the large-dimensional nonlinear dynamical systems.

A Appendix

A.1 Proof of Proposition 4

Consider the best response dynamics (3,4) for $u = 1, 2, \ldots, T$. Define $M_u = N(Q_N(x_u) - Q_N(x_{u-1}))$ for $u \geq 1$. Define m_u as the number of agents that increase x_i at u. Under Assumption 3, $M_u = m_u \phi \lambda$ holds. We first prove the following lemma.

Lemma 9 (Branching process in best response dynamics) *Under Assumptions 2 and 3, the process m_u, $u = 1, \ldots, T$, follows asymptotically as $N \to \infty$ a branching process where the number of initial parents m_1 follows a Poisson distribution with mean μ and the number of children each parent bears follows a Poisson distribution with mean $\phi\psi$.*

Proof: Define H_u as a set of agents i such that $x_{i,u} - x_{i,u-1} = \lambda$. Let c denote the upper bound of the support of ϵ_i. We define Condition U on a path m_v, $v = 1, \ldots, u-1$ for $u \geq 2$, as $(c/\lambda + \phi(\sum_{v=1}^{u-1} m_v))/N < 1$. U is a sufficient condition for $H_v \cap H_u = \emptyset$ for any $v < u$.

First, we examine the stochastic process m_u under U up to a finite step. The probability that an agent i belongs to H_1 is $\Pr(s_{i,0} + \epsilon_i/(N\lambda) \geq 1) = \int_0^c F_s(1) - F_s(1 - \epsilon_i/(N\lambda)) f(d\epsilon_i)$. Thus m_1 follows a binomial distribution with this probability and population N. Similarly, for $u \geq 2$, the probability that an agent $i \notin \bigcup_{v=1}^{u-1} H_v$ belongs to H_u is derived as follows. Define a short-hand $h_1 = (\epsilon_i/\lambda + \phi \sum_{v=1}^{u-1} m_v)/N$ and $h_2 = (\epsilon_i/\lambda + \phi \sum_{v=1}^{u-2} m_v)/N$.

$$\Pr(s_{i,u-1} + \phi m_{u-1}/N \geq 1 \mid \{m_v\}_{v=1}^{u-1},\, i \notin \bigcup_{v=1}^{u-1} H_v) \tag{A.1}$$

$$= \frac{\Pr(s_{i,u-1} + \phi m_{u-1}/N \geq 1,\, i \notin \bigcup_{v=1}^{u-1} H_v \mid \{m_v\}_{v=1}^{u-1})}{\Pr(i \notin \bigcup_{v=1}^{u-1} H_v \mid \{m_v\}_{v=1}^{u-1})} \tag{A.2}$$

$$= \frac{\int_0^c F_s(1-h_2) - F_s(1-h_1) f(d\epsilon_i)}{\int_0^c F_s(1-h_2) f(d\epsilon_i)} \tag{A.3}$$

The first equality holds by the multiplication rule of conditional probabilities. The second equality holds because $i \notin \bigcup_{v=1}^{u-1} H_v$ is equivalent to $s_{i,u-1} = s_{i,0} + (\epsilon_i/\lambda + \phi \sum_{v=1}^{u-2} m_v)/N < 1$. Hence, m_u given $\{m_v\}_{v=1}^{u-1}$ for $u \geq 2$ under U follows a binomial distribution with probability (A.3) and population $N - \sum_{v=1}^{u-1} m_v$.

Next, we derive an asymptotic process of m_u up to a finite step. We showed that m_u follows a stochastic process which is finite with probability one up to a finite step. Hence, by construction, U is satisfied with probability one when $N \to \infty$ up to a finite step. The asymptotic mean of m_1 is simply: $\int_0^c F_s(1) - F_s(1 - \epsilon_i/(N\lambda)) f(d\epsilon_i) N \to \mathrm{E}[\epsilon_i]\psi/\lambda = \mu$ as $N \to \infty$. Hence m_1 asymptotically follows a Poisson distribution with mean μ. The asymptotic mean of m_u given m_{u-1} for $u \geq 2$ is derived similarly. Note that $h_1 \to 0$ and $h_2 \to 0$ as $N \to \infty$ for any finite path of m_v. Then,

$$\frac{\int_0^c F_s(1-h_2) - F_s(1-h_1) f(d\epsilon_i)}{\int_0^c F_s(1-h_2) f(d\epsilon_i)} (N - \sum_{v=1}^{u-1} m_v)$$

$$\to (-\psi(\mathrm{E}[\epsilon_i]/\lambda + \phi \sum_{v=1}^{u-2} m_v) + \psi(\mathrm{E}[\epsilon_i]/\lambda + \phi \sum_{v=1}^{u-1} m_v)) = \phi\psi m_{u-1}. \tag{A.4}$$

Hence m_u given m_{u-1} asymptotically follows a Poisson distribution with mean $\phi\psi m_{u-1}$. Since a Poisson distribution is infinitely divisible, the process m_u given m_1 follows a branching process whose step distribution follows a Poisson with mean $\phi\psi$.

The stopping time T is finite with probability one, since the mean number of children born by a parent is $\phi\psi \leq 1$ by Assumption 3 [16]. Hence U is satisfied by an entire path m_v, $v = 1, 2, \ldots, T$, with probability one. □

The sum of the branching process $\sum_{v=1}^{T} m_v$ given $m_1 = l$ is known to follow a Borel-Tanner distribution [17, page 68], $\Pr(\sum_{u=1}^{T} m_u = w \mid m_1 = l) = (l/w) e^{-\phi w} (\phi w)^{w-l}/(w-l)!$, for $w = l, l+1, \ldots$. By Lemma 9, m_1 follows a Poisson distribution with mean μ. Thus $\sum_{v=1}^{T} m_v$ follows a compound Poisson distribution with Poisson mean μ and a Borel-Tanner for $m_1 = 1$. Calculating the compound distribution, we obtain the desired distribution function (7). This distribution is infinitely divisible since this is a compound Poisson distribution. Equation (8) is obtained by applying Stirling's formula $w! = \sqrt{2\pi} w^{w+0.5} e^{-w}$ to (7) for large w. □

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