Hitotsubashi University

Global COE Hi-Stat Discussion Paper Series 250

Research Unit for Statistical and Empirical Analysis in Social Sciences (Hi-Stat)

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October 2012

Hi-Stat Discussion Paper

Hi-Stat
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Estimating and Testing Multiple Structural Changes in Linear Models Using Band Spectral Regressions

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October 11, 2012

Abstract

We provide methods for estimating and testing multiple structural changes occurring at unknown dates in linear models using band spectral regressions. We consider changes over time within some frequency bands, permitting the coefficients to be different across frequency bands. Using standard assumptions, we show that the limit distributions obtained are similar to those in the time domain counterpart. We show that when the coefficients change only within some frequency band, we have increased efficiency of the estimates and power of the tests. We also discuss a very useful application related to contexts in which the data is contaminated by some low frequency process (e.g., level shifts or trends) and that the researcher is interested in whether the original non-contaminated model is stable. All that is needed to obtain estimates of the break dates and tests for structural changes that are not affected by such low frequency contaminations is to truncate a low frequency band that shrinks to zero at rate $\log(T)/T$. Simulations show that the tests have good sizes for a wide range of truncations so that the method is quite robust. We analyze the stability of the relation between hours worked and productivity. When applying structural change tests in the time domain we document strong evidence of instabilities. When excluding a few low frequencies, none of the structural change tests are significant. Hence, the results provide evidence to the effect that the relation between hours worked and productivity is stable over any spectral band that excludes the lowest frequencies, in particular it is stable over the business-cycle band.

JEL Classification Number: C14, C22

Keywords: multiple structural changes, band spectral regression, low frequency contaminations, hours-productivity, business-cycle.

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1 Introduction

This paper considers methods for estimating and testing multiple structural changes in linear models using band spectral regressions. Since the classic work by Hannan (1963), band spectral regressions have found wide applicability and have been useful for various problems when the coefficients of linear regression models are suspected to be frequency dependent. Engle (1974, 1978) adopted Hannan's insight to an econometric context and, for linear regression models, showed that the spectral least squares coefficients estimates and the associated test statistics have the same properties as in the standard time domain regressions. He also considered the classical Chow test for a change in the coefficients across frequency bands.

Our paper tackles the problem of structural changes from a different angle. First, as has become common now, we consider the possibility of multiple structural changes occurring at unknown dates. More importantly, instead of considering changes across frequencies, we consider changes over time within some frequency bands, permitting the coefficients to be different across frequency bands. We derive the appropriate methods to estimate the break dates and to construct the tests for structural changes. Using standard assumptions, we show that the limit distributions obtained are similar to those in the time domain counterpart as derived by Bai and Perron (1998) or Perron and Qu (2006); see Perron (2006) for a review. We show that when the coefficients change only within some frequency band (e.g., the business cycle) we can have increased efficiency of the estimates of the break dates and increased power for the tests provided, of course, that the user chosen band contains the band at which the changes occur. Our framework can therefore be very useful in various empirical applications. For instance, using an international data set consisting of series covering a long span, Basu and Taylor (1999) document that the cyclical behavior of the real wage (the relationship between aggregate output and real wages within some spectral band) may have been changing over time. Their analysis is based on changes in the correlation coefficients across different spectral bands and different time segments. We provide a general framework to analyze such issues in a rigorous and systematic manner.

We also discuss a very useful application of testing for structural changes via a band spectral approach. The framework we consider is one in which the data is contaminated by some low frequency process and that the researcher is interested in whether the original non-contaminated model is stable. For example, the dependent variable may be affected by some random level shift process (a low frequency contamination) but at the business cycle frequency the model of interest is otherwise stable. We show that all that is needed to obtain estimates of the break dates and tests for structural changes that are not affected by such low frequency contaminations is to truncate a low frequency band that shrinks to zero at rate $\log(T)/T$. Simulations show that the tests have good sizes for a wide range of truncations. The exact truncation does not really matter, as long as some of the very low frequencies are excluded. Hence, the method is quite robust. We also show that our method delivers more precise estimates of the break dates and tests with better power compare to using filtered series obtained via a band-pass filter or from a Hodrick-Prescott (1997) filter.

Along this line, our method has enhanced potential applicability in wide range of problems in macroeconomics, finance and other fields. Indeed, it has been shown for numerous problems that estimates and tests are sensitive to the low frequency components which are often driven by mean shifts or various types of trends. This feature also applies to issues related to structural changes. In a finance context, it has been documented that investigations of instabilities in stock returns predictive regressions is largely driven by low frequency components. For instance, mean shifts in dividends can lead one to conclude that the dividend/price ratio no longer has predictive power, e.g., Lettau and Neiuwerburgh (2008). Our framework allows one to draw conclusions about the stability of a relationship at some "business-cycle" frequency, say, without having to specify the nature of the low frequency movements. In a macroeconomic context, Fernald (2007) highlights the sensitivity of results about the effect of a productivity shock on hours worked based on vector autoregressions identified from long-run restrictions to the specifications of the low frequency components of hours worked and productivity. Our empirical application, reported in Section 5 sheds further light on this important issue. We analyze the stability of the relation between hours worked and productivity. When applying the structural change tests in the time domain, or equivalently the full set of frequencies, we document strong evidence of instabilities. When excluding a few low frequencies, none of the structural change tests are significant. Hence, the results provide evidence to the effect that the relation between hours worked and productivity is stable over any spectral band that excludes the lowest frequencies, in particular it is stable over the business-cycle band. This result has important implications for the analysis of the effect of a technological shock on hours worked. It indicates that the various structural-based methods used to assess the sign and magnitude of this effect should be carried using a frequency band that excludes the lowest frequencies or with a business-cycle band.

In view of this type of applications of our methods, our work is related to a recent strand in the literature that attempts to deliver tests and estimates that are robust to low frequency contaminations. One example pertains to estimation of the long-memory parameter. It is by now well known that spurious long-memory can be induced by level shifts or various kinds of low frequency contaminations. Perron and Qu (2007, 2010), McCloskey and Perron (2012) and Iacone (2010) exploit the fact that the level shifts or time trends will produce high peaks of the periodograms at a very few low frequencies, and suggests procedures that are robust by eliminating such low frequencies. Tests for spurious versus genuine long-memory have been proposed by Qu (2011) (see also Shimotsu, 2006). McCloskey (2010) provides a general method applicable to the estimation of various time series models, such as ARMA, GARCH and stochastic volatility models.

The structure of the paper is the following. Section 2 introduces the framework adopted, the basic model, the assumptions imposed, the asymptotic distributions of the estimates of the break dates and of the tests for structural changes. Section 3 considers models with low frequency contaminations and show how the trimming of some low frequencies delivers estimates and tests having the same limit distribution as in the non-contaminated models. Section 4 presents simulation evidence showing that the procedures suggested have good properties in small samples and performs better than using filtered data. Section 5 illustrates the usefulness of our methods by considering the stability of the relation between hours worked and productivity. Section 6 provides brief concluding comments and an appendix contains the technical derivations.

2 The framework and assumptions

2.1 The model

Consider a general multiple linear regression model with m breaks or $m+1$ regimes. There are T observations and m is assumed known for now. The break dates occur at $\{T_1, ..., T_m\}$. Let $y = (y_1, ..., y_T)'$ be the dependent variable and X a T by p matrix of regressors. Define $\bar{X} = diag(X_1, ..., X_{m+1})$, a T by $(m+1)p$ matrix with $X_i = (x_{T_{i-1}+1}, ..., x_{T_i})'$ for $i = 1, ..., m+1$, with the convention that $T_0 = 1$ and $T_{m+1} = T$ (each matrix X_i is a subset of the regressor matrix X corresponding to regime i). The matrix $\bar{X}$ is a diagonal partition of X, the partition being taken with respect to the set of break points $\{T_1, ..., T_m\}$. It will also be convenient to define $Y_i = (y_{T_{i-1}+1}, ..., y_{T_i})'$. The vector $U = (u_1, ..., u_T)'$ is the set of disturbances and $\beta = (\beta_1', ..., \beta_{m+1}')'$ is the $(m+1)p$ vector of coefficients. We consider the general pure structural change model with restrictions on the coefficients, i.e.

$$Y = \bar{X}\beta + U, \tag{1}$$

where $R\beta = r$ with R a k by $(m+1)p$ matrix with rank k and r a k dimensional vector of constant. Note that this framework includes the case of a partial structural change model by an appropriate choice of the restrictions on the parameters (see Perron and Qu, 2006).

2.2 The band spectral regression

Band spectral regressions were early proposed by Hannan (1963) and have been adopted subsequently in the econometric literature, see in particular Engle (1974, 1978). The framework is useful in estimating linear regression models for which the coefficients are frequency dependent. Many economic applications fit in this framework. For example, consider a consumption function for which consumers are assumed to react to the transitory and permanent income in different ways as the classical permanent income hypothesis suggests. Here, the relationship between income and consumption can be different for higher (transitory) and lower (permanent) frequency variations. More recently the technique was found to be useful in estimating cointegrating relations by Phillips (1991). Also, Corbae et. al. (2002) suggest that the removal of time trends should be conducted in the frequency domain by estimating frequency dependent coefficients using band spectral regressions.

We first provide a brief description of the basic principles underlying band spectral regressions. Consider a generic model where Y^* is the dependent variable and Z is the matrix of regressors. The starting point is to apply a discrete Fourier transformation to the data. Let W be an orthogonal $T \times T$ matrix with $w_{j,k} = T^{-1/2}\exp\left(ij(k-1)(2\pi/T)\right)$ for its (j,k) component where, as usual, $i = \sqrt{-1}$. Then the transformed data are, say, $\widetilde{Z} = WZ$ and $\widetilde{Y}^* = WY^*$. To have the analysis pertain to a particular band of interest, we follow the technique suggested by Corbae et. al. (2002). Let the band of interest be $B_A = [\omega_l, \omega_h]$ $(0 \leq \omega_l < \omega_h \leq \pi)$[1]. It is often easier to describe a certain frequency in terms of the position of the observation in the vector. Hence, we define $j_l = [\omega_l T/\pi]$ and $j_h = [\omega_h T/\pi]$ with $[\cdot]$ returning the integer of the argument. The band selection can then be applied with another linear operator consisting of a $T \times T$ selection matrix A with ones for the jth diagonal elements for $j_l \leq j \leq j_h$ and zeros for all other elements. The transformed dependent variable

[1] Following the convention since Engle (1974), in practice we produce the observations in dual bands corresponding to $\omega \in [-\pi, \pi]$ and choose symmetric bands, i.e. $\pm\omega$ at the same time. This way, the discrete Fourier transform will include both $\cos\omega + i\sin\omega$ and $\cos\omega - i\sin\omega$ and we can avoid complex valued quantities.

is now $A\widetilde{Y}^*$, the transformed regressors $A\widetilde{Z}$ and the OLS estimate is

$$\begin{aligned}\widetilde{\beta}^A &= (Z'\Phi Z)^{-1}(Z'\Phi Y^*), \\ &= \left[\textstyle\sum_{B_A} I_{z,T}(\omega_j)\right]^{-1}\left[\textstyle\sum_{B_A} I_{zy*,T}(\omega_j)\right], \end{aligned} \tag{2}$$

where $\Phi = W'AW$, $I_{z,T}(\omega)$ is the matrix of sample cross periodgrams of z_t and $I_{zy*,T}(\omega)$ is a vector of cross periodgrams of z_t and y_t^*, both evaluated at frequency ω. If serial correlation in the errors is suspected, we can account for it using a matrix A defined with an estimate of $f_u^{-1/2}(\omega)$ in the diagonal of A instead of one, where $f_u(\omega)$ is the power spectrum of the error term at frequency ω. As a matter of notation, define $N_A = j_h - j_l + 1$, the number of non-zero data points in the variables transformed by the operators A and W. Finally, we consider for now the case of fixed bands in the asymptotic analysis so that ω_l and ω_h are fixed. This implies that j_l and j_h increase at the same rate as T so that N_A remains a fixed portion of the sample size T. This framework is standard and does provide a useful approximation in finite samples, as will be shown later.

With this background description, the estimates of the break dates in our model are defined as the solutions of the global least square minimization problem applied to band spectral regressions such that:

$$(\hat{T}_1, ..., \hat{T}_m) = \arg\min_{T_1,..,T_m} SSR_T^{B_A}(T_1, ..., T_m), \tag{3}$$

where

$$SSR_T^{B_A}(T_1, ..., T_m) = \sum_{i=1}^{m+1}(AWY_i - AWX_i\widetilde{\beta}_i^A)'(AWY_i - AWX_i\widetilde{\beta}_i^A), \tag{4}$$

with $\widetilde{\beta}_i^A$ $(i = 1, ..., m+1)$ the band spectral least squares coefficient estimates for the selected band B_A defined by (2) with $Y^* = Y_i$ and $Z = X_i$, which contain the observations $t = T_{i-1} + 1, ..., T_i$.

2.3 Assumptions

In order to derive the limit distribution of our estimates, we impose the following standard assumptions on the data, the errors and the break dates.

- **Assumption A1:** For each segment $i = 1, ..., m+1$, $\zeta_t^i = \{\zeta_t\}_{t=T_{i-1}^0+1}^{T_i^0} = \{(x_t', u_t)'\}_{t=T_{i-1}^0+1}^{T_i^0}$ is a jointly stationary time series, that is, $\zeta_t^i = \sum_{j=0}^{\infty} c_j^i \eta_{t-j}^i$, with $\eta_t^i \sim i.i.d.(0, \Sigma_i)$ with finite fourth moments and coefficients c_j^i satisfying $\sum_{j=0}^{\infty} j^{1/2} \left\| c_j^i \right\| < \infty$. Partitioned conformably, the spectral density matrix $f_\zeta^i(\omega)$ of ζ_t^i is

$$f_\zeta^i(\omega) = \begin{bmatrix} f_x^i(\omega) & 0 \\ 0 & f_u^i(\omega) \end{bmatrix},$$

with $f_x^i(\omega)$ a non-random positive definite matrix and $f_u^i(\omega)$ a positive constant for any $\omega \in [\omega_l, \omega_h]$.

- **Assumption A2:** There exists an $l_0 > 0$ such that for all $l > l_0$, the minimum eigenvalues of the sample periodgram matrix $I_x(\omega)$ constructed by $\{x_t\}_{t=T_i^0+1}^{T_i^0+l}$ and that by $\{x_t\}_{t=T_i^0-l}^{T_i^0}$ are bounded away from zero $(i = 1, ..., m)$ for some $\omega \in [\omega_l, \omega_h]$.

- **Assumption A3:** The sample periodgram matrix associated with the spectral band $\omega \in [\omega_l, \omega_h]$, $\sum_{\omega=\omega_l}^{\omega_h} I_x(\omega)$, constructed using $\{x_t\}_{t=k}^{l}$ for $l - k \geq \epsilon T$, is invertible for some $\epsilon > 0$.

- **Assumption A4**: Let the L_r-norm of a random matrix Z be defined by $\|Z\|_r = (\sum_i \sum_j E\left|Z_{ij}\right|^r)^{1/r}$ for $r \geq 1$. (Note that $\|Z\|$ is the usual matrix norm or the Euclidean norm of a vector.) With $\{\mathcal{F}_i : i = 1, 2, ..\}$ a sequence of increasing σ-fields, we assume that $\{x_i u_i, \mathcal{F}_i\}$ forms a L^r-mixingale sequence with $r = 2 + \varepsilon$ for some $\varepsilon > 0$. That is, there exist nonnegative constants $\{\kappa_i : i \geq 1\}$ and $\{\psi_j : j \geq 0\}$ such that $\psi_j \downarrow 0$ as $j \to \infty$ and for all $i \geq 1$ and $j \geq 0$, we have: (a) $\|E(x_i u_i|\mathcal{F}_{i-j})\|_r \leq \kappa_i \psi_j$, (b) $\|x_i u_i - E(x_i u_i|\mathcal{F}_{i+j})\|_r \leq \kappa_i \psi_{j+1}$. Also assume (c) $\max_i \kappa_i \leq K < \infty$, (d) $\sum_{j=0}^{\infty} j^{1+k} \psi_j < \infty$, (e) $\|x_i\|_{2r} < M < \infty$ and $\|u_i\|_{2r} < N < \infty$ for some $K, M, N > 0$.

- **Assumption A5:** $T_i^0 = [T\lambda_i^0]$, where $0 < \lambda_1^0 < ... < \lambda_m^0 < 1$.

- **Assumption A6**: Let $\Delta_{T,i} = \beta_{T,i+1} - \beta_{T,i}$. Assume $\Delta_{T,i} = v_T \Delta_i$ for some Δ_i independent of T, where $v_T > 0$ is a scalar satisfying $v_T \to 0$ and $T^{(1/2)-\vartheta} v_T \to \infty$ for some $\vartheta \in (0, 1/2)$. In addition, we assume $E\left\|x_t\right\|^2 < K$ and $E\left|u_t\right|^{2/\vartheta} < K$ for some $K < \infty$ and all t.

These assumptions are standard in the literature. They follow Bai and Perron (1998) and Perron and Qu (2006) for the structural change problem and Corbae et. al. (2002) for the band spectral regression framework. Assumption A1 corresponds to Assumption 1 in Corbae et. al. (2002) and it imposes stationarity within each regime. It also implies Assumption

A1 in Perron and Qu (2006). Assuming the cross-spectrum of u_t and x_t essentially rules out endogeneity. It can be relaxed by interpreting the coefficients as the pseudo-true values, i.e., as the limit in probability of the inconsistent estimates. As shown in Perron and Yamamoto (2012), this still permits consistent estimation of the break fractions and the confidence intervals for the estimates that can be constructed in the usual manner. Assumptions A2 and A3 impose conditions that are the frequency domain analogs of A2 and A3 in Bai and Perron (1998). Assumption A4 imposes mild conditions on the regressors and errors which permit a wide class of potential correlation structures in the errors and regressors. It also allows lagged dependent variables as regressors when the errors are a martingale difference sequence. A5 imposes the break points to be asymptotically distinct, a standard condition needed to have non-degenerate limit distributions. A6 is also standard in the literature. It dictates an asymptotic framework whereby the magnitudes of the breaks decrease as the sample size increases, a feature needed to derive a limit distribution of the estimates of the break dates that does not depend on the exact distribution of the errors.

2.4 Asymptotic properties

We now establish the consistency, rate of convergence and asymptotic distribution of the estimates of the break dates defined by (3) and (4). We start with the following important lemma.

Lemma 1 *For the full spectrum case, that is $A = I$, the following equivalence holds:*

$$SSR_T^{B_A}(\hat{T}_1, ..., \hat{T}_m) \equiv SSR_T(\hat{T}_1, ..., \hat{T}_m),$$

where $SSR_T(\hat{T}_1, ..., \hat{T}_m)$ is the overall sum of squared residuals when the structural change model is applied using a standard time domain procedure for model (1), viz.,

$$SSR_T(\hat{T}_1, ..., \hat{T}_m) = \sum_{i=1}^{m+1} (Y_i - X_i\hat{\beta}_i)'(Y_i - X_i\hat{\beta}_i),$$

with $\hat{\beta}_i = (X_i'X_i)^{-1}(X_i'Y_i)$.

This lemma shows that the global minimization problem (3) applied to the full spectrum reduces to the standard time domain structural change problem for model (1). This is an intuitive and useful property and a short proof is given in the appendix. This equivalence will be useful in deriving the asymptotic results when the analysis is restricted in a certain band spectrum. To see this, consider the following time-domain data generating process instead of (1),

$$y_t^* = x_t^* \beta_i + u_t^*, \qquad t = 1, ..., T \tag{5}$$

for $i = 1, ..., m+1$, where $\zeta_t^* = \{x_t^*, u_t^*\}$ is a process with the same spectral density as that of ζ_t at the Fourier frequencies $\omega \in B_A$ and has no variation for $\omega \notin B_A$. In matrix notation,

$$Y^* = X^* \beta + U^*,$$

where $X^* = W^{-1}AW\bar{X}, U^* = W^{-1}AWU$, and $Y^* = W^{-1}AWY$. Note first that premultiplying by W^{-1} applies an inverse Fourier transform to the variables so that we are back to the time domain and, second, that the coefficient vector β is also not affected by this transformation. As discussed in the appendix, the asymptotic properties of the series x_t^* are investigated using the following structure sometimes called ideal (but infeasible) band-pass filter

$$x_t^* = \textstyle\sum_{k=-\infty}^{\infty} b_k x_{t-k},$$

with $b_0 = [(\omega_h - \omega_l)/2\pi]$ and $b_k = [(\sin(\omega_h j) - \sin(\omega_l j))/(2\pi j)]$. By applying Lemma 1 to (5), we obtain an equivalence of the global sum sum of squared residuals pertaining to model (5) in the time domain and that pertaining to (3) with an arbitrary selector matrix A. This implies that the the asymptotic properties of the estimates of the structural change model involving a band spectral regression can be analyzed by investigating its time domain counterpart (5). To this effect, we now state a lemma applicable to the variables in the time domain model (5).

Lemma 2 *Let $\Delta T_i^0 = T_i^0 - T_{i-1}^0$ and suppose A1-A5 hold. With ζ_t^* defined for any nonempty band B_A, the followings hold:*

(a) $(\Delta T_i^0)^{-1} \sum_{t=T_{i-1}^0+1}^{T_{i-1}^0+[s\Delta T_i^0]} x_t^* x_t^{*\prime} \xrightarrow{p} sQ_i^*$;

(b) $(\Delta T_i^0)^{-1} \sum_{t=T_{i-1}^0+1}^{T_{i-1}^0+[s\Delta T_i^0]} u_t^{*2} \xrightarrow{p} s\sigma_i^{*2}$;

(c) $(\Delta T_i^0)^{-1} \sum_{r=T_{i-1}^0+1}^{T_{i-1}^0+[s\Delta T_i^0]} \sum_{t=T_{i-1}^0+1}^{T_{i-1}^0+[s\Delta T_i^0]} E(x_r^* x_t^{*\prime} u_r^* u_t^*) \xrightarrow{p} s\Omega_i^*$ *uniformly in s;*

(d) $(\Delta T_i^0)^{-1/2} \sum_{t=T_{i-1}^0+1}^{T_{i-1}^0+[s\Delta T_i^0]} x_t^* u_t^* \Rightarrow B_i^*(s)$;

where Q_i^* and Ω_i^* are $p \times p$ positive definite matrices, σ_i^{*2} is a positive scalar and $B_i^*(s)$ is a multivariate Gaussian process on [0,1] with mean zero and covariance $EB_i^*(s)B_i^*(u) = \min\{s,u\}\,\Omega_i^*$.

Lemma 2 plays an essential role in establishing the asymptotic distribution of the estimates of the break dates and test statistics. The main result is stated in the following theorem.

Theorem 1 *Let $\hat{T}_i$ be the estimates defined by (3) and $\hat{\lambda}_i = \hat{T}_i/T$ for $i = 1, ..., m$. Then, under A1-A6, we have for any nonempty choice of the band B_A. (a) For every $\nu > 0$, there exists a $C < \infty$, such that for all large T, $P(|T(\hat{\lambda}_i - \lambda_i^0)| > C) < \nu$. (b)*

$$(\Delta_i' Q_i^* \Delta_i) v_T^2 (\hat{T}_i - T_i^0) \Rightarrow \arg\max_s Z^{(i)}(s) (i = 1, ..., m),$$

where

$$Z^{(i)}(s) = \begin{cases} \phi_{i,1} W_1^{(i)}(-s) - |s|/2, & \text{for} \quad s \leq 0 \\ \sqrt{\xi_i}\phi_{i,2} W_2^{(i)}(-s) - \xi_i |s|/2, & \text{for} \quad s > 0 \end{cases},$$

with

$$\begin{aligned} \xi_i &= \Delta_i' Q_{i+1}^* \Delta_i / \Delta_i' Q_i^* \Delta_i, \\ \phi_{i,1}^2 &= \Delta_i' \Omega_i^* \Delta_i / \Delta_i' Q_i^* \Delta_i, \\ \phi_{i,2}^2 &= \Delta_i' \Omega_{i+1}^* \Delta_i / \Delta_i' Q_{i+1}^* \Delta_i, \end{aligned}$$

and $W_j^{(i)}(j = 1, 2)$ are independent Wiener processes defined on $[0, \infty)$.

Remark 1 *Note that Q_i^* and Ω_i^* can also be expressed as $Q_i^* = \int_{\omega_l}^{\omega_h} f_x^i(\omega) d\omega$ and $\Omega_i^* = \int_{\omega_l}^{\omega_h} f_x^i(\omega) f_u^i(\omega) d\omega$, which are fixed matrices under assumption A1 for any ω_l and ω_h, $0 \leq \omega_l < \omega_h \leq \pi$.*

2.5 Testing for structural change

We now consider the problem of testing the null hypothesis of no break versus a fixed number (m) of breaks and show that the conventional Sup$F$ test applied to band spectral regression, has the same limit distribution as in the standard time domain setup (see Andrews, 1993, and Bai and Perron, 1998). Note that, as pointed out by Engle (1974), the number of degrees of freedom is N_A, the number of observations for AWX and not T. The $SupF_T$ test is then defined by

$$SupF_T = \sup_{(\lambda_1, ..., \lambda_m) \in \Lambda_m} F_T(\lambda_1, ..., \lambda_m),$$

where

$$F_T(\lambda_1, ..., \lambda_m) = \left(\frac{N_A - (m+1)p - k}{mp}\right) \frac{\widetilde{\beta}^{A\prime} R'(R(\bar{X}^{*\prime}\bar{X}^*)^{-1}R')^{-1}R\widetilde{\beta}^A,}{SSR_T^{B_A}(T_1, ..., T_m)} \tag{6}$$

with R the usual matrix such that $(R\beta)' = (\beta_1 - \beta_2, ..., \beta_m - \beta_{m+1})$, $SSR_T^{B_A}(T_1, ..., T_m)$ is as defined in (4) and $\Lambda_\epsilon^m = \{(\lambda_1, ..., \lambda_m) : |\lambda_{i+1} - \lambda_i| \geq \epsilon, \lambda_1 \geq \epsilon, \lambda_k \leq 1 - \epsilon\}$ for some small trimming value ϵ. The limiting distribution of the $SupF_T$ statistic is described in the following Theorem.

Theorem 2 *Under A1-A6, $supF_T \Rightarrow \sup_{(\lambda_1,...,\lambda_m)\in\Lambda_m} F(\lambda_1, ..., \lambda_m)$ where*

$$F(\lambda_1, ..., \lambda_m) = \frac{1}{mp} \sum_{i=1}^{m} \frac{\left\|\lambda_i W_p(\lambda_{i+1}) - \lambda_{i+1} W_p(\lambda_i)\right\|^2}{\lambda_{i+1}\lambda_i(\lambda_{i+1} - \lambda_i)}$$

The proof is straightforward and presented in the appendix. Note the sequential tests for l versus $l+1$ breaks (which permits estimating the number of breaks m) and the double maximum tests investigated in Bai and Perron (1998) can also be constructed with appropriate changes for the regressors, residuals, coefficient estimates and the number of observations as described above. They have the same limit distributions as those stated in Bai and Perron (1998). Serial correlation in the errors is accounted for using heteroskedastic robust standard errors in the frequency domain as pointed out by Engle (1974).

3 Estimating and testing structural changes with contaminated models

In this section, we discuss a very useful application of testing for structural changes via a band spectral approach. The framework we consider is one in which the data is contaminated by some low frequency process and that the researcher is interested in whether the original non-contaminated model is stable. For example, the dependent variable may be affected by some random level shift process (a low frequency contamination) but at the business cycle frequencies the model of interest is otherwise stable.

Let $\{d_t\}$ be an unobservable contaminating component whose exact form is not known to the researcher. The specification of the data-generating process is then

$$y_t^D = x_t\beta_j + d_t + u_t, \tag{7}$$

for $j = 1, ..., m+1$, or equivalently

$$Y^D = \bar{X}\beta + D + U,$$

in vector form with $D = (d_1, ..., d_T)'$ a $T \times 1$ vector. The interest is in testing whether the coefficient vector β is stable over time without requiring a particular model for the contaminating component d_t. The only requirement is that the contaminating component is dominant at low frequency and that it is uncorrelated with the regressors and the errors, which are standard conditions in this literature and reasonable given the types of contaminations analyzed (see below). The specific conditions required are stated in the following assumptions.

- **Assumption A7:** The cross spectral density $f^i_{\zeta d}(\omega)$ of ζ_t and d_t is 0 for any $\omega \in [\omega_l, \omega_h]$.
- **Assumption A8:** $I_{d,T}(\omega_j) = O_p(Tj^{-2})$ for all $j = 1, ..., [T/2]$.

The assumption A7 ensures the strict exogeneity of the process d_t in the model. Assumption A8 states that the contaminating component has a periodogram that is divergent for $j < T^{1/2}$ but is negligible for $j > T^{1/2}$. Hence, by restricting the analysis to a set of frequencies that exclude a neighborhood around zero, one can obtain results that are not affected by the contamination. Many processes of interest satisfy A8. The following is a non-exhaustive list: a) a random level shift process of the form

$$d_t = \sum_{j=1}^{t} \delta_{T,j}, \;\; \delta_{T,j} = \pi_{T,j}\eta_j, \tag{8}$$

where $\eta_j \sim i.i.d.(0, \sigma^2_\eta)$ with finite moments of all orders, $\pi_{T,j} \sim i.i.d.$ Bernoulli$(p/T, 1)$ for some $p \geq 0$, and with the components $\pi_{T,j}$ and η_j being mutually independent; b) deterministic level shifts of the form

$$d_t = \sum_{n=1}^{N} c_n I(T_{n-1} < t \leq T_n), \tag{9}$$

where N is a fixed positive integer and $I(\cdot)$ is the indicator function; c) deterministic trends of the form

$$d_t = \Psi(t/T), \tag{10}$$

where $\Psi(\cdot)$ is a deterministic nonconstant function on [0,1] that is either Lipschitz continuous or monotone and bounded.

The fact that A8 is satisfied for the random level shift process (8) was shown in Perron and Qu (2010), for the deterministic level shifts process it was shown in McCloskey and Perron (2012), while Qu (2011), building on results by Künsh (1986) showed it for the general deterministic trend function. To have more generality and methods with increased efficiency, we allow ω_l to approach 0 at some rate so that what is excluded is only a shrinking frequency band near zero. Recall that the lower bound of the truncation is $j_l = [\omega_l T/\pi]$. We start with a result that states the relationship between the global sums of squared residuals from the band spectral regressions obtained from the original and contaminated models.

Lemma 3 *Consider model (7) with $\{d_t\}$ satisfying A7 and A8. With $\omega_h > 0$, let $j_l \to \infty$ and $j_l/\log(T) \to \infty$ as $T \to \infty$. Then,*

$$SSR_T^{D,B_A}(T_1, ..., T_m) = SSR_T^{B_A}(T_1, ..., T_m) + \phi_T + o_p(1),$$

with $\phi_T = o_p(T)$, where

$$SSR_T^{D,B_A}(T_1, ..., T_m) = \sum_{i=1}^{m}(AWY_i^D - AWX_i\widetilde{\beta}_i^{D,A})'(AWY_i^D - AWX_i\widetilde{\beta}_i^{D,A}).$$

with $\widetilde{\beta}_i^{D,A} = (X_i'\Phi X_i)^{-1}(X_i'\Phi Y_i^D)$.

Since $SSR_T^{B_A}(T_1, ..., T_M) = O_p(T)$, the lemma shows that, under the stated assumptions, one obtains the asymptotic equivalence of the sum of squared residuals given a set of break dates between the models with and without the contaminating term. What is required is that a certain low frequency band that shrinks to zero at rate $\omega_l \propto \log(T)/T$ is truncated, the band spectrum estimates of the break dates are then not affected, at least in large samples, by an unknown contaminating component $\{d_t\}$ specified by A7 and A8. If one restricts the analysis to a fixed band $B_A = [\omega_l, \omega_h]$ with ω_l any fixed positive number, then $j_l = O(T)$ and the requirement is automatically satisfied. This provides a method to obtain estimates and tests that are robust to such misspecifications. The results are formally stated in the following proposition.

Proposition 1 *Consider the contaminated model (7) with A1-A8 holding. With $\omega_h > 0$, let $j_l \to \infty$ and $j_l/\log(T) \to \infty$ as $T \to \infty$, then the band spectrum estimates of the multiple structural changes*

$$(\hat{T}_1, ..., \hat{T}_m) = \arg\min_{T_1,...,T_m} SSR_T^{D,j_l}(T_1, ..., T_m),$$

satisfy the properties stated in Theorem 1. Define the SupF test statistic by

$$F_T = \left(\frac{T - j_l - (m+1)p - k}{mp}\right)\frac{\widetilde{\beta}^{A,j_l\prime}R'(R(\bar{X}^{*\prime}\bar{X}^*)^{-1}R')^{-1}R\widetilde{\beta}^{A,j_l}}{SSR_T^{D,j_l}(T_1, ..., T_m)},$$

where $\beta^{A,j_l} = (\beta_1^{A,j_l\prime}, ..., \beta_{m+1}^{A,j_l\prime})'$ *with* $\widetilde{\beta}_i^{A,j_l} = (X_i'\Phi_{j_l}X_i)^{-1}(X_i'\Phi_{j_l}Y_i^D)$ *and* Φ_{j_l} *is a selection matrix with zeros in the first* j_l *diagonal elements and ones in the other diagonals. Then, the SupF test has the limiting distribution as stated in Theorem 2.*

Remark 2 *To be more precise, one could state the requirement on the rate of growth of* j_l *as a function of the effective sample size within each regime, namely* $j_l/\log(\Delta T_i) \to \infty$ *as* $T \to \infty$*, where* $\Delta T_i = T_i - T_{i-1}$*,* $i = 1, ..., m+1$*. However, this make no difference theoretically since* $\Delta T_i = O(T)$*.*

4 Monte Carlo simulations

In this section, we present simulation results about the properties of the estimates of the break dates obtained from the band spectral regression (bias, standard errors, coverage rate of the asymptotic distribution) and the size and power of the test for structural change. We start in Section 4.1 with the case of no low frequency contamination and in Section 4.2 we consider models with such contaminations. In Section 4-3, we compare the proposed band spectral approach with the standard Bai and Perron (1998) method using filtered data, via a band-pass filter as suggested by Baxter and King (1999) or after applying a Hodrick-Prescott (1997) filter[2].

4.1 Models without contamination

The data generating process used is

$$y_t = x_t\beta_t + u_t, \qquad t = 1, ..., T,$$

where the regressor x_t is a stationary ARMA(1,1) process with a constant mean κ:

$$\begin{aligned} x_t &= \kappa + z_t, \\ z_t &= \rho z_{t-1} + e_t + \theta e_{t-1}, \end{aligned}$$

with e_t and u_t sequences of *i.i.d.* $N(0,1)$ random variables independent of each other. We consider a single break model

$$\beta_t = \begin{cases} -c & \text{for} \quad t < T_b \\ c & \text{for} \quad t \geq T_b \end{cases}.$$

[2]To have increased computational efficiency and to avoid potential problems associated with complex numbers, we adopted the finite Fourier transforms in real term proposed by Harvey (1978).

We consider three cases for the type of regressors: for case 1, x_t is uncorrelated so that $(\rho, \theta) = (0, 0)$; for case 2, x_t is MA(1) with $\rho = 0$ and $\theta = 0.5$; for case 3, x_t is an AR(1) process with $\rho = 0.5$ and $\theta = 0$. We set $\kappa = 1$ in all cases. For the choice of the bands, we consider several cases which are popular in empirical applications. The first group pertains to low frequency bands with $(\omega_l, \omega_h) = (0, \pi/4)$ and $(0, \pi/2)$. The second group corresponds to typical seasonal and business cycles bands with quarterly data given by $(\omega_l, \omega_h) = ([1/2-0.15]\pi, [1/2+0.15]\pi)$ and $(\pi/16, \pi/2)$, respectively. The third group consists of high frequency bands with $(\omega_l, \omega_h) = (\pi/4, \pi)$ and $(\pi/2, \pi)$. We tried several other types of bands and the results were similar. We set $T = 100$ and 200 and the break date is at mid-sample, so that $T_b = 50$ for $T = 100$ and $T_b = 100$ for $T = 200$. The number of replications is 1,000.

We first consider the properties of the estimates of the break fraction $\hat{T}_B/T$ with $c = 0.1$. Table 1 reports results for the bias, standard error and coverage rate of the asymptotic distribution. The bias is very close to zero for all cases considered, which supports the consistency result for the break fractions in Theorem 1(a). Table 1 also reports the coverage rates obtained from the asymptotic distribution for 90% nominal confidence intervals. The results show the exact coverage rates to be very close to the nominal level in all cases confirming the adequacy of the limiting distribution as an approximation to the finite sample distribution. The results with $T = 200$ are similar with, as expected, a reduction in the variance of the estimates. We also computed the size of the heteroskedastic robust Sup$F$ test when the errors follow an AR(1) process and obtained broadly similar results.

Table 2 shows the finite sample size properties of the Sup$F$ test with $c = 0$ for a 5% nominal size. The results show that the exact size is very close to 5% in all cases. We next consider its power. Figure 1 shows the rejection frequency as a function of the magnitude of the break c. The three panels correspond to the cases with low, middle, and high frequency bands. In all cases, the results show good power, which approaches one quickly. As expected, using the full spectrum gives tests with the highest power. This is due to the fact that the data is generated with coefficients that are the same across frequencies. Of more interest are cases for which the coefficients change only in some particular frequency band, a problem we address next.

We now consider the power of the Sup$F$ test when the true data generating process has a structural break only in a particular spectral band B_A^0. In such cases, we expect that power would be highest when the band used in constructing the test B_A is the same as B_A^0, showing that tests for structural change based on our band regression framework can yield higher power. As we shall see, this is indeed the case. To this end, the data-generating process used is

$$y_t = x_t^A \beta_t + x_t^C \beta + u_t, \qquad t = 1, ..., T,$$

where

$$x_t^A = \begin{cases} x_t, & \text{if } \omega \in B_A^0 \\ 0, & \text{otherwise} \end{cases},$$

$$x_t^C = \begin{cases} x_t, & \text{if } \omega \notin B_A^0 \\ 0, & \text{otherwise} \end{cases},$$

and β_t is the same as in the previous experiments but β is a constant (set at $\beta = 0$) so that a structural change is present only in the frequency band B_A^0. The following four bands are considered for B_A^0: a) a low frequency band $(0, \pi/4)$, b) a high frequency band $(\pi/4, \pi)$, c) a seasonal frequency band $([1/2-0.15]\pi, [1/2+0.15]\pi)$ and d) a business cycle frequency band $(\pi/16, \pi/2)$. In Figure 2, results are presented for the power of the Sup$F$ test using the true spectral band B_A^0 and the full spectrum since the latter is equivalent to the standard time domain structural break test. The results show that important power gains can be achieved using tests based on a band spectral regression if one uses the correct band in which the change occurs. Note that the power gains are more important when the band in which the change occurs consists of higher frequencies. As expected, if the band considered is one in which no break occurs, then power is equal to the size of the test, see panels (a) and (b).

4.2 Models with contaminating components

We now consider models with a contaminating component and evaluate how the truncation of the low frequencies helps in obtaining tests with good size and power properties. The data are generated by

$$y_t = x_t \beta_t + d_t + u_t.$$

We consider the following four cases for the contaminating component d_t, which all satisfy assumptions A7 and A8:

- Case D1, Deterministic Level Shifts: $d_t = c_1 I(1 \leq t < T_D) + c_2 I(T_D \leq t \leq T)$;
- Case D2, Random Level Shifts: $d_t = \sum_{j=1}^{n} \pi_{T,j} \eta_j$, where $\eta_j \sim i.i.d.\ N(0,1)$ and $\pi_{T,j} \sim i.i.d.\ B(p/T, 1)$, η_j and $\pi_{T,j}$ are independent;
- Case D3, Linear Trend with a Break: $d_t = \gamma_1 t I(1 \leq t < T_D) + \gamma_2 t I(T_D \leq t \leq T)$;

- Case D4, Quadratic Trend**:** $d_t = \psi t^2$.

The parameter values were selected in order to have the long run variance of all four processes be of similar magnitude. To that effect, we set $(c_1, c_2) = (-1, 1)$, $p = 5$, $(\gamma_1, \gamma_2) = (0.02, 0.01)$, and $\psi = 0.01$. Although these values are arbitrary, simulations using other values yielded qualitatively similar results. The break date of the contaminating processes (1) and (3) was set to $T_D = 50$. We only report the results for $T = 100$ (those for $T = 200$ were qualitatively similar). The specifications for the other components $x_t\beta_t$ and u_t are as in the previous sub-section. We consider the following truncations j_l: the integer values of $1, 5, \log(T)$, $\log(T)^2$, $T^{0.5}$, and $T^{0.6}$.

Table 3 provides the exact sizes of the Sup$F$ test for a nominal 5% size test according to the pattern of $\{d_t\}$, the truncations and the DGP for x_t. Of importance is the fact that for all cases serious size distortions are present when no truncation is applied. However, the exact size is much closer to the nominal level when a truncation is applied. For power, Table 4-1 presents the non size adjusted powers and Table 4-2 displays the size adjusted ones. We only report the case with a white noise regressor (case 1) given that the results were qualitatively similar for the other cases. First, the size-adjusted power of the test without truncation is comparatively very small. Second, when a truncation is applied the power is improved considerably. Third, in general power is not much sensitive to the particular choice for the truncation rule.

The size and power results are comforting since any reasonable choice of the truncation rule, say greater than or equal to $\log(T)$ and less than $T^{.6}$, will lead to test with similar properties. What is important is that some truncation be done, even truncating a single frequency yields dramatic improvements over the full sample-based tests.

4.3 Comparisons with filtered series

An issue of interest is how our method compares to simply using filtered data prior to estimating and testing for structural changes. To provide some answers to this question, we compare the properties of the break date estimates and the structural change tests based on our band spectral approach with standard methods applied to filtered series. For the latter, band-pass filters as well as the Hodrick-Prescott (1997) filter (HP) are considered. For an original series y_t, the filtered series obtained using Baxter and King's (1999) approximate band-pass filter (BP) with frequency band $\omega \in [\omega_l, \omega_h]$, denoted y_t^{BP}, is defined by:

$$y_t^{BP} = \beta(L)y_t = [\beta_h(L) - \beta_l(L)]y_t,$$

where

$$\begin{aligned}
\beta_h(L) &= \sum\nolimits_{k=-K}^{K} b_k^h L^k, \quad b_0^h = \frac{\omega_h}{\pi}\ (k=0),\ b_k^h = \frac{\sin(k\omega_h)}{k\pi}\ \ (k \neq 0), \\
\beta_l(L) &= \sum\nolimits_{k=-K}^{K} b_k^l L^k, \quad b_0^l = \frac{\omega_l}{\pi}\ (k=0),\ b_k^l = \frac{\sin(k\omega_l)}{k\pi}\ \ (k \neq 0).
\end{aligned}$$

For the truncation parameter, we consider $K = 4$ and 12. The HP filtered series, y_t^{HP}, is defined by:

$$y_t^{HP} = \left[\frac{\lambda(1-L)^2(1-L^{-1})^2}{1+\lambda(1-L)^2(1-L^{-1})^2}\right] y_t.$$

For the parameter λ, we consider two popular choices, namely $1,600$ and 6.25.

In Table 5-1, we present the bias and standard deviation of the break fraction estimates, and the exact coverage rate of the asymptotic 90% confidence intervals when the DGPs are the non-contaminated models of Section 4.1. Throughout, 1,000 replications are used. For brevity, we only consider three spectral bands: $[\pi/16, \pi/2]$ (band 1), $[0, \pi/16]$ (band 2), and $[\pi/2, \pi]$ (band 3). Here, our comparisons are only with the BP filtered series. The results show that when using the BP filtered-based estimates the bias remains small but the standard deviations are larger than when using the band spectral approach. Also, the exact coverage rate of the asymptotic confidence intervals are near 90% in all cases using the band spectral approach but there is severe under-coverage when using the BP filtered approach.

The next experiment pertains to a comparison our $\log(T)$ truncation method with standard methods applied to HP filtered series in the case of the contaminated processes considered in Section 4.2. The results presented in Table 6 show that using HP filtered series leads to estimates with larger variance and exact coverage rates below the 90% nominal level.

The last experiment pertains to the power of the Sup$F$ test for a single structural change. We use the non-contaminated models with a break in all frequencies when comparing with the BP filter (results reported in Figure 3, panels a-c) and with a break in the frequency band $[\pi/16, \pi/2]$ when comparing with the HP filter (results reported in Figure 3, panel d). The power functions are for tests with a 5% nominal size. In all cases, our band spectral method leads to tests with higher power.

These simulations illustrate the relative efficiency and flexibility of our proposed method over standard methods based on filtered series[3].

[3] We also compared the power of the test for structural change using our truncation method with the HP filtered approach when the models are contaminated as in Section 4.2. In this case, the power functions are comparable.

5 Empirical example

At the core of real business cycle theories is the prediction that labour supply rises following a technological shock. A large body of literature has tackled this problem empirically. One of the first and most influential is the study by Gali (2001) who found that, if hours worked and productivity are specified as integrated processes, hours worked instead falls after a technological shock. On the other hand, Christiano et al. (2004) argued that hours worked should be considered stationary, in which case hours worked do increase after a technological shock. However, as argued by Fernald (2007), the results appear largely driven by low frequency components such as types of time trend and structural breaks in the data. The aim here is to assess whether the relation between hours worked and productivity is stable over time when allowing for possible low frequency contaminations and also whether it is stable over the business cycle frequencies.

Note that we are not concerned about addressing the issue about whether technological shock have a positive or negative impact on hours worked. This would require a full structural model that is well identified. Our concern is on the stability of the relationship between the two variables, which is a valuable starting point to analyze the structural issues of interest. To that effect, one does not have to specify a structural model. One can indeed simply use a reduced form estimated by OLS even if it involves correlation between the errors and the regressors, as shown in Perron and Yamamoto (2012).

The data used is the same as in Gali (2001) and was downloaded directly from the U.S. Department of Labor website. Labor productivity is Output per Hour of All Persons, the hours worked series is Hours of All Persons in the Business Sector and the population is measured by the Civilian Noninstitutional Population over 16 Years. All series are transformed into their natural logarithms. The data used is from 1948Q1 to 2009Q4. We consider the following reduced form equation:

$$n_t = \sum_{j=1}^{4} a_j \Delta p_{t-j} + \sum_{k=1}^{4} b_k n_{t-k} + u_t, \tag{11}$$

where n_t standards for hours worked per capita and p_t is productivity, both series being in logarithmic forms. When we use the level specification, n_t is linear detrended. When we use the difference specification, the first differences Δn_t are used for the regression[4]. Note

[4] We use demeaned data instead of including a constant in the regression. If a constant is included, this will imply a rank deficiency of the regressor matrix given that applying a discrete Fourier transform and truncating the zero frequency implies that the constant becomes zero.

that this is a part of the system estimated in Shapiro and Watson (1988). We consider possible breaks in the autoregressive coefficients a_j and report the Sup$F$, double maximum (UD max), and Sup$F(l+1|l)$ tests. If tests for breaks are significant, the estimated break dates resulting from the sequential procedure are reported. The trimming used for the permissible break dates is $\epsilon = 0.15$ and the maximum number of breaks allowed is 5. Table 7-1 reports the results of the specification (11) and Table 7-2 is for the model without lagged n_t (or Δn_t) but allowing for possible serial correlations in u_t using a heteroskedasticity and autocorrelation robust covariance matrix estimate. Each table presents results for the "full spectrum" ($0 \leq \omega \leq \pi$), the "truncated spectrum" ($L_T = \lceil \log T \rceil$), and the "business cycle band" ($\pi/16 \leq \omega \leq \pi/2$), the latter including frequencies corresponding to periods ranging from 1 to 8 years.

Consider first the results in Table 7-1 using the full spectrum (the usual time domain tests). With the level specification, the Sup$F$ test is significant at the 1% level suggesting strong evidence of at least one break in the coefficients. Since the Sup$F(2|1)$ test is insignificant, we conclude that there is a one-time structural change at 1986:Q1. With the difference specification, the results are similar although now the Sup$F$ test is significant at the 10% level and the Sup$F(2|1)$ test is significant at the 5% level. The two estimated break dates are 1967:Q1 and 1981:Q2, quite different from those obtained with the level specification. The results make it difficult to give a relevant economic interpretation. A possibility is that the tests are significant because of some low frequency components in the series, suggesting the need to apply the tests with a truncation and within the business-cycle band. When doing so, the results are very different. None of the structural change tests are significant using either the level or difference specification for hours. Hence, the results provide evidence to the effect that the relation between hours worked and productivity is stable over any spectral band that excludes the lowest frequencies, in particular it is stable over the business-cycle band.

Table 7-2 provides the results using a model without the lagged dependent variables. Here we find no break with the level specification and one break with the difference specification. The break date is estimated at 1976:Q1, which is not consistent with the previous specification with lagged dependent variables. What is noteworthy is how different the results are between the two specifications when using a full frequency regression. The differences can be ascribed to possible low frequency components in the hours and productivity series, which has been extensively discussed in the literature (e.g., Fernald, 2007, Francis and Ramey, 2009, and Gospodinov et al., 2011). However, once we exclude the low frequency contamination using a small trimming, there is no evidence of structural changes in the relation between hours and productivity in either models. Furthermore, there is no significant change when considering a business-cycle band. These results have important implications for the analysis of the effect of a technological shock on hours worked. It indicates that the various structural-based methods used to assess the sign and magnitude of this effect should be carried using a frequency band that excludes the lowest frequencies or within a business-cycle band.

6 Conclusion

We investigated methods for estimating and testing multiple structural changes using band spectral regressions. We showed that all standard results in Bai and Perron (1998) and Qu and Perron (2006) continue to hold with appropriate modifications. We documented the fact that the tests have good size in finite samples and that the estimates of the break dates obtained have good properties, including the adequacy of the limit distributions as approximations to the finite sample distributions of the estimates of the break dates. Structural change tests using band spectral regressions were shown to be more powerful than their time domain counterparts when breaks occur only within some frequency band, provided of course that the user-chosen band contains the appropriate subset. An important advantage of using a band spectral framework is that tests and estimates that are robust to low frequency contaminations can easily be obtained. We have shown that inference can be made robust to various contaminations (trends, random level shifts, etc.) by simply excluding a few frequencies near zero. We illustrated our methods by showing that the relationship between hours worked and productivity is stable if one uses estimates that are robust to such low frequency contaminations but not otherwise. This example sheds light on the importance of a careful consideration of the frequency band in estimating and testing multiple structural changes and highlights the usefulness of the methods developed in this paper.

Appendix

Proof of Lemma 1: Denote $SSR_T^{B_A}$ with $A = I$ by SSR_T^{full}. Since

$$\begin{aligned} SSR_T &= \textstyle\sum_i [Y_i - X_i\hat{\beta}_i]'[Y_i - X_i\hat{\beta}_i], \\ SSR_T^{full} &= \textstyle\sum_i [WY_i - WX_i\widetilde{\beta}_i]'[WY_i - WX_i\widetilde{\beta}_i], \\ &= \textstyle\sum_i [Y_i - X_i\widetilde{\beta}_i]'[Y_i - X_i\widetilde{\beta}_i], \end{aligned}$$

by using $W'W = I$, all we need to show is $\hat{\beta}_i = \widetilde{\beta}_i$. The result follows from the fact that

$$\begin{aligned} \widetilde{\beta}_i &= (X_i'W'WX_i)^{-1}(X_i'W'WY_i), \\ &= (X_i'X_i)^{-1}(X_i'Y_i) = \hat{\beta}_i. \end{aligned}$$

Proof of Lemma 2: For parts (a) and (b), it is easy to show from A1 that the process ζ_t^* constructed with any band has a constant spectral density at any ω. This implies the covariance stationarity of ζ_t^* from which the results follow applying standard law of large number. For parts (c) and (d), given the fact that the series ζ_t^* can be represented as a band-pass process of ζ_t, the former can be expressed as an infinite order moving average of the latter such that $\zeta_t^* = \sum_{j=-\infty}^{\infty} b_j \zeta_{t-j} = \sum_{j=-\infty}^{\infty} c_j^* \eta_{t-j}$ and

$$c_j^* = \frac{1}{2\pi}\left[\int_{\omega_l}^{\omega_h} c_j e^{i\omega j} d\omega\right] = \frac{c_j}{2\pi i j}\left(e^{i\omega_h j} - e^{i\omega_l j}\right) = c_j \frac{\sin(\omega_h j) - \sin(\omega_l j)}{2\pi j}$$

Following Phillips and Solo (1992), we need to show that $\sum_{j=0}^{\infty} j^{1/2} \left\|c_j^*\right\| < \infty$ for the invariance principle to hold. This follows given that

$$j^{1/2}\left\| c_j \left(\frac{\sin(\omega_h j) - \sin(\omega_l j)}{2\pi j}\right)\right\| \leq j^{1/2}\left\|c_j\right\| \left|\frac{\sin(\omega_h j) - \sin(\omega_l j)}{2\pi j}\right|,$$

and

$$\left|\frac{\sin(\omega_h j) - \sin(\omega_l j)}{2\pi j}\right| \leq \left|\frac{\omega_h - \omega_l}{2\pi}\right| \leq 1.$$

Proof of Theorems 1 and 2: We show that the assumptions for the original model with ζ_t are also satisfied with the model involving the series ζ_t^*. In particular, we need to show that A1-A4 hold. It is obvious that A1 holds for ζ_t^* since $f_\zeta(\omega) = f_{\zeta^*}(\omega)$ for $\omega \in [\omega_l, \omega_h]$. For A2 and A3, we know that $\sum x_t^* x_t^{*\prime} = \sum_{\omega_j \in B_A} I_x(\omega_j)$. Since $\sum x_t^* x_t^{*\prime}$ is symmetric, for any non zero $p \times 1$ vector ν,

$$\begin{aligned} \min \nu' \left[\textstyle\sum x_t^* x_t^{*\prime}\right] \nu &= \min \nu' \left[\textstyle\sum_{\omega \in B_A} I_x(\omega)\right] \nu, \\ &\geq \min \textstyle\sum_{\omega \in B_A} \nu' I_x(\omega) \nu, \\ &\geq (\mu_0 + \mu_1)\nu'\nu \end{aligned}$$

with μ_0 and μ_1 the minimum eigenvalue of $I(\omega_0)$ and $I(\omega_1)$. By A2, these are bounded away from zero. For A3, the minimum eigenvalue of $\sum_{t=k}^{l} x_t^* x_t^{*\prime}$ is shown to be strictly positive using the same argument. Since it is symmetric and the minimum eigenvalue is strictly positive, the invertibility is guaranteed. For A4, the fact that $||c_j^*|| \leq \|c_j\|$ for all j implies that properties from (a) to (e) in A4 are satisfied with $\{x_t^*, u_t^*\}$. The time domain estimates of the break dates based on model (5) can be expressed as

$$(\hat{T}_1^*, ..., \hat{T}_m^*) = \arg\max SST_T^*(T_1, ..., T_m)$$

with

$$SST_T^*(T_1, ..., T_m) = \textstyle\sum_i [W^{-1}AWY_i - W^{-1}AWX_i\widetilde{\beta}_i]'[W^{-1}AWY_i - W^{-1}AWX_i\widetilde{\beta}_i].$$

The equivalence $SST_T^{B_A} \equiv SST_T^*$ is then shown using Lemma 1 so that the Theorem 1 follows. Also, given that assumptions A1-A6 applies to the model (5), Lemma 2 implies that the limiting distributions of the Sup F test is as stated in Theorem 2 following the arguments in Perron and Qu (2006).

Proof of Lemma 3: Let $\widetilde{\delta} = \widetilde{\beta}^A - \widetilde{\beta}^{D,A}$ where $\widetilde{\beta}^{D,A}$ and $\widetilde{\beta}^A$ are the band spectral regression estimate of β obtained from the model (7) and the model (1). Then for the ith segment

$$\begin{aligned}
SSR_{T,i}^{D,B_A} &= [AWU_i + AWX_i(\beta_i - \widetilde{\beta}_i^A) + AWX_i\widetilde{\delta}_i + AWD_i]' \\
&\quad \times[AWU_i + AWX_i(\beta_i - \widetilde{\beta}_i^A) + AWX_i\widetilde{\delta}_i + AWD_i], \\
&= SSR_{T,i}^{B_A} + D_i'\Phi D_i + \widetilde{\delta}_i' X_i'\Phi X_i\widetilde{\delta}_i + 2U_i'\Phi X_i\widetilde{\delta}_i + 2(\beta_i - \widetilde{\beta}_i^A)' X_i'\Phi X_i\widetilde{\delta}_i \\
&\quad +2U_i'\Phi D_i + 2(\beta_i - \widetilde{\beta}_i^A)' X_i'\Phi D_i + 2\widetilde{\delta}_i' X_i'\Phi D_i, \\
&= SSR_{T,i}^{B_A} + D_i'\Phi D_i + I + II + III + IV + V + VI
\end{aligned}$$

By assumption A7, $X_i'\Phi D_i = o_p(1)$ and $U_i'\Phi D_i = o_p(1)$. We also have

$$\widetilde{\delta}_i = (X_i'\Phi X_i)^{-1}(X_i'\Phi D_i) = o_p(T^{-1}).$$

Hence we obtain $I = o_p(T^{-1}) \cdot O_p(T) \cdot o_p(T^{-1}) = o_p(T^{-1})$, $II = O_p(T^{1/2}) \cdot o_p(T^{-1}) = o_p(T^{-1/2})$, $III = O_p(1) \cdot O_p(T) \cdot o_p(T^{-1}) = o_p(1)$, $IV, V = o_p(1)$ and $VI = o_p(T^{-1}) \cdot o_p(1) = o_p(T^{-1})$. Consider now the term $D'\Phi D$. We have $D'\Phi D = \sum_{j_l}^{j_h} I_{d,T}(\omega_j)$. Given Assumption A8, this is bounded in probability by $CT\sum_{j_l}^{j_h}(1/j^2)$ for some large enough $C > 0$. Now, $\sum_{j_l}^{j_h}(1/j^2) \leq (1/j_l)\sum_{j_l}^{j_h}(1/j) = O(\log(T)/j_l)$ since $j_h = [\omega_h T/\pi] = O(T)$. Hence, $D'\Phi D = O_p(T\log(T)/j_l) = o_p(T)$ if $\log(T)/j_l \to 0$ as $T \to \infty$.

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Table 1. Finite sample properties of the estimates of the break dates

(T=100)

	ω_l	0	0	0	π/2-.15	π/16	π/4	π/2
	ω_h	π	π/4	π/2	π/2+.15	π/2	π	π
bias	DGP-1 for x_t	0.002	0.002	-0.002	0.001	0.003	0.011	0.002
	DGP-2 for x_t	0.012	-0.002	-0.004	0.006	-0.004	-0.012	-0.001
	DGP-3 for x_t	0.005	0.006	0.003	-0.003	-0.005	-0.011	-0.007
s.e.	DGP-1 for x_t	0.198	0.204	0.189	0.208	0.195	0.207	0.201
	DGP-2 for x_t	0.193	0.195	0.189	0.205	0.189	0.209	0.204
	DGP-3 for x_t	0.207	0.205	0.190	0.208	0.193	0.212	0.210
coverage rate	DGP-1 for x_t	0.884	0.892	0.904	0.900	0.932	0.932	0.925
	DGP-2 for x_t	0.886	0.886	0.910	0.903	0.947	0.924	0.910
	DGP-3 for x_t	0.859	0.877	0.897	0.891	0.933	0.928	0.926

(T=200)

	ω_l	0	0	0	π/2-.15	π/16	π/4	π/2
	ω_h	π	π/4	π/2	π/2+.15	π/2	π	π
bias	DGP-1 for x_t	-0.007	-0.015	-0.005	-0.001	-0.005	0.004	0.004
	DGP-2 for x_t	0.000	-0.002	-0.001	0.009	0.000	-0.016	-0.005
	DGP-3 for x_t	-0.002	-0.003	0.000	-0.001	0.000	-0.015	-0.010
s.e.	DGP-1 for x_t	0.176	0.189	0.180	0.201	0.208	0.197	0.203
	DGP-2 for x_t	0.169	0.182	0.170	0.194	0.197	0.205	0.208
	DGP-3 for x_t	0.182	0.181	0.183	0.204	0.199	0.207	0.209
coverage rate	DGP-1 for x_t	0.873	0.883	0.905	0.920	0.932	0.932	0.918
	DGP-2 for x_t	0.872	0.899	0.900	0.926	0.924	0.929	0.919
	DGP-3 for x_t	0.860	0.884	0.902	0.931	0.915	0.937	0.943

Table 2. Exact size of the SupF test; 5% nominal size

(T=100)

ω_l	0	0	0	π/2-.15	π/16	π/4	π/2
ω_h	π	π/4	π/2	π/2+.15	π/2	π	π
DGP-1 for x_t	0.07	0.08	0.05	0.11	0.07	0.05	0.06
DGP-2 for x_t	0.07	0.08	0.05	0.10	0.06	0.05	0.06
DGP-3 for x_t	0.07	0.08	0.05	0.12	0.08	0.06	0.07

(T=200)

ω_l	0	0	0	π/2-.15	π/16	π/4	π/2
ω_h	π	π/4	π/2	π/2+.15	π/2	π	π
DGP-1 for x_t	0.05	0.06	0.05	0.09	0.06	0.04	0.05
DGP-2 for x_t	0.06	0.06	0.05	0.08	0.07	0.05	0.06
DGP-3 for x_t	0.06	0.06	0.05	0.07	0.05	0.05	0.07

Table 3. Exact size of the SupF test with contaminated models (5% nominal size)

Case D1: Deterministic Level Shift

j_l	0	1	5	logT	$\log T^2$	$T^{.5}$	$T^{.6}$
DGP-1 for x_t	1.00	0.04	0.04	0.05	0.05	0.05	0.05
DGP-2 for x_t	1.00	0.03	0.04	0.03	0.04	0.03	0.05
DGP-3 for x_t	0.99	0.06	0.07	0.07	0.07	0.07	0.07

Case D2: Random Level Shifts

j_l	0	1	5	logT	$\log T^2$	$T^{.5}$	$T^{.6}$
DGP-1 for x_t	0.65	0.04	0.05	0.05	0.05	0.05	0.05
DGP-2 for x_t	0.61	0.06	0.05	0.06	0.06	0.05	0.06
DGP-3 for x_t	0.58	0.10	0.07	0.08	0.08	0.09	0.08

Case D3: Linear Trend with a Break

j_l	0	1	5	logT	$\log T^2$	$T^{.5}$	$T^{.6}$
DGP-1 for x_t	0.70	0.03	0.05	0.05	0.06	0.06	0.07
DGP-2 for x_t	0.59	0.05	0.05	0.06	0.06	0.06	0.06
DGP-3 for x_t	0.58	0.04	0.06	0.05	0.05	0.05	0.06

Case D4: Quadratic Trend

j_l	0	1	5	logT	$\log T^2$	$T^{.5}$	$T^{.6}$
DGP-1 for x_t	0.46	0.04	0.04	0.04	0.05	0.05	0.05
DGP-2 for x_t	0.41	0.04	0.05	0.05	0.06	0.06	0.07
DGP-3 for x_t	0.40	0.05	0.06	0.07	0.07	0.06	0.07

Table 4-1. Finite sample power of the SupF test with contaminated models (non size adjusted)

Case D1: Deterministic Level Shift

c	j_l						
	0	1	5	logT	$\log T^2$	$T^{.5}$	$T^{.6}$
0.0	1.00	0.04	0.04	0.05	0.05	0.05	0.05
0.1	1.00	0.07	0.09	0.09	0.10	0.10	0.10
0.2	1.00	0.29	0.31	0.31	0.31	0.30	0.30
0.3	1.00	0.60	0.60	0.61	0.60	0.59	0.58
0.4	1.00	0.86	0.84	0.85	0.84	0.83	0.81
0.5	1.00	0.97	0.96	0.97	0.95	0.95	0.95
0.6	1.00	1.00	1.00	1.00	1.00	1.00	0.99
0.7	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.9	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Case D2: Random Level Shifts

c	j_l						
	0	1	5	logT	$\log T^2$	$T^{.5}$	$T^{.6}$
0.0	0.65	0.04	0.05	0.05	0.05	0.05	0.05
0.1	0.67	0.08	0.12	0.12	0.13	0.12	0.13
0.2	0.71	0.25	0.29	0.29	0.29	0.29	0.27
0.3	0.78	0.52	0.60	0.60	0.60	0.59	0.56
0.4	0.82	0.74	0.84	0.83	0.82	0.82	0.80
0.5	0.84	0.93	0.96	0.96	0.96	0.96	0.95
0.6	0.87	0.97	0.99	0.99	0.99	0.99	0.99
0.7	0.90	0.99	1.00	1.00	1.00	1.00	1.00
0.8	0.94	1.00	1.00	1.00	1.00	1.00	1.00
0.9	0.96	1.00	1.00	1.00	1.00	1.00	1.00
1.0	0.96	1.00	1.00	1.00	1.00	1.00	1.00

Case D3: Linear Trend with a Break

c	j_l						
	0	**1**	**5**	**logT**	**logT^2**	**T$^{.5}$**	**T$^{.6}$**
0.0	0.70	0.03	0.05	0.05	0.06	0.06	0.07
0.1	0.92	0.09	0.09	0.10	0.09	0.10	0.09
0.2	0.99	0.26	0.28	0.28	0.27	0.28	0.26
0.3	1.00	0.61	0.61	0.63	0.61	0.62	0.58
0.4	1.00	0.85	0.84	0.85	0.84	0.84	0.81
0.5	1.00	0.97	0.97	0.98	0.97	0.97	0.96
0.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.7	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.9	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Case D4: Quadratic Trend

c	j_l						
	0	**1**	**5**	**logT**	**logT^2**	**T$^{.5}$**	**T$^{.6}$**
0.0	0.46	0.04	0.04	0.04	0.05	0.05	0.05
0.1	0.85	0.09	0.11	0.11	0.11	0.12	0.11
0.2	0.97	0.29	0.30	0.31	0.30	0.30	0.29
0.3	1.00	0.62	0.62	0.64	0.60	0.60	0.57
0.4	1.00	0.87	0.87	0.87	0.86	0.85	0.83
0.5	1.00	0.98	0.97	0.97	0.95	0.95	0.94
0.6	1.00	1.00	0.99	1.00	0.99	0.99	0.99
0.7	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.9	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 4-2. Finite sample power of the Sup F test with contaminated models (size adjusted)

Case D1: Deterministic Level Shift

c	j_l						
	0	1	5	logT	$\log T^2$	$T^{.5}$	$T^{.6}$
0.0	0.05	0.05	0.05	0.05	0.05	0.05	0.05
0.1	0.19	0.11	0.11	0.10	0.11	0.10	0.10
0.2	0.49	0.37	0.34	0.31	0.32	0.31	0.30
0.3	0.77	0.68	0.63	0.61	0.61	0.60	0.58
0.4	0.90	0.91	0.87	0.85	0.85	0.83	0.82
0.5	0.98	0.99	0.97	0.97	0.95	0.95	0.95
0.6	0.99	1.00	1.00	1.00	1.00	1.00	0.99
0.7	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.9	0.99	1.00	1.00	1.00	1.00	1.00	1.00
1.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Case D2: Random Level Shifts

c	j_l						
	0	1	5	logT	$\log T^2$	$T^{.5}$	$T^{.6}$
0.0	0.05	0.05	0.05	0.05	0.05	0.05	0.05
0.1	0.05	0.09	0.13	0.13	0.13	0.12	0.13
0.2	0.10	0.28	0.31	0.32	0.29	0.29	0.27
0.3	0.13	0.55	0.61	0.64	0.60	0.59	0.55
0.4	0.24	0.75	0.85	0.85	0.82	0.82	0.80
0.5	0.30	0.94	0.96	0.96	0.96	0.96	0.95
0.6	0.42	0.98	0.99	0.99	0.99	0.99	0.99
0.7	0.53	0.99	1.00	1.00	1.00	1.00	1.00
0.8	0.63	1.00	1.00	1.00	1.00	1.00	1.00
0.9	0.71	1.00	1.00	1.00	1.00	1.00	1.00
1.0	0.78	1.00	1.00	1.00	1.00	1.00	1.00

Case D3: Linear Trend with a Break

c	j_l						
	0	1	5	logT	$\log T^2$	$T^{.5}$	$T^{.6}$
0.0	0.05	0.05	0.05	0.05	0.05	0.05	0.05
0.1	0.21	0.13	0.10	0.10	0.07	0.09	0.07
0.2	0.53	0.32	0.28	0.29	0.22	0.26	0.22
0.3	0.84	0.68	0.62	0.65	0.55	0.61	0.52
0.4	0.96	0.89	0.85	0.85	0.80	0.83	0.77
0.5	1.00	0.99	0.98	0.98	0.96	0.97	0.94
0.6	1.00	1.00	1.00	1.00	1.00	1.00	0.99
0.7	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.9	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Case D4: Quadratic Trend

c	j_l						
	0	1	5	logT	$\log T^2$	$T^{.5}$	$T^{.6}$
0.0	0.05	0.05	0.05	0.05	0.05	0.05	0.05
0.1	0.22	0.11	0.13	0.13	0.11	0.12	0.11
0.2	0.56	0.33	0.32	0.34	0.30	0.30	0.29
0.3	0.89	0.67	0.66	0.66	0.61	0.61	0.57
0.4	0.99	0.91	0.88	0.88	0.86	0.86	0.83
0.5	1.00	0.98	0.97	0.98	0.95	0.95	0.94
0.6	1.00	1.00	1.00	1.00	0.99	0.99	0.99
0.7	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.8	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.9	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 5. Comparisons with filtered series using an approximate band-pass filter

		band1			band2			band3		
		BSR	BP(K=4)	BP(K=12)	BSR	BP(K=4)	BP(K=12)	BSR	BP(K=4)	BP(K=12)
bias	DGP-1 for x_t	0.004	0.001	-0.002	0.010	0.012	0.002	0.011	0.002	0.000
	DGP-2 for x_t	-0.004	0.007	0.004	-0.004	0.004	-0.002	0.002	0.000	0.008
	DGP-3 for x_t	-0.002	-0.002	-0.007	-0.002	0.002	0.014	0.004	0.005	0.006
s.e.	DGP-1 for x_t	0.192	0.225	0.224	0.204	0.222	0.211	0.214	0.231	0.234
	DGP-2 for x_t	0.187	0.223	0.220	0.203	0.220	0.206	0.207	0.224	0.228
	DGP-3 for x_t	0.190	0.228	0.220	0.201	0.220	0.211	0.207	0.226	0.233
coverage rate	DGP-1 for x_t	0.904	0.688	0.602	0.825	0.878	0.612	0.869	0.718	0.548
	DGP-2 for x_t	0.895	0.687	0.613	0.812	0.849	0.621	0.892	0.759	0.566
	DGP-3 for x_t	0.894	0.680	0.618	0.811	0.829	0.624	0.877	0.743	0.555

Table 6. Comparisons with filtered series using the Hodrick and Prescott filter

		Case D1			Case D2			Case D3			Case D4		
		BSR	HP		BSR	HP		BSR	HP		BSR	HP	
		log$T$	l=1600	l=6.25	log$T$	l=1600	l=6.25	log$T$	l=1600	l=6.25	log$T$	l=1600	l=6.25
bias	DGP-1 for x_t	0.001	-0.001	-0.002	-0.001	0.005	0.005	-0.019	0.000	0.001	0.002	0.002	0.004
	DGP-2 for x_t	-0.001	0.003	0.001	-0.002	-0.005	0.001	-0.012	0.003	0.003	0.002	0.001	0.005
	DGP-3 for x_t	-0.001	-0.006	-0.002	0.002	0.005	-0.001	-0.016	-0.001	-0.002	0.002	-0.001	0.003
s.e.	DGP-1 for x_t	0.123	0.212	0.219	0.200	0.219	0.222	0.189	0.220	0.222	0.197	0.218	0.221
	DGP-2 for x_t	0.124	0.206	0.221	0.194	0.218	0.222	0.188	0.215	0.218	0.195	0.218	0.221
	DGP-3 for x_t	0.126	0.206	0.222	0.196	0.218	0.226	0.191	0.217	0.225	0.195	0.216	0.223
coverage rate	DGP-1 for x_t	0.926	0.812	0.737	0.928	0.783	0.720	0.922	0.779	0.725	0.908	0.787	0.738
	DGP-2 for x_t	0.916	0.811	0.746	0.918	0.776	0.743	0.915	0.777	0.743	0.898	0.774	0.746
	DGP-3 for x_t	0.907	0.813	0.754	0.915	0.773	0.746	0.897	0.779	0.737	0.895	0.787	0.756

Table 7-1. Empirical results (1): model with lagged dependent variables

	Hours in Levels			Hours in First-Differences		
	full	truncated	cycle	full	truncated	cycle
SupF	23.84***	7.58	2.77	14.94*	0.69	3.48
SupF(2|1)	10.27	1.12	3.66	19.72**	8.80	6.17
SupF(3|2)	8.12	1.55	11.03	7.95	6.34	5.28
SupF(4|3)	3.49	1.12	0.47	7.13	5.09	1.64
UD max	23.84***	7.70	5.03	14.94*	7.03	6.57
Dates	1986:Q1	-	-	1967:Q1	-	-
	-	-	-	1981:Q2	-	-

Table 7-2. Empirical results (2): model without lagged dependent variables

	Hours in Levels			Hours in First-Differences		
	full	truncated	cycle	full	truncated	cycle
SupF	5.75	6.10	11.29	18.92**	0.34	5.93
SupF(2|1)	7.70	13.99	1.42	10.59	8.68	2.33
SupF(3|2)	29.36***	1.92	4.33	8.78	7.94	20.58**
SupF(4|3)	0.30	0.00	0.00	4.87	1.24	4.52
UD max	7.91	6.61	11.29	18.92**	6.36	7.00
Dates	-	-	-	1976:Q1	-	-

Note: 1. *, **, *** denote significance at the 10%, 5% and 1% levels respectively.

2. For Table 7-2, we use a heteroskedasticity and autocorrelation robust covariance estimate with a Bartlett kernel and the bandwidth chosen using Andrews' (1991) AR(1) approximation method for the full frequency results. For the truncated and the cycle results, White's (1980) heteroskedasticity robust covariance matrix estimate is used.

Figure 1. Finite sample power of the SupF test with a break common to all frequencies

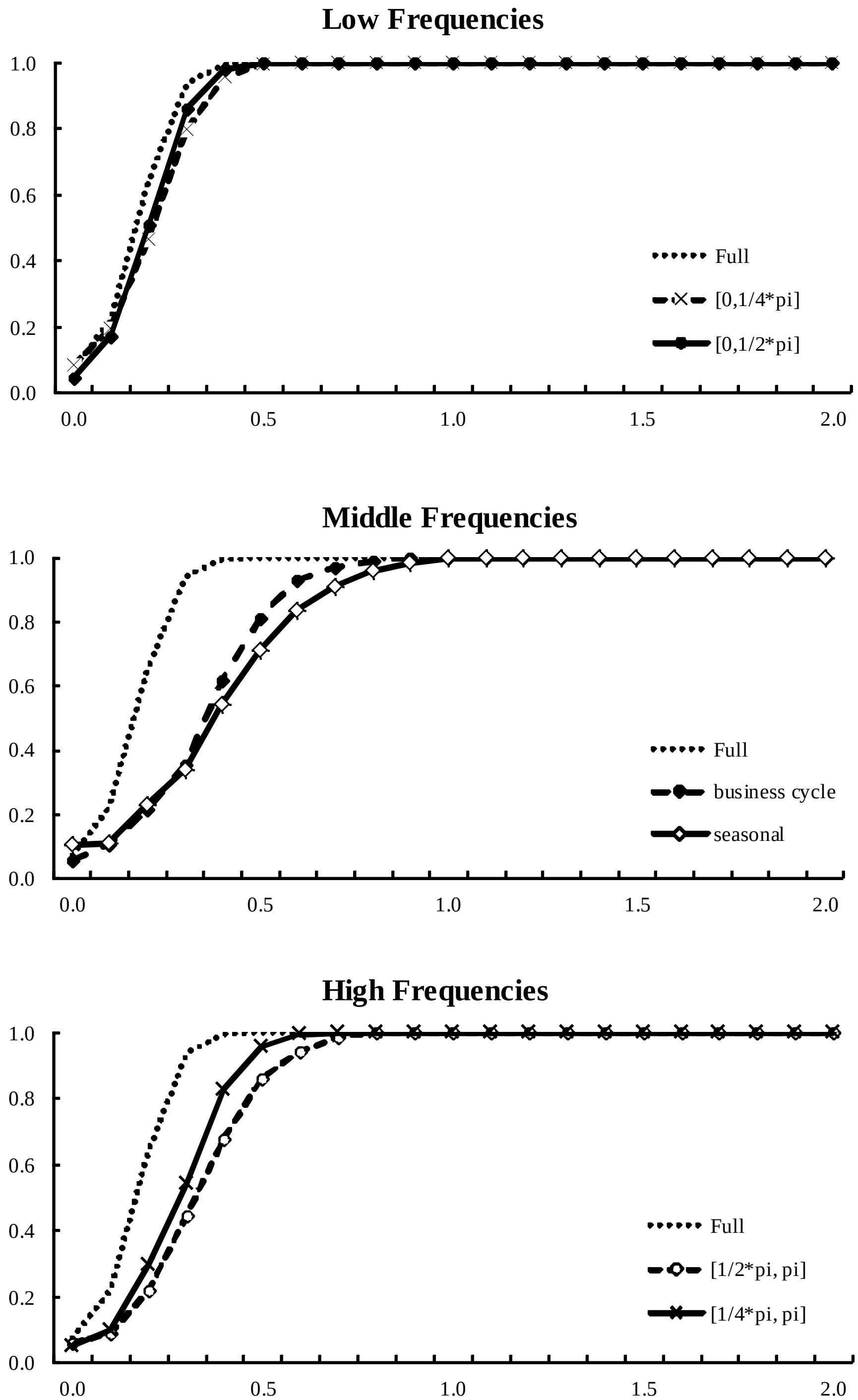
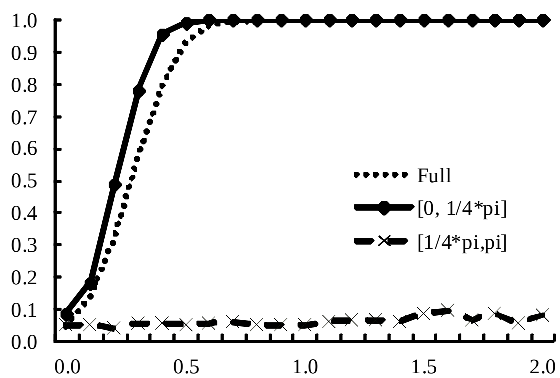
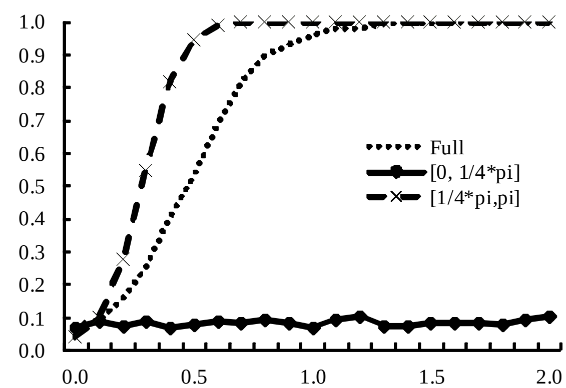


Figure 2. Finite sample power of the SupF test with a break in a particular spectral band

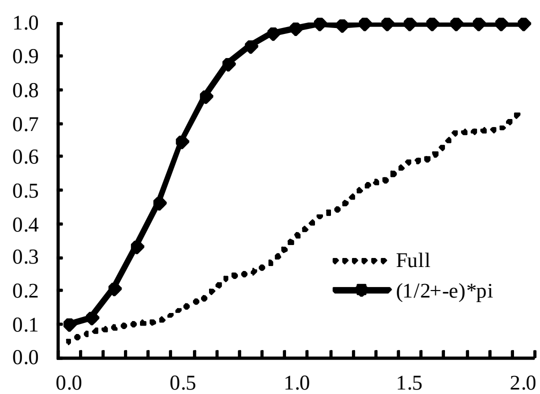
a) Break in $\omega = (0,\, \pi/4)$



b) Break in $\omega = (\pi/4,\, \pi)$



c) Break in $\omega = (\pi/2 - .15\pi,\, \pi/2 + .15\pi)$



d) Break in $\omega = (\pi/16,\, \pi/2)$

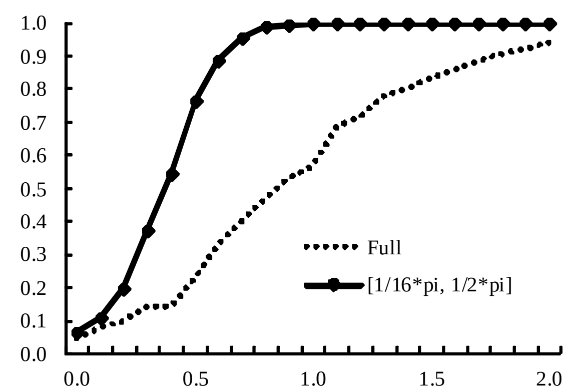
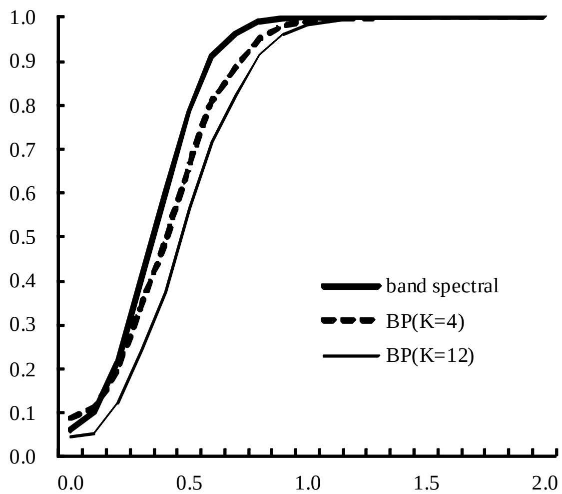
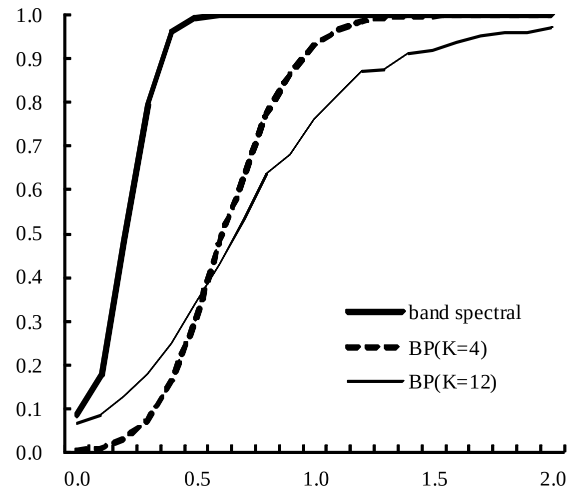


Figure 3. Finite sample power of the SupF tests: Comparisons with filtered series

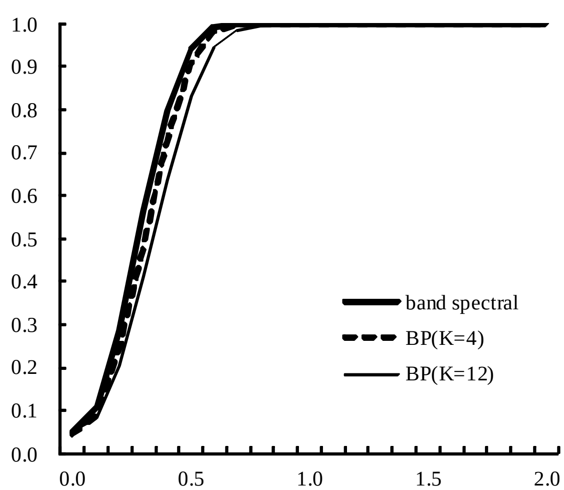
a) BP filters, band 1



b) BP filters, band 2



c) BP filters, band 3



d) HP filters