An Application of Continuous-time Stochastic Control to a Problem of Dynamic Portfolio Choice

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1. *Introduction*

The process of portfolio management is a dynamic process, that is, a sequence of decisions over time. Therefore, it is quite natural that the theory of portfolio selection, which was originally formulated as the problem of maximizing a target function at a particular date in the future by means of a single decision at the present, has recently been generalized to the analysis of intertemporal choices on asset holding.

Mossin (1968) has asked the following question: Suppose an investor is trying to maximize the expected utility of the wealth at the end of th n'th period ($n \geqq 2$). If he makes the decision to maximize the expected utility of the wealth at the end of the present period, is the decision optimal from the standpoint of the long-run optimization to maximize the expected utility of the wealth at the end of the n'th period? He has found the necessary and sufficient condition on the shape of the utility function in order for the myopic decision to be optimal from the long-run point of view. The condition is that the utility function of wealth belongs to a family of functions which includes quadratic, isoelastic functions as special cases.[1)]

The purpose of this note is to ask an equivalent question in the continuous-time model, and to obtain a similar result. In other words, this note is to provide an alternative proof to the conclusion obtained by Mossin. Since the proof in his discrete model is rather complicated, we consider our attempt to prove it in a continuous-time model worth doing.

In section 2, we shall exposite briefly the method of stochastic control with continuous-time variable. Particularly, the sufficient conditions for optimality will be examined explicitly. In section 3, we shall illustrate the use of the method by solving the problem outlined above.

2. *The Method of continuous-time Optimal Control*

Now we shall explain in an elementary way the continuous-time approach to dynamic stochastic programming.

The continuous-time formulation of programming problem under certainty, is written as follows : The state variable $x(t)$……which may be a vector in R^n……is generated by a system of ordinary differential equations,

$$\frac{dx}{dt} = f(x(t), c(t), t), \tag{1}$$

where $c(t)$ is a piecewise continuous control vector applied to the system at time t. As the target function one takes the functional

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1) See also Hamada (1969). Fama (1970) argues that the multiperiod decision rule is qualitatively similar to the single period decision rule. But, we are concerned with quantitative as well as qualitative equivalence.

$$J(c) = \int_s^T u(x(t), c(t), t)\,dt + U(x(T), T) \tag{2}$$

where s is the beginning and T is the end of the planning horizon. In the maximum principle $J(c)$ is to be maximized (or minimized) subject to : fixed initial data $(x(s), s)$, terminal data $(x(T), T)$ and the constraints on the control vector of the form $c(t) \in K$ for $s \leq t \leq T$.

For a stochastic optimal control problem, we are faced with a system of *stochastic* differential equations

$$\frac{dx}{dt} = f(x(t), c(t), t) + v(t) \tag{3}$$

where $v(t)$ is a white noise possibly multiplied by a coefficient matrix depending on the state and possibly on the control of the system at time t. For the target function we take the average or expected value of (2), namely

$$E[J(c)] = E\left[\int_s^T u(x(t), c(t), t)\,dt + U(x(T), T)\right]. \tag{4}$$

Before proceeding to solving the optimization problem, we recall the fundamental properties of the stochastic differential equation following Flemming (1970).

(3) can be written in a more rigorous form

$$dx(t) = f(x(t), c(t), t)\,dt + \sigma(x(t), c(t), t)\,dz \tag{5}$$

where z is n-dimensional brownian motion.[2)]

The meaning of (5) can be explained as follows : (5) is equivalent to the integral equation

$$x(t) = x(s) + \int_s^t f(x(\tau), c(\tau), \tau)\,d\tau + \int_s^t \sigma(x(\tau), c(\tau), \tau)\,dz(\tau). \tag{5'}$$

The $\int_s^t \sigma dz$ should be interpreted in Ito's sense that

$$\int_s^t \sigma(x(\tau), c(\tau), \tau)\,dz(\tau) = \underset{h\to 0}{\text{l.i.m.}} \sum_{j=1}^{N-1} (x(\tau_j), c(\tau_j), \tau_j)(z(\tau_{j+1}) - z(\tau_j)) \tag{6}$$

where $s = \tau_1 < \tau_2 < \cdots\cdots < \tau_N = t$, $h = \max(\tau_{j+1} - \tau_j)$.[3)]

Suppose we can observe the state vector without lag. Let us assume that there exists the optimal value of $E[J(c)]$. Then it must be the function of the initial data, that is

$V(x(s), s) = \max E[J(c)]$. Then according to the principle of Dynamic programming (Bellman 1957), the following functional equation should hold.[4,5)]

$$\frac{\partial V}{\partial s} + \max_c [A(x, c, s)V + u(x, c, s)] = 0. \tag{7}$$

2) A real stochastic process $z(t)$ on (s, T) is called one-dimensional brownian motion

i) z is a continuous process (namely, continuous with probability one

ii) $z(t) - z(s)$ has a normal distribution with mean 0 and variance $t-s$,

iii) z has independent increments, i. e. if $s \leq t_1 \leq t_2 \leq \cdots \leq t_N \leq T$, $z(t_{j+1}) - z(t_j)$ are independent for $j=1, \cdots, N-1$.

$z = (z_1, \cdots, z_n)$ is called an n-dimensional brownian motion if $z_1 \cdots z_n$ are independent one-demensional brownian motions.

3) We should note that this integral consists of the limit of the terms which evaluate σ at the initial point of the small time-interval.

4) For example, Florentin (1961), Fleming (1970).

5) From now on we omit the time variable in $x(t)$ and $c(t)$ if it does not lead to confusion.

where $\Lambda(x, c, s)$ is a partial differential operator such that

$$\Lambda(x, c, s) = \sum_{i,j=1}^{n} a_{ij}(x, c, s)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} f_i(x, c, s)\frac{\partial}{\partial x_j} \qquad \text{where } \|a_{ij}\| = \sigma\sigma^T/2. \tag{8}$$

Equation (7) should be solved with the boundary data

$$V(x, T) = U(x, T). \tag{9}$$

The derivation of (7) is illustrated in the one-dimensional case in the following way: According to the principle of optimality, the following relationship should hold for the small Δs.

$$V(x, s) = \max_c \left[u(s, c, s)\Delta s + \int V(x+\Delta x, s+\Delta s)\, dP(\Delta x/x, s) \right] + o(\Delta x) \tag{10}$$

where $P(\Delta x/x, s)$ is the conditional probability distribution given $x(s)$ and s defined by the stochastic differential equation (5).

Let us expand V under the integral sign formally in the Taylor's series,

$$V(x+\Delta x, s+\Delta s) = V(x, s) + \frac{\partial V}{\partial x}\Delta x + \frac{\partial V}{\partial s}\Delta s + \frac{\partial^2 V}{\partial x^2}\frac{(\Delta x)^2}{2!} + \frac{\partial^2 V}{\partial s^2}\frac{(\Delta s)^2}{2!} + \frac{\partial^2 V}{\partial x \partial s}\Delta x \Delta s + \cdots \tag{11}$$

If we substitute (11) into (10), noticing

$$\int dP(\Delta x/x, s) = 1$$

we obtain

$$\begin{aligned} V(x, s) = \max_c \Bigg[& u(x, c, t) + V(x, s) + \frac{\partial V}{\partial x}\int \Delta x dP(\Delta x/x, s) + \frac{\partial V}{\partial s}\Delta s \\ & + \frac{\partial^2 V}{\partial x^2}\int (\Delta x)^2 dP(\Delta x/x, s) + \frac{\partial^2 V}{\partial s^2}\frac{(\Delta s)^2}{2} + \Delta s \frac{\partial^2 V}{\partial x \partial s}\int \Delta x dP(\Delta x/x, s) + \cdots \Bigg] + o(\Delta s) \end{aligned} \tag{12}$$

Since from (5)

$$\int \Delta x dP(\Delta x/x, s) = f\Delta t + o(\Delta t)$$

$$\int (\Delta x)^2 dP(\Delta x/x, s) = \sigma^2 \Delta t + o(\Delta t)$$

by dividing (12) by Δt and putting $\Delta t \to 0$, the higher order of expression than two can be neglected. Therefore we obtain in the one-dimensional case,

$$\frac{\partial V}{\partial s} + \max_c \left[f(x, c, s)\frac{\partial V}{\partial x} + \frac{1}{2}\left\{\sigma(x, c, s)\right\}^2 \frac{\partial^2 V}{\partial x^2} + u(x, c, s) \right] = 0 \tag{13}$$

which is a special case of (7). Similarly, one can deduce formally the functional equation (7) in the case of higher dimensions.

It is not always easy, however, to find the solution of the nonlinear partial differential equation (7). In order to prove the existence of a solution, that is, to prove the existence of the optimal control, some additional conditions are required. Setting aside the existence problem, let us consider here the sufficiency conditions for the optimality of the solution obtained from the functional equation above.

To prove the sufficiency of the solution satisfying (7) and (9), we need the generalized version of the Ito's lemma that states : *If x satisfies the stochastic dffierential equation* (5) *and $\Phi(x, s)$ has continuous partial derivatives Φ_{x_i}, Φ_s, $\Phi_{x_i x_j}$ $i, j = 1, \cdots, n$, then $\Phi(x, s)$ satisfies the equation*

$$d\Phi(x(t), t) = (\Phi_s + \Lambda\Phi)\, dt + \Phi_x \sigma dz, \tag{14}$$

operator Λ being defined by (8).

By integrating (14) from s to T and taking expected values, we obtain the formula

$$(15)\qquad E\Phi(x(T), T) - E\Phi(x(s), s) = E(\Phi_s + \Lambda\Phi)\,dt,$$

provided that a certain conditions[6] are satisfied which are needed to reduce $E\int \Phi_x\,\sigma dz$ to zero.

If $V(x, s)$ satisfies equation (13) under the control policy $\hat{c}$, then for any other feasible control c,

$$(16)\qquad V_s + \Lambda(x, c, t)\,V + u(x, c, t) \leqq V_s + \Lambda(x, \hat{c}, t)\,V + u(x, \hat{c}, t) = 0.$$

By integrating

$$(17)\qquad \int_s^T (V_s + \Lambda^c V)\,dt + \int_s^T u^c dt \leqq \int_s^T (V_s + \Lambda^{\hat{c}} V)\,dt + \int_s^T u^{\hat{c}} dt = 0.$$

where $\Lambda^{\hat{c}} = \Lambda(x, \hat{c}, t)$, $u^{\hat{c}} = u(x, \hat{c}, t)$ and so on.

By taking expectation and using (15), we get, (E^c being expectation corresponding to control c)

$$(18)\qquad E^c[V(x(T), T) - V(x(s), s)] + E^c\int_s^T u(x(t), c(t), t)\,dt$$
$$\leqq E^{\hat{c}}[V(x(T), T) - V(x(s), s)] + E^{\hat{c}}\int_s^T u(x(t), \hat{c}(t), t)\,dt = 0$$

That is, by the boundary condition (9),

$$(19)\qquad E^c\left[\int_s^T u(x(t), c(t), t)\,dt + U(x(T), T)\right] \leqq E^{\hat{c}}\left[\int_s^T u(x(t), \hat{c}(t), t)\,dt + U(x(T), T)\right.$$
$$= V(x(s), s),$$

Therefore, $\hat{c}(t)$ is optimal.

Thus we can state the following sufficiency theorem :

Suppose $V(x, s)$ is a solution of (7) and (9) such that

a) *V is continuous on $Q = (T_0, T) \times B$, B being a subset in R,*

b) *the partial derivatives V_s, V_{x_i}, $V_{x_i x_j}$ are continuous on Q, $i, j, = 1, \cdots, n$*

c) *in case B is bounded there exist positive constants C, γ such that*

$$(20)\qquad V(x, s) \leqq C(1 + |x|)^{\gamma}.$$

Moreover, suppose that c is a control policy such that for almost all $(x, s) \in Q$, $\Lambda(x, c, s)\,V + u(x, c, s)$ is maximum on K when $c = \hat{c}(x, s)$.

Then $\hat{c}$ is the optimal policy.

Let us now turn to the infinite horizon problem. If the maximand is the following discounted sum

$$(21)\qquad E\int_0^{\infty} u(x(t), c(t))\,e^{-\beta t} dt,$$

this includes familiar problems in economics. In the case of infinite horizon, if the generating function is autonomous, the control and the optimand become functions independent of time. The functional equation is reduced to

$$(22)\qquad \max[u(x, c) + \Lambda(x, c)\,V - \beta V] = 0.$$

Thus the parabolic partial differential equation is reduced to an elliptic partial differential equation. If this is one-dimensional problem, then the equation is an ordinary differential equatoin of the second order, which is easier to solve. However, since we do not have the boundary condition like (9), we need

6) These conditions are stated in the sufficiency theorem below as (a), (b), (c). One can see that the definition of stochastic integral (6) plays a crucial role in cancelling $E\int \Phi_x \sigma\, dz = 0$

an additional transversality condition. This fact gives difficulty in some cases. The following sufficiency theorem is known (Kushner 1967).

Suppose $V(x)$ is a solution of (22) with continuous second derivatives, and suppose that

$$\lim_{T\to\infty} E^{\hat{c}} V(x) e^{-\beta T} = 0 \tag{23}$$

Then $V(x)$ and the corresponding control $\hat{c}(x)$ gives the maximand among the all admissible $c(x)$ satisfying

$$\lim_{T\to\infty} E^{c}[V(x) e^{-\beta T}] = 0 \tag{24}$$

The outline of proof is as follows: Applying the Ito's lemma to $V(x)e^{-\beta t}$ instead of $V(x)$, we can get from (22) the following inequality for a large T,

$$E^c \int_0^T u^c e^{-\beta t} dt + E^c V(x) e^{-\beta T} \leqq E^{\hat{c}} \int_0^T u^{\hat{c}} e^{-\beta T} dt + E^{\hat{c}} V(x) e^{-\beta T} = V(x) \tag{25}$$

If (23) and (24) are satisfied

$$\begin{aligned} &E^{\hat{c}} \int_0^T u(\hat{x}, \hat{c}) e^{-\beta t} dt - E^c \int_0^T u(x, c) e^{-\beta t} dt \\ &\geqq E(V(x) - V(\hat{x})) e^{-\beta T} \xrightarrow[T\to\infty]{} 0 \end{aligned} \tag{26}$$

This proves the optimality of $\hat{c}(x)$.

3. *An Alternative Proof to the Mossin's Problem*

Suppose an individual or a mutual fund is trying to maximize the expected value of the utility $U(x(T))$ defined on the stock of wealth x at the end of the planning period T. He is assumed to choose a combination of a risky asset and a riskless asset with a fixed rate of return. Let us assume that the rate of return on the riskless asset is i and that the rate of return on the risky asset is generated by the Gaussian process with stationary mean μ and variance σ. Then if we denote the proportion of risky asset in the total wealth as θ, the generating function of the wealth in the continuous version is written,

$$dx = \{(\mu - 1\}\theta + i\} x\, dt + \theta \sigma x\, dz, \tag{27}$$

where z is the normalized brownian motion in a single dimension. The choice problem for the decision maker is to maximize $E[U(x(T))]$ with respect to $\theta(t)$ subject to (27) and the given value of $x(s)$, s being the initial time. Then the Bellman equation can be written, $V = \max E[V(x(T))]$

$$\frac{\partial V}{\partial s} + \max_{\theta} \left\{ ((\mu - 1)\theta + i) x \frac{\partial V}{\partial x} + \frac{\theta^2 x^2 \sigma^2}{2} \frac{\partial^2 V}{\partial x^2} \right\} = 0 \tag{28}$$

The question to be asked in this section is: What restrictions on the utility function U are needed in order that the "myopic" decision, that is, the decision to maximize $E[U(x(T'))]$ $(T' < T)$ is optimal also for long-run maximization of $E[(U(x(T))]$? In other words, the question is whether the length of time to the final date influences the decision. This is what Mossin has called the presence or absence of the time effect.

Formally, the question is under what conditions the two problems: max: $E[U(x(T'))]$ and max: $E[U(x(T))]$ give identical optimal solutions. This question is again equivalent to the question of under what conditions the solution $V(x, t)$ and $V(x, s)$ yield the same θ for $t \neq s$. The conclusion is true if and only if the optimal θ is independent of time.

Assuming the internal solution, we obtain

(29) $$\theta = -\frac{\mu - i}{\sigma^2}\frac{1}{x}\frac{\partial V}{\partial x}\bigg/\frac{\partial^2 V}{\partial x^2}$$

If θ is independent of time, then

(30) $$\frac{1}{x}\frac{\partial V}{\partial x}\bigg/\frac{\partial^2 V}{\partial x^2} = g(x)$$

should be solved.

Following solution is easily found,

(31) $$V = A(s)\int e^{\int \frac{dx}{xg(x)}}dx + B(s)$$

(28) is rewritten in view of (29)

(32) $$\frac{\partial V}{\partial s} + ix\frac{\partial V}{\partial x} - \frac{(\mu - i)^2}{2\sigma^2}\left(\frac{\partial V}{\partial x}\right)^2\bigg/\frac{\partial^2 V}{\partial x^2} = 0.$$

Then by substituting (31) into (32) we obtain

(33) $$A'(s)\int e^{\int \frac{dx}{xg(x)}}dx + B'(s) + ixA(s)e^{\int \frac{dx}{xg(x)}} - \frac{(\mu - i)^2}{2\sigma^2}xg(x)A(s)e^{\int \frac{dx}{xg(x)}} = 0$$

Differentiating (33) by x and dividing by $A(s)e^{\int \frac{dx}{xg(x)}}$ we get

$$\frac{A'(s)}{A(s)} = i\left\{1 + \frac{1}{g(x)}\right\} - \frac{(\mu - i)^2}{2\sigma^2}\{1 + g(x) + xg'(x)\}$$

Since both sides are functions of s and x independently, they must be constant. Therefore we should find $g(x)$ which satisfies

(34) $$i\left(1 + \frac{1}{g(x)}\right) - \frac{(\mu - i)^2}{2\sigma^2}(1 + g(x) + xg'(x)) = \text{const.}$$

Here we should look for the solutions of (34) that are independent of parameters i, μ or σ. For we are not interested in the utility functions which permit the myopic decision only when parameters of distribution take some specific values. Thus we obtain the following type of solutions to (34):

If $i = 0$, then we have

(35) $$g(x) + xg'(x) = \text{const.}$$

that is $g(x) = a + \frac{b}{x}$, which generate the following type of $F(x)$.

(36) $$F(x) = \begin{cases} A(ax + b)^r + B & \text{with its limiting forms of} \\ A\log(a'x + b') + B & \\ Ae^{a''x} + B. & A, B \text{ being integration constant.} \end{cases}$$

If $i \neq 0$, we have two types of solutions.

(37) $$g(x) = \text{const.}$$

(38) $$C + \log x = -\frac{1}{p+q}[p\log(y - p) + q\log(y + q)]^{7)}$$

or

$$[(y - p)^p(y + q)^q]^{-\frac{1}{p+q}} = Cx,$$

where $y = gx$, C is an integration constant, and p and $(-q)$ — p, $q > 0$ — are two roots of the equation

7) Use the transformation $\log x = x'$.

$$\lambda^2-\gamma\lambda-\frac{2\sigma^2 i}{(\mu-i)^2}=0,\quad \gamma \text{ being an arbitrary constant.}$$

The second type of solutions generates $F(x)$ which is crucially dependent on the value of parameters. Therefore if $i\neq 0$, we have $g(x)=$const. which generates the type of $F(x)$ as,

$$F(x)=\begin{cases} Ax^{\gamma}+B \\ A\log x+B \end{cases} \tag{39}$$

If we combine the solution $F(x)$ of (32) and $A(s)$ to form $V(x, s)$ and if we note that

$$V(x, T)=U(x, T)$$

then we can conclude that the utility functions $U(x)$ which allow the myopic decision to be optimal coincides with the type of $F(x)$.

Thus the utility function which permits myopic decision, or which excludes the time effect is either one of the above type of functions given by (36) and (39). This result is the same as Mossin obtained in his discrete-time model.

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