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MAXIMAL FUNCTIONS FOR DISTRIBUTIONS ON OPEN SETS

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Notation

The following notations will be used throughout this note.
If Ω is an open subset of $\mathbf{R}^n$, then $C_0^\infty(\Omega)$ denotes the set of smooth functions with compact support in Ω, and $\mathscr{D}'(\Omega)$ denotes the set of distributions on Ω. If $f\in\mathscr{D}'(\Omega)$ and $\phi\in C_0^\infty(\Omega)$, then $\langle f, \phi\rangle$ denotes the value of f evaluated at ϕ.
A multi-index is an n-tuple of nonnegative integers. If $\alpha=(\alpha_1, \dots, \alpha_n)$ is a multi-index and $x=(x_1, \cdots, x_n)\in\mathbf{R}^n$, then

$$|\alpha|=\alpha_1+\cdots+\alpha_n,$$
$$x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n},$$
$$|x|=(\sum_{j=1}^n x_j^2)^{1/2},$$
$$\partial_x^\alpha f(x)=(\partial^\alpha f)(x)=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_1)^{\alpha_n}f(x).$$

If s is a real number, $[s]$ denotes the integer which is defined by $[s]\leqq s<[s]+1$.
$\mathbf{N}$ denotes the set of positive integers.
Let Ω be an open subset of $\mathbf{R}^n$ and $0<p\leqq\infty$. For measurable functions f on Ω, we set

$$||f||_{p,\Omega}=\begin{cases}\left(\int_\Omega |f(x)|^p dx\right)^{1/p} & \text{if } 0<p<\infty\\ \text{ess. sup } \{|f(x)|;\ x\in\Omega\} & \text{if } p=\infty.\end{cases}$$

We set $L^p(\Omega)=\{f;\ ||f||_{p,\Omega}<\infty\}$. If $\Omega=\mathbf{R}^n$, we denote $||f||_{p,\Omega}$ simply by $||f||_p$.
$L^1_{\text{loc}}(\mathbf{R}^n)$ denotes the set of locally integrable functions on $\mathbf{R}^n$.
Let Ω' and Ω be open subsets of $\mathbf{R}^n$ with $\Omega'\subset\Omega$. If $f\in\mathscr{D}'(\Omega)$ or if f is a function defined on Ω, then $f|\Omega'$ denotes the restriction of f to Ω'.
If E is a measurable subset of $\mathbf{R}^n$, then $|E|$ denotes the Lebesgue measure of E and χ_E denotes the characteristic function of E.
supp f denotes the support of f.
$f*g$ denotes the convolution of f and g.

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I. *Introduction*

One of the results of C. Fefferman and E.M. Stein [5] reads as follows. Take a function $\phi\in C_0^\infty(\mathbf{R}^n)$ with $\int\phi(x)dx=1$. For any $f\in\mathscr{D}'(\mathbf{R}^n)$, define $f^{+\phi}(x)$, $x\in\mathbf{R}^n$, by

$$f^{+\phi}(x)=\sup_{t>0}|(f*t^{-n}\phi(t^{-1}\cdot))(x)|.$$

For $N\in\mathbf{N}$, $x\in\mathbf{R}^n$ and $t>0$, we denote by $\mathscr{B}_N(x, t)$ the set of those $\psi\in C_0^\infty(\mathbf{R}^n)$ such that $\operatorname{supp}\psi\subset\{y\in\mathbf{R}^n;\ |y-x|<t\}$ and $|\partial_y{}^\alpha\psi(y)|\leqq t^{-n-|\alpha|}$ for multi-indices α with $|\alpha|\leqq N$. For any $f\in\mathscr{D}'(\mathbf{R}^n)$, define $f^{*N}(x)$, $x\in\mathbf{R}^n$, by

$$f^{*N}(x)=\sup\ \{|\langle f, \psi\rangle|;\ \psi\in\bigcup_{t>0}\mathscr{B}_N(x, t)\}.$$

Then the following holds: If $p>0$ and N is sufficiently large, the inequality

$$||f^{*N}||_p\leqq C_{n,p,N,\phi}||f^{+\phi}||_p \tag{1.1}$$

holds for all tempered distributions f. (See [5; §11].)

In proving the above result, Fefferman and Stein used Fourier transform; this is the reason why their result was restricted to tempered distributions.

In [11] and [12], Uchiyama gave an alternate proof to the above result, which did not use Fourier transform, and at the same time extended the result. Among other things, he proved, for all $f\in\mathscr{D}'(\mathbf{R}^n)$ and for appropriate $q>0$ and N, the following pointwise estimate:

$$f^{*N}(x)\leqq C_{n,q,N,\phi}M_q(f^{+\phi})(x),$$

where $M_q(\cdot)$ is defined by

$$M_q(g)(x)=\sup_{t>0}(t^{-n}\int_{|y-x|<t}|g(y)|^q dy)^{1/q}.$$

The inequality (1.1) is a consequence of this estimate.

The purpose of this note is to further extend Uchiyama's result. In particular, we shall consider distributions on arbitrary open subsets of $\mathbf{R}^n$.

In the next section, Section II, we shall give our results and explain the relationship between Uchiyama's results and our results in detail.

Sections III and IV will be devoted to the proofs of our results.

II. *Main Results*

Let $\{A(t);\ t>0\}$ be a set of linear transformations of $\mathbf{R}^n$ with the following properties: $t\to A(t)x$ is continuous for every $x\in\mathbf{R}^n$, $A(t)A(s)=A(ts)$, $A(1)=I$ (the identity operator), and $|A(t)x|\geqq t|x|$ if $t\geqq 1$. Let γ be the positive number for which $\det A(t)=t^\gamma$. For each $x\in\mathbf{R}^n\setminus\{0\}$, we denote by $\rho(x)$ the uniquet $t>0$ for which $|A(t^{-1})x|=1$; we set $\rho(0)=0$. This function ρ has the following properties:

$$\rho(x+y)\leqq\rho(x)+\rho(y),$$

$$\rho(-x)=\rho(x),$$

$$\rho(A(t)x)=t\rho(x),$$

$$\rho(x)\leqq 1 \quad \text{if and only if} \quad |x|\leqq 1,$$
$$|x|\leqq\rho(x) \quad \text{if} \quad |x| \quad \text{or} \quad \rho(x)\leqq 1,$$
$$\rho(x)\leqq|x| \quad \text{if} \quad |x| \quad \text{or} \quad \rho(x)\geqq 1.$$

For $x\in\mathbf{R}^n$ and $t>0$, we set

$$B(x, t)=\{y\in\mathbf{R}^n;\ \rho(y-x)<t\}.$$

It holds that $|B(x, t)|=|B(0, 1)|t^r$. For $x\in\mathbf{R}^n$ and $E\subset\mathbf{R}^n$, we set

$$\operatorname{dis}(x, E)=\begin{cases}\inf\{\rho(x-y);\ y\in E\} & \text{if} \quad E\neq\phi\\ \infty & \text{if} \quad E=\phi.\end{cases}$$

For any function ϕ on $\mathbf{R}^n$ and for $t>0$, we define $(\phi)_t$ by

$$(\phi)_t(x)=t^{-r}\phi(A(t^{-1})x).$$

As for these matters, see Calderón-Torchinsky [2].

Hereafter, the letter C denotes a constant, which may be different in each occasion. The constant C depends only on the dimension n, the group $\{A(t)\}$ and other explicitly indicated parameters.

In the following, $\mathscr{P}_m$ denotes the set of polynominals on $\mathbf{R}^n$ of order not exceeding m.

Let $a>0$ and f a function defined on $\mathbf{R}^n$. If $f\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, then we set

$$||f||_{\Lambda(a)}=\sup_{x,t}\left\{\inf_h t^{-r-a}\int_{B(x,t)}|f(y)-h(y)|dy\right\},$$

where the infimum is taken over all $h\in\mathscr{P}_{[a]}$ and the supremum is taken over all $x\in\mathbf{R}^n$ and all $t>0$. If $f\notin L^1_{\mathrm{loc}}(\mathbf{R}^n)$, then we set $||f||_{\Lambda(a)}=\infty$. We denote by $\Lambda(a)$ the set of those f for which $||f||_{\Lambda(a)}<\infty$.

Let $a>0$, Ω an open subset of $\mathbf{R}^n$ and f a function defined on Ω. We denote by $\tilde{f}$ the extension of f to $\mathbf{R}^n$ whose value is equal to zero outside Ω. If $\Omega\neq\mathbf{R}^n$, we set

$$||f||_{\Lambda(a;\Omega)}=||\tilde{f}||_{\Lambda(a)}+\sup_{x\in\Omega}\{|f(x)|(\operatorname{dis}(x, \Omega^c))^{-a}\}.$$

If $\Omega=\mathbf{R}^n$, we set $||f||_{\Lambda(a;\Omega)}=||f||_{\Lambda(a)}$. We denote by $\Lambda(a;\Omega)$ the set of those f for which $||f||_{\Lambda(a;\Omega)}<\infty$.

REMARK 2.1. If $A(t)=tI$, the following hold.

(1) The space $\Lambda(a)$ coincides, modulo polynominals, with the homogeneous Besov space $B_{\infty,\infty}{}^a$. As for this fact, see, for example, [8].

(2) Let $a>0$ and $a\notin\mathbf{N}$. Then functions of class $\Lambda(a)$ can be characterized by Lipschitz continuity of order $a-[a]$ of their derivatives of order $[a]$ (see, *e.g.*, [8; §6.2]). Using this characterization, we easily see that the inequalities

$$||f||_{\Lambda(a)}\leqq||f|\Omega||_{\Lambda(a;\Omega)}\leqq C_a||f||_{\Lambda(a)}$$

hold for any Ω, open subset of $\mathbf{R}^n$, and for any function f on $\mathbf{R}^n$ with $\operatorname{supp} f\subset\Omega$. (As for a related result, see 307 of the next section.)

(3) The above result does not hold if $a\in\mathbf{N}$. This can be seen from the following example. Let $n=1$. Set

$$f(x)=\begin{cases}x\log x & \text{if} \quad x>1\\ 0 & \text{if} \quad |x|\leqq 1\\ x\log|x| & \text{if} \quad x<-1.\end{cases}$$

Then, $f \in \Lambda(1)$ and $\operatorname{supp} f \subset \{x; |x|>1/2\}=\Omega$, whereas the inequality

$$|f(x)| \leqq C\|f\|_{\Lambda(1)} \operatorname{dis}(x, \Omega^c)$$

does not hold.

Let Φ be a function defined on $\mathbf{R}^n \times \mathbf{R}^n \times (0, \infty)$. Let $0<L\leqq\infty$ and K a positive function defined on the interval $(0,L)$. We assume Φ, L and K satisfy the conditions (i)~(viii) below.

(i) $\Phi(x, \cdot, t) \in C_0^\infty(B(x, t))$ for all $x \in \mathbf{R}^n$ and $t>0$.

(ii) $\|\Phi(x, \cdot, t)\|_{\Lambda(b)} \leqq K_b t^{-\gamma-b}$ for all $x \in \mathbf{R}^n$, $t>0$ and $b \in (0, L)$.

(iii) $\Phi(\cdot, y, t) \in L^1(\mathbf{R}^n)$ and $\int_{\mathbf{R}^n} \Phi(x, y, t)dx=1$ for all $y \in \mathbf{R}^n$ and $t>0$.

(iv) For a bounded function k on $\mathbf{R}^n$ with compact support and for $t>0$, we define $k\#\Phi(t)$ by

$$(k\#\Phi(t))(y)=\int k(x)\Phi(x, y, t)dx;$$

then, for all $b \in (0, L)$ and $t>0$, we have

$$\|k\#\Phi(t)\|_{\Lambda(b)} \leqq K_b \|k\|_{\Lambda(b)}.$$

(v) If k is a bounded function with compact support and $t>0$, then $k\#\Phi(t)$ is a smooth function.

(vi) For any $f \in \mathscr{D}'(\mathbf{R}^n)$ and for any $t>0$, the function $x \to \langle f, \Phi(x, \cdot, t)\rangle$ is locally integrable on $\mathbf{R}^n$.

(vii) If k is a bounded function with compact support, $f \in \mathscr{D}'(\mathbf{R}^n)$ and $t>0$, then

$$\langle f, k\#\Phi(t)\rangle=\int k(x)\langle f, \Phi(x, \cdot, t)\rangle dx.$$

(viii) For any open subset Ω of $\mathbf{R}^n$ and for any $f \in \mathscr{D}'(\Omega)$, we define the *radial maximal function* $M_{\Phi,\Omega}{}^+(f)(x)$, $x \in \Omega$, by

$$M_{\Phi,\Omega}{}^+(f)(x)=\sup\{|\langle f, \Phi(x, \cdot, t)\rangle|; 0<t<\operatorname{dis}(x, \Omega^c)\};$$

the final condition is that $M_{\Phi,\Omega}{}^+(f)$ is measurable for any Ω and f as mentioned above.

REMARK 2.2. (1) The estimate $|\Phi(x, y, t)| \leqq C_b K_b t^{-\gamma}$ holds. This can be proved by the use of 307 of the next section.

(2) The condition (iv) is satisfied if, together with the other conditions, the following condition is satisfied: For multi-indices α with $|\alpha|<L$, the integral $\int \Phi(x, y, t)(x-y)^\alpha dx$ does not depend on y. As for a proof of this fact, see Uchiyama [11; Proof of Lemma 4]. (In the notation of [11], our space $\Lambda(a)$ is denoted by $\Lambda_{a/\gamma}$.)

(3) If $\phi \in C_0^\infty(B(0, 1))$ and $\int \phi(x)dx=1$, and if we set $\Phi(x, y, t)=(\phi)_t(x-y)$, then this Φ satisfies the conditions (i)~(viii) with $L=\infty$.

Before we state our results, we introduce a terminology. Let $b>0$, Ω an open subset of $\mathbf{R}^n$, and $f \in \mathscr{D}'(\Omega)$. Then we say f is of *order b* if there exists a constant M such that

$$|\langle f, \phi\rangle| \leqq M\|\phi\|_{\Lambda(b;\Omega)} \text{ for all } \phi \in C_0^\infty(\Omega).$$

If Ω' is an open subset of Ω, we say f is of order b on Ω' if the restriction $f|\Omega'$ is of order b.

The following is our main theorem.

THEOREM. *Let Ω be an open subset of $\mathbf{R}^n$ and let $f\in\mathscr{D}'(\Omega)$. Suppose for every relatively compact open subset Ω' of Ω there exists $b_{\Omega'}\in(0, L)$ for which f is of order $b_{\Omega'}$ on Ω'. Then the inequality*

$$|\langle f, \psi\rangle|\leqq C_{a,L,K}||\psi||_{\Lambda(a;\Omega)}||M_{\Phi,\Omega}{}^+(f)||_{\gamma/(\gamma+a),\Omega}$$

holds for all $a\in(0, L)$ and all $\psi\in C_0^\infty(\Omega)$.

REMARK 2.3. From the definition of distribution and 306.2 of the next section, we see that every $f\in\mathscr{D}'(\Omega)$ is of finite order (*i.e.*, of order b, in our terminology, for some $b\in(0, \infty)$) on every relatively compact open subset of Ω. Hence, if $L=\infty$, every $f\in\mathscr{D}'(\Omega)$ satisfies the assumption of the Theorem.

In order to give some corollaries to the Theorem, we introduce some maximal functions. Let Ω be an open subset of $\mathbf{R}^n$ and let $f\in\mathscr{D}'(\Omega)$.

First, for any $a>0$ we define the *grand maximal function* $M_{a,\Omega}{}^*(f)(x)$, $x\in\Omega$, by

$$M_{a,\Omega}{}^*(f)(x)=\sup_{\psi}|\langle f, \psi\rangle|,$$

where the supremum is taken over those ψ for which there exist $y\in\Omega$ and $t>0$ such that

$$\begin{cases} x\in B(y, t)\subset\Omega, \\ \psi\in C_0^\infty(B(y, t)), \\ ||\psi||_{\Lambda(a)}\leqq t^{-\gamma-a}. \end{cases} \tag{2.1}$$

This function $M_{a,\Omega}{}^*(f)$ is lower semicontinuous and hence measurable.

Secondly, for any $a>0$ and $\delta>0$, we define the *truncated grand maximal function* $M_{a,\Omega,\delta}{}^*(f)(x)$, $x\in\Omega$, by

$$M_{a,\Omega,\delta}{}^*(f)(x)=\sup_{\psi}|\langle f, \psi\rangle|,$$

where the supremum is taken over those ψ for which there exist $y\in\Omega$ and t satisfying $0<t<\delta$ and (2.1).

Thirdly, for $\delta>0$ we define the *truncated radial maximal function* $M_{\Phi,\Omega,\delta}{}^+(f)(x)$, $x\in\Omega$, by

$$M_{\Phi,\Omega,\delta}{}^+(f)(x)=\sup\{|\langle f, \Phi(x, \cdot, t)\rangle|;\ 0<t<\min\{\delta, \operatorname{dis}(x, \Omega^c)\}\}.$$

Finally, for any measurable function g on $\mathbf{R}^n$ and for $s>0$, we define $M_s[g](x)$, $x\in\mathbf{R}^n$, by

$$M_s[g](x)=\sup_{t>0}\left\{t^{-\gamma}\int_{B(x,t)}|g(y)|^s dy\right\}^{1/s}.$$

Now the following hold.

COROLLARY 1. *Let Ω and f be the same as in the Theorem, and let $0<a<L$. Then*

$$M_{a,\Omega}{}^*(f)(x)\leqq C_{a,L,K}M_{\gamma/(\gamma+a)}[(M_{\Phi,\Omega}{}^+(f))^\sim](x),\ x\in\Omega. \tag{2.2}$$

If, in addition, $\gamma/(\gamma+a)<p\leqq\infty$, then

$$||M_{a,\Omega}{}^*(f)||_{p,\Omega}\leqq C_{a,L,K,p}||M_{\Phi,\Omega}{}^+(f)||_{p,\Omega}. \tag{2.3}$$

COROLLARY 2. *Let Ω and f be the same as in the Theorem, and let $0<a<L$ and $\delta>0$. Suppose $M_{\Phi,\Omega,\delta}{}^+(f)$ is a measurable function. Then*

$$M_{a,\Omega,\delta}{}^*(f)(x)\leqq C_{a,L,K}M_{\gamma/(\gamma+a)}[(M_{\Phi,\Omega,\delta}{}^+(f))^\sim](x),\ x\in\Omega.$$

If, in addition, $\gamma/(\gamma+a)<p\leqq\infty$, *then*

$$||M_{a,\Omega,\delta}^{*}(f)||_{p,\Omega}\leqq C_{a,L,K,p}||M_{\Phi,\Omega,\delta}^{+}(f)||_{p,\Omega}.$$

REMARK 2.4. These results are refinements of the results of Uchiyama [11] and [12]. In [11], Uchiyama proved the inequalities of our Corollary 1 under the *a priori* assumption that $f\in H^{q}(\mathbf{R}^{n})$ for some $q\geqq\gamma/(\gamma+a)$, where $H^{q}(\mathbf{R}^{n})$ is the parabolic H^{q} space studied by Calderón and Torchinsky [3]. In [12], he treated the case where $A(t)=tI$ and $\Phi(x,y,t)=(\phi)_{t}(x-y)$, where $\phi\in C_{0}^{\infty}(B(0,1))$ and $\int\phi(x)dx=1$, and obtained the following results: For any $f\in\mathscr{D}'(\mathbf{R}^{n})$ and $s>0$, let

$$f_{\phi,s}^{+}(x)=\sup_{0<t\leqq s}|(f*(\phi)_{t})(x)|;$$

then if $\varepsilon>0$, $a>0$, $f\in\mathscr{D}'(\mathbf{R}^{n})$, $x_{0}\in\mathbf{R}^{n}$, $t_{0}>0$ and $\varphi\in C_{0}^{\infty}(B(x_{0},t_{0}))$, it holds that

$$|\langle f,\varphi\rangle|\leqq C_{\phi,\varepsilon,a}||\varphi||_{A(a)}||f_{\phi,\varepsilon t_{0}}^{+}||_{n/(n+a),B(x_{0},(1+\varepsilon)t_{0})}.$$

(Note that $\gamma=n$ if $A(t)=tI$.) In view of Remark 2.3 and 307 of the next section, one can easily see that this result is covered by our Theorem.

III. *Preliminaries*

Before we proceed to the proofs of our results, we shall summarize in this section some preliminary results.

The presentation of the material in this section is arranged as follows. Paragraphs 301 to 308 contain statements. The remaining paragraphs contain the proofs or references for proofs. The statements of paragraph x, $301\leqq x\leqq 308$, are proved (or references for proofs are given) in paragraph $x+50$.

Throughout this section, a stands for an arbitrary positive number unless the contrary is explicitly stated.

301. Let Ω be an open subset of $\mathbf{R}^{n}$ and suppose $\Omega\neq\mathbf{R}^{n}$. Then there exist sequences $\{x_{\nu}\}$, $\{r_{\nu}\}$ and $\{\phi_{\nu}\}$ with the following properties.

(i) $x_{\nu}\in\Omega$, $r_{\nu}>0$.

(ii) $\bigcup_{\nu}B(x_{\nu},r_{\nu})=\Omega$.

(iii) $B(x_{\nu},20r_{\nu})\subset\Omega$.

(iv) $B(x_{\nu},60r_{\nu})\cap\Omega^{c}\neq\phi$.

(v) $\sum_{\nu}\chi_{B(x_{\nu},10r_{\nu})}(x)\leqq C$ for all $x\in\mathbf{R}^{n}$.

(vi) If $x\in B(x_{\nu},10r_{\nu})$, then $10r_{\nu}\leqq\mathrm{dis}(x,\Omega^{c})\leqq 70r_{\nu}$.

(vii) $\phi_{\nu}\in C_{0}^{\infty}(B(x_{\nu},2r_{\nu}))$.

(viii) $\sum_{\nu}\phi_{\nu}(x)=1$ for all $x\in\Omega$.

(ix) If $0<t\leqq r_{\nu}$, then $|\partial_{x}^{\alpha}\phi_{\nu}(A(t)x)|\leqq C_{\alpha}(t/r_{\nu})^{|\alpha|}$.

302. Let μ be a measure on $\mathbf{R}^{n}\times(0,\infty)$, $v\geqq 0$ and $M\geqq 0$. Suppose the inequality

$$\mu(B(x,t)\times(0,t))\leqq M|B(x,t)|^{1+v}$$

holds for all $x\in\mathbf{R}^n$ and $t>0$. Then, for $p>1$, it holds that

$$\iint G(f, x, t)^{p(1+v)}d\mu(x, t)\leqq C_{v,p}M\|f\|_p^{p(1+v)},$$

where

$$G(f, x, t)=t^{-\gamma}\int_{B(x,t)}|f(y)|dy.$$

303. Let J be an open cube in $\mathbf{R}^n$ with sides parallel to the coordinate axes. We denote by $l(J)$ the side length of J. Let $m\in\mathbf{N}$, $m\geqq 2$, $M_0>0$, $M_m>0$ and f a function of class C^m defined on J. Suppose that $\|f\|_{\infty,J}\leqq M_0$ and $\|\partial^\alpha f\|_{\infty,J}\leqq M_m$ if $|\alpha|=m$, and that

$$l(J)^m M_m\geqq 2^{m^2}M_0.$$

Then for multi-indices α with $0<|\alpha|<m$, it holds that

$$\|\partial^\alpha f\|_{\infty,J}\leqq 2^{m\ \alpha}\, M_0^{1-|\alpha|/m}M_m^{|\alpha|/m}.$$

304. For $x\in\mathbf{R}^n$ and $t>0$, we denote by $\mathscr{A}_a(x, t)$ the set of those $\phi\in C_0^\infty(B(x, t))$ such that $\|\phi\|_\infty\leqq t^{-\gamma-a}$ and

$$\int\phi(x)x^\alpha dx=0 \quad\text{if}\quad |\alpha|\leqq[a].$$

Then for all $f\in L_{\rm loc}^1(\mathbf{R}^n)$ it holds that

$$\|f\|_{\Lambda(a)}=\sup\ \{|\langle f, \phi\rangle|;\ \phi\in\mathscr{A}_a(x, t)\text{ for some } x\in\mathbf{R}^n,\ t>0\}.$$

305.1. Let η be a function in $C_0^\infty(B(0, 1))$ with the property that

$$\int\eta(x)x^\alpha dx=\begin{cases}1 & \text{if}\quad \alpha=0\\ 0 & \text{if}\quad 0<|\alpha|\leqq[a].\end{cases}$$

For $f\in\Lambda(a)$, set $F(x, t)=(f*(\eta)_t)(x)$. Then $F(x, t)$ is a smooth function in $x\in\mathbf{R}^n$ and $t>0$ and has the following estimates:

$$|\partial_x{}^\alpha F(A(t)x, t)|\leqq C_{\alpha,\eta}\|f\|_{\Lambda(a)}t^a \quad\text{if}\quad |\alpha|=[a]+1, \tag{3.1}$$

$$|\partial_t F(x, t)|\leqq C_\eta\|f\|_{\Lambda(a)}t^{a-1}. \tag{3.2}$$

305.2. Conversely, suppose $M\geqq 0$ and $F(x, t)$ is a smooth function in $x\in\mathbf{R}^n$ and $t>0$ with the estimates

$$|\partial_x{}^\alpha F(A(t)x,t)|\leqq Mt^a \quad\text{if}\quad |\alpha|=[a]+1,$$

$$|\partial_t F(x, t)|\leqq Mt^{a-1}.$$

Then, as t tends to zero $F(x, t)$ converges uniformly in $x\in\mathbf{R}^n$, and for $f(x)=\lim_{t\downarrow 0}F(x, t)$ we have $\|f\|_{\Lambda(a)}\leqq C_aM$.

306.1. If f is a bounded function of calss $C^{[a]+1}$ and if the derivatives of f of order $[a]+1$ are bounded, then f belongs to $\Lambda(a)$ and

$$\|f\|_{\Lambda(a)}\leqq C_a(\|f\|_\infty+\sum_{|\alpha|=[a]+1}\|\partial^\alpha f\|_\infty).$$

306.2. Let β be a positive number for which $\|A(t)\|\leqq t^\beta$ for $t\geqq 1$ (cf. Calderón-Torchinsky [2; p.2]). Let k be a non-negative integer satisfying $k\beta<a$. Let E be an arbitrary

compact subset of $\mathbf{R}^n$. Then the following holds. If $f \in \Lambda(a)$ and $\operatorname{supp} f \subset E$, then we can modify f on a set of measure zero to obtain a C^k function, which we shall still denote by f, and we have

$$\sum_{|\alpha| \leqq k} \|\partial^\alpha f\|_\infty \leqq C_{a,k,E} \|f\|_{\Lambda(a)}.$$

307. If Ω is an open convex subset of $\mathbf{R}^n$, then the inequalities

$$\|f\|_{\Lambda(a)} \leqq \|f|\Omega\|_{\Lambda(a;\Omega)} \leqq C_a \|f\|_{\Lambda(a)}$$

hold for functions f on $\mathbf{R}^n$ which vanish outside Ω. (Note that the above constant C_a does not depend on Ω.)

308. Let Ω be an open subset of R^n with $\Omega \neq \mathbf{R}^n$. Let $\{x_\nu\}$, $\{r_\nu\}$ and $\{\phi_\nu\}$ be as mentioned in 301.

308.1. Suppose that $f \in \Lambda(a; \Omega)$ and that $\operatorname{supp} \tilde{f}$ is a compact subset of Ω. Set $f_\nu(x) = \tilde{f}(x)\phi_\nu(x)$. Then

$$\operatorname{supp} f_\nu \subset B(x_\nu, 2r_\nu),$$
$$\|f_\nu\|_{\Lambda(a)} \leqq C_a \|f\|_{\Lambda(a;\Omega)},$$
$$f_\nu = 0 \text{ except for a finite number of } \nu\text{'s},$$
$$f = \sum_\nu f_\nu \text{ on } \Omega.$$

308.2. Conversely, suppose $M \geqq 0$ and suppose g and $\{g_\nu\}$ are functions on $\mathbf{R}^n$ such that

$$\operatorname{supp} g_\nu \subset B(x_\nu, 6r_\nu),$$
$$\|g_\nu\|_{\Lambda(a)} \leqq M,$$
$$g_\nu = 0 \text{ except for a finite number of } \nu\text{'s},$$
$$g = \sum_\nu g_\nu.$$

Then $\operatorname{supp} g \subset \Omega$ and

$$\|g|\Omega\|_{\Lambda(a;\Omega)} \leqq C_a M.$$

351. Using a 'dyadic grid' of balls we can carry out an argument regarding the Whitney decomposition. Cf. Latter [7] and Stein [9; Chap. VI, §1].

352. See Duren [4] or Uchiyama [10; Lemma 1].

353. For a proof of this result for the case $n=1$ and $m=2$, see Landau [6] or Bourbaki [1; Chap. I, §3, Exercice 12]. In the case $n=1$ and $m>2$, we can prove the result by induction on m. (In the case $n=1$, we can replace the constant $2^{m|\alpha|}$ by $2^{(m-|\alpha|)|\alpha|}$.) The result for the case $n>1$ can be proved by repeated application of the result for $n=1$.

354. The dual space of the quotient Banach space $L^1(B(x,t))/\mathscr{P}_{[a]}$ can be identified with the annihilator of $\mathscr{P}_{[a]}$ in $L^\infty(B(x,t))$. This fact and the Hahn-Banach theorem, combined with a limiting argument, give the desired result.

355.1. It holds that

$$\partial_x{}^\alpha F(A(t)x, t) = \int f(y) t^{-\gamma} \partial_x{}^\alpha \eta(x - A(t^{-1})y) dy.$$

It is easy to see that if $|\alpha| = [a]+1$, the function

$$g(y)=(C_{a,\eta}t^a)^{-1}t^{-\gamma}\partial_x{}^\alpha\eta(x-A(t^{-1})y)$$

belongs to $\mathscr{A}_a(A(t)x, t)$. Hence by 304 we obtain (3.1). We can prove (3.2) in a similar way.

355.2. Let $x\in\mathbf{R}^n$ and $t>0$. If $P\in\mathscr{P}_{[a]}$ and if $P(y)$ coincides, up to the terms of degree$\leqq[a]$, with the Taylor series of the function $F(A(t)y, t)$ expanded about $y=A(t^{-1})x$, then

$$|F(A(t)y, t)-P(y)|\leqq C_a Mt^a|y-A(t^{-1})x|^{[a]+1}.$$

From this, we see that

$$|F(y, t)-P(A(t^{-1})y)|\leqq C_a Mt^a \quad \text{if} \quad y\in B(x, t).$$

On the other hand, we have

$$|F(y, t)-f(y)|\leqq\int_0^t|\partial_s F(y, s)|ds\leqq C_a Mt^a. \tag{3.3}$$

Combining the above estimates, we obtain

$$|f(y)-P(A(t^{-1})y)|\leqq C_a Mt^a \quad \text{if} \quad y\in B(x, t).$$

From this follows the estimate $||f||_{A(a)}\leqq C_a M$. The uniform convergence of $F(y, t)\to f(y)$ follows from (3.3).

356.1. This can be immediately proved by the use of Taylor's formula.

356.2. This can be proved by the same method as given in 355.1 and 355.2. We shall omit the details. Cf. also [8; §6.2].

357. We may and shall assume $\Omega\neq\mathbf{R}^n$. It is sufficient to show the second inequality. Define $F(x, t)$ in the same way as in 305.1. Then (3.1) and (3.2) hold. Let $x_0\in\Omega$ and $t_0=\operatorname{dis}(x_0, \Omega^c)$. Take $p\in\Omega^c$ such that $t_0=\rho(x_0-p)$ and set $y_0=2p-x_0$. Then, since Ω is an open convex set, it follows that $B(y_0, t_0)\cap\overline{\Omega}=\phi$, where $\overline{\Omega}$ denotes the closure of Ω. From this we see that $y_0\notin\operatorname{supp} F(\cdot, t_0)$. Hence (3.1), combined with Taylor's formula, gives

$$|F(A(t_0)x, t_0)|\leqq C_a||f||_{A(a)}t_0{}^a|x-A(t_0{}^{-1})y_0|^{[a]+1}.$$

In particular, we have

$$|F(x_0, t_0)|\leqq C_a||f||_{A(a)}t_0{}^a.$$

From this and (3.2) we obtain

$$|f(x_0)|\leqq C_a||f||_{A(a)}t_0{}^a.$$

This implies the result.

358.1. Fix a function η with the properties of 305.1. Set $F(x, t)=(\tilde{f}*(\eta)_t)(x)$. Let $x_0\in\Omega$ and $t_0=\operatorname{dis}(x_0, \Omega^c)$. If $0<t<t_0$ and $A(t)x\in B(x_0, 2t_0)$, then

$$|\partial_x{}^\alpha F(A(t)x, t)|\leqq C_a||f||_{A(a;\Omega)}t^a \quad \text{if} \quad |\alpha|=[a]+1 \tag{3.4}$$

and

$$|F(A(t)x, t)|\leqq C_a||f||_{A(a;\Omega)}t_0{}^a. \tag{3.5}$$

(The inequality (3.4) is given in 305.1, and the inequality (3.5) follows from the estimate $|\tilde{f}(y)|\leqq C_a||f||_{A(a;\Omega)}t_0{}^a$ for $y\in B(x_0, 3t_0)$.) If $0<t<t_0$ and $A(t)y\in B(x_0, t_0)$, then the cube

$$\{(x_1, \ldots, x_n)\in\mathbf{R}^n;\ |x_j-y_j|<t_0/\sqrt{n}t,\ j=1, \ldots, n\}$$

is included in $B(A(t^{-1})x_0, 2t_0/t)$, and hence (3.4) and (3.5) hold in this cube. Hence, we can use 303 to see that the inequalities

$$|\partial_x{}^\alpha F(A(t)x,\ t)|\leqq C_a||f||_{\Lambda(a;\Omega)}t^{a|\alpha|/([a]+1)}t_0{}^{a(1-|\alpha|/([a]+1))} \tag{3.6}$$

hold if $0\leqq|\alpha|\leqq[a]+1$, $0<t<t_0$ and $A(t)x\in B(x_0,\ t_0)$. By 305.1 we also have

$$|\partial_t F(x,\ t)|\leqq C_a||f||_{\Lambda(a;\Omega)}t^{a-1}. \tag{3.7}$$

Take a smooth function h on **R** with the properties that $0\leqq h(t)\leqq 1$, $h(t)=1$ if $t\leqq 1/2$, and $h(t)=0$ if $t>1$. Set

$$F_\nu(x,\ t)=F(x,\ t)\phi_\nu(x)h(t/r_\nu).$$

Notice that $f_\nu(x)=\lim_{t\downarrow 0} F_\nu(x,\ t)$. By (3.6) and 301 (ix), we have

$$|\partial_x{}^\alpha F_\nu(A(t)x,\ t)|\leqq C_a||f||_{\Lambda(a;\Omega)}t^a \quad \text{if} \quad |\alpha|=[a]+1.$$

By (3.5) and (3.7), we have

$$|\partial_t F_\nu(x,\ t)|\leqq C_a||f||_{\Lambda(a;\Omega)}t^{a-1}.$$

Hence, by using 305.2, we obtain

$$||f_\nu||_{\Lambda(a)}\leqq C_a||f||_{\Lambda(a;\Omega)}.$$

Other claims are clear.

358.2. Let η and h be the same as in 358.1. Set

$$G_\nu(x,\ t)=(g_\nu*(\eta)_t)(x)h(t/r_\nu).$$

By using 305.1 and 307, we have

$$|\partial_x{}^\alpha G_\nu(A(t)x,\ t)|\leqq C_a||g_\nu||_{\Lambda(a)}t^a\leqq C_aMt^a \quad \text{if} \quad |\alpha|=[a]+1$$

and

$$|\partial_t G_\nu(x,\ t)|\leqq C_a||g_\nu||_{\Lambda(a)}t^{a-1}\leqq C_aMt^{a-1}.$$

Since supp $G_\nu(\cdot,\ t)\subset B(x_\nu,\ 7r_\nu)$ and the overlaps of the balls $B(x_\nu,\ 7r_\nu)$ are bounded (301 (v)), we can deduce, from the above estimates, that

$$|\partial_x{}^\alpha \sum_\nu G_\nu(A(t)x,\ t)|\leqq C_aMt^a \quad \text{if} \quad |\alpha|=[a]+1$$

and

$$|\partial_t \sum_\nu G_\nu(x,\ t)|\leqq C_aMt^{a-1}.$$

We now use 305.2 to obtain

$$||\sum_\nu g_\nu||_{\Lambda(a)}\leqq C_aM.$$

Moreover, using 307, we have $|g_\nu(x)|\leqq C_aMr_\nu{}^a$. Hence

$$|g(x)|\leqq \sum_\nu C_aMr_\nu{}^a\chi_{B(x_\nu,6r_\nu)}(x)\leqq C_aM(\text{dis}\ (x,\ \Omega^c))^a.$$

Thus $||g||_{\Lambda(a;\Omega)}\leqq C_aM$. The other claim, supp $g\subset\Omega$, is clear.

IV. *Proofs of the Main Results*

Before we proceed to the proofs of the main results, we give two lemmas, which are fundamental in our arguments.

LEMMA 1. *Let* $1>\delta>0$, $T>0$, $N\in\mathbf{N}$, $0<b<L$, $g\in L_{\mathrm{loc}}^1(\mathbf{R}^n)$, *and* $\eta\in C_0^\infty(B(0,1))$. *Suppose* $g(x)\geqq 0$ *and*

$$\int\eta(x)x^\alpha dx=\begin{cases}1 & \text{if } \alpha=0\\ 0 & \text{if } 0<|\alpha|\leqq[b].\end{cases}$$

Let $\phi\in C_0^\infty(\mathbf{R}^n)$, $x_0\in\mathbf{R}^n$ *and* $t_0>0$, *and suppose* $\operatorname{supp}\phi\subset B(x_0, t_0)$. *Set*

$$\begin{aligned}
&k_0=\phi*(\eta)_{t_0},\\
&k_j=\phi*((\eta)_{t_02^{-j}}-(\eta)_{t_02^{-j+1}})\ (j\in\mathbf{N}),\\
&\Omega_j=\{x\in\mathbf{R}^n;\ g(x)\leqq TG(g, x, \delta t_0 2^{-j})\}\ (j\in\mathbf{N}\cup\{0\}),\\
&h_j=k_j\chi_{\Omega_j}\ (j\in\mathbf{N}\cup\{0\}),\\
&\phi'=\sum_{j=0}^{N}h_j\#\Phi(\delta t_0 2^{-j}).
\end{aligned}$$

Then for all a with $0<a<b$ *the following estimates hold*:

$$\begin{aligned}
&\|h_j\|_\infty\leqq C_{a,\eta}\|\phi\|_{A(a)}(t_02^{-j})^a,\\
&\operatorname{supp} h_j\subset B(x_0, 2t_0),\\
&h_j(x)=0 \quad\text{if}\quad x\notin\Omega_j,\\
&\operatorname{supp}\phi'\subset B(x_0, 3t_0),\\
&\|\phi*(\eta)_{t_02^{-N}}-\phi'\|_{A(a)}\leqq C_{a,b,\eta}K_b(\delta^{1-a/b}+T^{-1}\delta^{-a})\|\phi\|_{A(a)},
\end{aligned}$$

where $j\in\mathbf{N}\cup\{0\}$.

This lemma is a modification of Lemma 5 of Uchiyama [11]. We shall not give a proof to this lemma since we can prove it by only slightly modifying the argument of Uchiyama [*ibid.*].

LEMMA 2. *Let* Ω *be an open subset of* $\mathbf{R}^n$ *with* $\Omega\neq\mathbf{R}^n$. *Let* $\{x_\nu\}$, $\{r_\nu\}$ *and* $\{\phi_\nu\}$ *be as given in* 301. *Let* δ, T, N, b, g *and* η *be the same as in Lemma* 1. *For* $\phi\in C_0^\infty(\Omega)$, *we set* $\phi_\nu=\phi\phi_\nu$. *Since* $\operatorname{supp}\phi_\nu\subset B(x_\nu, 2r_\nu)$, *we can apply the process* $(\phi, x_0, t_0)\rightarrow((h_j), \phi')$ *as given in Lemma* 1 *to each* $(\phi_\nu, x_\nu, 2r_\nu)$; *we denote by* $h_{\nu,j}$ *and* ϕ_ν' *the corresponding* h_j *and* ϕ'. *We set* $\phi'=\sum_\nu\phi_\nu'$. *Then, if* $0<a<b$ *and if N is sufficiently large, the following hold*:

$$\begin{aligned}
&\phi'\in C_0^\infty(\Omega),\\
&\|h_{\nu,j}\|_\infty\leqq C_{a,\eta}\|\phi\|_{A(a;\Omega)}(2^{-j}r_\nu)^a,\\
&\operatorname{supp} h_{\nu,j}\subset B(x_\nu, 4r_\nu),\\
&h_{\nu,j}(x)=0 \quad\text{if}\quad g(x)>TG(g, x, \delta 2^{-j+1}r_\nu),
\end{aligned}$$

$$||\psi-\psi'||_{\Lambda(a;\Omega)}\leqq\left(\frac{1}{4}+C_{a,b,\eta}K_b(\delta^{1-a/b}+T^{-1}\delta^{-a})\right)||\psi||_{\Lambda(a;\Omega)},$$

where $j\in\mathbf{N}\cup\{0\}$.

PROOF. Since supp ψ is a compact subset of Ω, there are only a finite number of ν's for which $\psi_\nu\neq 0$. Hence if we take N sufficiently large, we have

$$\sum_\nu||\psi_\nu-\psi_\nu*(\eta)_{2^{-N+1}r_\nu}||_{\Lambda(a;\Omega)}\leqq\frac{1}{4}||\psi||_{\Lambda(a;\Omega)}.$$

Now the desired results follow from 308.1, 308.2 and Lemma 1.

Now we shall prove the results stated in Section 2.

PROOF OF THE THEOREM. Let $0<a<L$. We may and shall assume $M_{\Phi,\Omega}{}^+(f)\in L^{\gamma/(\gamma+a)}(\Omega)$. We shall denote the constants $C_{a,\eta}$ and $C_{a,b,\eta}$ in Lemma 2 by $C_{a,\eta}{}^0$ and $C_{a,b,\eta}{}^0$ respectively.

First, we assume f is of order a. Set

$$g(x)=\begin{cases}(M_{\Phi,\Omega}{}^+(f)(x))^{\gamma/2(\gamma+a)} & \text{if} \quad x\in\Omega\\ 0 & \text{if} \quad x\notin\Omega.\end{cases}$$

Then $g\in L^2(\mathbf{R}^n)\subset L_{\mathrm{loc}}{}^1(\mathbf{R}^n)$. Take b and η such that $a<b<L$, $\eta\in C_0{}^\infty(B(0,1))$ and

$$\int\eta(x)x^\alpha dx=\begin{cases}1 & \text{if} \quad \alpha=0\\ 0 & \text{if} \quad 0<|\alpha|\leqq[b].\end{cases}$$

Take δ and T such that $0<\delta<1$, $T>0$ and

$$C_{a,b,\eta}{}^0K_b(\delta^{1-a/b}+T^{-1}\delta^{-a})\leqq\frac{1}{4}. \tag{4.1}$$

Then repeated application of Lemma 2 gives functions $\psi^{(i)}$ and $h_{\nu,j}{}^{(i)}$, $i\in\mathbf{N}$, with the following properties:

$$\psi^{(i)}=\sum_\nu\sum_{j=0}^{N(i)}h_{\nu,j}{}^{(i)}\#\Phi(\delta 2^{-j+1}r_\nu)\in C_0{}^\infty(\Omega), \tag{4.2}$$

$$||h_{\nu,j}{}^{(i)}||_\infty\leqq C_{a,\eta}{}^0 2^{-i+1}||\psi||_{\Lambda(a;\Omega)}(2^{-j}r_\nu)^a, \tag{4.3}$$

$$\text{supp } h_{\nu,j}{}^{(i)}\subset B(x_\nu,4r_\nu), \tag{4.4}$$

$$h_{\nu,j}{}^{(i)}(x)=0 \quad \text{if} \quad g(x)>TG(g,x,\delta 2^{-j+1}r_\nu), \tag{4.5}$$

$$||\psi-\sum_{i=1}^M\psi^{(i)}||_{\Lambda(a;\Omega)}\leqq 2^{-M}||\psi||_{\Lambda(a;\Omega)}(M\in\mathbf{N}). \tag{4.6}$$

Moreover, as the proof of Lemma 2 shows, for each i there are only finitely many ν's for which $h_{\nu,j}{}^{(i)}\neq 0$ with at least one j. Now, since f is of order a, (4.2) and (4.6) imply that

$$\langle f,\psi\rangle=\lim_{M\to\infty}\langle f,\sum_{i=1}^M\psi^{(i)}\rangle=\sum_{i=1}^\infty\sum_\nu\sum_{j=0}^{N(i)}\int h_{\nu,j}{}^{(i)}(x)\langle f,\Phi(x,\cdot,\delta 2^{-j+1}r_\nu)\rangle dx. \tag{4.7}$$

If $h_{\nu,j}{}^{(i)}(x)\neq 0$, then from (4.4) and (4.5) we have

$$|\langle f,\Phi(x,\cdot,\delta 2^{-j+1}r_\nu)\rangle|\leqq M_{\Phi,\Omega}{}^+(f)(x)$$
$$=g(x)^{2(1+a/\gamma)}\leqq(TG(g,x,\delta 2^{-j+1}r_\nu))^{2(1+a/\gamma)}.$$

From this and (4.7) we obtain

$$|\langle f, \phi\rangle| \leqq \iint G(g, x, t)^{2(1+a/r)} d\mu(x, t),$$

where μ is the measure on $\mathbf{R}^n \times (0, \infty)$ given by

$$\iint \phi(x, t) d\mu(x, t) = \sum_{i=1}^{\infty} \sum_{\nu} \sum_{j=0}^{\infty} T^{2(1+a/r)} \int |h_{\nu,j}^{(i)}(x)| \phi(x, \delta 2^{-j+1} r_\nu) dx.$$

Using (4.3) and (4.4), we easily see that this μ satisfies the estimate

$$\mu(B(x, t) \times (0, t)) \leqq C_{a,\eta} \delta^{-a} T^{2(1+a/r)} \|\phi\|_{A(a;\Omega)} |B(x, t)|^{1+a/r}.$$

Hence by 302 we obtain

$$(4.8) \quad |\langle f, \phi\rangle| \leqq C_{a,\eta,\delta,T} \|\phi\|_{A(a;\Omega)} \left(\int g(x)^2 dx\right)^{1+a/r} = C_{a,\eta,\delta,T} \|\phi\|_{A(a;\Omega)} \|M_{\Phi,\Omega}^{+}(f)\|_{r/(r+a),\Omega}.$$

This is the desired inequality since η, δ and T can be taken depending only on a, L and K.

Secondly, assume that f is of order a' for some $a' \in (0, L)$. We shall prove that (4.8) still holds (with the same constant $C_{a,\eta,\delta,T} = C_{a,L,K}$). In order to prove this, it is sufficient to prove f is in fact of order a. (If this is proved, we can use the results of the first case to obtain (4.8).) Take b, b' and η such that $a < b < L$, $a' < b' < L$, $\eta \in C_0^\infty(B(0, 1))$ and

$$\int \eta(x) x^\alpha dx = \begin{cases} 1 & \text{if } \alpha = 0 \\ 0 & \text{if } 0 < |\alpha| \leqq \max \{[b], [b']\}. \end{cases}$$

Take δ and T satisfying $0 < \delta < 1$, $T > 0$, (4.1) and

$$C_{a',b',\eta}{}^0 K_{b'} (\delta^{1-a'/b'} + T^{-1} \delta^{-a'}) \leqq \frac{1}{4}.$$

Let $g(x)$ be the same as in the first case. With these δ, T, η and g, we repeat the argument of the first case to obtain $\phi^{(i)}$ and $h_{\nu,j}^{(i)}$ which have, in addition to the properties mentioned in the first case, the property

$$\|\phi - \sum_{i=1}^{M} \phi^{(i)}\|_{A(a';\Omega)} \leqq 2^{-M} \|\phi\|_{A(a';\Omega)}, \quad M \in \mathbf{N}.$$

Since f is now of order a', the equalities (4.7) hold again. Thus, the same argument as in the first case gives (4.8). Note that in this case, η, δ and T depends on a', and hence (4.8) obtained above does not immediately imply the conclusion of the Theorem but it implies, at any rate, f is of order a, which we wanted to show. This completes the proof for the second case.

Finally, we shall consider the general case. Suppose f satisfies the condition of the Theorem. Take an increasing sequence of open sets $\{\Omega_j\}$ such that $\overline{\Omega}_j$ is compact, $\overline{\Omega}_j \subset \Omega$ and $\bigcup_{j=1}^{\infty} \Omega_j = \Omega$. Since

$$M_{\Phi,\Omega_j}^{+}(f|\Omega_j)(x) \leqq M_{\Phi,\Omega}^{+}(f)(x) \quad \text{for} \quad x \in \Omega_j$$

and since $f|\Omega_j$ is of order b_j with some $b_j \in (0, L)$, the result in the second case implies that the inequality

$$(4.9) \qquad |\langle f, \phi\rangle| \leqq C_{a,L,K} \|\phi\|_{A(a;\Omega_j)} \|M_{\Phi,\Omega}^{+}(f)\|_{r/(r+a),\Omega}$$

holds for all $\phi \in C_0^\infty(\Omega_j)$. If $\phi \in C_0^\infty(\Omega)$, then $\phi \in C_0^\infty(\Omega_j)$ for all sufficiently large j and $\|\phi\|_{A(a;\Omega_j)} \to \|\phi\|_{A(a;\Omega)}$ as $j \to \infty$. Hence letting $j \to \infty$ in (4.9) we obtain the desired inequality. This completes the proof of the Theorem.

PROOF OF COROLLARY 1. Suppose $x \in B(y, t) \subset \Omega$, $\psi \in C_0^\infty(B(y, t))$ and $\|\psi\|_{\Lambda(a)} \leqq t^{-\gamma-a}$. By 307, we have

$$\|\psi\|_{\Lambda(a;B(y,t))} \leqq C_a t^{-\gamma-a}.$$

Hence, using the inequality of the Theorem, we obtain

$$\begin{aligned}|\langle f, \psi\rangle| &\leqq C_{a,L,K} \|\psi\|_{\Lambda(a;B(y,t))} \|M_{\Phi,B(y,t)}{}^+(f|B(y, t))\|_{\gamma/(\gamma+a),B(y,t)} \\ &\leqq C_{a,L,K} t^{-\gamma-a} \|M_{\Phi,\Omega}{}^+(f)\|_{\gamma/(\gamma+a),B(y,t)} \\ &\leqq C_{a,L,K} M_{\gamma/(\gamma+a)}[(M_{\Phi,\Omega}{}^+(f))^\sim](x).\end{aligned}$$

Taking supremum over ψ we obtain (2.2). The Hardy-Littlewood maximal theorem (see *e.g.* Stein [9; Chapt. I, §1]) gives the L^p inequality (2.3). This completes the proof.

Corollary 2 can be proved in a similar way. We shall omit the details.

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