Hitotsubashi Journal of Economics 53 (2012), pp.177-200.

COMPARING COURNOT DUOPOLY AND MONOPOLY WITH ASYMMETRIC DIFFERENTIATED GOODS*

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Received October 2011, Accepted September 2012

Abstract

This study compares a differentiated Cournot duopoly with a two-product monopoly by using the socially optimal solution as a reference point. Each solution is allowed to be either an interior or a corner solution. We establish that the ranking regarding each individual price is clear-cut and normal. In contrast, every one of the rankings regarding individual outputs and industry output can go either way. More importantly, the duopoly may be less welfare-efficient than the monopoly. For example, when demands are linear, lower welfare is achieved if the asymmetry between firms is strong enough. One reason is that when firms are asymmetric, the output structure in the duopoly is distorted with probability one, whereas the output structure in the monopoly is generally socially optimal.

Keywords: Cournot duopoly, multi-product monopoly, output structure, horizontal merger, anti-trust policy
JEL classification: L12, L13, L41

* We would like to thank the National Science Council, Taiwan, for its financial support under Grant 98-2410-H-008-024. The authors are grateful for the very helpful and detailed comments of the referee, Wen-Jung Liang, Yan-Shu Lin, Fu-Chuan Lai and Dachrahn Wu.
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I. *Introduction*

This study aims at comparing the welfare in the Cournot equilibrium with that in the monopoly solution in an industry with two substitute goods. Recently, Zanchettin (2006) demonstrates that some standard results obtained by the work of Singh and Vives (1984), a classic contribution comparing the Cournot and Bertrand equilibria,[1] are reversed when the cost and demand asymmetry between firms is strong. This means that the results are sensitive to the degree of asymmetry between firms. Accordingly, we allow the solutions to be either interior or corner, because when the solutions are assumed to be interior, the degree of asymmetry is implicitly assumed to be small enough.

Note that, in an asymmetric model, for a given industry output, production efficiency at the industry level depends on the distribution of industry output among firms, e.g., it might be raised when production shifts from a less to a more efficient firm (Lahiri and Ono, 1988, p.1200; Ushio, 2000, p.268; Zanchettin, 2006, p.1007; Chang, 2010, Section 3.5). Accordingly, we emphasize the distinction between *output level* (a weighted sum of individual outputs, e.g., industry output) and *output structure* (the distribution of industry output among firms). Moreover, we will use the socially optimal solution as a reference point in the comparison.

Tirole (1988, p.70) argues that a multi-product monopolist will charge higher prices than separate firms each producing a single good if all the goods are gross substitutes. This argument is one version of the widespread belief that “relative efficiency in resource allocation increases monotonically as the number of firms expands” (Baumol, 1982, p.2).

The Tirole argument has not received the attention it deserves in that it is relevant to anti-trust policy. This belief of course works against horizontal mergers in an oligopoly. Moreover, the issue is not straightforward, and there is some confusion in the literature. There are three reasons for this. First, as pointed out by Andriychenko *et al*. (2006, p.375), Tirole implicitly assumes that (A) the optimal price of each good exceeds its marginal cost and that (B) each “own elasticity of demand” has the same value when it is evaluated at different equilibrium prices. Second, Chang (2010, Propositions 3.2, 3.6, 4.1 and 4.2) demonstrates that lower prices are not necessarily better in welfare terms. This is because, in either one of the Cournot and Bertrand equilibria, a more efficient firm may have a higher markup rate (Chang, 2010, Proposition 3.5 and Corollary 3.2).[2] The higher markup rate of the more efficient firm leads to a considerable increase in demand for the less efficient good, suggesting that the output structure is distorted in favor of the less efficient firm. A small increase in the less efficient firm’s price thus mitigates the output-structure distortion since the price increase shifts production from the less to the more efficient firm. Third, as a matter of fact, Lahiri and Ono (1988, Proposition 2) have already established that under a homogeneous Cournot oligopoly, welfare increases if a firm with a sufficiently low market share is removed from the market. This is because removing this firm raises production efficiency at the industry level.

It is important to conduct the comparison in a setting with differentiated goods. This is because, according to Chang (2010, Section 3.5), when goods are differentiated, there exists an

[1] For a review of the literature, please refer to Chang and Peng (2012).

[2] Proposition 3.5 and Corollary 3.2 deal with the Bertrand competition. However, as pointed out by Chang (2010, p.89), the counterparts of Proposition 3.5 and Corollary 3.2 for the Cournot competition hold.

effect, namely, the so called *diversity-value effect*, which works against the above mentioned *production-efficiency effect*. The diversity-value effect captures the desirability of variety as emphasized by Dixit and Stiglitz (1977, p. 297): the diversity value is maximized by letting the total output be equally distributed among all goods. Therefore, from the viewpoint of the diversity-value effect, it is socially desirable to keep a less efficient firm in the market. As a matter of fact, Chang (2010, Propositions 4.3 and 5.1) demonstrates that if demands arise from a quasi-linear utility function for a representative consumer, if demands are linear, and if each solution is interior (i.e., each good is active in each solution), then outputs and welfare are both higher in the Bertrand equilibrium than in the monopoly solution. This result is in contrast with that obtained by Lahiri and Ono (1988).

Chang (2010) points out an important difference between the Bertrand and Cournot equilibria. As mentioned, if some conditions are satisfied, then each Bertrand output exceeds its corresponding monopoly output. In contrast, under the same conditions, the Cournot competition may entail output reversals, e.g., the output of the most efficient good may be lower in the Cournot equilibrium than in the monopoly solution. Therefore, it is important to analyze whether the Cournot equilibrium is more welfare-efficient than the monopoly solution, since the output reversal suggests a possible welfare reversal.

The rest of this paper is structured as follows. The model is described in Section II. Section III provides a preliminary analysis. In section IV, we conduct the analysis in the quantity space, whereas, in section V, we conduct the analysis in the price space. In Section VI, some conclusions are offered.

II. *The Model*

There are two substitutes goods in an industry. The quantities are x_1 and x_2; the prices are p_1 and p_2. Firm i ($i=1, 2$) produces good i at a constant marginal cost $c_i \geq 0$. Throughout this study, each lower-case bold notation denotes an 2×1 vector, e.g., $\mathbf{x}\equiv(x_1, x_2)'$ and $\mathbf{p}\equiv(p_1, p_2)'$ where "$'$" denotes the transpose. In contrast, each upper-case bold notation stands for a set, e.g., $\mathbf{R}_+$ is the set of positive real numbers (including zero). If a capitalized notation (except for the bold ones) has a subscript, then the subscript represents a partial differentiation. For example, $P^i(\mathbf{x})$ stands for the inverse demand function for good i, and P^i_j denotes $\partial P^i/\partial x_j$. [Because P^i is a function of x_1 and x_2, it should be clear that the subscript j refers to x_j.] $\Pi^i(\mathbf{x})\equiv x_i[P^i(\mathbf{x})-c_i]$ denotes the profit function of good i; $\Pi(\mathbf{x})\equiv\Pi^1(\mathbf{x})+\Pi^2(\mathbf{x})$ denotes the industry profit function.

When a lower-case notation has the superscript C, it denotes Cournot. Similarly, M refers to the monopoly solution; S, the socially optimal solution (i.e., the perfectly competitive equilibrium). For example, $\mathbf{x}^C\equiv(x^C_1, x^C_2)'$ denotes the output vector under the Cournot equilibrium; $\mathbf{x}^M\equiv(x^M_1, x^M_2)'$ denotes the monopoly solution; $\mathbf{x}^S\equiv(x^S_1, x^S_2)'$ is the socially optimal solution. Following Amir and Jin (2001, p.308), we call $P^i(\mathbf{0})-c_i$ good i's primary markup where $\mathbf{0}=(0, 0)'$. We use i and j to refer to the goods, and it is understood that if i denotes 1 in an expression, then j represents 2 and *vice versa*.

The demands arise from a representative consumer's quasi-linear utility function $U(\mathbf{x})+z$ where z represents the outside good, and it is provided by a competitive sector with price being equal to 1. The utility-maximizing problem is $\max_{\mathbf{x}\geq\mathbf{0},\, z\geq 0} U(\mathbf{x})+z$ s.t. $\mathbf{p}\cdot\mathbf{x}+z=y$ where y is the income of the representative consumer. It is assumed that the income is sufficiently large so that some amount of the outside good is always consumed, implying that the marginal utility of income is unity (Chang and Peng, 2012, footnote 2). Hence,

$$P^i(\mathbf{x})=U_i(\mathbf{x}),\ \forall\mathbf{x}\in\mathbf{R}^2_+,\ i=1,\ 2. \tag{1}$$

Moreover, the welfare (or total surplus) is

$$W(\mathbf{x})\equiv U(\mathbf{x})-\mathbf{c}\cdot\mathbf{x},\ \forall\mathbf{x}\in\mathbf{R}^2_+. \tag{2}$$

Throughout this paper, it is assumed that $P^i(\mathbf{x})$ is twice continuously differentiable on $\mathbf{R}^2_+$, $i=1, 2$. Moreover, the following five assumptions are made:

Positive primary markups (PPM): $P^i(\mathbf{0})-c_i>0$, $i=1, 2$.
Strict dependence (SD): $P^i_j(\mathbf{x})<0$, $i=1, 2$, $\forall\mathbf{x}\in\mathbf{R}^2_+$.
Strategic substitutes (SS): *(i)* $\Pi_{ij}(\mathbf{x})<0$, $i=1, 2$, $\forall\mathbf{x}\in\mathbf{R}^2_+$. *(ii)* $\Pi^i_{ij}(\mathbf{x})<0$, $i=1, 2$, $\forall\mathbf{x}\in\mathbf{R}^2_+$.
Second-order conditions (SOC): *(i)* $U(\mathbf{x})$ *is strictly concave with respect to* $\mathbf{x}$ *on* $\mathbf{R}^2_+$. *Hence,* $W(\mathbf{x})$ *is strictly concave with respect to* $\mathbf{x}$ *on* $\mathbf{R}^2_+$. *(ii)* $\Pi(\mathbf{x})$ *is strictly concave with respect to* $\mathbf{x}$ *on* $\mathbf{R}^2_+$. *(iii)* $\Pi^i_{ii}(\mathbf{x})<0$, $i=1, 2$, $\forall\mathbf{x}\in\mathbf{R}^2_+$. *(iv)* $\Pi^1_{11}(\mathbf{x})\Pi^2_{22}(\mathbf{x})-\Pi^1_{12}(\mathbf{x})\Pi^2_{21}(\mathbf{x})>0$, $\forall\mathbf{x}\in\mathbf{R}^2_+$.
Finiteness (FN): *(i) There exists a unique finite number* $\tilde{x}^S_i>0$*, a quantity of good i, such that* $P^i(\tilde{x}^S_i, 0)=c_i$, $i=1, 2$. *(ii) There exists a unique finite number* $x^L_j>0$*, a quantity of good j, such that* $P^i(0, x^L_j)=c_i$, $i=1, 2$.

PPM means that each firm can survive if its rival is inactive.[3] Hence, PPM is not a restrictive assumption, because it is only used to exclude irrelevant goods.[4] SD is used to exclude the trivial case where the goods are independent. Either one of SS and SOC is a standard assumption. We will point out the geometric meaning of FN at the end of this section.

Note that if demands are linear, then SS, SOC-(ii)-(iv) and FN automatically follow from PPM, SD and SOC-(i), and hence these assumptions are reasonable.[5] Next we prove this result. Let $P(\mathbf{x})$ stand for $(P^1(\mathbf{x}), P^2(\mathbf{x}))'$, $\forall\mathbf{x}\in\mathbf{R}^2_+$. Eq. (1) implies that $\partial P/\partial\mathbf{x}'=\partial^2U/\partial\mathbf{x}^2$. SOC-(i) thus guarantees that $\partial P(\mathbf{x})/\partial\mathbf{x}'$ is negative definite, $\forall\mathbf{x}\in\mathbf{R}^2_+$, i.e.,

$$P^1_2(\mathbf{x})=P^2_1(\mathbf{x}),\ \forall\mathbf{x}\in\mathbf{R}^2_+, \tag{3}$$

$$P^i_i(\mathbf{x})<0,\ i=1,\ 2,\ \forall\mathbf{x}\in\mathbf{R}^2_+, \tag{4}$$

$$P^1_1(\mathbf{x})P^2_2(\mathbf{x})-P^1_2(\mathbf{x})P^2_1(\mathbf{x})>0,\ \forall\mathbf{x}\in\mathbf{R}^2_+. \tag{5}$$

If demands are linear, then from (4) and SD it follows that $\lim_{x_i\to\infty}P^i(x_i, 0)=-\infty$ and $\lim_{x_j\to\infty}P^i(0, x_j)=-\infty$, implying that PPM⇒FN. Partially differentiating Π and Π^i with respect to x_i yields

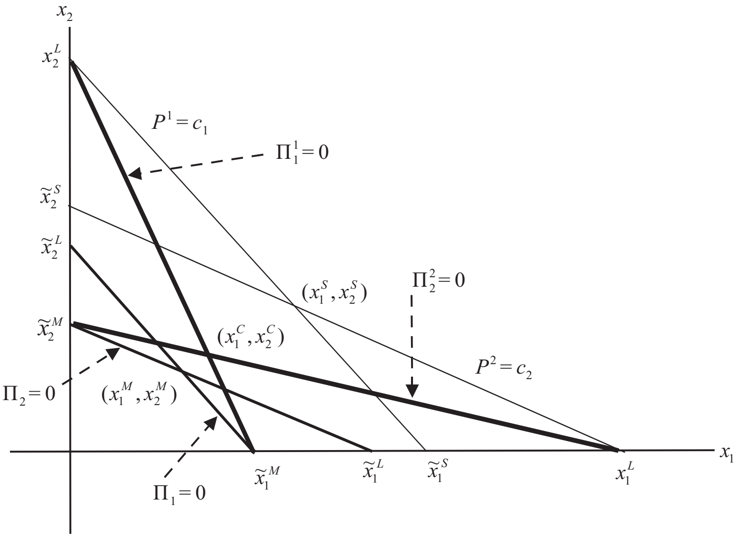
$$\Pi_i=x_iP^i_i+x_jP^i_i+P^i-c_i,\ i=1,\ 2, \tag{6}$$

[3] For example, when firm 2 is somehow inactive (i.e., $x_2=0$), firm 1 will choose a positive quantity, and thereby earn a positive profit if and only if $P^1(\mathbf{0})>c_1$.

[4] For example, if $P^1(\mathbf{0})-c_1\leq0$, then, according to (4), $x_1>0\Rightarrow P^1(x_1, x_2)<c_1$, and hence good 1 should be inactive in any solution.

[5] As argued by Cheng (1985, p.147) and Chang and Peng (2012, footnote 5), any property that is satisfied when demands are linear can be a reasonable assumption.

FIG. 1. A CASE WITH AN INTERIOR MONOPOLY SOLUTION



$$\Pi_i^i = x_i P_i^i + P^i - c_i, \; i = 1, 2. \tag{7}$$

Eqs. (6) and (7), together with (3), imply that if demands are linear, then

$$\frac{\partial^2 \Pi}{\partial \mathbf{x}^2} = \begin{pmatrix} 2P_1^1 & P_2^1 + P_1^2 \\ P_1^2 + P_2^1 & 2P_2^2 \end{pmatrix} = 2 \times \begin{pmatrix} P_1^1 & P_2^1 \\ P_1^2 & P_2^2 \end{pmatrix}, \tag{8}$$

$$\begin{pmatrix} \Pi_{11}^1 & \Pi_{12}^1 \\ \Pi_{21}^2 & \Pi_{22}^2 \end{pmatrix} = \begin{pmatrix} 2P_1^1 & P_2^1 \\ P_1^2 & 2P_2^2 \end{pmatrix}, \tag{9}$$

and SS thereby follows from SD; (ii)-(iv) of SOC from (4)-(5).

We are in a position to present the geometric meanings of the five assumptions. The analysis is conducted geometrically using the following six curves, each related to one particular kind of first-order condition: $\mathbf{W}_i(0) \equiv \{\mathbf{x} \in \mathbf{R}_+^2 : W_i(\mathbf{x}) = 0\}$, $\mathbf{\Pi}_i(0) \equiv \{\mathbf{x} \in \mathbf{R}_+^2 : \Pi_i(\mathbf{x}) = 0\}$, $\mathbf{\Pi}_i^i(0) \equiv \{\mathbf{x} \in \mathbf{R}_+^2 : \Pi_i^i(\mathbf{x}) = 0\}$, $i = 1, 2$. Eqs. (1)-(2) imply that

$$W_i(\mathbf{x}) = P^i(\mathbf{x}) - c_i, \; i = 1, 2, \; \forall \mathbf{x} \in \mathbf{R}_+^2. \tag{10}$$

Hence, $\mathbf{W}_i(0)$ is an iso-price curve, namely, $\mathbf{W}_i(0) = \mathbf{P}^i(c_i)$ where $\mathbf{P}^i(c_i) \equiv \{\mathbf{x} \in \mathbf{R}_+^2 : P^i(\mathbf{x}) = c_i\}$. It is easy to show that, in the (x_1, x_2) space, the slope of $\mathbf{P}^i(c_i)$ is given by $-P_1^i/P_2^i$; $\mathbf{\Pi}_i(0)$, $-\Pi_{i1}/\Pi_{i2}$; and $\mathbf{\Pi}_i^i(0)$, $-\Pi_{i1}^i/\Pi_{i2}^i$. Therefore, SD, SS and SOC imply that, as shown in Fig. 1, each curve is downward sloping. Moreover, each good-1 curve is steeper than its corresponding good-2 curve. For example, as shown in Fig. 1, $\mathbf{\Pi}_1^1(0)$ is steeper than $\mathbf{\Pi}_2^2(0)$.

FN means that, in the (x_1, x_2) space, as shown in Fig. 1, $\mathbf{P}^1(c_1)$ meets the x_1- and x_2-axis at $(\tilde{x}_1^S, 0)$ and $(0, x_2^L)$, respectively, whereas $\mathbf{P}^2(c_2)$ meets the x_1- and x_2-axis at $(x_1^L, 0)$ and $(0, \tilde{x}_2^S)$, respectively. Eqs. (6)-(7) imply that

$$\Pi_i(\mathbf{x}) \leq \Pi_i^i(\mathbf{x}),\ \forall \mathbf{x} \in \mathbf{R}_+^2,\ \text{with strict inequality if } x_j > 0,\ i=1, 2, \tag{11}$$

$$\Pi_i^i(\mathbf{x}) \leq P^i(\mathbf{x}) - c_i,\ \forall \mathbf{x} \in \mathbf{R}_+^2,\ \text{with strict inequality if } x_i > 0,\ i=1, 2, \tag{12}$$

$$\Pi_i(\mathbf{0}) = \Pi_i^i(\mathbf{0}) = P^i(\mathbf{0}) - c_i. \tag{13}$$

Accordingly, FN leads to the following fact:

Fact 1. (i) There exists a unique number $\tilde{x}_i^M \in (0, \tilde{x}_i^S)$ such that $\Pi_i(\tilde{x}_i^M, 0) = \Pi_i^i(\tilde{x}_i^M, 0) = 0$, $i=1, 2$. (ii) $\Pi_i^i(0, x_j^L) = 0$, $i=1, 2$. (iii) There exists a unique number $\tilde{x}_j^L \in (0, x_j^L)$ such that $\Pi_i(0, \tilde{x}_j^L) = 0$, $i=1, 2$.

Proof: According to FN-(i), $\tilde{x}_i^S > 0$ and $P^i(\tilde{x}_i^S, 0) - c_i = 0$. Eqs. (11) -(12) thus imply that $\Pi_i(\tilde{x}_i^S, 0) = \Pi_i^i(\tilde{x}_i^S, 0) < 0$. Eq. (13), together with PPM, implies that $\Pi_i(\mathbf{0}) = \Pi_i^i(\mathbf{0}) > 0$. Part (i) thus follows from (ii)-(iii) of SOC. According to (12), if $x_i = 0$, then $\Pi_i^i = P^i - c_i$, implying that Part (ii) follows from FN-(ii). Eq. (11), together with Part (ii), implies that $\Pi_i(0, x_j^L) < 0$. Hence, the inequality $\Pi_i(\mathbf{0}) > 0$, together with SOC-(ii), yields Part (iii). Q.E.D.

Fact 1 implies that, as shown in Fig. 1, every one of the four curves $\mathbf{\Pi}_1^1(0)$, $\mathbf{\Pi}_2^2(0)$, $\mathbf{\Pi}_1(0)$ and $\mathbf{\Pi}_2(0)$ meets the two axes. For example, (i) and (ii) of Fact 1 imply that $\mathbf{\Pi}_1^1(0)$ meets the x_1- and x_2-axis at $(\tilde{x}_1^M, 0)$ and $(0, x_2^L)$, respectively. Dixit (1979, pp.22-23) refers to $\tilde{x}_1^M$ as the "monopoly output" for firm 1 and x_2^L as the "limit quantity" for firm 2.

III. *Preliminaries*

In subsection III.1, we characterize the basic properties of the three solutions based on the six curves. In subsection III.2, we present a basic lemma to be used to conduct the welfare comparison. In subsection III.3, a specific example with linear demands is discussed.

1. Basic Properties of the Three Solutions

The first-order condition of the program $\max_{x_i} \Pi^i$ s.t. $x_i \geq 0$ is

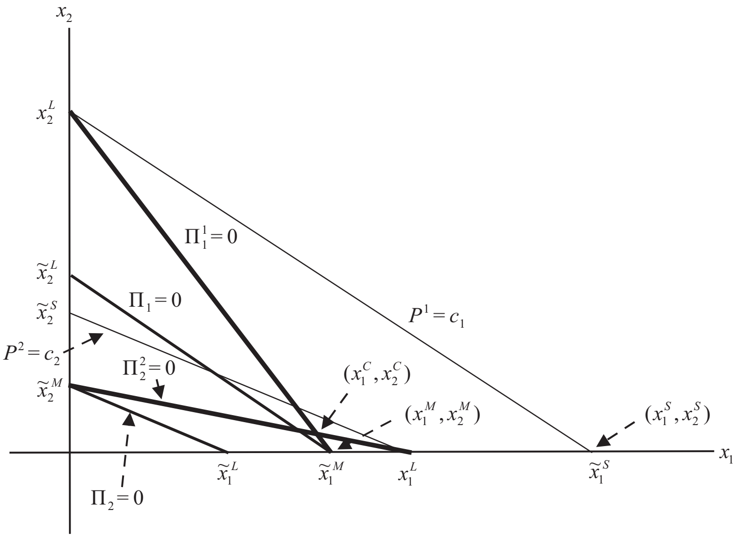
$$x_i \geq 0,\ \Pi_i^i \leq 0,\ x_i \Pi_i^i = 0. \tag{14}$$

According to Fact 1, $\Pi_i^i(0, x_j^L) = 0$, which, together with SS, implies that $\Pi_i^i(0, x_j) < 0$, $\forall x_j > x_j^L$. Therefore, $\{\mathbf{x} \in \mathbf{R}_+^2 : x_i = 0, x_j > x_j^L\}$ is a subset of firm i's Cournot reaction curve, and hence firm i's Cournot reaction curve is $\mathbf{\Pi}_i^i(0) \cup \{\mathbf{x} \in \mathbf{R}_+^2 : x_i = 0, x_j > x_j^L\}$, $i=1, 2$. Note that we also can regard this curve as $\mathbf{\Pi}_i^i(0) \cup \{\mathbf{x} \in \mathbf{R}_+^2 : x_i = 0, x_j \geq x_j^L\}$ because $(0, x_j^L)' \in \mathbf{\Pi}_i^i(0)$. Moreover, $\mathbf{x}^C$ is a Cournot equilibrium if and only if it is an intersection point between the above two Cournot reaction curves.

The condition for the socially optimal solution is $x_i \geq 0$, $W_i \leq 0$, and $x_i W_i = 0$, $i=1, 2$, whereas the condition for the monopoly solution is $x_i \geq 0$, $\Pi_i \leq 0$, and $x_i \Pi_i = 0$, $i=1, 2$. Therefore, we can similarly demonstrate that $\mathbf{x}^S$ is a socially optimal solution if and only if it is the intersection point of the two reaction curves $\mathbf{P}^i(c_i) \cup \{\mathbf{x} \in \mathbf{R}_+^2 : x_i = 0, x_j \geq x_j^L\}$, $i=1, 2$.[6]

[6] We may refer to $\mathbf{W}_i(0) \cup \{\mathbf{x} \in \mathbf{R}_+^2 : x_i = 0, x_j \geq x_j^L\}$ as firm i's *public* reaction curve. This curve can be used to analyze a mixed duopoly. To illustrate, assume that firm 1 is a welfare-maximizing public firm and firm 2 is a profit-

FIG. 2. A CASE WITH A CORNER MONOPOLY SOLUTION



Moreover, $\mathbf{x}^M$ is a monopoly solution if and only if it is the intersection point between the two curves $\mathbf{\Pi}_i(0)\cup\{\mathbf{x}\in\mathbf{R}_+^2 : x_i=0,\ x_j\geq\tilde{x}_j^L\}$, $i=1, 2$.[7]

It is easy to establish that under each regime, there exists a unique solution, and at least one of the two goods is active. Moreover, under the socially optimal solution, if good j is inactive, then $x_i=\tilde{x}_i^S$; whereas under either one of the Cournot equilibrium and the monopoly solution, if good j is inactive, then $x_i=\tilde{x}_i^M$. At last, whether good i ($i=1, 2$) is active or not is determined by

Lemma 1. (i) $x_i^S>0\Leftrightarrow\tilde{x}_j^S<x_j^L$. (ii) $x_i^C>0\Leftrightarrow\tilde{x}_j^M<x_j^L$. (iii) $x_i^M>0\Leftrightarrow\tilde{x}_j^M<\tilde{x}_j^L$.

In the next step the above results are proven for the socially optimal solution. There are three cases. First, if $\tilde{x}_i^S\leq x_i^L$, $i=1, 2$, then, as shown in Fig. 1, $\mathbf{P}^1(c_1)$ and $\mathbf{P}^2(c_2)$ meet at a unique point. [Since $\mathbf{P}^1(c_1)$ is steeper than $\mathbf{P}^2(c_2)$, they meet at a unique point if they meet at all.] In particular, if $\tilde{x}_i^S<x_i^L$, $i=1, 2$, then the intersection point is an interior one. Second, if $\tilde{x}_1^S\geq x_1^L$, then, as shown in Fig. 2, $\mathbf{P}^1(c_1)$ meets $\{\mathbf{x}\in\mathbf{R}_+^2 : x_2=0,\ x_1\geq x_1^L\}$ at the point $(\tilde{x}_1^S, 0)$. [As shown in Fig. 2, from $\tilde{x}_1^S\geq x_1^L$ it automatically follows that $\tilde{x}_2^S<x_2^L$ because $\mathbf{P}^1(c_1)$ is steeper than $\mathbf{P}^2(c_2)$.] Third, if $\tilde{x}_2^S\geq x_2^L$, then $\mathbf{P}^2(c_2)$ meets $\{\mathbf{x}\in\mathbf{R}_+^2 : x_1=0,\ x_2\geq x_2^L\}$ at the point $(0, \tilde{x}_2^S)$. [Again, $\tilde{x}_2^S\geq x_2^L\Rightarrow\tilde{x}_1^S<x_1^L$.] The desired results thus follow. We can apply the same procedure to prove the counterparts of the above results for the other two solutions.

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maximizing private firm. Equilibrium is represented by the intersection point between firm 1's public reaction curve and firm 2's Cournot reaction curve.

[7] The monopoly solution coincides with two firms maximizing their collective profits. We thus may refer to $\mathbf{\Pi}_i(0)\cup\{\mathbf{x}\in\mathbf{R}_+^2 : x_i=0,\ x_j\geq\tilde{x}_j^L\}$ as firm i's *collusive* reaction curve.

2. The Basic Lemma for Welfare Comparison

Let $\mathbf{W}^+$ stand for $\{\mathbf{x}\in\mathbf{R}^2_+ : W_i(\mathbf{x})>0,\ i=1, 2\}$ (the superscript $+$ denotes that each derivative is positive), the region in which higher quantities are always better in welfare terms. The welfare ranking is analyzed based on the following fact (Chang, 2010, Lemma 5.3):

Fact 2. Assume that $\mathbf{W}^+$ is a connected set. If $\mathbf{x}\in\mathbf{W}^+$ and $\mathbf{y}\in\mathbf{W}^+$, then "$\mathbf{x}\geq\mathbf{y}$ and $\mathbf{x}\neq\mathbf{y}$" $\Rightarrow W(\mathbf{x})>W(\mathbf{y})$.
Eq. (10) implies that, in the (x_1, x_2) space, $\mathbf{W}^+$ is the region lying to the left of $\mathbf{P}^1(c_1)$ and below $\mathbf{P}^2(c_2)$. As a result, $\mathbf{W}^+$ is a connected set. Fact 2 thus leads to the following lemma:[8]

Lemma 2. Assume that $P^i(\mathbf{x})>c_i$ and $P^i(\mathbf{y})>c_i$, $i=1, 2$. If $\mathbf{x}\geq\mathbf{y}$ and $\mathbf{x}\neq\mathbf{y}$, then $W(\mathbf{x})>W(\mathbf{y})$.

3. The Linear Demand Model

The following example will be used to conduct the analysis when general functional forms are not tractable:

Example 1. $U(\mathbf{x})=\alpha_1x_1+\alpha_2x_2-(\beta_1x_1^2+2\gamma x_1x_2+\beta_2x_2^2)/2$, $\forall\mathbf{x}\in\mathbf{R}^2_+$. Moreover, $c_i=0$, $i=1, 2$. In order to guarantee that PPM, SD and SOC-(i) hold, it is assumed that $\alpha_i>0$, $\beta_i>0$, $i=1, 2$, $\gamma>0$, and $\beta_1\beta_2-\gamma^2>0$.
In this example, according to (1), the demands are linear:

$$P^i(\mathbf{x})=\alpha_i-\beta_ix_i-\gamma x_j,\ i=1, 2,\ \forall\mathbf{x}\in\mathbf{R}^2_+.$$

Accordingly, the five assumptions mentioned in section II are satisfied. Moreover,

$$\tilde{x}_i^M=\frac{\alpha_i}{2\beta_i},\ \tilde{x}_i^L=\frac{\alpha_j}{2\gamma},\ i=1, 2, \tag{15}$$

$$\tilde{x}_i^S=2\tilde{x}_i^M,\ x_i^L=2\tilde{x}_i^L,\ i=1, 2. \tag{16}$$

Lemma 1, together with (16), implies that the monopolist and the social planner are the same in the area of product selection:

Proposition 1. Consider Example 1. For each i, $x_i^S>0\Leftrightarrow x_i^M>0$.
Define

$$\eta_i^C\equiv\frac{\gamma}{2\beta_j},\ \eta_i^S\equiv\frac{\gamma}{\beta_j},\ \eta_i^*\equiv\frac{3\beta_i\gamma}{2\beta_1\beta_2+\gamma^2},\ i=1, 2.$$

We demonstrate that[9]

$$\eta_1^C<\eta_1^S<\eta_1^*<\frac{1}{\eta_2^*}<\frac{1}{\eta_2^S}<\frac{1}{\eta_2^C}. \tag{17}$$

[8] According to this lemma, Assumption A mentioned in the introduction plays an important role in determining the welfare ranking.

[9] Relying on the symmetry of the firms' positions, it is sufficient to establish that $\eta_1^*<1/\eta_2^*<1/\eta_2^S<1/\eta_2^C$. We can show that $\eta_1^*<1/\eta_2^*\Leftrightarrow 3\beta_1\beta_2\gamma^2<9\beta_1^2\beta_2^2+\gamma^4$. Moreover, $\gamma^2<\beta_1\beta_2\Rightarrow 3\beta_1\beta_2\gamma^2<3\beta_1^2\beta_2^2$, and hence $3\beta_1\beta_2\gamma^2<9\beta_1^2\beta_2^2+\gamma^4$. It is easy to show that $1/\eta_2^*<1/\eta_2^S\Leftrightarrow\beta_1\beta_2>\gamma^2$. It is trivial that $1/\eta_2^S<1/\eta_2^C$. Q.E.D.

It is easy to show that, for each i, η_i^S (η_i^C) can be used to determine whether x_i^S (x_i^C) is positive:

$$\alpha_i \gtreqless \eta_i^S \alpha_j, \quad \text{as } \tilde{x}_j^S \lesseqgtr x_j^L, \tag{18}$$
$$\alpha_i \gtreqless \eta_i^C \alpha_j, \quad \text{as } \tilde{x}_j^M \lesseqgtr x_j^L. \tag{19}$$

Note that (18), together with (16), implies that η_i^S also can be used to determine whether x_i^M ($i=1, 2$) is positive:

$$\alpha_i \gtreqless \eta_i^S \alpha_j, \quad \text{as } \tilde{x}_j^M \lesseqgtr \tilde{x}_j^L. \tag{20}$$

According to Propositions 3.3, 4.2 and 4.4 of Chang (2010), η_i^* ($i=1, 2$) can be used to determine whether there exist two kinds of quantity reversals:

Fact 3. Consider Example 1, and assume that each solution is interior (i.e., $\mathbf{x}^S \gg \mathbf{0}$, $\mathbf{x}^C \gg \mathbf{0}$ and $\mathbf{x}^M \gg \mathbf{0}$). If $\alpha_i \gtreqless \eta_i^* \alpha_j$, then $x_i^S \gtreqless x_i^C$ and $x_j^M \lesseqgtr x_j^C$.

Note that, according to (17)-(20), each solution is interior if and only if $\eta_1^S < \alpha_1/\alpha_2 < 1/\eta_2^S$ (i.e., $\alpha_i > \eta_i^S \alpha_j$, $i=1, 2$).[10]

Eq. (19), together with (17), implies that there are three cases for the Cournot equilibrium. First, if $\alpha_1/\alpha_2 \in [\eta_1^C, 1/\eta_2^C]$, then $\mathbf{\Pi}_1^1(0)$ meets $\mathbf{\Pi}_2^2(0)$, and hence $\mathbf{x}^C$ is obtained by solving $\Pi_i^i=0$, $i=1, 2$:

$$x_i^C = \frac{2\beta_j \alpha_i - \gamma \alpha_j}{4\beta_1 \beta_2 - \gamma^2}, \; i=1, 2. \tag{21}$$

Second, if $\alpha_1/\alpha_2 \geq 1/\eta_2^C$, then $x_1^C = \tilde{x}_1^M$ and $x_2^C = 0$. Third, if $\alpha_1/\alpha_2 \leq \eta_1^C$, then $x_1^C = 0$ and $x_2^C = \tilde{x}_2^M$.

Similarly, Eqs. (18) and (20), together with (17), imply that there are three cases for $\mathbf{x}^S$ and $\mathbf{x}^M$. First, if $\alpha_1/\alpha_2 \in [\eta_1^S, 1/\eta_2^S]$, then $\mathbf{x}^S$ is obtained by solving $P^i = c_i$, $i=1, 2$, and $\mathbf{x}^M$ is obtained by solving $\Pi_i=0$, $i=1, 2$:

$$x_i^S = \frac{\beta_j \alpha_i - \gamma \alpha_j}{\beta_1 \beta_2 - \gamma^2}, \; i=1, 2, \tag{22}$$
$$x_i^M = \frac{\beta_j \alpha_i - \gamma \alpha_j}{2(\beta_1 \beta_2 - \gamma^2)}, \; i=1, 2. \tag{23}$$

Second, if $\alpha_1/\alpha_2 \geq 1/\eta_2^S$, then $x_1^S = \tilde{x}_1^S$, $x_1^M = \tilde{x}_1^M$ and $x_2^S = x_2^M = 0$. Third, if $\alpha_1/\alpha_2 \leq \eta_1^S$, then $x_1^S = x_1^M = 0$, $x_2^S = \tilde{x}_2^S$ and $x_2^M = \tilde{x}_2^M$. As a result, $\mathbf{x}^S$ and $\mathbf{x}^M$ are parallel regardless of whether they are interior or not:

Proposition 2. For Example 1, $\mathbf{x}^M = \mathbf{x}^S/2$.

IV. *The Quantity Space*

In this section, the analysis is conducted in the quantity space. The main purpose is to analyze whether the Cournot equilibrium is more welfare-efficient than the monopoly solution or not. However, in order to explain the welfare ranking, we begin with comparing them in the

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[10] Eqs. (18) and (20) imply that $\mathbf{x}^S \gg \mathbf{0}$ and $\mathbf{x}^M \gg \mathbf{0}$ if and only if $\eta_1^S < \alpha_1/\alpha_2 < 1/\eta_2^S$. According to (19), $\mathbf{x}^C \gg \mathbf{0} \Leftrightarrow \eta_1^C < \alpha_1/\alpha_2 < 1/\eta_2^C$. Moreover, from (17) it follows that $\eta_1^S < \alpha_1/\alpha_2 < 1/\eta_2^S \Rightarrow \eta_1^C < \alpha_1/\alpha_2 < 1/\eta_2^C$.

areas of product selection, individual outputs, output level and output structure. This study follows Chang and Peng (2012) by using the socially optimal solution as a reference point when conducting the analysis because it is interesting to verify whether the Cournot equilibrium is closer to the socially optimal solution (i.e., the perfectly competitive equilibrium) than the monopoly solution.

1. Product Selection

As mentioned in the introduction, under the Cournot competition, a more efficient firm may have a higher markup rate. This suggests that it is easier for the less efficient good to be active under the Cournot equilibrium, compared with the perfectly competitive equilibrium. This conjecture, together with Proposition 1, suggests that the following proposition is true:

Proposition 3. (i) For each i, if $x_i^S>0$, then $x_i^C>0$, and the converse is not true. (ii) For each i, if $x_i^M>0$, then $x_i^C>0$, and the converse is not true.
According to Fact 1, $\tilde{x}_i^M<\tilde{x}_i^S$ and $\tilde{x}_i^L<x_i^L$, $i=1, 2$. Lemma 1 thus implies that $x_i^S>0\Rightarrow x_i^C>0$ and $x_i^M>0\Rightarrow x_i^C>0$. It remains to establish that "the converse is not true." It is sufficient to demonstrate that, for Example 1, it is possible for the inequality $x_i^C>0$ to coexist with either one of $x_i^M=0$ and $x_i^S=0$. The six numbers in (17) divide $(0, +\infty)$ into seven intervals. However, relying on the symmetry of the firms' positions, we concentrate on the following four intervals:[11]

Case I: $\eta_1^* \leq \alpha_1/\alpha_2 \leq 1/\eta_2^*$,
Case II: $1/\eta_2^* < \alpha_1/\alpha_2 < 1/\eta_2^S$,
Case III: $1/\eta_2^S \leq \alpha_1/\alpha_2 < 1/\eta_2^C$,
Case IV: $\alpha_1/\alpha_2 \geq 1/\eta_2^C$.

According to (17)-(20), in Cases I and II, each solution is an interior solution; in Case III, $\mathbf{x}^C$ is an interior solution, but either one of $\mathbf{x}^S$ and $\mathbf{x}^M$ is a corner solution (i.e., $x_2^S=x_2^M=0$); in Case IV, each solution is a corner solution, namely, $\mathbf{x}^C=\mathbf{x}^M=(\tilde{x}_1^M, 0)'$ and $\mathbf{x}^S=(\tilde{x}_1^S, 0)'$. The existence of Case III completes the proof of Proposition 3.

Proposition 3-(i) indicates that the Cournot competition may allow a socially undesirable good to be active. In contrast, Proposition 1 means that, for Example 1, the monopoly solution and the socially optimal solution are the same in terms of selecting products. Hence, it seems that, in the area of product selection, the monopoly outperforms the Cournot duopoly.

As mentioned, at least one of the two goods is active in each solution. Hence, without loss of generality it can be assumed that $x_1^M>0$. Proposition 3-(ii) thus implies that there are the following three cases: (A) as shown in Fig. 1, both $\mathbf{x}^M$ and $\mathbf{x}^C$ are interior solutions; (B) as shown in Fig. 2, $\mathbf{x}^C$ is an interior solution, but $\mathbf{x}^M$ is a corner solution: $x_1^M=\tilde{x}_1^M$ and $x_2^M=0$; and (C) both $\mathbf{x}^M$ and $\mathbf{x}^C$ are corner solutions: $x_1^C=x_1^M=\tilde{x}_1^M$ and $x_2^C=x_2^M=0$. In Case C, it is trivial to compare $\mathbf{x}^C$ and $\mathbf{x}^M$. Accordingly, in order to simplify the presentation, hereafter we restrict our

[11] We need not consider the following three cases: Case II′: $\eta_1^S<\alpha_1/\alpha_2<\eta_1^*$, Case III′: $\eta_1^C<\alpha_1/\alpha_2\leq\eta_1^S$ and Case IV′: $\alpha_1/\alpha_2\leq\eta_1^C$. This is because, in essence, Case II′ for example is the same as Case II. Note that $\eta_1^S<\alpha_1/\alpha_2<\eta_1^*\Leftrightarrow 1/\eta_1^*<\alpha_2/\alpha_1<1/\eta_1^S$. Accordingly, the same results obtained for Case II apply to Case II′, provided that 1 and 2 are interchanged.

attention to Cases A and B, i.e., it is assumed that the Cournot equilibrium is an interior solution:

$$\tilde{x}_i^M < x_i^L,\ i=1,\ 2. \tag{24}$$

2. Individual Outputs

Since the Cournot equilibrium is an interior solution, $\mathbf{\Pi}_1^1(0)$ and $\mathbf{\Pi}_2^2(0)$ intersect at an interior point of $\mathbf{R}_+^2$, and hence these two curves divide the quantity space into four connected regions. To which one of these four regions does $\mathbf{x}^M$ belong? Eqs. (11)-(12) indicate that the rankings between the three kinds of curves are clear-cut:

Fact 4. Consider the (x_1, x_2) space. (i) $\mathbf{\Pi}_1^1(0)$ lies to the right of $\mathbf{\Pi}_1(0)$, and $\mathbf{\Pi}_2^2(0)$ lies above $\mathbf{\Pi}_2(0)$. (ii) $\mathbf{\Pi}_1^1(0)$ lies to the left of $\mathbf{P}^1(c_1)$, and $\mathbf{\Pi}_2^2(0)$ lies below $\mathbf{P}^2(c_2)$.
As a consequence of Fact 4-(i), as shown in Figs. 1-2, $\mathbf{x}^M$ belongs to the southwest region regardless of whether $\mathbf{x}^M$ is interior or not:

Lemma 3. Assume that $x_1^M > 0$ and the Cournot equilibrium is an interior solution. (i) If the monopoly solution is an interior solution, then, as shown in Fig. 1, $\mathbf{x}^M$ strictly lies to the left of $\mathbf{\Pi}_1^1(0)$ and below $\mathbf{\Pi}_2^2(0)$. (ii) If the monopoly solution is a corner solution, then, as shown in Fig. 2, $\mathbf{x}^M$ is the intersection point between $\mathbf{\Pi}_1^1(0)$ and the x_1-axis (i.e., $x_1^M = \tilde{x}_1^M$ and $x_2^M = 0$). Hence, $x_1^M > x_1^C > 0$ (an output reversal) and $x_2^C > x_2^M = 0$.

Proof: If $\mathbf{x}^M \gg \mathbf{0}$, then $\mathbf{x}^M$ is the intersection point between $\mathbf{\Pi}_1(0)$ and $\mathbf{\Pi}_2(0)$, and hence Part (i) follows from Fact 4-(i). Otherwise, as mentioned in subsection III.1, $x_1^M = \tilde{x}_1^M$. Because $\mathbf{\Pi}_1^1(0)$ slopes downward, from $\mathbf{x}^C \in \mathbf{\Pi}_1^1(0)$ it follows that, as shown in Fig. 2, $\tilde{x}_1^M > x_1^C$. Therefore, Part (ii) holds. Q.E.D.

In contrast, Fact 4-(ii) indicates that $\mathbf{x}^S$ belongs to the northeast region regardless of whether $\mathbf{x}^S$ is interior or not:

Lemma 4. Assume that $x_1^S > 0$ and the Cournot equilibrium is an interior solution. (i) If the socially optimal solution is an interior solution, then, as shown in Fig. 1, $\mathbf{x}^S$ lies strictly to the right of $\mathbf{\Pi}_1^1(0)$ and above $\mathbf{\Pi}_2^2(0)$. (ii) If the socially optimal solution is a corner solution, then, as shown in Fig. 2, $\mathbf{x}^S$ is the intersection point of $\mathbf{P}^1(c_1)$ and the x_1-axis (i.e., $x_1^S = \tilde{x}_1^S$ and $x_2^S = 0$). Hence, $x_1^S > x_1^C > 0$ and $x_2^C > x_2^S = 0$ (an output reversal).

Lemmas 3-(i) and 4-(i) straightforwardly yield the following proposition:

Proposition 4. If the goods are symmetric (i.e., U is symmetric[12] and $c_1 = c_2$), then each good has a normal output ranking: $0 < x_i^M < x_i^C < x_i^S$, $i=1, 2$.
Proof: It is clear that $x_1^k = x_2^k > 0$, $k = M, C, S$. The desired result thus follow from Lemma 3-(i) and Lemma 4-(i). Q.E.D.

Since both $\mathbf{\Pi}_1^1(0)$ and $\mathbf{\Pi}_2^2(0)$ slope downward, from Lemmas 3-(i) and 4-(i) it does not necessarily follow that each good has a normal output ranking. As a matter of fact, according to Lemmas 3-(ii) and 4-(ii), if either $\mathbf{x}^M$ or $\mathbf{x}^S$ is a corner solution, then an output reversal should

[12] Let $\hat{\mathbf{x}}$ be any permutation of $\mathbf{x}$, then $U(\hat{\mathbf{x}}) = U(\mathbf{x})$, $\forall\, \mathbf{x}$.

exist. By continuity, these results suggest that when all the solutions are interior solutions, an output reversal still exists as long as the asymmetry is strong enough. This conjecture is true for Example 1:

Proposition 5. Consider Example 1. (i) In Case I, $x_1^S \geq x_1^C \geq x_1^M > 0$ and $x_2^S \geq x_2^C \geq x_2^M > 0$, where all of the inequalities are strict if α_1/α_2 is not equal to either one of η_1^* and $1/\eta_2^*$. (ii) In Case II, $x_1^S > x_1^M > x_1^C > 0$ and $x_2^C > x_2^S > x_2^M > 0$. (iii) In Case III, $x_1^S > x_1^M > x_1^C > 0$ and $x_2^C > x_2^S = x_2^M = 0$.
Proof: As mentioned, in Cases I and II, each solution is an interior solution, whereas, in Case III, $\mathbf{x}^C$ is an interior solution, but either one of $\mathbf{x}^S$ and $\mathbf{x}^M$ is a corner solution (i.e., $x_2^S = x_2^M = 0$). Therefore, Part (i) follows from Fact 3,[13] whereas Part (ii) follows from Fact 3 and Proposition 2.[14] Part (iii) follows from Proposition 2 and Lemma 3-(ii). Q.E.D.

Note that, in Cases II and III, there are two output reversals: $x_1^M > x_1^C$ and $x_2^C > x_2^S$.

3. Output Level

It is well known that industry output supplied under the Cournot duopoly is greater than industry output supplied under the monopoly if two producers sell identical products (Sonnenschein, 1968, p.317). In this subsection, we aim at comparing the solutions from the viewpoint of industry output. To this end, the following condition is introduced:[15]

Diagonal dominance (DD): $|\Pi_{ii}^i| > |\Pi_{ij}^i|$, $i=1, 2$.
According to SS-(ii) and SOC-(iii), $\Pi_{ik}^i < 0$, $i, k = 1, 2$. Hence, DD means that

$$\frac{\Pi_{11}^1}{\Pi_{12}^1} > 1 > \frac{\Pi_{21}^2}{\Pi_{22}^2}. \tag{25}$$

Moreover, DD is sufficient but not necessary for SOC-(iv). DD is not required except when it is specifically mentioned.

The following proposition establishes that if DD holds, then the industry-output ranking is normal regardless of whether $\mathbf{x}^M$ and $\mathbf{x}^S$ are interior or not:

Proposition 6. Assume that the Cournot equilibrium is an interior solution. If DD holds, then the industry-output ranking is normal: $x_1^M + x_2^M < x_1^C + x_2^C < x_1^S + x_2^S$.
Proof: Since the Cournot equilibrium is interior, $\mathbf{\Pi}_1^1(0)$ meets $\mathbf{\Pi}_2^2(0)$ at an interior point. The Cournot equilibrium $\mathbf{x}^C$ belongs to the iso-industry-output curve $\{\mathbf{x} \in \mathbf{R}_+^2 : x_1 + x_2 = x_1^C + x_2^C\}$, which is a line segment. Eq. (25) implies that, as shown in Fig. 3, $\mathbf{\Pi}_1^1(0)$ is steeper than the line segment, whereas $\mathbf{\Pi}_2^2(0)$ is flatter than the line segment. Lemmas 3-4 thus guarantee that $\mathbf{x}^M$ strictly lies below the line segment, whereas $\mathbf{x}^S$ strictly lies above the line segment. It follows that $x_1^M + x_2^M < x_1^C + x_2^C < x_1^S + x_2^S$. Q.E.D.

If demands are symmetric (i.e., U is symmetric), then $\Pi_{11}^1 = \Pi_{22}^2$ and $\Pi_{12}^1 = \Pi_{21}^2$, implying

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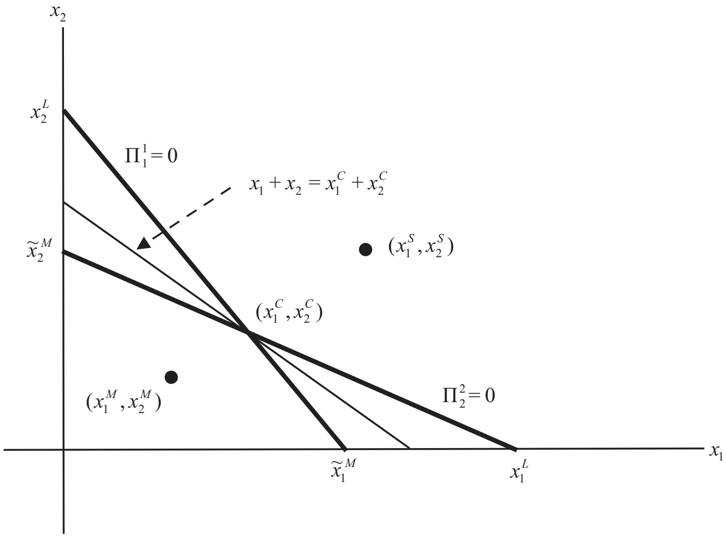
[13] In Case I, $\alpha_1 \geq \eta_1^* \alpha_2$ and $\alpha_2 \geq \eta_2^* \alpha_1$. According to Fact 3, from $\alpha_1 \geq \eta_1^* \alpha_2$ it follows that $x_1^S \geq x_1^C$ and $x_2^M \leq x_2^C$. Similarly, from $\alpha_2 \geq \eta_2^* \alpha_1$ it follows that $x_2^S \geq x_2^C$ and $x_1^M \leq x_1^C$.

[14] In Case II, $\alpha_2 < \eta_2^* \alpha_1$, which, according to Fact 3, implies that $x_2^S < x_2^C$ and $x_1^M > x_1^C$. According to Proposition 2, $x_2^M = x_2^S/2$ and $x_1^S = 2x_1^M$, implying that $x_2^M < x_2^S$ and $x_1^S > x_1^M$.

[15] Eq. (9) implies that when demands are linear, DD is weaker than the condition stating that the Jacobian matrix for the inverse demand functions is diagonally dominated (i.e., $|P_i^i| > |P_j^i|$, $i=1, 2$).

FIG. 3. A CASE WITH A DIAGONAL DOMINANCE



that SOC-(iv) is sufficient and necessary for DD. [DD has nothing to do with either c_1 or c_2.] Therefore, Proposition 6 yields the following corollary:

Corollary 1. Assume that the Cournot equilibrium is an interior solution. If the demands are symmetric, then $x_1^M+x_2^M<x_1^C+x_2^C<x_1^S+x_2^S$.

We are now in a position to discuss the economic meaning of DD. For Example 1, DD means that $\gamma<\min(2\beta_1, 2\beta_2)$. [DD has nothing to do with either α_1 or α_2.] According to (5), it is always true that $\max(\beta_1, \beta_2)>\gamma$. As a result, if $\beta_1=\beta_2$,[16] which is a popular assumption in many studies in the literature [e.g., Qiu (1997, p.215), Hsu and Wang (2005, p.186) and Zanchettin (2006, p.1001)], then DD automatically holds. In general, DD may fail. When DD is not satisfied, there are only the following two alternative cases: (1) $2\beta_1\leq\gamma<\beta_2$ and (2) $2\beta_2\leq\gamma<\beta_1$.[17] It is important to distinguish between two types of quality: First, good i's quality increases with α_i (type I quality).[18] Second, good i's quality decreases with β_i (type II quality).[19] In Case 1 (2), good 1 is much better (worse) than good 2 in terms of the type II quality. Therefore, DD can fail only if the two goods differ too much in terms of the type II quality and the degree of substitutability (i.e., γ) is high enough. In other words, DD means that the two goods do not differ too much in terms of the type II quality or the degree of substitutability is low enough. In order to emphasize the above abnormal cases, let us define the following two conditions:

16 This condition is weaker than the assumption that demands are symmetric (i.e., $\alpha_1=\alpha_2$ and $\beta_1=\beta_2$).

17 If DD fails, then $\exists i, \ni 2\beta_i\leq\gamma$, which, together with $\gamma<\max[\beta_1, \beta_2]$, implies that $\gamma<\beta_j$.

18 In Hur's model (2006, p.199), firm i ($i=1, 2$) invests in product R&D to raise α_i.

19 In the model studied by Symeonidis (2003, pp.42-3), firm i ($i=1, 2$) invests in product R&D to reduce both β_i and γ.

A1: $2\beta_1 < \gamma < \beta_2$,
A2: $2\beta_2 < \gamma < \beta_1$.

We demonstrate that DD is crucial: If DD is not satisfied, then either one of the following two cases is possible: (a) $x_1^C + x_2^C < x_1^M + x_2^M < x_1^S + x_2^S$; (b) $x_1^M + x_2^M < x_1^S + x_2^S < x_1^C + x_2^C$. Proposition 5-(i) guarantees that, in Case I, the industry-output ranking is normal regardless of whether DD is satisfied or not. We thus can restrict our attention to Cases II and III:

Proposition 7. Consider Example 1. (i) Consider Case II. (a) If A1 holds, then $x_1^S + x_2^S > x_1^C + x_2^C$, and $\exists \tilde{\alpha} \in (1/\eta_2^*, 1/\eta_2^S)$, $\ni \text{sign}((x_1^C + x_2^C) - (x_1^M + x_2^M)) = \text{sign}(\tilde{\alpha} - \alpha_1/\alpha_2)$, $\forall \alpha_1/\alpha_2 \in (1/\eta_2^*, 1/\eta_2^S)$. (b) If A2 holds, then $x_1^C + x_2^C > x_1^M + x_2^M$, and $\exists \hat{\alpha} \in (1/\eta_2^*, 1/\eta_2^S)$, $\ni \text{sign}((x_1^S + x_2^S) - (x_1^C + x_2^C)) = \text{sign}(\hat{\alpha} - \alpha_1/\alpha_2)$, $\forall \alpha_1/\alpha_2 \in (1/\eta_2^*, 1/\eta_2^S)$. (ii) In Case III, $x_1^S + x_2^S > x_1^C + x_2^C$. Moreover, $(x_1^C + x_2^C) - (x_1^M + x_2^M)$ shares the same sign with $2\beta_1 - \gamma$. Hence, if and only if A1 holds, then $x_1^C + x_2^C < x_1^M + x_2^M$. (Proof in Appendix.)
Note that in either one of (i.a) and (ii), it is guaranteed that $x_1^S + x_2^S > x_1^C + x_2^C$, whereas it is not necessarily true that $x_1^C + x_2^C > x_1^M + x_2^M$. In contrast, in (i.b), it is true that $x_1^C + x_2^C > x_1^M + x_2^M$, whereas it is not guaranteed that $x_1^S + x_2^S > x_1^C + x_2^C$. Note that, in (i.b), there may exist an overshooting: $x_1^C + x_2^C$ not only exceeds $x_1^M + x_2^M$, but may also be higher than $x_1^S + x_2^S$.

4. Output Structure

The output vector is called a Ramsey output vector if there exists $k \in [\Pi(\mathbf{x}^S), \Pi(\mathbf{x}^M)]$ such that it solves the following Ramsey problem:[20]

$$\max_{\mathbf{x}} W(\mathbf{x}) \text{ s.t. } \Pi(\mathbf{x}) \geq k. \tag{26}$$

The output vector is referred to as having an "optimal output structure" if and only if it is a Ramsey output vector. There are two obvious Ramsey output vectors. First, when $k = \Pi(\mathbf{x}^S)$, $\mathbf{x}^S$ solves Program (26), and hence it is a Ramsey output vector. Second, when $k = \Pi(\mathbf{x}^M)$, only $\mathbf{x}^M$ can satisfy the constraint, implying that it is a Ramsey output vector.

When the goods are symmetric, the Cournot competition does not distort the output structure:

Proposition 8. If the goods are symmetric, then $\mathbf{x}^C$ is a Ramsey output vector.

Proof: Because of SOC-(i)-(ii), both $\{\mathbf{x} \in \mathbf{R}_+^2 : W(\mathbf{x}) \geq W(\mathbf{x}^C)\}$ and $\{\mathbf{x} \in \mathbf{R}_+^2 : \Pi(\mathbf{x}) \geq \Pi(\mathbf{x}^C)\}$ are strictly convex sets. Therefore, from the fact that W is symmetric it follows that the iso-industry-output curve $\{\mathbf{x} \in \mathbf{R}_+^2 : x_1 + x_2 = x_1^C + x_2^C\}$ is a line tangential to the indifference curve $\{\mathbf{x} \in \mathbf{R}_+^2 : W(\mathbf{x}) = W(\mathbf{x}^C)\}$ at $\mathbf{x}^C$. Similarly, the iso-industry-output curve is also a line tangential to the iso-industry-profit curve $\{\mathbf{x} \in \mathbf{R}_+^2 : \Pi(\mathbf{x}) = \Pi(\mathbf{x}^C)\}$ at $\mathbf{x}^C$. It thus follows that the indifference curve is tangential to the iso-industry-profit curve at $\mathbf{x}^C$, implying that $\mathbf{x}^C$ is a Ramsey output vector. Q.E.D.

In contrast, Chang and Peng (2009, p.687) and Chang (2010, p.80) argue that if the goods

[20] In the Ramsey pricing literature, social planner problems are usually formulated in the price space; e.g., see program (1) in Chang and Peng (2009). Program (26) is a counterpart for the quantity space of the Ramsey program.

are asymmetric, then oligopolistic competition distorts the output structure, except when there are coincidences. Furthermore, this distortion arises from the horizontal externalities: each firm will not take into account the impacts on other firms of its choice. In order to verify this argument, consider Example 1. In appearance, if τ is a positive number, then $\tau\mathbf{x}^S$ shares the same structure with $\mathbf{x}^S$, suggesting that $\tau\mathbf{x}^S$ is a Ramsey output vector. Actually, this is true regardless of whether $\mathbf{x}^S$ is an interior solution or not:

Lemma 5. Consider Example 1, and assume that the Cournot equilibrium is an interior solution. If $\hat{\mathbf{x}}$ belongs to $\{\mathbf{x}\in\mathbf{R}^2_+ : \mathbf{x}=t\mathbf{x}^M+(1-t)\mathbf{x}^S,\ 0\leq t\leq 1\}$, the line segment joining $\mathbf{x}^M$ and $\mathbf{x}^S$, then it is a Ramsey output vector.
Proof: According to Proposition 2, $\mathbf{x}^M=\mathbf{x}^S/2$, implying that there exists $\tau\in[1/2, 1]$ such that $\hat{\mathbf{x}}=\tau\mathbf{x}^S$. Let us begin by studying the case where $\mathbf{x}^S$ is interior. Eq. (22) can be written as $\mathbf{x}^S=B^{-1}\mathbf{a}$ where

$$\mathbf{a}=\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix},\ B=\begin{pmatrix}\beta_1 & \gamma\\ \gamma & \beta_2\end{pmatrix}.$$

It is easy to show that $W=U=\mathbf{a}\cdot\mathbf{x}-\mathbf{x}'B\mathbf{x}/2$ and $\Pi=\mathbf{x}\cdot\partial U/\partial\mathbf{x}=\mathbf{x}\cdot(\mathbf{a}-B\mathbf{x})=\mathbf{a}\cdot\mathbf{x}-\mathbf{x}'B\mathbf{x}$. It follows that $\partial W/\partial\mathbf{x}=\mathbf{a}-B\mathbf{x}$ and $\partial\Pi/\partial\mathbf{x}=\mathbf{a}-2B\mathbf{x}$. Therefore, $\partial W(\tau\mathbf{x}^S)/\partial\mathbf{x}=(1-\tau)\mathbf{a}$ and $\partial\Pi(\tau\mathbf{x}^S)/\partial\mathbf{x}=(1-2\tau)\mathbf{a}$, implying that the two vectors $\partial W(\tau\mathbf{x}^S)/\partial\mathbf{x}$ and $\partial\Pi(\tau\mathbf{x}^S)/\partial\mathbf{x}$ are parallel, namely, $W_1(\tau\mathbf{x}^S)/W_2(\tau\mathbf{x}^S)=\Pi_1(\tau\mathbf{x}^S)/\Pi_2(\tau\mathbf{x}^S)=\alpha_1/\alpha_2$. This means that $\tau\mathbf{x}^S$ is a tangent point: in the (x_1, x_2) space, the indifference curve $\{\mathbf{x}\in\mathbf{R}^2_+ : W(\mathbf{x})=W(\tau\mathbf{x}^S)\}$ and the iso-industry-profit curve $\{\mathbf{x}\in\mathbf{R}^2_+ : \Pi(\mathbf{x})=\Pi(\tau\mathbf{x}^S)\}$ meet only at $\tau\mathbf{x}^S$ (the slope of the common tangent is $-\alpha_1/\alpha_2$). Hence, $\tau\mathbf{x}^S$ is a Ramsey output vector.

The case where $\mathbf{x}^S$ is a corner solution is studied next. It is sufficient to examine the case where $x_1^S=\tilde{x}_1^S$ and $x_2^S=0$ (i.e., $\alpha_1/\alpha_2\geq 1/\eta_2^S$). Proposition 2, together with Lemma 3-(ii), implies that $x_1^M=\tilde{x}_1^M=x_1^S/2$ and $x_2^M=0$, and hence $x_1^M\leq\hat{x}_1\leq\tilde{x}_1^S$ and $\hat{x}_2=0$. Since both $\mathbf{x}^M$ and $\mathbf{x}^S$ are Ramsey output vectors, we can assume that $x_1^M<\hat{x}_1<\tilde{x}_1^S$. According to Fact 1-(i), $\Pi_1(\tilde{x}_1^M, 0)=0$, which, together with SOC-(ii), implies that $\Pi_1(\hat{x}_1, 0)<0$. According to Lemma 1-(iii), from $x_2^M=0$ it follows that $\tilde{x}_1^M\geq\tilde{x}_1^L$. According to Fact 1-(iii), $\Pi_2(\tilde{x}_1^L, 0)=0$. Hence, SS-(i) guarantees that $\Pi_2(\hat{x}_1, 0)<0$. The two inequalities $\Pi_1(\hat{x}_1, 0)<0$ and $\Pi_2(\hat{x}_1, 0)<0$ imply that the iso-industry-profit curve $\{\mathbf{x}\in\mathbf{R}^2_+ : \Pi(\mathbf{x})=\Pi(\hat{\mathbf{x}})\}$ has a downward slope. According to Lemma 1-(i), from $x_2^S=0$ it follows that $\tilde{x}_1^S\geq x_1^L$, which, together with (24), implies that $\tilde{x}_1^S\geq x_1^L>x_1^M$. [Note that $x_1^M=\tilde{x}_1^M$.] By definition, $P^1(\tilde{x}_1^S, 0)=0$. [In Example 1, $c_i=0$, $i=1, 2$.] Hence, $P^1(\hat{x}_1, 0)>0$. By definition, $P^2(x_1^L, 0)=0$. Accordingly, if $\hat{x}_1\geq x_1^L$, then $P^2(\hat{x}_1, 0)\leq 0$, which, together with $P^1(\hat{x}_1, 0)>0$, implies that the indifference curve $\{\mathbf{x}\in\mathbf{R}^2_+ : U(\mathbf{x})=U(\hat{\mathbf{x}})\}$ is upward sloping, and the direction of increasing utility is to the right. As a result, $\hat{\mathbf{x}}$ is socially superior to any other point belonging to the iso-profit curve. If $\hat{x}_1<x_1^L$, then the indifference curve slopes downward. We can show that the indifference curve is steeper than the iso-profit curve, i.e.,[21]

$$P^1(\hat{x}_1, 0)/P^2(\hat{x}_1, 0)\geq\Pi_1(\hat{x}_1, 0)/\Pi_2(\hat{x}_1, 0),\ \forall\,\hat{x}_1\in(x_1^M, x_1^L), \tag{27}$$

[21] As mentioned, $P^i(\hat{x}_1, 0)>0$ and $\Pi_i(\hat{x}_1, 0)<0$, $i=1, 2$. Therefore, (27) means that $P^1(\hat{x}_1, 0)\Pi_2(\hat{x}_1, 0)\leq P^2(\hat{x}_1, 0)\Pi_1(\hat{x}_1, 0)$. Eq. (6) implies that $\Pi_i(\hat{x}_1, 0)=P^i(\hat{x}_1, 0)+\hat{x}_1P^i_i(\hat{x}_1, 0)$, $i=1, 2$. Therefore, (27) means that $P^1(\hat{x}_1, 0)P^1_2(\hat{x}_1, 0)\leq P^2(\hat{x}_1, 0)P^1_1(\hat{x}_1, 0)$. For Example 1, the above inequality means that $(\alpha_1-\beta_1\hat{x}_1)(-\gamma)\leq(\alpha_2-\gamma\hat{x}_1)(-\beta_1)$, which is satisfied if and only if $\alpha_1\gamma\geq\alpha_2\beta_1$ (i.e., $\alpha_1/\alpha_2\geq 1/\eta_2^S$).

with strict inequality if $\alpha_1/\alpha_2 > 1/\eta_2^S$. Hence $\hat{\mathbf{x}}$ is socially superior to any other point belonging to the iso-industry-profit curve. Q.E.D.

Lemma 5 yields the following proposition:

Proposition 9. Consider Example 1, and assume that the Cournot equilibrium is interior. (i) If the socially optimal solution is a corner solution, then the Cournot competition distorts the output structure. (ii) If the socially optimal solution is an interior solution, then the Cournot competition distorts the output structure, except when

$$\frac{\beta_1}{\beta_2} = \frac{\alpha_1^2}{\alpha_2^2}. \tag{28}$$

Proof: If either x_1^S or x_2^S is zero, then it is clear that there does not exist a positive number τ such that $\mathbf{x}^C = \tau \mathbf{x}^S$. Therefore, it is sufficient to examine the case with $\mathbf{x}^S \gg \mathbf{0}$. It is easy to apply (21) and (22) to establish that $x_1^C/x_2^C = x_1^S/x_2^S$ if and only if (28) is satisfied. Q.E.D.

Hence, when the goods are asymmetric, the Cournot competition distorts the output structure "with probability one." [22]

In which direction does the Cournot competition distort the output structure? In order to address this issue, consider the following example:

Example 2. This is a sub-example of Example 1, but with $\beta_1 = \beta_2 = 1$ and $\alpha_1 > \alpha_2$ (good 1 is more efficient than good 2).

Let us characterize the output structure by using the following Herfindahl index:

$$H(\mathbf{x}) \equiv \sum_i s_i^2 \text{ where } s_i \text{ donotes the unit share of good } i : s_i \equiv x_i / \sum_k x_k,$$

which measures the degree of concentration of production with the larger firm. We have a clear-cut result regardless of whether $\mathbf{x}^S$ is interior or not:

Proposition 10. Consider Example 2, and assume that the Cournot equilibrium is interior (i.e., $1 < \alpha_1/\alpha_2 < 1/\eta_2^C = 2/\gamma$). (i) In each solution, the more efficient good has a higher unit share: $s_1 > s_2$. (ii) $H(\mathbf{x}^C) < H(\mathbf{x}^M) = H(\mathbf{x}^S)$.

Proof: Proposition 2 implies that $H(\mathbf{x}^M) = H(\mathbf{x}^S)$. We can thus restrict our attention to comparing $\mathbf{x}^C$ and $\mathbf{x}^S$. According to Proposition 6.2 from Chang and Peng (2012), $\mathbf{x}^k \gg \mathbf{0} \Rightarrow x_1^k > x_2^k$, $k = C, S$. If $\mathbf{x}^S$ is a corner solution, then $1/\eta_2^S \leq \alpha_1/\alpha_2 < 1/\eta_2^C$, implying that $x_1^S > 0$ and $x_2^S = 0$. This completes the proof of Part (i). If $\mathbf{x}^S$ is an interior solution, then, according to Proposition 6.3-(iii) in Chang and Peng (2012), $H(\mathbf{x}^C) < H(\mathbf{x}^S)$. If $\mathbf{x}^S$ is a corner solution, then $H(\mathbf{x}^S) = 1$, implying that $H(\mathbf{x}^C) < H(\mathbf{x}^S)$. [From $\mathbf{x}^C \gg \mathbf{0}$ it follows that $H(\mathbf{x}^C) < 1$.] This completes the proof of Part (ii). Q.E.D.

This proposition extends Propositions 6.2 and 6.3 in Chang and Peng (2012) by allowing the solutions to be either interior or corner solutions. Part (i) of this proposition implies that, as pointed out by Chang and Peng (2012, Section 3.3), the Herfindahl index also measures the

[22] The parameter space is $\mathbf{\Omega} \equiv \{(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma) \in \mathbf{R}_+^5 : \alpha_i > 0, \beta_i > 0, i = 1, 2, \gamma > 0, \beta_1\beta_2 - \gamma^2 > 0\}$. $\mathbf{\Omega}$ has a dimension of 5. However, $\{(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma) \in \mathbf{\Omega} : \text{(28) is satisfied}\}$ has a dimension of 4.

degree of concentration of production with the more efficient firm. Therefore, Part (ii) means that under the Cournot competition, the output structure is unambiguously distorted in favor of the less efficient firm.

5. Welfare Analysis

In this subsection, the welfare ranking is analyzed. However, according to Lemma 2, we should begin with examining whether the profit margins are positive or not. The equilibrium condition (14) for the Cournot competition can be rewritten as follows:

$$x_i \geq 0,\ P^i - c_i + x_i P^i_i \leq 0,\ x_i(P^i - c_i + x_i P^i_i) = 0,\ i=1,\ 2.$$

This condition straightforwardly leads to the following lemma:

Lemma 6. If good i ($i=1, 2$) is active under the Cournot equilibrium, then $p^C_i > c_i$. Otherwise, $p^C_i \leq c_i$.
Therefore, it is guaranteed that $\mathbf{p}^C \gg \mathbf{c}$ because, as mentioned, it is assumed that the Cournot equilibrium is interior. This is also the case for the monopoly solution regardless of whether it is an interior solution or not:[23]

Lemma 7. If the Cournot equilibrium is an interior solution, then $\mathbf{p}^M \gg \mathbf{c}$.
Proof: According to Fact 4-(ii), $\mathbf{\Pi}^1_1(0)$ lies to the left of $\mathbf{P}^1(c_1)$, and $\mathbf{\Pi}^2_2(0)$ lies below $\mathbf{P}^2(c_2)$. Therefore, Lemma 3 indicates that, as shown in Figs. 1-2, $\mathbf{x}^M$ always strictly lies to the left of $\mathbf{P}^1(c_1)$ and below $\mathbf{P}^2(c_2)$ regardless of whether $\mathbf{x}^M$ is an interior solution or not.[24] Eq. (4) thus guarantees that $p^M_i > c_i$ ($i=1, 2$). Q.E.D.

In summary, it is guaranteed that $\mathbf{p}^C \gg \mathbf{c}$ and $\mathbf{p}^M \gg \mathbf{c}$. Lemma 2 thus implies that higher quantities are always better in welfare terms. Therefore, Proposition 4 and Proposition 5-(i) yield the following corollary:

Corollary 2. (i) If goods are symmetric, then $W(\mathbf{x}^C) > W(\mathbf{x}^M)$. (ii) Consider Example 1. In Case I, $W(\mathbf{x}^C) > W(\mathbf{x}^M)$.

Corollary 2-(ii) implies that, for Example 1, if the demand asymmetry is weak enough, then the welfare ranking is normal. In contrast, the welfare ranking is abnormal if the demand asymmetry is strong enough:

Proposition 11. Consider Example 1. (i) In Case II, $W(\mathbf{x}^C) > W(\mathbf{x}^M)$. (ii) Consider Case III. There exists $\bar{\alpha} \in (1/\eta^S_2, 1/\eta^C_2)$ such that $\mathrm{sign}(W(\mathbf{x}^M) - W(\mathbf{x}^C)) = \mathrm{sign}(\alpha_1/\alpha_2 - \bar{\alpha})$. (Proof in Appendix.)
Proposition 11, together with Corollary 2-(ii), yields an elegant result for Cases I-III:

Corollary 3. Consider Example 1, and assume that $\alpha_1/\alpha_2 \in [\eta^*_1, 1/\eta^C_2)$. There exists $\bar{\alpha} \in (1/\eta^S_2, 1/\eta^C_2)$ such that $W(\mathbf{x}^C) \gtreqqless W(\mathbf{x}^M)$ as $\alpha_1/\alpha_2 \lesseqqgtr \bar{\alpha}$.

Corollary 3 is explained next. When the demand asymmetry exists, as demonstrated in

[23] Lemma 7 can be used to justify Assumption A mentioned in the introduction.

[24] When $\mathbf{x}^M = (\tilde{x}^M_1, 0)'$, $\mathbf{x}^M$ does not strictly lie to the left of $\mathbf{\Pi}^1_1(0)$, but it does indeed strictly lie to the left of $\mathbf{P}^1(c_1)$ because $\tilde{x}^M_1 < \tilde{x}^S_1$.

Proposition 9, the Cournot competition distorts the output-structure with probability one. Moreover, Proposition 5 suggests that the output-structure distortion gets stronger when the degree of demand asymmetry increases. This is because, in Cases II and III, the demand asymmetry is strong, and the output structure in the Cournot equilibrium is highly distorted in the sense that the output of good 1 is very low: $x_1^C < x_1^M$, but the output of good 2 is very high: $x_2^C > x_2^S$. In contrast, in Case I, the demand asymmetry is weak, and each good has an unambiguously normal output ranking. This explains why welfare reversal exists if and only if the demand asymmetry is strong enough.

V. *Individual Price Rankings*

In this section, we conduct an analysis in the price space to establish that $\mathbf{p}^M \gg \mathbf{p}^C$. Write $\mathbf{T}$ for the range of P, i.e., $\mathbf{T} \equiv \{\mathbf{p} \in \mathbf{R}^2 : \exists\, \mathbf{x} \in \mathbf{R}^2_+, \ni \mathbf{p} = P(\mathbf{x})\}$. Let $P(\mathbf{\Pi}^i_i(0))$ $(i=1, 2)$ stand for the image of $\mathbf{\Pi}^i_i(0)$ under the mapping $P: \mathbf{R}^2_+ \mapsto \mathbf{T}$, i.e., $P(\mathbf{\Pi}^i_i(0)) \equiv \{\mathbf{p} \in \mathbf{T} : \exists\, \mathbf{x} \in \mathbf{R}^2_+, \ni \mathbf{p} = P(\mathbf{x}) \text{ and } \Pi^i_i(\mathbf{x}) = 0\}$. $P(\mathbf{\Pi}_i(0))$ is defined similarly. Throughout this section, the following two additional assumptions are made:

Relative slopes (RS): $\Pi^i_{ii}(\mathbf{x})/\Pi^i_{ij}(\mathbf{x}) > P^i_i(\mathbf{x})/P^i_j(\mathbf{x})$, $i=1, 2$, $\forall\, \mathbf{x} \in \mathbf{R}^2_+$.
One to one (OTO): *(i)* $P: \mathbf{R}^2_+ \mapsto \mathbf{T}$ *is twice continuously differentiable and one to one on* $\mathbf{R}^2_+$. *(ii)* $X: \mathbf{T} \mapsto \mathbf{R}^2_+$ *is twice continuously differentiable on* $\mathbf{T}$ *where* X *is the inverse function of* P.
RS means that, in the (x_1, x_2) space, $\mathbf{\Pi}^1_1(0)$ is steeper than $\mathbf{P}^1(c_1)$, whereas $\mathbf{\Pi}^2_2(0)$ is flatter than $\mathbf{P}^2(c_2)$. Fact 4-(ii), together with FN-(ii) and Fact 1-(ii), suggests that RS is a plausible assumption.[25] If demands are linear, then, according to (9), $\Pi^i_{ii}/\Pi^i_{ij} = 2P^i_i/P^i_j$, and hence RS indeed holds. Chang and Peng (2012, Section 2) have already established that if demands are linear, then OTO is satisfied. Therefore, both RS and OTO are reasonable assumptions.

Write $\partial\mathbf{T}^i$ $(i=1, 2)$ for $P(\{\mathbf{x} \in \mathbf{R}^2_+ : x_i = 0\})$, the image of the x_j-axis under the mapping $P: \mathbf{R}^2_+ \mapsto \mathbf{T}$. $\partial\mathbf{T}^1$ and $\partial\mathbf{T}^2$ are the two boundaries of $\mathbf{T}$. As shown in Fig. 4, in the (p_1, p_2) space, both $\partial\mathbf{T}^1$ and $\partial\mathbf{T}^2$ are upward sloping, and $\partial\mathbf{T}^1$ is steeper than $\partial\mathbf{T}^2$.[26] [The shaded region represents $\mathbf{T}$.]

We next analyze what will happen when point $\mathbf{x}$ moves along $\mathbf{\Pi}^i_i(0)$. For each i, the following fact holds:

Fact 5. Assume that $\mathbf{x} \in \mathbf{\Pi}^i_i(0)$, and $\mathrm{d}\mathbf{x}$ is an infinitesimal variation such that $\mathrm{d}x_j > 0$ (i.e., firm j raises its output) and $\mathbf{x} + \mathrm{d}\mathbf{x} \in \mathbf{\Pi}^i_i(0)$. Then $\mathrm{d}x_i < 0$, $\mathrm{d}p_i < 0$, and $\mathrm{d}p_j < 0$. (Proof in Appendix.)
Fact 5 straightforwardly leads to the following lemma:

Lemma 8. In the (p_1, p_2) space, $P(\mathbf{\Pi}^i_i(0))$ slopes upward, $i=1, 2$.

As shown in Fig. 4, either one of $P(\mathbf{\Pi}^1_1(0))$ and $P(\mathbf{\Pi}^2_2(0))$ meets the two boundaries. For

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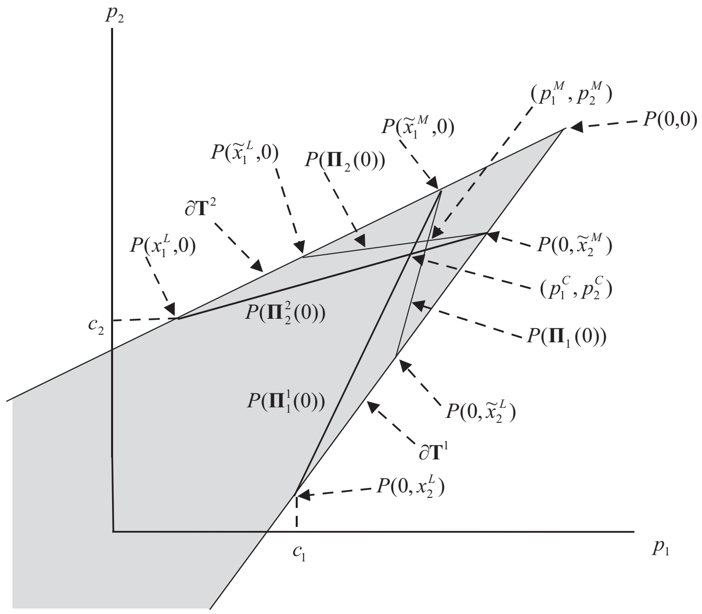
[25] Take $i=1$. According to FN-(ii) and Fact 1-(ii), $\mathbf{\Pi}^1_1(0)$ and $\mathbf{P}^1(c_1)$ meet the x_2-axis at the same point, namely, $(0, \tilde{x}^L_2)$. According to Fact 4-(ii), $\mathbf{\Pi}^1_1(0)$ lies to the left of $\mathbf{P}^1(c_1)$, suggesting that $\mathbf{\Pi}^1_1(0)$ is steeper than $\mathbf{P}^1(c_1)$.

[26] Let $X^i: \mathbf{T} \mapsto \mathbf{R}_+$ stand for the demand function for good i, $i=1, 2$. Hence, $\partial\mathbf{T}^i = \{\mathbf{p} \in \mathbf{T} : X^i(\mathbf{p}) = 0\}$, $i=1, 2$, implying that, in the (p_1, p_2) space, the slope of $\partial\mathbf{T}^i$ is $-X^i_1/X^i_2$. According to (4) of Chang and Peng (2012), $\partial X/\partial\mathbf{p}' = [\partial P/\partial\mathbf{x}']^{-1}$. Therefore, SD and (3)-(5) imply that $X^1_1 < 0$, $X^2_2 < 0$, $X^1_2 = X^2_1 > 0$ and $X^1_1 X^2_2 - X^1_2 X^2_1 > 0$. It follows that $-X^1_1/X^1_2 > -X^2_1/X^2_2 > 0$.

FIG. 4. COURNOT REACTION CURVES IN THE PRICE SPACE



example, $P(\mathbf{\Pi}_1^1(0))$ meets $\partial\mathbf{T}^1$ and $\partial\mathbf{T}^2$ at $P(0, x_2^L)$ and $P(\tilde{x}_1^M, 0)$, respectively. [In Fig. 4, it is emphasized that $P^i(0, x_j^L) = c_i$, $i = 1, 2$.]

The following lemma demonstrates that, as shown in Fig. 4, there is a normal ranking between $P(\mathbf{\Pi}_i^i(0))$ and $P(\mathbf{\Pi}_i(0))$:

Lemma 9. In the (p_1, p_2) space, $P(\mathbf{\Pi}_1(0))$ lies to the right of $P(\mathbf{\Pi}_1^1(0))$ and $P(\mathbf{\Pi}_2(0))$ lies above $P(\mathbf{\Pi}_2^2(0))$. (Proof in Appendix.)

As mentioned, without loss of generality it is assumed that $\mathbf{x}^C$ is an interior solution, and hence we have a clear-cut ranking:

Proposition 12. Assume that the Cournot equilibrium is an interior solution. Each price is higher under the monopoly solution than under the Cournot equilibrium, i.e., $\mathbf{p}^M \gg \mathbf{p}^C$.

Proof: Because $\mathbf{x}^C$ is an interior solution, $\mathbf{p}^C$ is the intersection point of $P(\mathbf{\Pi}_1^1(0))$ and $P(\mathbf{\Pi}_2^2(0))$. When $\mathbf{x}^M \gg \mathbf{0}$, as shown in Fig. 4, $\mathbf{p}^M$ is the intersection point of $P(\mathbf{\Pi}_1(0))$ and $P(\mathbf{\Pi}_2(0))$, and hence, according to Lemma 9, $\mathbf{p}^M$ strictly lies to the right of $P(\mathbf{\Pi}_1^1(0))$ and above $P(\mathbf{\Pi}_2^2(0))$. Lemma 8 thus guarantees that $\mathbf{p}^M \gg \mathbf{p}^C$. If $\mathbf{x}^M$ is a corner solution, then $\mathbf{x}^M$ is equal to either $(\tilde{x}_1^M, 0)'$ or $(0, \tilde{x}_2^M)'$. Note that both $P(\tilde{x}_1^M, 0)$ and $\mathbf{p}^C$ belong to $P(\mathbf{\Pi}_1^1(0))$. Moreover, since $P(\mathbf{\Pi}_1^1(0))$ slopes upward, as shown in Fig. 4, $P(\tilde{x}_1^M, 0)$ strictly lies to the northeast of $\mathbf{p}^C$. Similarly, $P(0, \tilde{x}_2^M)$ strictly lies to the northeast of $\mathbf{p}^C$. Therefore, it is guaranteed that $\mathbf{p}^C \ll \mathbf{p}^M$.

Q.E.D.

VI. *Conclusions*

It is demonstrated that if a good is active in either the socially optimal solution or the monopoly solution, then it is active in the Cournot equilibrium; but, the converse is not true. Therefore, it is easier for a less efficient good to be active under the Cournot competition. When demands are linear, a good is active in the monopoly solution if and only if it is active in the socially optimal solution.

An output vector is called a Ramsey output vector if it maximizes the social welfare subject to the constraint that industry profit cannot fall below a given level. Moreover, if an output vector is a Ramsey output vector it is referred to as having a socially optimal output structure. It is demonstrated that the output structure of the monopoly solution is socially optimal. In contrast, for the case with linear demands, the Cournot competition distorts the output structure with probability one.

All of the above results suggest that, from the viewpoints of product selection and output structure, the monopoly solution is socially better than the Cournot equilibrium.

As mentioned in the introduction, in previous studies, it has been demonstrated that the rankings regarding individual outputs can go either way. We establish that the industry-output ranking is normal when the Cournot reaction curve of firm 1 (2) is steeper (flatter) than any iso-industry-output curve. When demands are linear, this condition means that the two goods do not differ too much in the slope of inverse demand or that the degree of substitutability is low enough. If this condition is violated, then either case is possible.

The results suggest that it is important to examine whether it is possible to have a welfare reversal or not. The results regarding product selection indicate that we can assume that the Cournot equilibrium is an interior solution. This is because if the Cournot equilibrium is a corner solution, then the monopoly solution is also a corner solution. Moreover, we have demonstrated that, in this case, the two solutions are the same, and hence it is trivial to compare them. Therefore, there are two cases which need to be analyzed. First, when both of the two solutions are interior solutions. Second, when the Cournot equilibrium is an interior solution, but the monopoly solution is a corner solution.

Assume that the demands are linear. For the first case, if the asymmetry between goods is weak enough, then each ranking is normal: individual outputs and welfare are both higher in the Cournot equilibrium than in the monopoly solution. If the asymmetry is strong enough, then the rankings regarding individual outputs may be abnormal, but, the welfare ranking still is normal. In contrast, in the second case, abnormal rankings regarding individual outputs always exist, and these abnormal rankings are able to reverse the welfare ranking: the Cournot equilibrium yields lower welfare than the monopoly solution if and only if the asymmetry is strong enough.

We establish that, in general, each Cournot price falls below its corresponding monopoly price regardless of whether the monopoly solution is an interior solution or not. This result does not rely upon Assumptions A and B mentioned in the introduction. Unfortunately, as mentioned, lower prices are not necessarily better in welfare terms.

Appendix

Proof of Proposition 7

(i): In Case II, each solution is an interior solution, implying that (21)-(23) are valid. Therefore,

$$x_1^S+x_2^S=\frac{\alpha_1(\beta_2-\gamma)+\alpha_2(\beta_1-\gamma)}{\beta_1\beta_2-\gamma^2}, \tag{A1}$$

$$x_1^M+x_2^M=\frac{1}{2}\times\frac{\alpha_1(\beta_2-\gamma)+\alpha_2(\beta_1-\gamma)}{\beta_1\beta_2-\gamma^2}, \tag{A2}$$

$$x_1^C+x_2^C=\frac{\alpha_1(2\beta_2-\gamma)+\alpha_2(2\beta_1-\gamma)}{4\beta_1\beta_2-\gamma^2}. \tag{A3}$$

Subtracting (A3) from (A1) yields

$$(x_1^S+x_2^S)-(x_1^C+x_2^C)=\frac{[\beta_1(2\beta_2-\gamma)(\alpha_1\beta_2-\alpha_2\gamma)+\beta_2(2\beta_1-\gamma)(\alpha_2\beta_1-\alpha_1\gamma)]}{(\beta_1\beta_2-\gamma^2)(4\beta_1\beta_2-\gamma^2)}$$

which shares the same sign with $[\cdot]$. Subtracting (A2) from (A3) yields

$$(x_1^C+x_2^C)-(x_1^M+x_2^M)=\frac{\gamma\{(2\beta_1-\gamma)(\alpha_1\beta_2-\alpha_2\gamma)+(2\beta_2-\gamma)(\alpha_2\beta_1-\alpha_1\gamma)\}}{2(\beta_1\beta_2-\gamma^2)(4\beta_1\beta_2-\gamma^2)}$$

which shares the same sign with $\{\cdot\}$. In Case II, $\alpha_1\beta_2-\alpha_2\gamma>0$ and $\alpha_2\beta_1-\alpha_1\gamma>0$. If DD holds, then $2\beta_1-\gamma>0$ and $2\beta_2-\gamma>0$, implying both $[\cdot]$ and $\{\cdot\}$ are positive. [This confirms Proposition 6.] It remains to study the case without DD.

We can show that

$$[\cdot]=\alpha_1\beta_2(2\beta_1\beta_2+\gamma^2-3\beta_1\gamma)+\alpha_2\beta_1(2\beta_1\beta_2+\gamma^2-3\beta_2\gamma),$$
$$\{\cdot\}=\alpha_1(2\beta_1\beta_2+\gamma^2-3\beta_2\gamma)+\alpha_2(2\beta_1\beta_2+\gamma^2-3\beta_1\gamma).$$

Therefore, $[\cdot]=\alpha_2\times g(\alpha_1/\alpha_2)$ and $\{\cdot\}=\alpha_2\times h(\alpha_1/\alpha_2)$ where functions g and h are defined as follows:

$$g(t)\equiv\beta_2(2\beta_1\beta_2+\gamma^2-3\beta_1\gamma)t+\beta_1(2\beta_1\beta_2+\gamma^2-3\beta_2\gamma), \tag{A4}$$

$$h(t)\equiv(2\beta_1\beta_2+\gamma^2-3\beta_2\gamma)t+(2\beta_1\beta_2+\gamma^2-3\beta_1\gamma). \tag{A5}$$

Because $\alpha_2>0$, $\text{sign}([\cdot])=\text{sign}(g)$ and $\text{sign}(\{\cdot\})=\text{sign}(h)$. According to (A4) and (A5), both g and h are linear functions: $g'(t)=\beta_2(2\beta_1\beta_2+\gamma^2-3\beta_1\gamma)$, $g''(t)=0$, $h'(t)=2\beta_1\beta_2+\gamma^2-3\beta_2\gamma$ and $h''(t)=0$. Moreover, when g and h are evaluated at two endpoints, they take the following values:

$$g(1/\eta_2^*)=(4\beta_1\beta_2-\gamma^2)(\beta_1\beta_2-\gamma^2)/3\gamma,\ g(1/\eta_2^S)=\beta_1(2\beta_2-\gamma)(\beta_1\beta_2-\gamma^2)/\gamma,$$
$$h(1/\eta_2^*)=(4\beta_1\beta_2-\gamma^2)(\beta_1\beta_2-\gamma^2)/3\beta_2\gamma,\ h(1/\eta_2^S)=(2\beta_1-\gamma)(\beta_1\beta_2-\gamma^2)/\gamma.$$

Note that both $g(1/\eta_2^*)$ and $h(1/\eta_2^*)$ have a clear-cut sign: positive.

We can show that if A1 (A2) holds, then $2\beta_1-\gamma<(>)0$, $2\beta_2-\gamma>(<)0$, $2\beta_1\beta_2+\gamma^2-3\beta_1\gamma>(<)0$ and $2\beta_1\beta_2+\gamma^2-3\beta_2\gamma<(>)0$.

We are in a position to draw conclusions. First, consider the case where A1 holds. In this case, $g'(t)>0$, $h'(t)<0$, $\forall t$, and $h(1/\eta_2^S)<0$. From $g(1/\eta_2^*)>0$ and $g'>0$ it follows that $g(t)>0$, $\forall t\in(1/\eta_2^*, 1/\eta_2^S)$. From $h(1/\eta_2^*)>0$, $h'<0$ and $h(1/\eta_2^S)<0$ it follows that $\exists\tilde{\alpha}\in(1/\eta_2^*, 1/\eta_2^S)$,

$\ni \text{sign}(h(t)) = \text{sign}(\tilde{\alpha} - t)$, $\forall t \in (1/\eta_2^*, 1/\eta_2^S)$.

Second, consider the case where A2 holds. In this case, $g'(t) < 0$, $h'(t) > 0$, $\forall t$, and $g(1/\eta_2^S) < 0$. From $g(1/\eta_2^*) > 0$, $g'(t) < 0$ and $g(1/\eta_2^S) < 0$ it follows that $\exists \hat{\alpha} \in (1/\eta_2^*, 1/\eta_2^S)$, $\ni \text{sign}(g(t)) = \text{sign}(\hat{\alpha} - t)$, $\forall t \in (1/\eta_2^*, 1/\eta_2^S)$. From $h(1/\eta_2^*) > 0$ and $h' > 0$ it follows that $h(t) > 0$, $\forall t \in (1/\eta_2^*, 1/\eta_2^S)$.

(ii): In Case III,

$$x_1^S = \tilde{x}_1^S = \frac{\alpha_1}{\beta_1},\ x_1^M = \tilde{x}_1^M = \frac{\alpha_1}{2\beta_1},\ x_2^S = x_2^M = 0. \tag{A6}$$

It follows that

$$x_1^S + x_2^S = \tilde{x}_1^S = \frac{\alpha_1}{\beta_1},\ x_1^M + x_2^M = \tilde{x}_1^M = \frac{\alpha_1}{2\beta_1},$$

which, together with (A3), implies that

$$(x_1^S + x_2^S) - (x_1^C + x_2^C) = \frac{[\alpha_1(2\beta_1\beta_2 - \gamma^2 + \beta_1\gamma) - \alpha_2\beta_1(2\beta_1 - \gamma)]}{\beta_1(4\beta_1\beta_2 - \gamma^2)}, \tag{A7}$$

$$(x_1^C + x_2^C) - (x_1^M + x_2^M) = \frac{\{(2\beta_1 - \gamma)(2\alpha_2\beta_1 - \alpha_1\gamma)\}}{2\beta_1(4\beta_1\beta_2 - \gamma^2)}. \tag{A8}$$

According to (A7), $(x_1^S + x_2^S) - (x_1^C + x_2^C)$ shares the same sign with $[\cdot]$. From $2\beta_1\beta_2 - \gamma^2 + \beta_1\gamma > 0$ it follows that $\alpha_1/\alpha_2 > \hat{I} \Leftrightarrow [\cdot] > 0$ where $\hat{I} = \beta_1(2\beta_1 - \gamma)/(2\beta_1\beta_2 - \gamma^2 + \beta_1\gamma)$. We can show that $\hat{I} < \beta_1/\gamma$. In Case III, $\alpha_1/\alpha_2 \geq \beta_1/\gamma$, implying that $\alpha_1/\alpha_2 > \hat{I}$.

According to (A8), $(x_1^C + x_2^C) - (x_1^M + x_2^M)$ shares the same sign with $\{\cdot\}$. In Case III, $\alpha_1/\alpha_2 < 2\beta_1/\gamma$, implying that $2\alpha_2\beta_1 - \alpha_1\gamma > 0$. Therefore, $\{\cdot\}$ has the same sign with $2\beta_1 - \gamma$. Q.E.D.

Proof of Proposition 11

(i): In Case II, both $\mathbf{x}^C$ and $\mathbf{x}^M$ are interior solutions, implying that (21) and (23) are valid. Substituting (21) into (2) leads to

$$W(\mathbf{x}^C) = \frac{(\alpha_2^2\beta_1 + \alpha_1^2\beta_2 - 2\alpha_1\alpha_2\gamma)(12\beta_1\beta_2 - \gamma^2) + 8\alpha_1\alpha_2\beta_1\beta_2\gamma}{2(4\beta_1\beta_2 - \gamma^2)^2}. \tag{A9}$$

Similarly, substituting (23) into (2) yields

$$W(\mathbf{x}^M) = \frac{3(\alpha_2^2\beta_1 + \alpha_1^2\beta_2 - 2\alpha_1\alpha_2\gamma)}{8(\beta_1\beta_2 - \gamma^2)}.$$

Let us define

$$\delta \equiv \beta_1\beta_2 - \gamma^2,\ \Delta_i \equiv \alpha_i\beta_j - \alpha_j\gamma,\ i = 1, 2.$$

These notations have the following meanings: First, (15) implies that $\Delta_i > 0 \Leftrightarrow \tilde{x}_j^M < \tilde{x}_j^L$, $i = 1, 2$. Therefore, according to Lemma 1-(iii), $\Delta_i > 0 \Leftrightarrow x_i^M > 0$, $i = 1, 2$. Second, if $\mathbf{x}^S$ is an interior solution, then, as shown in (14), $x_i^S = \Delta_i/\delta$, $i = 1, 2$.

Note that $W(\mathbf{x}^M)$ and $W(\mathbf{x}^C)$ share the following term: $\alpha_2^2\beta_1 + \alpha_1^2\beta_2 - 2\alpha_1\alpha_2\gamma$, which is equal to $\alpha_1(\alpha_1\beta_2 - \alpha_2\gamma) + \alpha_2(\alpha_2\beta_1 - \alpha_1\gamma) = \alpha_1\Delta_1 + \alpha_2\Delta_2$. We can show that

$$W(\mathbf{x}^C) - W(\mathbf{x}^M) = \frac{(\alpha_1\Delta_1 + \alpha_2\Delta_2)\gamma^4 + 28\beta_1\beta_2\gamma\Delta_1\Delta_2 + 4\alpha_1\alpha_2\beta_1\beta_2\gamma\delta}{8(4\beta_1\beta_2 - \gamma^2)^2\delta}. \tag{A10}$$

For Example 1, it is always true that $\alpha_i > 0$, $\beta_i > 0$, $i=1, 2$, $\gamma > 0$ and $\delta > 0$. In case II, $\Delta_i > 0$, $i=1, 2$. Therefore, (A10) implies that $W(\mathbf{x}^C) - W(\mathbf{x}^M) > 0$.

(ii): From (A6) it follows that

$$W(\mathbf{x}^M) = \frac{3\alpha_1^2}{8\beta_1}. \tag{A11}$$

Subtracting (A9) with (A11) yields

$$W(\mathbf{x}^C) - W(\mathbf{x}^M) = \frac{\alpha_2^2}{8\beta_1(4\beta_1\beta_2 - \gamma^2)} \times f(\alpha_1/\alpha_2)$$

where function f is defined as the following:

$$f(t) = \gamma^2(20\beta_1\beta_2 - 3\gamma^2)t^2 - (8\beta_1\gamma)(8\beta_1\beta_2 - \gamma^2)t + 4\beta_1^2(12\beta_1\beta_2 - \gamma^2).$$

Therefore, sign $(W(\mathbf{x}^C) - W(\mathbf{x}^M))$ = sign (f). It remains to analyze sign (f). It is easy to show that $f'' = 2\gamma^2(20\beta_1\beta_2 - 3\gamma^2)$, which is positive because $\beta_1\beta_2 - \gamma^2 > 0$. Write t^c for the t which solves $f'(t) = 0$. We can show that $t^c = 4\beta_1(8\beta_1\beta_2 - \gamma^2)/\gamma(20\beta_1\beta_2 - 3\gamma^2)$, $\beta_1/\gamma < t^c < 2\beta_1/\gamma$, $f(t^c) = -4\beta_1[8\beta_1\beta_2(2\beta_1\beta_2 - \gamma^2) + \gamma^4]/(20\beta_1\beta_2 - 3\gamma^2) < 0$, $f(\beta_1/\gamma) = \beta_1^2(4\beta_1\beta_2 + \gamma^2) > 0$ and $f(2\beta_1/\gamma) = \beta_1^2 \times 0 = 0$. Therefore, there exists $\bar{\alpha} \in (\beta_1/\gamma, t^c)$, such that sign$(f(t))$=sign$(\bar{\alpha} - t)$, $\forall t \in (\beta_1/\gamma, 2\beta_1/\gamma)$. Q.E.D.

Proof of Fact 5 Take $i=1$. According to RS, $\Pi_{11}^1/\Pi_{12}^1 > P_1^1/P_2^1$. Moreover, Eq. (5) means that $P_1^1/P_2^1 > P_1^2/P_2^2$. It follows that $\Pi_{11}^1/\Pi_{12}^1 > P_1^2/P_2^2$. Since $-\Pi_{11}^1/\Pi_{12}^1$ is the slope of $\mathbf{\Pi}_1^1(0)$,

$$\frac{dx_2}{dx_1} = -\frac{\Pi_{11}^1}{\Pi_{12}^1} < 0,$$

$$\frac{dp_1}{dx_1} = P_1^1 + P_2^1\frac{dx_2}{dx_1} = P_1^1 - P_2^1\frac{\Pi_{11}^1}{\Pi_{12}^1} = \left(\frac{P_1^1}{P_2^1} - \frac{\Pi_{11}^1}{\Pi_{12}^1}\right)P_2^1 > 0,$$

$$\frac{dp_2}{dx_1} = P_1^2 + P_2^2\frac{dx_2}{dx_1} = P_1^2 - P_2^2\frac{\Pi_{11}^1}{\Pi_{12}^1} = \left(\frac{P_1^2}{P_2^2} - \frac{\Pi_{11}^1}{\Pi_{12}^1}\right)P_2^2 > 0.$$

Q.E.D.

Proof of Lemma 9 OTO implies that $\mathbf{p} \in P(\mathbf{\Pi}_1^1(0))$ if and only if $\mathbf{p} \in \mathbf{T}$ and $\Pi_1^1(X(\mathbf{p})) = 0$.[27] Write $F^1(\mathbf{p})$ for $\Pi_1^1(X(\mathbf{p}))$. It is clear that $F_1^1 = \Pi_{11}^1 X_1^1 + \Pi_{12}^1 X_1^2$. Hence, $F_1^1 = (\Pi_{11}^1 P_2^2 - \Pi_{12}^1 P_1^2)/\det[P_j^i]$. According to the proof of Fact 5, $\Pi_{11}^1/\Pi_{12}^1 > P_1^2/P_2^2$. Moreover , (5) means that $\det[P_j^i] > 0$. It follows that $F_1^1 > 0$. Therefore, if $\Pi_1^1(X(\mathbf{p}))$ is positive (negative), then $\mathbf{p}$ lies to the right (left) of $P(\mathbf{\Pi}_1^1(0))$.

OTO similarly implies that $\mathbf{p} \in P(\mathbf{\Pi}_1(0))$ if and only if $\mathbf{p} \in \mathbf{T}$ and $\Pi_1(X(\mathbf{p})) = 0$. Assume that $\tilde{\mathbf{p}} \in P(\mathbf{\Pi}_1(0))$, implying that $\Pi_1(X(\tilde{\mathbf{p}})) = 0$. Therefore, Eq. (11) implies that $\Pi_1^1(X(\tilde{\mathbf{p}})) \geq 0$, with strict inequality if $X^2(\tilde{\mathbf{p}}) \neq 0$. If $X^2(\tilde{\mathbf{p}}) = 0$, then $X(\tilde{\mathbf{p}}) = (\tilde{x}_1^M, 0)'$, implying that $\tilde{\mathbf{p}} \in P(\mathbf{\Pi}_1^1(0))$. Otherwise, $\Pi_1^1(X(\tilde{\mathbf{p}})) > 0$, implying that $\tilde{\mathbf{p}}$ strictly lies to the right of $P(\mathbf{\Pi}_1^1(0))$. Q.E.D.

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[27] If $\hat{\mathbf{p}} \in P(\mathbf{\Pi}_1^1(0))$, then $\hat{\mathbf{p}} \in \mathbf{T}$ and there exists $\hat{\mathbf{x}} \in \mathbf{R}_+^2$ such that $\Pi_1^1(\hat{\mathbf{x}}) = 0$ and $P(\hat{\mathbf{x}}) = \hat{\mathbf{p}}$. OTO implies that $\hat{\mathbf{x}} = X(\hat{\mathbf{p}})$, and hence $\Pi_1^1(X(\hat{\mathbf{p}})) = 0$. If $\tilde{\mathbf{p}} \in \mathbf{T}$ and $\Pi_1^1(X(\tilde{\mathbf{p}})) = 0$, then $X(\tilde{\mathbf{p}}) \in \mathbf{R}_+^2$. Moreover, OTO guarantees that $P(X(\tilde{\mathbf{p}})) = \tilde{\mathbf{p}}$, implying that $\tilde{\mathbf{p}} \in P(\mathbf{\Pi}_1^1(0))$.

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