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# IMPOSSIBILITIES OF PARETIAN SOCIAL WELFARE FUNCTIONS FOR INFINITE UTILITY STREAMS WITH DISTRIBUTIVE EQUITY*

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## *Abstract*

This paper examines the existence of social welfare functions satisfying some reasonable axioms of distributive equity and Pareto efficiency in aggregating infinite utility streams. Our main results show that there exist social welfare functions that satisfy both the Pigou-Dalton transfer principle and weak dominance—a weak version of the Pareto principle—, but there exists no social welfare function that satisfies any of our distributive equity axioms and weak Pareto principle simultaneously. Thus, we prove that no Paretian ranking can satisfy numerical representability and any of distributive equity axioms in the setting of intertemporal social choice.

*Keywords:* intergenerational equity; distributional equity
*JEL Classification Codes*: D63, D71

## I. *Introduction*

In analysis of the intertemporal social choice problem, two classes of equity axioms have been considered in the previous literature. The first is a class of anonymity axioms that demands that each generation be treated equally in the evaluation of infinite utility streams (Diamond 1965; Svensson 1980; Campbell 1985; Epstein 1986; Lauwers 1997; Shinotsuka 1998; Basu and Mitra 2003; 2007b; Mitra and Basu 2007). The second is a class of distributive equity axioms that prefers a more equitable distribution of utilities (Fleurbaey and Michel 2001; 2003; Sakai 2003; 2006; Asheim and Tungodden 2004; Bossert, et al. 2007; Hara, et al. 2007).

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* This paper is part of my Ph.D. thesis at Hitotsubashi University. This paper was accepted by an editor of this journal when I was a Young Research Fellow of Hitotsubashi university. The main results of this paper (Propositions 3-4 and Corollaries 1-2) overlap with Alcantud (2010), Alcantud (2012), and Alcantud and Garcia-Sanz (2010). These works independently proved their propositions around the same time. I would like to thank Kotaro Suzumura, Motohiro Sato, Peter Lambert, Kiwako Sakamoto, Reiko Gotoh and Toyotaka Sakai for their helpful comments and suggestions. Financial support from the Twenty-First Century Centers of Excellence (COE) project on "Normative Evaluation and Social Choice of Contemporary Economic Systems" is gratefully acknowledged. Any remaining errors are my responsibility.

Basu and Mitra's pioneering study reveals some impossibility/possibility results about the existence of social welfare functions satisfying the finite anonymity axiom and several variations of the Pareto principle (Basu and Mitra 2007b). While the finite anonymity axiom treats each generation equally in their utility levels, it ignores the distributional aspect of utility levels.

Therefore, this paper focuses on distributive equity axioms and analyze the compatibility of these axioms with weaker versions of the Pareto principle. One of the most fundamental distributive equity axioms is the Pigou-Dalton transfer principle adapted to the infinite-horizon framework. In the setting of intertemporal social choice, the Pigou-Dalton transfer principle requires that a pure transfer of income from the better-off generation to the worse-off generation must increase social welfare given other things. We then show that there exist social welfare functions that satisfy the Pigou-Dalton transfer principle and a weak version of the Pareto principle (called the weak dominance axiom in the previous literature), but there exists no social welfare function that satisfies any of the distributive equity axioms and the weak Pareto principle at the same time. Thus, we prove that no Paretian social welfare function can satisfy any of the distributive equity axioms. These impossibility results form a contrast to a social ranking approach in which many possibility results can be gained by sacrificing the continuity requirement (Basu and Mitra 2007a; Bossert, Sprumont, and Suzumura 2007).

## II. *Basic Notations and Definitions*

Let $\mathbb{R}$ denote the set of real numbers. Let $\mathbb{Z}_+$ and $\mathbb{Z}_{++}$ denote, respectively, the set of non-negative integers and the set of positive integers. We write the set of all possible utility levels for each generation as $Y$, so the set of infinite utility streams is $X = Y^{\mathbb{Z}_{++}}$. For simplicity of notation, we denote the K times repetition of the finite vector $(x_1, ..., x_T)$ by $(x_1, ..., x_T)_{K\text{-rep}}$, where $T, K \in \mathbb{Z}_{++}$. If $K = \infty$, then we write $(x_1, ..., x_T)_{K\text{-rep}}$ as $(x_1, ..., x_T)_{\text{rep}}$. Throughout this paper, we interpret the utilities as either income levels or level-plus comparable ones following Blackorby et al. (1984).

There are at least two ways to evaluate infinite utility streams defined as above. One way is to use a *primitive binary relation* on $X$, and the other is to consider a *social welfare function, i.e.*, a real-valued function defined on $X$. In this paper, we use the latter approach and examine the existence of social welfare functions satisfying some appealing axioms.

As Basu and Mitra (2003; 2007b) show, the finite anonymity axiom contradicts several versions of the Pareto efficiency axiom in the social welfare function approach. The anonymity axiom, however, treats each generation equally in their utility levels and ignores the distributional aspect of utility levels.

Hence, this paper focuses on some axioms of distributional equity proposed by the previous studies (Fleurbaey and Michel 2001; 2003; Sakai 2003; 2006; Asheim and Tungodden 2004; Hara, et al. 2007) as a solution to the distributional problem in aggregating infinite utility streams.

First, we introduce the *strict equity preference* axiom, which is used by the axiomatic characterization of leximin rankings in Bossert et al. (2007).

Strict Equity Preference (SEP): $\forall \mathrm{x}, \mathrm{y} \in X$, $\forall i, j \in \mathbb{Z}_{++}$;

$$[y_i > x_i \geq x_j > y_j \ \& \ \forall k \neq i, j,\ x_k = y_k] \Rightarrow W(\mathrm{x}) > W(\mathrm{y}).$$

In this axiom, social welfare is improved whenever an income transfer from the better-off generation to the worse-off one is implemented; this may increase or decrease the total surplus between the loss of the rich and the gain of the poor.

Formally, this axiom can be divided into the following three variations according to the degree of inequality aversion over an income transfer.

The first axiom is the well-known condition called the *Pigou-Dalton transfer principle*, which is a standard axiom in the study of income inequality measures.

Pigou-Dalton Transfer Principle (PDT): $\forall \mathrm{x}, \mathrm{y} \in X$, $\forall \epsilon > 0$, $\forall i, j \in \mathbb{Z}_{++}$;

$$[x_i = y_i - \epsilon \geq y_j + \epsilon = x_j \ \& \ \forall k \neq i, j,\ x_k = y_k] \Rightarrow W(\mathrm{x}) > W(\mathrm{y}).$$

Intuitively, this axiom states that transferring income from the rich to the poor is always good as long as their relative positions are not reversed.

The definitions of the following two variations of the SEP depend on the total surplus between a loss of the rich and a gain of the poor.

Altruistic Equity-1 (AE-1): $\forall \mathrm{x}, \mathrm{y} \in X$, $\forall \epsilon > \delta > 0$, $\forall i, j \in \mathbb{Z}_{++}$;

$$[x_i = y_i - \delta \geq y_j + \epsilon = x_j \ \& \ \forall k \neq i, j,\ x_k = y_k] \Rightarrow W(\mathrm{x}) > W(\mathrm{y}).$$

Altruistic Equity-2 (AE-2): $\forall \mathrm{x}, \mathrm{y} \in X$, $\forall \epsilon > \delta > 0$, $\forall i, j \in \mathbb{Z}_{++}$;

$$[x_i = y_i - \epsilon \geq y_j + \delta = x_j \ \& \ \forall k \neq i, j,\ x_k = y_k] \Rightarrow W(\mathrm{x}) > W(\mathrm{y}).$$

*Altruistic equity-1*, the axiom proposed by Hara et al. (2007), states that a transfer from the rich to the poor should be done if the gain of the poor is greater than the loss of the rich.

In contrast, *altruistic equity-2* requires that society should accept a transfer from the rich to the poor even if the gain of the poor is smaller than the loss of the rich.

The last of the equity axioms is proposed and investigated by Asheim, Mitra, and Tungodden (2007) and Banerjee (2006).

Hammond Equity for Future Generations (HEF): $\forall \mathrm{x}, \mathrm{y} \in X$;

$$[y_1 > x_1 > \bar{x} > \bar{y} \ \& \ \forall i \geq 2,\ x_i = \bar{x} \ \& \ y_1 = \bar{y}] \Rightarrow W(\mathrm{x}) > W(\mathrm{y}).$$

This axiom states that social welfare must increase if inequity between generation 1 and all future generations is wholly improved. Note that this axiom is logically independent of the strict equity preference.

Now, we introduce efficiency axioms in the framework of intertemporal social choice. The first axiom is the standard Pareto criterion.

Pareto Principle (P): $\forall \mathrm{x}, \mathrm{y} \in X$;

$$[\forall i \in \mathbb{Z}_{++},\ x_i \geq y_i \ \ \& \ \ \exists j \in \mathbb{Z}_{++}\ x_j > y_j] \Rightarrow W(\mathrm{x}) > W(\mathrm{y}).$$

In the previous studies, this efficiency axiom is often too strong to gain possibility results. Therefore, this paper considers two weak forms of the Pareto principle, as in Basu and Mitra (2007b). The first version can be written as follows:

Weak Pareto Principle (WP): $\forall \mathrm{x}, \mathrm{y} \in X$; $\forall i \in \mathbb{Z}_{++}, x_i > y_i \Rightarrow W(\mathrm{x}) > W(\mathrm{y})$.

The *weak Pareto principle* requires that social welfare must be improved whenever all generations strictly increase their utilities. The second axiom of efficiency is defined as follows:

Weak Dominance (WD): $\forall \mathrm{x}, \mathrm{y} \in X$;

$$[\exists j \in \mathbb{Z}_{++}, x_j > y_j \ \& \ \forall i \neq j, x_i = y_i] \Rightarrow W(\mathrm{x}) > W(\mathrm{y}).$$

The *weak dominance* axiom states that an improvement of exactly one generation's utility increases social welfare. Under weak dominance, we can easily show that AE-2 implies PDT and that PDT implies AE-1[1].

## III. *Consistency of the Weak Dominance Axiom*

This section examines the compatibility of the weak dominance axiom -one of the Pareto efficiency axioms- with three variations of SEP in terms of the social welfare function approach.

First, we show that WD is incompatible with AE-2 which is the strongest requirement among our three equity axioms.

Proposition 1: *Let Y include a closed interval* [0, 1]. *Then, there exists no social welfare function satisfying AE-2 and WD.*

[Proof] Suppose the contrary. Let a social welfare function $W$ satisfy AE-2 and WD. Define the two utility streams $\mathrm{x}(\epsilon)$, $\mathrm{y}(\epsilon)$ as follows:
For $\epsilon \in (3/10, 1/2)$,

$$\mathrm{x}(\epsilon) = \left(2\epsilon, \frac{1}{2}\epsilon, (\delta)_{\mathrm{rep}}\right),$$

$$\mathrm{y}(\epsilon) = \left(\epsilon, \frac{1}{2}\epsilon, (\delta)_{\mathrm{rep}}\right),$$

where $\delta \in [0, 1]$. It is easy to show that for all $\epsilon$, both $\mathrm{x}(\epsilon)$ and $\mathrm{y}(\epsilon)$ are in $(0, 1)^{\mathbb{Z}_{++}}$. Now, for all $\epsilon \in (3/10, 1/2)$, the WD axiom implies $W(\mathrm{x}(\epsilon)) > W(\mathrm{y}(\epsilon))$. Then, for all $\epsilon, \epsilon' \in (3/10, 1/2)$, $\epsilon > \epsilon'$ implies $2\epsilon' > \epsilon > \epsilon/2 > \epsilon'/2$ and $2\epsilon' - \epsilon > \frac{1}{2}(\epsilon - \epsilon')$. Hence, we have $W(\mathrm{y}(\epsilon)) > W(\mathrm{x}(\epsilon'))$ because of AE-2. By the above argument, $W(\mathrm{x}(\epsilon')) > W(\mathrm{y}(\epsilon'))$ holds. Therefore, for all $\epsilon, \epsilon' \in$

[1] Note that the logical relationships among the three equity axioms are all independent under the weak Pareto principle.

(3/10, 1/2) with $\epsilon > \epsilon'$, both of the closed intervals $[W(\mathrm{y}(\epsilon)), W(\mathrm{x}(\epsilon))]$ and $[W(\mathrm{y}(\epsilon')), W(\mathrm{x}(\epsilon'))]$ are non-degenerate and their intersection must be empty. Thus, for all real numbers $\epsilon \in (3/10, 1/2)$, we can choose a distinct rational number $r(\epsilon)$ from the closed interval $[W(\mathrm{y}(\epsilon)), W(\mathrm{x}(\epsilon))]$. This implies, however, that the cardinality of the continuum is a countable cardinality, a contradiction. ‖

Since the SEP axiom implies AE-2, the following corollary can be immediately established by the above proposition.

Corollary 1[2]: *Let* $Y \supseteq [0, 1]$. *Then, there is no social welfare function satisfying SEP and WD.*

Hammond equity for future generations is logically independent of altruistic equity2, but Banerjee (2006) proves that this equity axiom is incompatible with weak dominance.

Proposition 2 (Banerjee 2006): *Let* $Y \supseteq [0, 1]$. *Then, there is no social welfare function satisfying HEF and WD.*

However, replacing WD with WP, we obtain a social welfare function satisfying both HEF and WP. For example, $W(\mathrm{x}) = \min\{x_1, x_2\}$ is a social welfare function satisfying these axioms (Asheim, Mitra and Tungodden 2007, Example 1). But this function is not particularly useful because it does not impose value on procedural and distributive equity over future generations except for generations 1 and 2. In this sense, HEF is so weak as a concept of distributive equity that a class of social welfare functions satisfying HEF cannot rule out inequitable social welfare functions.

The next proposition states that the Pigou-Dalton transfer principle is compatible with the weak dominance axiom[3].

Proposition 3: *There exist social welfare functions satisfying PDT and WD.*

[Proof] The following functions are variations of SWFs proposed by Basu and Mitra (2007b). For all infinite utility streams $\mathrm{x} \in X$, define a set $E(\mathrm{x})$ as follows:

$$E(\mathrm{x}) = \{\mathrm{y} \in X \mid \exists\, T \in \mathbb{Z}_{++}, \forall t \geq T, x_t = y_t\}.$$

That is, $E(\mathrm{x})$ is the set of the same utility streams for all generations beyond some generation $T$. The set $E$ clearly forms an equivalence class, and the universal set $X$ is partitioned by E. We denote the set of partitions of X as $\mathcal{E}$. Then, by the axiom of choice, there exists a function $g$ that assigns each $E \in \mathcal{E}$ into $\mathrm{x} \in X$ such that $g(E) \in E$ for all $E \in \mathcal{E}$. Using this function $g$, define a social welfare function as follows:

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[2] Under weak dominance, SEP is equivalent to the axiom of Hammond equity (Alcantud 2012, Lemma 1). Hence, this corollary means that there is no SWF satisfying the axioms of Hammond equity and weak dominance.

[3] Weak dominance is a kind of weakened efficiency axiom. However, there is another way to relax the Pareto principle. For example, monotonicity is a weaker version of the Pareto principle, which requires that weak improvement of all generations' utilities never decreases social welfare. However, the proofs of Propositions 4-6 in the next section show the non-existence of social welfare functions satisfying monotonicity and any of our three axioms of distributive equity.

$$W(\mathrm{x})=\lim_{n\to\infty}\sum_{t=1}^{n}[U(x_t)-U(g_t(E(\mathrm{x})))],$$

where $U(\bullet)$ is a strictly increasing, real-valued, and strictly concave function. The function $W(\mathrm{x})$ must have a limit for all $\mathrm{x}\in\mathrm{X}$ since both x and $g(E(\mathrm{x}))$ are in the $E(\mathrm{x})$. Therefore, this social welfare function is well-defined and satisfies WD by definition. Because of the strict concavity of $U(\bullet)$, this function satisfies PDT. ‖

Since combining WD with PDT implies AE-1, the social welfare function above readily satisfies AE-1.

Proposition 3 shows that the non-negativeness of the total surplus between two generations through a transfer leads to the existence of equitable social welfare functions satisfying weak dominance. That is, if society always prefers income transfers from a richer generation to a poorer one, which keep the total surplus between the loss of the rich and the gain of the poor non-negative, then we can construct social welfare functions satisfying the axioms of distributive equity and weak dominance.

The next section, however, proves that the weak Pareto principle is incompatible with any axiom of distributive equity regardless of whether the total surplus between the loss of the rich and the gain of the poor is negative or non-negative.

## IV. *Impossibilities of the Weak Pareto Principle*

This section investigates the existence of a social welfare function satisfying the axioms of distributive equity and the weak Pareto principle. In the following propositions, we show that any axiom of distributive equity is incompatible with the weak Pareto principle. Therefore, there is a conflict between the axioms of distributive equity and the weak Pareto principle in the social welfare function approach for an infinite generations setting.

First, we show that the weak Pareto principle is incompatible with the altruistic equity-1, which is the weakest requirement among our three equity axioms under weak dominance.

Proposition 4: *Let* $Y\supseteq[0, 1]$. *Then, there is no social welfare function satisfying AE-1 and WP.*

[Proof] Suppose the contrary. Let a social welfare function $W$ satisfy AE-1 and WP. Define the two utility streams $\mathrm{x}(\epsilon)$, $\mathrm{y}^k(\epsilon)$ as follows:
For $\epsilon\in(3/10, 1/2)$,

$$\mathrm{x}(\epsilon)=((\epsilon, 2\epsilon)_{\mathrm{rep}}),$$

$$\mathrm{y}^k(\epsilon)=((\epsilon+(2/3)^{k-1}(2/10), 2\epsilon-(1/3)^{k-1}(1/10))_{2^{k-1}\text{-rep}}, (\epsilon, 2\epsilon)_{\mathrm{rep}}).$$

By definition, both $\mathrm{x}(\epsilon)$ and $\mathrm{y}^k(\epsilon)$ are in $(0, 1)^{\mathbb{Z}_{++}}$ for all $\epsilon\in(3/10, 1/2)$ and all natural numbers $k$.

Now, we will show that AE-1 implies $W(\mathrm{x}(\epsilon))<W(\mathrm{y}^1(\epsilon))<W(\mathrm{y}^2(\epsilon))<$.... It is easy to check that $\mathrm{y}^1(\epsilon)$ is derived from $\mathrm{x}(\epsilon)$ by an income transfer where generation 2 loses (1/10) but generation 1 gains (2/10). Hence, we have $W(\mathrm{x}(\epsilon))<W(\mathrm{y}^1(\epsilon))$.

Next, we show that for all k, the stream $\mathrm{y}^{k+1}(\epsilon)$ is constructed from $\mathrm{y}^k(\epsilon)$ by repeated applications of an appropriate income transfer.

In the stream $y^k(\epsilon)$, odd-numbered generations are relatively poor but there are inequalities among them; that is, one has $\epsilon+(2/3)^{k-1}(2/10)$ but the other has only $\epsilon$. By implementing the transfer where the relatively rich odd-numbered generations lose $(2/3)^k(1/10)$ but the relatively poor odd-numbered generations gain $(2/3)^k(2/10)$, each generation has the same utility $\epsilon+(2/3)^k$ $(2/10)$. Specifically, for all $i=1, 2, ..., 2^{k-1}$, we take $(2/3)^k(1/10)$ from the $(2i-1)$-th generation and give $(2/3)^k(2/10)$ to the $(2^k+2i-1)$-th generation. Similarly, among even-numbered generations, which are relatively rich in $y^k(\epsilon)$, there are also inequalities. These generations can have the same utility level, $2\epsilon-(1/3)^k(1/10)$, by repeated applications of a transfer in which the rich, who have $2\epsilon$, lose $(1/3)^k(1/10)$ but the poor, who only have $(2\epsilon-(1/3)^{k-1}-1(1/10))$, gain $(1/3)^k(2/10)$. That is, for all $i=1, ..., 2^{k-1}$, we take $(1/3)^k(1/10)$ from the $2i$-th generation and give $(1/3)^k(2/10)$ to the $(2^k+2i)$-th generation. Therefore, for all $k$, $W(y^k(\epsilon))<W(y^{k+1}(\epsilon))$.

Then, for all distinct $\epsilon$, $\epsilon'\in(3/10, 1/2)$ with $\epsilon>\epsilon'$, there exists a natural number $K^*$ such that $\epsilon>\epsilon'+(2/3)^{K^*-1}(2/10)$. Obviously, $2\epsilon>2\epsilon'$ and $2\epsilon>2\epsilon'-(1/3)^{K^*-1}(1/10)$, so WP implies $W(x(\epsilon))>W(y^{K^*}(\epsilon'))$. Because $W(y^{K^*}(\epsilon'))>W(y^1(\epsilon'))$ by the above argument, for all $\epsilon$, $\epsilon'\in(3/10, 1/2)$ with $\epsilon>\epsilon'$, the two closed intervals $[W(x(\epsilon)), W(y^1(\epsilon))]$ and $[W(x(\epsilon')), W(y^1(\epsilon'))]$ are non-degenerate, and their intersection is empty.

Thus, for all real numbers $\epsilon\in(3/10, 1/2)$, we can choose a distinct rational number $r(\epsilon)$ from the closed interval $[W(x(\epsilon)), W(y^1(\epsilon))]$. This implies, however, that the cardinality of the continuum equals the countable cardinality, a contradiction. ‖

Since it holds true that AE-2 → PDT → AE-1 under weak dominance, Proposition 4 is interpreted as the strongest result for the non-existence of equitable social welfare functions satisfying the weak Pareto principle. In fact, the remaining two equity axioms are incompatible with the weak Pareto principle in the social welfare function approach. That is, whatever the total surplus gained by an income transfer from the rich to the poor, whether negative or non-negative, combining the distributive equity axioms and the weak Pareto principle always leads to impossibility results. The following two impossibility results are established by proofs similar to that of Proposition 4.

Proposition 5: *Let* $Y\supseteq[0, 1]$. *Then, there is no social welfare function satisfying PDT and WP*.

[Proof] Since the proof of this proposition is similar to that of Proposition 4, we omit some details of the proof. Consider the two utility streams $x(\epsilon)$, $y^k(\epsilon)$ as follows:
For $\epsilon\in(1/5, 1/2)$,

$$x(\epsilon)=((\epsilon, 2\epsilon)_{\text{rep}}),$$

$$y^k(\epsilon)=((\epsilon+(1/2)^{k-1}(1/10), 2\epsilon-(1/2)^{k-1}(1/10))_{2^{k-1}\text{-rep}}, (\epsilon, 2\epsilon)_{\text{rep}}).$$

The stream $y^1(\epsilon)$ is constructed from $x(\epsilon)$ by the income transfer 1/10 from generation 2 to generation 1. Hence, PDT implies $W(x(\epsilon))<W(y^1(\epsilon))$. Next, for all $k$, we show that the stream $y^{k+1}(\epsilon)$ is induced from $y^k(\epsilon)$ through the following transfer.

First, we transfer $(1/2)^k\times(1/10)$ from the better-off odd-numbered generations, who have $\epsilon+(1/2)^{k-1}(1/10)$, to the worse-off odd-numbered generations, who have only $\epsilon$ in the stream $y^k(\epsilon)$. Then, each generation has the same utility $\epsilon+(1/2)^k(1/10)$. Similarly, even-numbered generations can have the same utility level, $2\epsilon-(1/2)^k(1/10)$, by the transfer $(1/2)^k(1/10)$ from the better-off even-numbered generations, who have $2\epsilon$, to the worse-off even-numbered generations, who have only $(2\epsilon-(1/2)^{k-1}(1/10))$. Therefore, we show that $W(\mathrm{y}^k(\epsilon))<W(\mathrm{y}^{k+1}(\epsilon))$ for all $k$.

The remainder of this proof can be shown by the same argument used in the proof of Proposition 4. ‖

Proposition 6: *Let* $Y\supseteq[0, 1]$. *Then, there is no social welfare function satisfying AE-2 and WP.*

[Proof] Define the two utility streams $\mathrm{x}(\epsilon)$, $\mathrm{y}^k(\epsilon)$ as follows:
For $\epsilon\in(3/10, 1/2)$,

$$\mathrm{x}(\epsilon)=((\epsilon, 2\epsilon)_{\mathrm{rep}}),$$

$$\mathrm{y}^k(\epsilon)=((\epsilon+(1/3)^{k-1}(1/10), 2\epsilon-(2/3)^{k-1}(2/10))_{2^{k-1}\text{-rep}}, (\epsilon, 2\epsilon)_{\mathrm{rep}}).$$

The stream $\mathrm{y}^1(\epsilon)$ is gained from $\mathrm{x}(\epsilon)$ by the transfer in which generation 2 loses (2/10) but generation 1 gains (1/10). Next, we show that for all $k$, the stream $\mathrm{y}^{k+1}(\epsilon)$ is constructed from $\mathrm{y}^k(\epsilon)$ by repeated applications of an appropriate transfer. In the stream $\mathrm{y}^k(\epsilon)$, if we implement the transfer in which the relatively rich odd-numbered generations lose $(1/3)^k(2/10)$ but the relatively poor odd-numbered generations gain $(1/3)^k(1/10)$, then each odd-numbered generation has the same utility $\epsilon+(1/3)^k(1/10)$. In addition, even-numbered generations can have the same utility level, $2\epsilon-(2/3)^k(2/10)$, by repeated applications of a transfer in which the rich, who have $2\epsilon$, lose $(2/3)^k(2/10)$, but the poor, who only have $(2\epsilon-(2/3)^{k-1}(2/10))$, gain $(2/3)^k(1/10)$. Therefore, for all $k$, $W(\mathrm{y}^k(\epsilon))<W(\mathrm{y}^{k+1}(\epsilon))$.

The remainder of the proof can be shown by the same argument used in the proof of Proposition 4. ‖

Since SEP implies the axioms PDT, AE-1, and AE-2, the following corollary follows directly from our impossibility results (Propositions 4-6).

Corollary 2: *Let* $Y\supseteq[0, 1]$. *Then, there is no social welfare function satisfying SEP and WP*.

## V. *Concluding Remarks*

In this paper, we have examined the compatibility of the distributive equity axioms with two weak forms of the Pareto principle in aggregating infinite utility streams. We have shown that there exists no Paretian social welfare function satisfying any variation of distributive equity in our framework. Table 1 summarizes the results obtained in this paper.

As we can see from Table 1, we can construct social welfare functions satisfying two variations of distributive equity and weak dominance. These possibility results, however, should not be construed essentially as "positive" because we never construct a social welfare function simultaneously satisfying any form of the distributive equity axioms and the weak Pareto principle. Accordingly, our analysis reveals the difficulties of the social welfare function

TABLE 1. LOGICAL RELATIONSHIPS BETWEEN EFFICIENCY AND DISTRIBUTIVE EQUITY

| | AE-1 | PDT | AE-2 | SEP | HEF |
|---|---|---|---|---|---|
| WD | ○ | ○ | × | × | × |
| WP | × | × | × | × | ○ |

*Note*: The symbols "○" and "×" indicate, respectively, "possibility" and "impossibility" results.

approach in the context of intergenerational equity.

Now we discuss the implications.

Firstly, the non-existence of a social welfare function does not necessarily mean the non-existence of a social ranking for infinite utility streams. Indeed, Bossert, Suzumura and Sprumont (2007) characterize *quasi-orderings* satisfying the axioms of the strong Pareto principle, the finite anonymity and the Pigou-Dalton transfer principle. Since every quasi-ordering has an ordering extension of itself, without a numerical representation of social evaluation for intergenerational equity, we can construct a social *ordering* that satisfies the axioms of equity and efficiency.

Secondly, we can obtain some possibility results by weakening the domain restriction of $Y$. Indeed, if $Y=\mathbb{Z}_+$, then we can easily construct a social welfare function satisfying the Pigou-Dalton transfer principle and the Pareto principle following Basu and Mitra (2007b).

Lastly, the requirement of a numerical representation for social welfare functions is logically independent of the continuity of social rankings. If a social ranking on $X$ is continuous with some topology, then the numerical representation of this ranking must be continuous on $X$ as well. However, this does not imply the continuity of the social ranking on $X$ in which the numerical representation of the ranking is generally possible. In their seminal papers, Sakai (2006) and Hara et al. (2007) show that there is no continuous social ranking on $X$ satisfying both the Pigou-Dalton transfer principle and some axioms of collective rationality such as acyclicity. However, their impossibility results are independent of ours because the numerical representability of social rankings does not imply the continuity of these rankings. The results of this paper suggest a trade-off between intergenerational equity and efficiency in the objective functions of the usual dynamic optimization problems.

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