Characterizing Social Value of Information*

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Abstract

This paper examines the social value of information in symmetric Bayesian games with quadratic payoff functions and normally distributed public and private signals. The main results identify necessary and sufficient conditions for welfare to increase with public or private information. Using the conditions, we classify games into eight types by welfare effects of information. In the first type, welfare necessarily increases with both public and private information. In the second type, welfare can decrease, but only with public information. In the third type, welfare can decrease as well as increase with both public and private information. In the fourth type, welfare can decrease with both, but can increase only with private information. The remaining four types are the counterparts of the above four types with the opposite welfare effects of information. For each type, we characterize a socially optimal information structure and a socially optimal Bayesian correlated equilibrium.

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1 Introduction

In games with incomplete information, more information is not necessarily valuable, thus raising doubts over the desirability of transparency. A notable example is a beauty contest game of Morris and Shin (2002a) (henceforth MS). Players have access to normally distributed public and private signals on the state of fundamentals. They aim to take actions appropriate to the state, but also engage in a race to second-guess the opponents' actions. More specifically, a player's best response is a weighted mean of the conditional expectation of the state and that of the opponents' actions. Assume that welfare is measured by the negative of the mean squared error of an action from the state. Then, increased precision of public information is detrimental to welfare if players have access to sufficiently precise private information.

Many studies have challenged MS's anti-transparency result. Svensson (2006) doubts its quantitative significance: welfare is locally decreasing in the precision of public information only if it is implausibly low; even on a global analysis, welfare is higher with public information than without insofar as public information has precision no lower than the precision of private information. Angeletos and Pavan (2004) and Hellwig (2005) note that the anti-transparency result stems from the particular payoff function and welfare criterion: a public disclosure of more precise information is always beneficial to welfare in the models of technological spillovers (Angeletos and Pavan, 2004) and monopolistic competition (Hellwig, 2005).[1]

To explore the desirability of transparency given the above mixed results, we must study the social value of information in more general settings and identify the environments in which more precise information is beneficial or harmful to welfare. For this purpose, Angeletos and Pavan (2007) (henceforth AP) introduce a class of Bayesian games where a continuum of players have symmetric quadratic payoff functions and receive normally distributed public and private signals on the state of fundamentals.[2] This class of Bayesian games is tractable yet sufficiently flexible to encompass a number of applications, including a beauty contest game. AP characterize both the unique Bayesian Nash equilibrium and the socially optimal

[1]Several authors challenge the anti-transparency result considering different methods of information dissemination. See Colombo and Femminis (2008), Cornand and Heinemann (2008), Myatt and Wallace (2010), and Arato and Nakamura (2012), among others. In contrast, James and Lawler (2011) strengthen MS's conclusion.

[2]For the same purpose, Ui (2009) independently proposes a class of quadratic Bayesian potential games with a finite number of players, allowing asymmetric payoff functions and signal structures.

strategy profile, and analyze the social value of public and private information using the socially optimal strategy profile as an efficiency benchmark. They classify games according to the type of inefficiency exhibited by the equilibrium and find the following. In the first class of games, where the equilibrium is efficient under both complete and incomplete information, welfare necessarily increases with both public and private information. In the second class of games, where the equilibrium is inefficient only under incomplete information, welfare can decrease with either public or private information. In the third class of games, where the equilibrium is inefficient even under complete information, welfare can decrease with both public and private information—a possibility not present in the previous two classes of games.

Clearly, even in the third class of games, there exist games such that welfare necessarily increases with both public and private information. Which games are they? In the other games, exactly when does welfare decrease with public or private information? Does complete information remain socially optimal? If not, what information structure is socially optimal? Specifically, what is the optimal degree of transparency in public information?

This paper attempts to answer these questions using a finite-player version of AP's model. Assuming a finite number of players rather than a continuum has three advantages. First, it is straightforward to extend the finite case to the continuum case. Next, we can conduct comparative statics with respect to the number of players. In fact, the welfare effects of information can depend upon the number of players. Finally, the assumption of a continuum of players is inappropriate in some cases for studying the social value of information. For example, in voluntary provision of public goods, each player would make no contribution facing an infinite number of opponents, where information has no influence on welfare. This paper exploits these advantages.

Following AP, our welfare measure is the total ex ante expected payoff in the equilibrium. By the first order condition for equilibrium, welfare is represented as a linear combination of the variance and covariance of actions; however, it is more useful to view it as a linear combination of the covariance and the difference between the variance and covariance. The covariance is a measure of common variation of actions; it increases with both public and private information because more precise information causes more correlated actions. The difference between the variance and covariance is a measure of idiosyncratic variation of actions; it can decrease with both public and private information because a higher correlation of actions brings the covariance and variance closer. Therefore, welfare increases with both public and private information if the relative weight of the covariance in welfare is large, but can decrease if it is small. This key observation makes our analysis tractable and intuitive.

The main results of this paper are necessary and sufficient conditions for welfare to increase with public or private information. We state the conditions in terms of the precision of public and private information, the relative weights of the covariance and difference terms, and the (normalized) cross-derivative of the payoff function. Using the results, we classify games into eight types by the welfare effects of information. Four types of games, which we call types +I, +II, +III, and +IV, have a positive coefficient of the difference term. As the number increases, the coefficient of the covariance term decreases, thus changing the welfare effects of information from positive to negative. In type +I, welfare necessarily increases with both public and private information. In type +II, such as a beauty contest game, welfare can decrease, but only with public information. In type +III, welfare can decrease as well as increase with both public and private information; specifically, whenever private information is harmful, so is public information. In type +IV, welfare can decrease with both, but can increase only with private information. The remaining four types of games, which we call types −I, −II, −III, and −IV, are the counterparts of types +I, +II, +III, and +IV with the opposite welfare effects of information, respectively. In type −I, welfare necessarily decreases with both public and private information. In type −II, welfare can increase, but only with public information. In type −III, welfare can increase as well as decrease with both public and private information; specifically, whenever private information is beneficial, so is public information. In type −IV, welfare can increase with both, but can decrease only with private information.

Using the main results, we characterize the socially optimal information structure that maximizes welfare in each type.[3] In types +I, +II, and −IV, complete information is optimal. In types −I, −II, and −III, no information is optimal. In types +III and +IV, incomplete information only with appropriate noisy private signals is optimal. We also characterize the optimal precision of public information fixing the precision of private information, and the optimal precision of private information

[3]Bergemann and Morris (2012a) study a Cournot game with a continuum of players and characterize its optimal information structure, which is a special case of our result in the continuum case. See Section 5.6. In an auction with many bidders, Bergemann and Pesendorfer (2007) study its optimal information structure that maximizes revenue.

fixing the precision of public information. The former result provides the optimal degree of transparency in public information. In types +I, +II, and −IV, the highest transparency is optimal, whereas in types +IV and −I, the lowest transparency is optimal. In the other types, it depends upon the precision of private information. For example, in type +III, the highest transparency is optimal if the precision of private information is below a given threshold; otherwise, the lowest transparency is optimal.

There are several applications. A homogeneous-product Cournot game with linear demand and cost functions is type +I with two players, type +II with three, and type +III with four or more. A differentiated-product Bertrand game with linear demand and quadratic cost functions is either type +I or −IV depending upon the cross-price effect. We also consider public goods games with quadratic production and linear cost functions. If production is random, this game is type +I, whereas if cost is random, this game is type −I. Thus, the welfare effects of information can be opposite depending upon the source of uncertainty.

All the above results have their counterparts in games with a continuum of players, which are also classified into eight types with the same properties as those in the finite case. As an application, we reconsider the large Cournot and Bertrand games studied by AP and revise their results. They show the following: in the Cournot game, expected total profits necessarily increase with private information; in the Bertrand game, expected total profits necessarily increase with both public and private information. These results, however, are not true because of errors in the proofs. We show that the Cournot game can be type +III and the Bertrand game can be type −IV, where expected total profits can decrease with private information.

We then reconsider the class of games with a continuum of players studied by AP in which the equilibrium is inefficient only under incomplete information. They state that welfare can decrease with either public or private information, but not with both. We show that this is true if the socially optimal strategy profiles exist, which is a crucial assumption in AP, but without this assumption, there exists a game in which welfare can decrease with both public and private information. Our analysis does not require the existence of socially optimal strategy profiles, which enables us to study the social value of information in a broader class of games than AP's class.

The aforementioned characterization of the socially optimal information structures also holds in the continuum case and is useful in identifying socially optimal Bayesian *correlated* equilibria. Consider a mediator who knows the true state and makes private action recommendations to players having no information about the state. If each player has an incentive to follow the mediator's recommendation, we say that the resulting joint action distribution is a Bayesian correlated equilibrium. Bergemann and Morris (2012a) show that, in games with a continuum of players, the set of all Bayesian correlated equilibria coincides with the set of all action distributions of Bayesian Nash equilibria generated by the bivariate (public and private) signal structures. This finding implies that the action distribution of the Bayesian Nash equilibrium under the aforementioned optimal information structure is the optimal Bayesian correlated equilibrium that achieves the highest welfare. Thus, the recommended actions in the optimal Bayesian correlated equilibria are completely correlated in types +I, +II, and −IV, constant in types −I, −II, and −III, and conditionally independent given the state in types +III and +IV. This result complements that of Bergemann and Morris (2012a), who go in the opposite direction. Focusing on large Cournot games, they find two types of optimal Bayesian correlated equilibria, and then characterize the corresponding optimal information structures. Our result shows that there are three types of optimal Bayesian correlated equilibria in total and also provides a necessary and sufficient condition for each of them.

The literature on the social value of information in interactive contexts dates back at least to Hirshleifer (1971), who finds that public disclosure of information can make agents worse off by ruling out opportunities to insure. Recently, there have been several theoretical studies on the social value of information in Bayesian games extending Blackwell's theorem (Blackwell, 1953), most of which consider zero-sum games (Gossner and Mertens, 2001; Lehrer and Rosenberg, 2006; Pęski, 2008; De Meyer et al., 2010) or games with common interests (Lehrer et al., 2010).[4] In contrast, this paper considers non-zero-sum symmetric quadratic games with bivariate normal signals, and examines all possibilities of the welfare effects of information. Our results serve as a benchmark for exploring the social value of information in more general Bayesian games.

This paper is organized as follows. Section 2 introduces the model and obtains the expected payoff in the equilibrium. Section 3 presents the main results and characterizes the optimal information structures. Section 4 discusses applications. Section 5 studies the continuum case. Section 6 concludes.

[4] See also Kamien et al. (1990), Neyman (1991), Gossner (2000), and Bassan et al. (2003), among others.

2 Model

Consider a Bayesian game with n players. An individual player is indexed by $i \in N \equiv \{1, \dots, n\}$. Player i chooses an action $a_i \in \mathbb{R}$. Player i's payoff function is quadratic in an action profile $a \equiv (a_i)_{i \in N} \in \mathbb{R}^N$ and a payoff state $\theta \in \mathbb{R}$:

$$\begin{aligned} u_i(a, \theta) = &- a_i^2 + 2\alpha a_i \sum_{j \neq i} a_j + 2\beta\theta a_i \\ &+ \kappa \sum_{j \neq i} a_j^2 + \lambda \sum_{j<k: j,k \neq i} a_j a_k + \mu \sum_{j \neq i} \theta a_j + \nu \sum_{j \neq i} a_j + f(\theta), \end{aligned} \tag{1}$$

where $\alpha, \beta, \kappa, \lambda, \mu, \nu \in \mathbb{R}$ are constants[5] and $f : \mathbb{R} \to \mathbb{R}$ is a measurable function. Constants α and β are coefficients of terms including a_i, which determine player i's best response. This game exhibits strategic complementarity if $\alpha > 0$ and strategic substitutability if $\alpha < 0$. Assume $\beta > 0$ without loss of generality. Constants κ, λ, μ, and ν are coefficients of terms not including a_i, which have no influence on player i's best response. As we will see later, ν and $f(\theta)$ play no role in our welfare analysis.

Player i observes a private signal $x_i = \theta + \varepsilon_i$ and a public signal $y = \theta + \varepsilon_0$, where ε_i, ε_0, and θ are independently and normally distributed with

$$E[\theta] = \bar{\theta},\ E[\varepsilon_i] = E[\varepsilon_0] = 0,\ \mathrm{var}[\theta] = \tau_\theta^{-1},\ \mathrm{var}[\varepsilon_i] = \tau_x^{-1},\ \mathrm{var}[\varepsilon_0] = \tau_y^{-1},$$

and ε_i and ε_j are independent for $i \neq j$. Player i's signal vector is denoted by $\mathbf{s}_i = (x_i, y)^\top$. The mean vector and covariance matrices are denoted by

$$\bar{\mathbf{s}} \equiv E[\mathbf{s}_i],\ C \equiv \mathrm{var}[\mathbf{s}_i],\ D \equiv \mathrm{cov}[\mathbf{s}_i, \mathbf{s}_j],\ \mathbf{g} \equiv \mathrm{cov}[\theta, \mathbf{s}_i].$$

We refer to τ_x, τ_y, and $\tau \equiv (\tau_x, \tau_y)$ as the precision of private information, that of public information, and an information structure of the game, respectively.

Let $\sigma_i : \mathbb{R}^2 \to \mathbb{R}$ be player i's strategy for $i \in N$, which maps a signal vector $\mathbf{s}_i$ to an action $\sigma_i(\mathbf{s}_i)$. A strategy profile $(\sigma_i)_{i \in N}$ is a Bayesian Nash equilibrium if each player maximizes his interim expected payoff given the opponents' strategies; that is, $\sigma_i(\mathbf{s}_i) = \arg\max_{a_i} E[u_i((a_i, \sigma_{-i}), \theta)|\mathbf{s}_i]$ for all $\mathbf{s}_i \in \mathbb{R}^2$ and $i \in N$, where $\sigma_{-i} = (\sigma_j(\mathbf{s}_j))_{j \neq i}$. The first order condition for equilibrium is

$$-\sigma_i(\mathbf{s}_i) + \alpha \sum_{j \neq i} E[\sigma_j(\mathbf{s}_j)|\mathbf{s}_i] + \beta E[\theta|\mathbf{s}_i] = 0. \tag{2}$$

[5]By choosing θ appropriately, we can make a linear term of a_i equal zero.

Radner (1962) introduces a team in which each player has an identical payoff function[6]

$$v(a,\theta) \equiv -\sum_j a_j^2 + 2\alpha \sum_{j<k} a_j a_k + 2\beta \sum_j \theta a_j. \tag{3}$$

This payoff function is (1) with $\kappa = -1$, $\lambda = 2\alpha$, $\mu = 2\beta$, $\nu = 0$, and $f(\theta) = 0$. Radner (1962, Theorem 5) shows that if $v(a,\theta)$ is strictly concave in a then there exists a unique equilibrium, which coincides with a unique socially optimal strategy profile, and that each strategy in the equilibrium is an affine function. Because each player's best response is independent of κ, λ, μ, ν, and $f(\theta)$, a team with (3) has a unique equilibrium if and only if a Bayesian game with (1) has the same unique equilibrium. Thus, Radner's theorem implies the following result.

Lemma 1. *If $\alpha_n \equiv (n-1)\alpha < 1$, then a game with* (1) *has a unique Bayesian Nash equilibrium $(\sigma_i)_{i\in N}$ with*

$$\sigma_i(\mathbf{s}_i) = \mathbf{b}^\top(\mathbf{s}_i - \bar{\mathbf{s}}) + c \tag{4}$$

for all $\mathbf{s}_i \in \mathbb{R}^2$ and $i \in N$, where $\mathbf{b} = \beta(C - \alpha_n D)^{-1}\mathbf{g}$ and $c = \beta\bar{\theta}/(1-\alpha_n)$.

Proof. The leading minors of the Hessian matrix of $v(a,\theta)$ are $-(1+\alpha)^{k-1}(1-(k-1)\alpha)$ for $k = 1,\ldots,n$. This implies that $v(a,\theta)$ is strictly concave in a if and only if $-(n-1) < \alpha_n < 1$. Thus, we can obtain the above unique equilibrium using Theorem 5 of Radner (1962) if $-(n-1) < \alpha_n < 1$, but a weaker condition $\alpha_n < 1$ suffices by the symmetry of payoff functions. See Appendix A. □

This lemma exploits the property that the best response correspondence of a Bayesian game coincides with that of a team. A Bayesian game with this property is called a Bayesian potential game (Monderer and Shapley, 1996; Heumen et al., 1996).[7]

The equilibrium strategy is an affine function of x_i and y. The ratio of the coefficients of x_i and y is $(1-\alpha_n)\tau_x/\tau_y$, which can be verified by direct calculation.

[6] Radner (1962) allows asymmetry of payoff functions and information structures. A team with quadratic payoff functions and normally distributed signals is called a linear quadratic Gaussian (LQG) team.

[7] Basar and Ho (1974) was the first to use Radner's theorem in the study of Bayesian potential games, followed by many papers on information sharing in oligopoly. See Raith (1996) and references therein. Ui (2009) considers a more general class of Bayesian potential games with quadratic payoff functions and normally distributed signals. See also Ui (2000) for a characterization of potential games.

Thus, if α_n is close to one (i.e., a high degree of strategic complementarity) or τ_x/τ_y is small, the weight of a public signal is large, and if α_n is small (i.e., a high degree of strategic substitutability) or τ_x/τ_y is large, the weight of a private signal is large.

Given the equilibrium, we calculate the ex ante expected payoff:

$$\begin{aligned}E[u_i(\sigma,\theta)] =& - E[{\sigma_i}^2] + 2\alpha E[\sigma_i \sum_{j\neq i}\sigma_j] + 2\beta E[\theta\sigma_i]\\ &+ \kappa E[\sum_{j\neq i}{\sigma_j}^2] + \lambda E[\sum_{j<k:j,k\neq i}\sigma_j\sigma_k] + \mu E[\sum_{j\neq i}\theta\sigma_j] + \nu E[\sum_{j\neq i}\sigma_j] + E[f(\theta)].\end{aligned}$$

Using the symmetry ($\sigma_i(\cdot) = \sigma_j(\cdot)$ for $i \neq j$) and setting

$$\alpha_n = (n-1)\alpha,\ \kappa_n = (n-1)\kappa,\ \lambda_n = (n-1)(n-2)\lambda/2,\ \mu_n = (n-1)\mu,\ \nu_n = (n-1)\nu,$$

we have

$$\begin{aligned}E[u_i(\sigma,\theta)] =&(\kappa_n - 1)E[{\sigma_i}^2] + (2\alpha_n + \lambda_n)E[\sigma_i\sigma_j] + (2\beta + \mu_n)E[\theta\sigma_i] + \nu_n E[\sigma_i] + E[f(\theta)]\\ =&(\kappa_n - 1)\mathrm{var}[\sigma_i] + (2\alpha_n + \lambda_n)\mathrm{cov}[\sigma_i,\sigma_j] + (2\beta + \mu_n)\mathrm{cov}[\theta,\sigma_i]\\ &+ (2\alpha_n + \kappa_n + \lambda_n - 1)c^2 + ((2\beta + \mu_n)\bar{\theta} + \nu_n)c + E[f(\theta)].\end{aligned}$$

Thus, we adopt

$$W(\tau) \equiv (\kappa_n - 1)\mathrm{var}[\sigma_i] + (2\alpha_n + \lambda_n)\mathrm{cov}[\sigma_i,\sigma_j] + (2\beta + \mu_n)\mathrm{cov}[\theta,\sigma_i] \tag{5}$$

as a measure of welfare because $E[u_i(\sigma,\theta)]$ equals $W(\tau)$ plus a constant and the total expected payoff is $\sum_{i\in N} E[u_i(\sigma,\theta)] = n \times E[u_i(\sigma,\theta)]$. We rewrite $W(\tau)$ as follows using the first order condition for equilibrium.

Lemma 2. *It holds that*

$$\begin{aligned}W(\tau) &= (\zeta_n(\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i,\sigma_j]) + \eta_n\mathrm{cov}[\sigma_i,\sigma_j])/\beta\\ &= \left(\zeta_n\mathbf{b}^\top(C - D)\mathbf{b} + \eta_n\mathbf{b}^\top D\mathbf{b}\right)/\beta \end{aligned}\tag{6}$$

with $W(0,0) = 0$, *where*

$$\zeta_n \equiv \mu_n + \beta(\kappa_n + 1),\ \eta_n \equiv (1 - \alpha_n)\mu_n + \beta(\kappa_n + \lambda_n + 1). \tag{7}$$

Proof. Multiplying the first order condition by $\sigma_i(\mathbf{s}_i)$ and taking the expectation, we have

$$\begin{aligned}-E[\sigma_i^2]+&\alpha_n E[\sigma_i\sigma_j] + \beta E[\theta\sigma_i]\\ &= -\mathrm{var}[\sigma_i] + \alpha_n\mathrm{cov}[\sigma_i,\sigma_j] + \beta\mathrm{cov}[\theta,\sigma_i] - (1-\alpha_n)c^2 + \beta\bar{\theta}c\\ &= -\mathrm{var}[\sigma_i] + \alpha_n\mathrm{cov}[\sigma_i,\sigma_j] + \beta\mathrm{cov}[\theta,\sigma_i] = 0\end{aligned}$$

for $i \neq j$ because $c = \beta\bar{\theta}/(1-\alpha_n)$ by Lemma 1, and thus

$$\mathrm{cov}[\theta, \sigma_i] = \beta^{-1}\mathrm{var}[\sigma_i] - \alpha_n \beta^{-1}\mathrm{cov}[\sigma_i, \sigma_j]. \tag{8}$$

Plugging this into (5), we have (6) because $\mathrm{var}[\sigma_i] = \mathbf{b}^\top C \mathbf{b}$ and $\mathrm{cov}[\sigma_i, \sigma_j] = \mathbf{b}^\top D \mathbf{b}$ by Lemma 1. To prove $W(0,0) = 0$, we must evaluate these quadratic forms. See Appendix B. □

Note that $W(\tau)$ is a linear combination of the variance and covariance of actions. Thus, when players receive no information and choose constant strategies, we have $W(\tau) = W(0,0) = 0$. This implies that welfare is greater with information than without if $W(\tau) \geq 0$ for all τ, and that welfare is greater without information than with if $W(\tau) \leq 0$ for all τ. To find the sign of $W(\tau)$, it is more useful to write $W(\tau)$ as a linear combination of $\mathrm{cov}[\sigma_i, \sigma_j] \geq 0$ and $\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i, \sigma_j] \geq 0$, which are measures of common variation of actions and idiosyncratic variation of actions, respectively.[8] Their coefficients determine the sign of $W(\tau)$: if $\zeta_n, \eta_n > 0$, then $W(\tau) \geq 0$ for all τ, and if $\zeta_n, \eta_n < 0$, then $W(\tau) \leq 0$ for all τ because $\beta > 0$. Information has no influence on welfare if $\zeta_n = \eta_n = 0$.

The coefficients ζ_n and η_n determine not only the sign of $W(\tau)$ but also a local property of $W(\tau)$ because $\mathrm{cov}[\sigma_i, \sigma_j]$ increases with τ_x and τ_y, whereas $\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i, \sigma_j]$ can decrease. The intuition is as follows. As either τ_x or τ_y increases, the correlation of actions increases, so $\mathrm{cov}[\sigma_i, \sigma_j]$ increases with τ_x and τ_y. The increase in τ_y makes the covariance closer to the variance, so $\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i, \sigma_j]$ decreases with τ_y. For the same reason, $\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i, \sigma_j]$ can decrease with τ_x, but it can also increase with τ_x if τ_x is sufficiently small or α_n is sufficiently close to one. In either case, the weight of a public signal is large in the equilibrium strategy, and thus the correlation of actions is close to one. As a result, the increase in the covariance caused by τ_x is small and less than that in the variance, so $\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i, \sigma_j]$ increases with τ_x.

To see how ζ_n and η_n determine a local property of $W(\tau)$, assume that $\zeta_n, \eta_n > 0$, for example. If η_n is sufficiently large compared to ζ_n, then $\mathrm{cov}[\sigma_i, \sigma_j]$ dominates in $W(\tau)$, and thus $W(\tau)$ necessarily increases with both τ_x and τ_y. In contrast, if ζ_n is sufficiently large compared to η_n, then $\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i, \sigma_j]$ dominates in $W(\tau)$, and thus $W(\tau)$ can decrease with both τ_x and τ_y. Section 3 characterizes $W(\tau)$ using ζ_n and η_n, together with α_n.

[8] In games with a continuum of players, there is another interpretation of $\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i, \sigma_j]$ and $\mathrm{cov}[\sigma_i, \sigma_j]$. See Section 5.1.

3 Result

3.1 Social value of information

We present the main results of this paper.

Proposition 1. *Assume that* $\alpha_n < 1$, $\beta > 0$, *and* $(\zeta_n, \eta_n) \neq (0,0)$. *Define*

$$X \equiv \begin{cases} ((1-\alpha_n)\zeta_n - 2\eta_n)/\zeta_n & \text{if } \zeta_n \neq 0, \\ -\infty & \text{if } \zeta_n = 0, \end{cases}$$

$$Y \equiv (1-\alpha_n)\,(2(1-\alpha_n)\zeta_n - 3\eta_n)/\eta_n \quad \text{if } \eta_n \neq 0.$$

Then, the following holds for $\tau_x, \tau_y, \tau_\theta > 0$.

(i) *In a game with* $\zeta_n \geq 0$ *and* $\eta_n > 0$,

$$\frac{\partial W(\tau)}{\partial \tau_x} \gtreqless 0 \;\Leftrightarrow\; X \lesseqgtr (\tau_y + \tau_\theta)/\tau_x,$$
$$\frac{\partial W(\tau)}{\partial \tau_y} \gtreqless 0 \;\Leftrightarrow\; Y \lesseqgtr (\tau_y + \tau_\theta)/\tau_x,$$

where $X \leq Y$ *and the equality holds only if* $X, Y < 0$.

(ii) *In a game with* $\eta_n \leq 0 < \zeta_n$,

$$\frac{\partial W(\tau)}{\partial \tau_x} \gtreqless 0 \;\Leftrightarrow\; X \lesseqgtr (\tau_y + \tau_\theta)/\tau_x,$$
$$\frac{\partial W(\tau)}{\partial \tau_y} < 0 \text{ for all } \tau,$$

where $X > 0$.

(iii) *In a game with* $\zeta_n \leq 0$ *and* $\eta_n < 0$,

$$\frac{\partial W(\tau)}{\partial \tau_x} \gtreqless 0 \;\Leftrightarrow\; X \gtreqless (\tau_y + \tau_\theta)/\tau_x,$$
$$\frac{\partial W(\tau)}{\partial \tau_y} \gtreqless 0 \;\Leftrightarrow\; Y \gtreqless (\tau_y + \tau_\theta)/\tau_x,$$

where $X \leq Y$ *and the equality holds only if* $X, Y < 0$.

(iv) *In a game with* $\zeta_n < 0 \leq \eta_n$,

$$\frac{\partial W(\tau)}{\partial \tau_x} \gtreqless 0 \;\Leftrightarrow\; X \gtreqless (\tau_y + \tau_\theta)/\tau_x,$$
$$\frac{\partial W(\tau)}{\partial \tau_y} > 0 \text{ for all } \tau,$$

where $X > 0$.

Proof. See Appendix C. □

The sign of $\partial W/\partial \tau_x$ is determined by X and $(\tau_y + \tau_\theta)/\tau_x$, and that of $\partial W/\partial \tau_y$ is determined by Y and $(\tau_y + \tau_\theta)/\tau_x$. Note that $(\tau_y + \tau_\theta)/\tau_x$ is the ratio of the levels of idiosyncratic uncertainty $1/\tau_x$ and common uncertainty $1/(\tau_y + \tau_\theta)$.

If $\zeta_n, \eta_n > 0$ and $X, Y \leq 0$, then $W(\tau)$ is increasing in τ_x and τ_y; otherwise, $W(\tau)$ can be decreasing. For example, if $\zeta_n, \eta_n > 0$ and $X > 0$, then $W(\tau)$ is decreasing in τ_x if $\tau_x > (\tau_y + \tau_\theta)/X$. Thus, the sign combinations of ζ_n, η_n, X, and Y are important. The following combinations are possible.

(i) In a game with $\zeta_n \geq 0$ and $\eta_n > 0$, all possible sign combinations of X and Y are $X \leq Y \leq 0$, $X \leq 0 < Y$, and $0 < X < Y$. We call a game with each combination type +I, type +II, and type +III, respectively.

(ii) In a game with $\eta_n \leq 0 < \zeta_n$, it holds that $X > 0$. We call this game type +IV.

(iii) In a game with $\zeta_n \leq 0$ and $\eta_n < 0$, all possible sign combinations of X and Y are $X \leq Y \leq 0$, $X \leq 0 < Y$, and $0 < X < Y$. We call a game with each combination type −I, type −II, and type −III, respectively.

(iv) In a game with $\zeta_n < 0 \leq \eta_n$, it holds that $X > 0$. We call this game type −IV.

For each type, Table 1 summarizes the signs of $\partial W/\partial \tau_x$ and $\partial W/\partial \tau_y$; Figure 1 illustrates them. In each graph of Figure 1, the horizontal axis is the τ_y-axis (the precision of public information); the vertical axis is the τ_x-axis (the precision of private information).[9] Arrows indicate the direction in which $W(\tau)$ increases; Black lines are contour lines of $W(\tau)$.

Each type has the following properties.

+I (−I) Welfare increases (decreases) with the precision of both public and private information at any information structure. See Figure 1a (Figure 1e).

+II (−II) If $\tau_x < (\tau_y + \tau_\theta)/Y$, welfare increases (decreases) with the precision of both public and private information. If $\tau_x > (\tau_y + \tau_\theta)/Y$, welfare increases (decreases) with the precision of private information and decreases (increases) with that of public information. See Figure 1b (Figure 1f), where the dashed line is a graph of $\tau_x = (\tau_y + \tau_\theta)/Y$.

[9] This follows the choice of the axes in Figure 1 of MS.

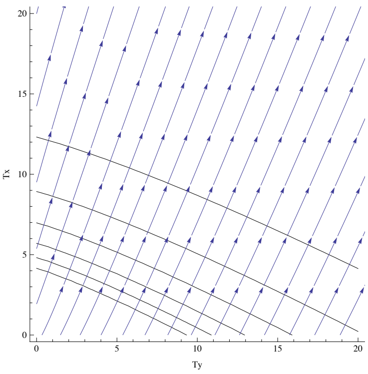
type	τ	$\partial W/\partial \tau_x$	$\partial W/\partial \tau_y$
+I	all τ	+	+
+II	$\tau_x < (\tau_y + \tau_\theta)/Y$	+	+
	$\tau_x > (\tau_y + \tau_\theta)/Y$	+	−
+III	$\tau_x < (\tau_y + \tau_\theta)/Y$	+	+
	$(\tau_y + \tau_\theta)/Y < \tau_x < (\tau_y + \tau_\theta)/X$	+	−
	$\tau_x > (\tau_y + \tau_\theta)/X$	−	−
+IV	$\tau_x < (\tau_y + \tau_\theta)/X$	+	−
	$\tau_x > (\tau_y + \tau_\theta)/X$	−	−
−I	all τ	−	−
−II	$\tau_x < (\tau_y + \tau_\theta)/Y$	−	−
	$\tau_x > (\tau_y + \tau_\theta)/Y$	−	+
−III	$\tau_x < (\tau_y + \tau_\theta)/Y$	−	−
	$(\tau_y + \tau_\theta)/Y < \tau_x < (\tau_y + \tau_\theta)/X$	−	+
	$\tau_x > (\tau_y + \tau_\theta)/X$	+	+
−IV	$\tau_x < (\tau_y + \tau_\theta)/X$	−	+
	$\tau_x > (\tau_y + \tau_\theta)/X$	+	+

Table 1: Eight types of games.

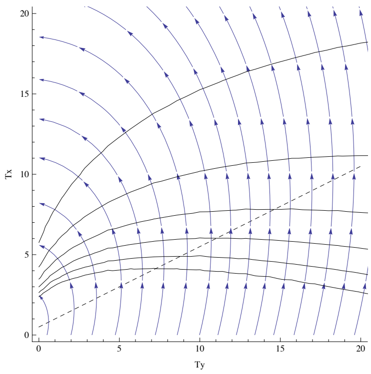
+III (−III) If $\tau_x < (\tau_y + \tau_\theta)/Y$, welfare increases (decreases) with the precision of both public and private information. If $(\tau_y + \tau_\theta)/Y < \tau_x < (\tau_y + \tau_\theta)/X$, welfare increases (decreases) with the precision of private information and decreases (increases) with that of public information. If $\tau_x > (\tau_y + \tau_\theta)/X$, welfare decreases (increases) with the precision of both public and private information. See Figure 1c (Figure 1g), where the lower and upper dashed lines are graphs of $\tau_x = (\tau_y + \tau_\theta)/Y$ and $\tau_x = (\tau_y + \tau_\theta)/X$, respectively.

+IV (−IV) If $\tau_x < (\tau_y + \tau_\theta)/X$, welfare increases (decreases) with the precision of private information and decreases (increases) with that of public information. If $\tau_x > (\tau_y + \tau_\theta)/X$, welfare decreases (increases) with the precision of both public and private information. See Figure 1d (Figure 1h), where the dashed line is a graph of $\tau_x = (\tau_y + \tau_\theta)/X$.

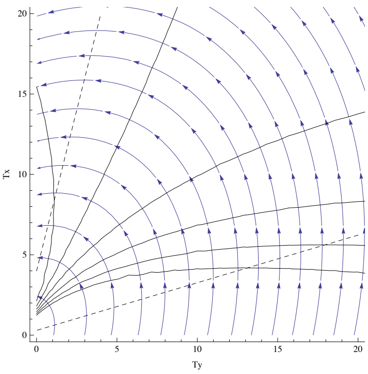
We can explain the properties of $\mathrm{cov}[\sigma_i, \sigma_j]$ and $\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i, \sigma_j]$ discussed in



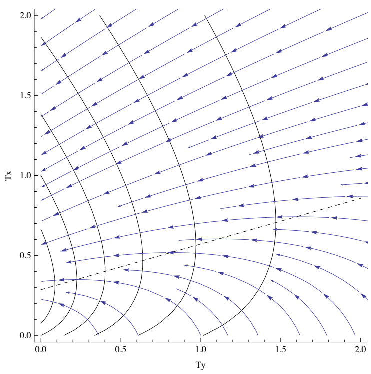
(a) +I



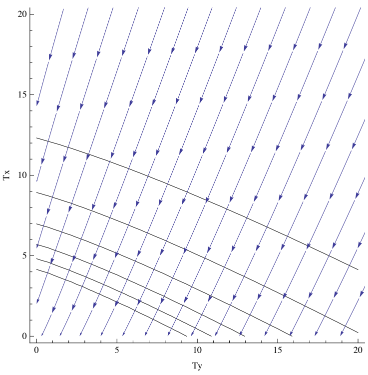
(b) +II



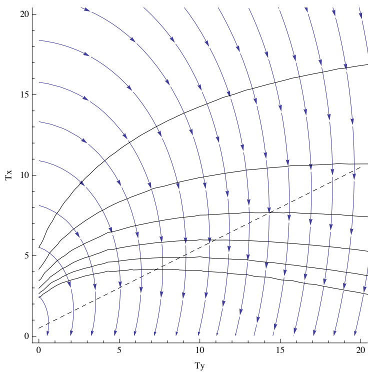
(c) +III



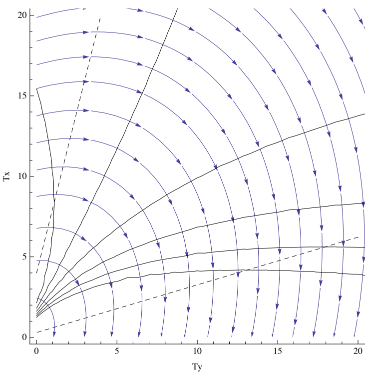
(d) +IV



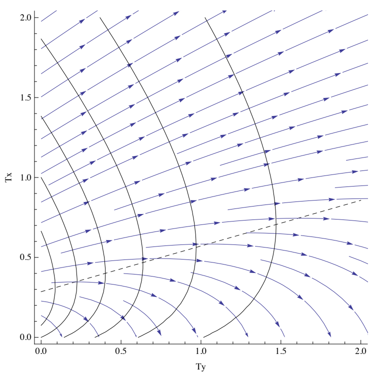
(e) −I



(f) −II



(g) −III



(h) −IV

Figure 1: Welfare and information structures in the (τ_y, τ_x)-plane.

Section 2 in terms of types +I and +IV. A game with $(\zeta_n, \eta_n) = (0, \beta)$ is type +I with $W(\tau) = \text{cov}[\sigma_i, \sigma_j]$, which increases with τ_x and τ_y. A game with $(\zeta_n, \eta_n) = (\beta, 0)$ is type +IV with $W(\tau) = \text{var}[\sigma_i] - \text{cov}[\sigma_i, \sigma_j]$, which decreases with τ_y for all τ and decreases with τ_x if and only if $\tau_x > (\tau_y + \tau_\theta)/X = (\tau_y + \tau_\theta)/(1 - \alpha_n)$. Note that if τ_x is sufficiently small or α_n is sufficiently close to one, then $\text{var}[\sigma_i] - \text{cov}[\sigma_i, \sigma_j]$ increases with τ_x.

To identify types +IV and −IV, it is sufficient to calculate ζ_n and η_n, but to identify the other types, we must also calculate X and Y. We provide a simpler characterization of these types by the value of $(1 - \alpha_n)\zeta_n/\eta_n$, which follows immediately from the definitions of X and Y.

Lemma 3. *A game with $\eta_n > 0$ is type +I if $0 \leq (1 - \alpha_n)\zeta_n/\eta_n \leq 3/2$, type +II if $3/2 < (1 - \alpha_n)\zeta_n/\eta_n \leq 2$, and type +III if $2 < (1 - \alpha_n)\zeta_n/\eta_n$. A game with $\eta_n < 0$ is type −I if $0 \leq (1 - \alpha_n)\zeta_n/\eta_n \leq 3/2$, type −II if $3/2 < (1 - \alpha_n)\zeta_n/\eta_n \leq 2$, and type −III if $2 < (1 - \alpha_n)\zeta_n/\eta_n$.*

To understand the role of $(1 - \alpha_n)\zeta_n/\eta_n$, assume that $\zeta_n \geq 0$ and $\eta_n > 0$, and consider the ratio of the two terms in $W(\tau) = \beta^{-1}\zeta_n(\text{var}[\sigma_i] - \text{cov}[\sigma_i, \sigma_j]) + \beta^{-1}\eta_n\text{cov}[\sigma_i, \sigma_j]$:

$$R = \frac{\zeta_n}{\eta_n} \cdot \frac{\text{var}[\sigma_i] - \text{cov}[\sigma_i, \sigma_j]}{\text{cov}[\sigma_i, \sigma_j]}.$$

If R is sufficiently small, then $\eta_n\text{cov}[\sigma_i, \sigma_j]$ dominates in $W(\tau)$, and thus $W(\tau)$ necessarily increases with τ_x and τ_y. If R is sufficiently large, then $\zeta_n(\text{var}[\sigma_i] - \text{cov}[\sigma_i, \sigma_j])$ dominates in $W(\tau)$, and thus $W(\tau)$ can decrease with τ_x and τ_y. Observe that $(\text{var}[\sigma_i] - \text{cov}[\sigma_i, \sigma_j])/\text{cov}[\sigma_i, \sigma_j]$ is decreasing in α_n because the correlation of actions $\text{cov}[\sigma_i, \sigma_j]/\text{var}[\sigma_i]$ is increasing in α_n; as α_n increases, players place a greater weight on a public signal,[10] and eventually the correlation of actions increases. This implies that R is decreasing in α_n and increasing in ζ_n/η_n. Therefore, if $(1 - \alpha_n)\zeta_n/\eta_n$ is sufficiently small, then R is so small that this game is type +I, and if $(1 - \alpha_n)\zeta_n/\eta_n$ is sufficiently large, then R is so large that this game is type +III. Type +II falls in between. If $\zeta_n > 0$ and $\eta_n \leq 0$, then $\zeta_n(\text{var}[\sigma_i] - \text{cov}[\sigma_i, \sigma_j])$ completely dominates in $W(\tau)$, and thus this game is type +IV. We can understand types −I, −II, −III, and −IV similarly by considering the case with $\zeta_n \leq 0$.

[10]Recall that the ratio of the coefficients of private and public signals in the equilibrium strategy is $(1 - \alpha_n)\tau_x/\tau_y$.

Remark 1. The above discussion suggests that a game with a high degree of strategic complementarity must be type +I. In fact, we have

$$\lim_{\alpha_n\to 1}(1-\alpha_n)\zeta_n/\eta_n = \lim_{\alpha_n\to 1}\frac{(1-\alpha_n)(\mu_n+\beta(\kappa_n+1))}{(1-\alpha_n)\mu_n+\beta(\kappa_n+\lambda_n+1)} = 0 \tag{9}$$

if $\lim_{\alpha_n\to 1}\eta_n = \beta(\kappa_n+\lambda_n+1)\neq 0$. Therefore, a game with a sufficiently high degree of strategic complementarity is type +I if $\zeta_n > 0$ and $\kappa_n+\lambda_n+1>0$ by Lemma 3. Similarly, we have

$$\lim_{\lambda_n\to\infty}(1-\alpha_n)\zeta_n/\eta_n = 0,\ \lim_{\mu_n\to\infty}(1-\alpha_n)\zeta_n/\eta_n = 1,$$

and thus a game with sufficiently large λ_n or μ_n is type +I if $\zeta_n,\eta_n>0$. The intuition is as follows. Because λ is a coefficient of $a_j a_k$, $\lambda_n\mathrm{cov}[\sigma_i,\sigma_j]$ plays a dominant role in $W(\tau)$ when λ_n is large. Because μ is a coefficient of θa_j, $\mu_n\mathrm{cov}[\theta,\sigma_j]$ plays a dominant role in $W(\tau)$ when μ_n is large. Both $\mathrm{cov}[\sigma_i,\sigma_j]$ and $\mathrm{cov}[\theta,\sigma_j]$ increase with τ_x and τ_y, so $W(\tau)$ increases with τ_x and τ_y when λ_n or μ_n is sufficiently large. In contrast, the effects of κ_n depend upon the other parameters. Because κ is a coefficient of a_j^2, it determines the coefficient of $\mathrm{var}[\sigma_i]$ in $W(\tau)$. Thus, as κ_n increases, the coefficients of both $\mathrm{var}[\sigma_i]-\mathrm{cov}[\sigma_i,\sigma_j]$ and $\mathrm{cov}[\sigma_i,\sigma_j]$ in $W(\tau)$ increase by the same amount, making the effects of κ_n depend upon the other parameters. In fact,

$$\lim_{\kappa_n\to\infty}(1-\alpha_n)\zeta_n/\eta_n = 1-\alpha_n,$$

which can take any positive number depending upon α_n.

Remark 2. A representative game of type +I is a team with an identical payoff function (3), where $\zeta_n = n\beta > 0$, $\eta_n = n(1-\alpha_n)\beta > 0$, and $(1-\alpha_n)\zeta_n/\eta_n = 1$. Adopting a team as a benchmark, we can read Lemma 3 to mean that more precise information can be detrimental to welfare if a game is far from a team in that $(1-\alpha_n)\zeta_n/\eta_n$ is sufficiently greater than one. To see this from a different perspective, consider a game with $\zeta_n>0$. Using (8), we can rewrite $W(\tau)$ as a linear combination of $\mathrm{cov}[\theta,\sigma_i]$ and $\mathrm{cov}[\sigma_i,\sigma_j]$:[11]

$$W(\tau) = (1-\alpha_n)\beta^{-1}\zeta_n\left(\frac{\beta}{1-\alpha_n}\mathrm{cov}[\theta,\sigma_i]+\left(\frac{\eta_n}{(1-\alpha_n)\zeta_n}-1\right)\mathrm{cov}[\sigma_i,\sigma_j]\right).$$

Let $D\equiv-(\eta_n/((1-\alpha_n)\zeta_n)-1)$ be the negative of the coefficient of $\mathrm{cov}[\sigma_i,\sigma_j]$. In a team, $D=0$, and thus $W(\tau)$ is a constant times $\mathrm{cov}[\theta,\sigma_j]$, which is increasing in τ_x and τ_y. Because $\mathrm{cov}[\sigma_i,\sigma_j]$ is also increasing in τ_x and τ_y, not only a team with $D=0$

[11] By (8), $\mathrm{var}[\sigma_i]=\beta\mathrm{cov}[\theta,\sigma_i]+\alpha_n\mathrm{cov}[\sigma_i,\sigma_j]$. By plugging this into (6), we obtain this formula.

but also any game with $D \leq 0$ must be type +I. However, if $D > 0$ is sufficiently large, this game is another type. In fact, Lemma 3 implies that a game is type +I if $D \leq 1/3$, type +II if $1/3 < D \leq 1/2$, type +III if $1/2 < D < 1$, and type +IV if $D \geq 1$. Thus, we can interpret D as a distance from a team regarding welfare effects of information.

3.2 Optimal information structure

As a corollary of Proposition 1, we obtain the socially optimal information structure that maximizes welfare in each type. Clearly, the most precise information is optimal in type +I, but it is not necessarily so in the other types.

Corollary 2. *In types +I, +II, and −IV,* $\sup_\tau W(\tau) = W(\tau_x, \infty) = W(\infty, \tau_y)$. *In types +III and +IV,* $\sup_\tau W(\tau) = W(\tau_\theta/X, 0)$. *In types −I, −II, and −III,* $\sup_\tau W(\tau) = W(0, 0)$.

Proof. See Appendix D. □

The highest precision is optimal not only in type +I but also in types +II and −IV, whereas the lowest precision is optimal in types −I, −II, and −III. In contrast, it is optimal to receive only noisy private signals in types +III and +IV. The optimal information structure $(\tau_\theta/X, 0)$ corresponds to the intercept of the dashed line $\tau_x = (\tau_y + \tau_\theta)/X$ in Figures 1c and 1d. In these types, the difference term $\zeta_n(\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i, \sigma_j])$ dominates in $W(\tau)$, which decreases with τ_y for all τ and decreases with τ_x if τ_x is sufficiently large. Therefore, we must have $\tau_y = 0$ and $\tau_x < \infty$ in the optimal information structure.

Next, we obtain the optimal precision of public information, fixing the precision of private information.

Corollary 3. *In types +I, +II, and −IV,* $\sup_{\tau_y} W(\tau) = W(\tau_x, \infty)$. *In type +III,*

$$\sup_{\tau_y} W(\tau) = \begin{cases} W(\tau_x, \infty) & \text{if } \tau_x < \eta_n\tau_\theta/((1-\alpha_n)X\zeta_n), \\ W(\tau_x, 0) & \text{if } \tau_x \geq \eta_n\tau_\theta/((1-\alpha_n)X\zeta_n). \end{cases}$$

In types +IV and −I, $\sup_{\tau_y} W(\tau) = W(\tau_x, 0)$. *In types −II and −III,*

$$\sup_{\tau_y} W(\tau) = \begin{cases} W(\tau_x, 0) & \text{if } \tau_x < \tau_\theta/Y, \\ W(\tau_x, Y\tau_x - \tau_\theta) & \text{if } \tau_x \geq \tau_\theta/Y. \end{cases}$$

Proof. See Appendix E. □

This result provides the optimal degree of transparency in public information. In types +I, +II, and −IV, the highest precision is optimal, while in types +IV and −I, the lowest precision is optimal. In the other types, the optimal precision depends upon the precision of private information. In type +III, the highest precision is optimal if private information has low precision, and the lowest precision is optimal if private information has high precision. Thus, no disclosure of public information is optimal if players have sufficiently precise private information. In types −II and −III, the lowest precision is optimal if private information has low precision, and the intermediate precision $Y\tau_x - \tau_\theta$ is optimal if private information has high precision, which corresponds to a point $(\tau_x, Y\tau_x - \tau_\theta)$ on the dashed line $\tau_x = (\tau_y + \tau_\theta)/Y$ in Figures 1f and 1g. Thus, a disclosure of appropriate noisy public information is optimal if players have sufficiently precise private information, where the optimal precision of public information is increasing in the precision of private information.

Finally, we obtain the optimal precision of private information, fixing the precision of public information.

Corollary 4. *In types +I, +II, and −IV,* $\sup_{\tau_x} W(\tau) = W(\infty, \tau_y)$*. In types +III and +IV,* $\sup_{\tau_x} W(\tau) = W((\tau_y + \tau_\theta)/X, \tau_y)$*. In types −I, −II, and −III,* $\sup_{\tau_x} W(\tau) = W(0, \tau_y)$*.*

Proof. See Appendix F. □

In types +I, +II, and −IV, the highest precision is optimal, while in type −I, −II, and −III, the lowest precision is optimal. In types +III and +IV, the optimal precision is $(\tau_y + \tau_\theta)/X$, which corresponds to a point $(\tau_y, (\tau_y + \tau_\theta)/X)$ on the dashed line $\tau_x = (\tau_y + \tau_\theta)/X$ in Figures 1c and 1d. To see what this implies, imagine that, for each $i \in N$, player i is allowed to choose the precision of his private information between the optimal precision and the highest precision (with no cost) as well as his strategy σ_i given the opponents' precision and strategies. This modified game has a unique equilibrium in which all players choose the highest precision, but they are better off by choosing the optimal precision. Thus, Corollary 4 could be useful in studying welfare implications of endogenous information acquisition.[12]

[12] For example, Hellwig and Veldkamp (2009) and Myatt and Wallace (2012) study endogenous information acquisition using quadratic Bayesian games with normally distributed signals.

4 Application

4.1 Cournot competition

Let $\kappa = \lambda = \mu = 0$ in (1). Cournot games with homogeneous or differentiated products conform to this formulation if a demand function is linear with a random demand intercept and a cost function is either linear or quadratic (cf. Raith, 1996). Because $\zeta_n = \eta_n = \beta > 0$ and $(1-\alpha_n)\zeta_n/\eta_n = 1-\alpha_n$, this game is type +I if $-1/2 \le \alpha_n < 1$, type +II if $-1 \le \alpha_n < -1/2$, and type +III if $\alpha_n < -1$ by Lemma 3. Thus, a game exhibiting strategic complementarity is type +I, whereas a game exhibiting a sufficiently high degree of strategic substitutability is type +III.

For example, consider a Cournot game with a homogeneous product and assume that demand and cost functions are linear. The inverse demand function is $\theta' - \rho\sum_i a_i$, where $\rho > 0$ and θ' is normally distributed, and the marginal cost is $c > 0$. Then, player i's profit is

$$(\theta' - \rho\sum_j a_j)a_i - ca_i = -\rho a_i^2 - \rho a_i \sum_{j\neq i} a_j + (\theta' - c)a_i.$$

Dividing this by ρ and setting $\theta = (\theta' - c)/\rho$, we have (1) with $\alpha = -1/2$, $\beta = 1/2$, and $\kappa = \lambda = \mu = \nu = f(\theta) = 0$. Because $\alpha_n = -(n-1)/2$, this game is type +I if $n = 2$, type +II if $n = 3$, and type +III if $n \ge 4$. While expected producer surplus can decrease with the precision of both public and private information for all $n \ge 4$, expected total surplus necessarily increases with the precision of both public and private information for all $n \ge 2$, as is also shown by Proposition 1.[13] The expected producer surplus is $n\rho\mathrm{var}[\sigma_i]$ plus a constant by (6); the expected consumer surplus is

$$\begin{aligned} E\left[\frac{\rho}{2}\left(\sum_{i\in N}\sigma_i\right)^2\right] &= \frac{\rho}{2}\left(nE[\sigma_i^2] + n(n-1)E[\sigma_i\sigma_j]\right) \\ &= \frac{n\rho}{2}\mathrm{var}[\sigma_i] + \frac{n(n-1)\rho}{2}\mathrm{cov}[\sigma_i,\sigma_j] + k_1, \end{aligned}$$

where k_1 is a constant; the expected total surplus is

$$ETS \equiv \frac{3n\rho}{2}(\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i,\sigma_j]) + \frac{n(n+2)\rho}{2}\mathrm{cov}[\sigma_i,\sigma_j] + k_2,$$

where k_2 is a constant. Note that $ETS - k_2$ is a linear combination of $\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i,\sigma_j]$ and $\mathrm{cov}[\sigma_i,\sigma_j]$. Thus, we can apply Proposition 1 to study the effects of information on ETS by considering a game with $(\alpha, \beta, \zeta_n, \eta_n) = (-1/2, 1/2, 3n\rho/2, n(n+$

[13] Vives (1984) show this for the case of $n = 2$.

$2)\rho/2)$ and $W(\tau) = (ETS - k_2)/\beta$. Such a game exists because, for arbitrary $(\alpha, \beta, \zeta_n, \eta_n)$, there exist κ, λ, and μ such that (7) holds.[14] Because $(1-\alpha_n)\zeta_n/\eta_n = 3/2 \times (n+1)/(n+2) < 3/2$, this game is type +I by Lemma 3, which implies that the expected total surplus necessarily increases with the precision of both public and private information.

4.2 Bertrand competition

If demand and cost functions are linear and a demand intercept is random, Bertrand games with differentiated products conform to the formulation in Section 4.1 (cf. Raith, 1996). Thus, we consider another formulation with a random marginal cost. Player i produces good i and chooses its price a_i. The demand function is $1 - a_i + \rho\sum_{j\neq i} a_j$ and the marginal cost is a normally distributed random variable θ'.[15] Then, player i's profit is

$$(1 - a_i + \rho\sum_{j\neq i} a_j)(a_i - \theta') = -a_i^2 + \rho a_i \sum_{j\neq i} a_j + (\theta'+1)a_i - \rho\sum_{j\neq i}\theta' a_j - \theta',$$

which is (1) with $\theta = \theta' + 1$, $\alpha = \rho/2$, $\beta = 1/2$, $\kappa = \lambda = 0$, $\mu = -\rho$, $\nu = \rho$, and $f(\theta) = -\theta + 1$. We write $\rho_n = (n-1)\rho$ and assume $\rho_n = 2\alpha_n < 2$ to guarantee the uniqueness of equilibrium. Then, we have $\zeta_n = (1-2\rho_n)/2$ and $\eta_n = (\rho_n - 1)^2/2$. Therefore, if $1/2 < \rho_n < 2$, then $\zeta_n < 0$ and $\eta_n \geq 0$, and thus this game is type −IV. If $\rho_n \leq 1/2$, then $\zeta_n \geq 0$, $\eta_n > 0$, and

$$(1-\alpha_n)\zeta_n - 3\eta_n/2 = -(\rho_n^2 - \rho_n + 1)/4 = -((\rho_n - 1/2)^2 + 3/4)/4 < 0.$$

By Lemma 3, this game is type +I because $(1-\alpha_n)\zeta_n/\eta_n < 3/2$.

4.3 Voluntary provision of public goods

Consider a public goods game with a quadratic production function. Player i's contribution level is a_i, and its marginal cost is a constant $c > 0$. Each player receives a common benefit $-(\sum_j a_j)^2 + \theta'\sum_j a_j$, where θ' is normally distributed. Then, player i's payoff is

$$\left(-\Big(\sum_j a_j\Big)^2 + \theta'\sum_j a_j\right) - ca_i,$$

[14] This implies that Proposition 1 is useful in studying any measure of welfare that is a linear combination of the variance and covariance of actions.

[15] If all players use the same input, the input price is common to all players.

which is (1) with $\theta = \theta' - c$, $\alpha = -1$, $\beta = 1/2$, $\kappa = -1$, $\lambda = -2$, $\mu = 1$, and $\nu = c$. We have $\zeta_n = n/2 > 0$, $\eta_n = n^2/2 > 0$, and $(1 - \alpha_n)\zeta_n/\eta_n = 1$. Thus, this game is type +I by Lemma 3.

Consider another formulation of a public goods game. A marginal cost is a normally distributed random variable θ' and a common benefit is $-(\sum_j a_j)^2 + c\sum_j a_j$, where $c > 0$ is a constant. Then, player i's payoff is

$$\left(-\Big(\sum_j a_j\Big)^2 + c\sum_j a_j\right) - \theta' a_i,$$

which is (1) with $\theta = c - \theta'$, $\alpha = -1$, $\beta = 1/2$, $\kappa = -1$, $\lambda = -2$, $\mu = 0$, and $\nu = c$. We have $\zeta_n = -(n-2)/2 \leq 0$, $\eta_n = -n(n-2)/2 \leq 0$, and $(1 - \alpha_n)\zeta_n/\eta_n = 1$ if $n \geq 3$. Thus, this game is type −I if $n \geq 3$ by Lemma 3. If $n = 2$, information has no influence on welfare because $\zeta_n = \eta_n = 0$ and $W(\tau) = 0$ for all τ.

To summarize, welfare necessarily increases with the precision of both public and private information if production is random, but necessarily decreases if cost is random. Teoh (1997) studies the social value of public information in a public goods game with binary states and random production, and shows that welfare is greater without information than with it. Thus, the welfare effects of information in public goods games crucially depend upon game formulation.

5 A continuum of players

This section considers AP's model with a continuum of players and discusses how to apply Proposition 1 and its consequences to the continuum model.

5.1 Model

Let $[0, 1]$ be a set of players with an individual player indexed by $i \in [0, 1]$. Player i chooses an action $a_i \in \mathbb{R}$ and an action profile is denoted by $a = (a_i)_{i\in[0,1]}$. Player i's payoff function is

$$\begin{aligned}\bar{u}_i(a, \theta) = &- a_i^2 + 2\alpha a_i \int_0^1 a_j dj + 2\beta\theta a_i \\ &+ \kappa \int_0^1 a_j^2 dj + \lambda \left(\int_0^1 a_j dj\right)^2 + \mu\theta \int_0^1 a_j dj + \nu \int_0^1 a_j dj + f(\theta). \qquad (10)\end{aligned}$$

Player i observes the same bivariate signal vector as that in Section 2. A strategy profile is a Bayesian Nash equilibrium if each player maximizes his interim expected payoff given the opponents' strategies. Angeletos and Pavan (2007, 2009) show that a unique equilibrium exists if $\alpha < 1$ and obtain it. The equilibrium strategy coincides with that in Lemma 1, where α_n is replaced with α.[16] We assume $\alpha < 1$.

The ex ante expected payoff in the equilibrium is

$$\begin{aligned}
&-E[{\sigma_i}^2] + 2\alpha E[\sigma_i\sigma_j] + 2\beta E[\theta\sigma_i] \\
&\qquad + \kappa E\Big[\int E[\sigma_j^2|\theta]dj\Big] + \lambda E\Big[\Big(\int E[\sigma_j|\theta]dj\Big)^2\Big] + \mu E\Big[\theta\int E[\sigma_j|\theta]dj\Big] \\
&\qquad + \nu E\Big[\int E[\sigma_j|\theta]dj\Big] + E[f(\theta)] \\
&= \overline{W}(\tau) + (2\alpha + \kappa + \lambda - 1)c^2 + ((2\beta + \mu)\bar{\theta} + \nu)c + E[f(\theta)],
\end{aligned} \tag{11}$$

where $\overline{W}(\tau) \equiv (\kappa - 1)\mathrm{var}[\sigma_i] + (2\alpha + \lambda)\mathrm{cov}[\sigma_i, \sigma_j] + (2\beta + \mu)\mathrm{cov}[\theta, \sigma_i]$, because the symmetry ($\sigma_i(\cdot) = \sigma_j(\cdot)$ for $i \neq j$) implies that

$$E\Big[\int E[\sigma_j^2|\theta]dj\Big] = E\Big[E[\sigma_j^2|\theta]\Big] = E[\sigma_j^2],$$

$$E\Big[\Big(\int E[\sigma_j|\theta]dj\Big)^2\Big] = E\Big[\int E[\sigma_j|\theta]dj \int E[\sigma_k|\theta]dk\Big] = E\Big[E[\sigma_j|\theta]E[\sigma_k|\theta]\Big] = E[\sigma_j\sigma_k],$$

$$E\Big[\theta\int E[\sigma_j|\theta]dj\Big] = E\Big[\theta E[\sigma_j|\theta]\Big] = E[\theta\sigma_j],$$

$$E\Big[\int E[\sigma_j|\theta]dj\Big] = E\Big[E[\sigma_j|\theta]\Big] = E[\sigma_j].$$

Thus, the total expected payoff equals $\overline{W}(\tau)$ plus a constant. By the same argument as that in Lemma 2, $\overline{W}(\tau)$ is rewritten as

$$\overline{W}(\tau) = \left(\zeta(\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i, \sigma_j]) + \eta\mathrm{cov}[\sigma_i, \sigma_j]\right)/\beta, \tag{12}$$

where $\zeta = \mu + \beta(\kappa + 1)$ and $\eta = (1 - \alpha)\mu + \beta(\kappa + \lambda + 1)$. Therefore, we can apply Proposition 1 and its consequences to the continuum model by replacing $(\alpha_n, \zeta_n, \eta_n)$ with $(\alpha, \zeta, \eta) \equiv (\alpha, \mu + \beta(\kappa + 1), (1 - \alpha)\mu + \beta(\kappa + \lambda + 1))$.

In the continuum model, there is another interpretation of $\overline{W}(\tau)$. Bergemann and Morris (2012a) consider the variance of the average action $\int a_j dj$ and that of the idiosyncratic difference $a_i - \int a_j dj$ in the continuum model and refer to them as volatility and dispersion, respectively. They show that the dispersion equals $\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i, \sigma_j]$ and the volatility equals $\mathrm{cov}[\sigma_i, \sigma_j]$. Thus, $\overline{W}(\tau)$ is a linear combination of

[16] Angeletos and Pavan (2007, 2009) do not use the theorem of Radner (1962), but the theorem implies not only Lemma 1 but also their result. See Ui and Yoshizawa (2013).

the dispersion and the volatility by (12). Note that, in the finite case, $W(\tau)$ is not a linear combination of dispersion and volatility in the corresponding sense.[17]

AP are the first to use the terms "volatility" and "dispersion" in this context. Dispersion is the same, but volatility is different. AP refer to the variance of $\int a_j dj - x(\theta)$ as volatility, where $x(\theta)$ is an action in the equilibrium under complete information, and write welfare as a linear combination of the dispersion, the volatility, and the other term, the last of which plays an important role in their analysis. In contrast, we write welfare as a linear combination of the dispersion and the volatility in the sense of Bergemann and Morris (2012a). This is a major methodological difference in the continuum case between AP and this paper.

In the remainder of this section, we discuss several applications of our results in the continuum case.

5.2 Beauty contest

Let $\alpha = r \in (0,1)$ and $\beta = 1-r$ in (10). By the first order condition for equilibrium, a player's best response is a weighted mean of the conditional expectation of the state and that of the opponents' actions, i.e., $rE[\sigma_j|\mathbf{s}_i] + (1-r)E[\theta|\mathbf{s}_i]$. Because it induces strategic behavior in the spirit of a Keynesian beauty contest, this game is referred to as a beauty contest game.

MS study a beauty contest game with a payoff function

$$-(1-r)(a_i-\theta)^2 - r\Big(\int (a_j-a_i)^2 dj - \int (a_j-a_k)^2 dj dk\Big),$$

which is (10) with $\alpha = r \in (0,1)$, $\beta = 1-r$, $\kappa = r$, $\lambda = -2r$, $\mu = \nu = 0$, and $f(\theta) = -(1-r)\theta^2$. The expected payoff is $-(1-r)E[(\sigma_i-\theta)^2]$, a constant times the mean squared error of σ_i from θ. Because $\zeta = 1-r^2 > 0$, $\eta = (1-r)^2 > 0$, and $(1-\alpha)\zeta/\eta = 1+r$, this game is type +I if $r \le 1/2$ and type +II if $r > 1/2$ by Lemma 3, which is MS's result.

On the face of it, MS's result seems inconsistent with (9), which suggests that a game with sufficiently large $\alpha = r$ is type +I. As r increases, however, not only α but also $\zeta/\eta = (1+r)/(1-r)$ increases because $\kappa = r$ and $\lambda = -2r$. This clearly contrasts with (9), where κ_n and λ_n are independent of α_n. Because κ and λ are coefficients of terms not including a_i, the beauty contest game with $r > 1/2$ is type

[17] The variance of the average action $\sum_i a_i/n$ is $\mathrm{var}[\sigma_i]/n + (n-1)\mathrm{cov}[\sigma_i,\sigma_j]/n$, which converges to $\mathrm{cov}[\sigma_i,\sigma_j]$ as $n \to \infty$.

+II not because of a high degree of strategic complementarity but because of strong externalities.[18]

Morris and Shin (2002b) consider the following payoff function:

$$-r_1 \int (a_j - a_i)^2 dj - r_2(a_i - \theta)^2 - r_3\Big(a_i - \int a_j dj\Big)^2 \\ + r_4 \int\int (a_j - a_k)^2 dj dk - r_5\Big(\int a_j dj - \theta\Big)^2, \tag{13}$$

where $r_1, r_2, r_3, r_4, r_5 \geq 0$, $r_1 + r_3 > 0$, and $r_2 > 0$. MS's beauty contest game is a special case with $r_1 = r_4 = r$, $r_2 = 1 - r$, and $r_3 = r_5 = 0$. This specification allows differing weights to the losses arising from the distances between a_i, θ, and the average actions.

Dividing (13) by $r_1 + r_2 + r_3$, we have (10) with $\alpha = r = (r_1 + r_3)/(r_1 + r_2 + r_3)$, $\beta = 1 - r$, $\kappa = (-r_1 + 2r_4)/(r_1 + r_2 + r_3)$, $\lambda = (-r_3 - 2r_4 - r_5)/(r_1 + r_2 + r_3)$, $\mu = 2r_5/(r_1 + r_2 + r_3)$, $\nu = 0$, and $f(\theta) = -(r_2 + r_5)\theta^2/(r_1 + r_2 + r_3)$. Thus,

$$\zeta = (r_2^2 + (r_3 + 2r_4 + 2r_5)r_2 + (2r_1 + 2r_3)r_5)/(r_1 + r_2 + r_3)^2 > 0,$$
$$\eta = r_2(r_2 + r_5)/(r_1 + r_2 + r_3)^2 > 0,$$
$$(1 - \alpha)\zeta - 3\eta/2 = r_2((r_1 + r_2 + r_3)r_5 + 4r_2r_4 - r_2(3r_1 + r_2 + r_3))/(2(r_1 + r_2 + r_3)^3)),$$
$$(1 - \alpha)\zeta - 2\eta = r_2^2(2r_4 - 2r_1 - r_2 - r_3)/(r_1 + r_2 + r_3)^3.$$

Therefore, by Lemma 3, this game is type +I if $(r_1+r_2+r_3)r_5+4r_2r_4 \leq r_2(3r_1+r_2+r_3)$, type +II if $(r_1 + r_2 + r_3)r_5 + 4r_2r_4 > r_2(3r_1 + r_2 + r_3)$ and $r_4 \leq (2r_1 + r_2 + r_3)/2$, and type +III if $r_4 > (2r_1 + r_2 + r_3)/2$.

Focusing on public information, Morris and Shin (2002b) show that welfare can decrease with the precision of public information if and only if $(r_1+r_2+r_3)r_5+4r_2r_4 > r_2(3r_1 + r_2 + r_3)$. The result above considers both public and private information and shows that welfare can decrease with the precision of both public and private information if and only if $r_4 > (2r_1 + r_2 + r_3)/2$.

5.3 Cournot competition

Let $\kappa = \lambda = \mu = 0$ in (10). Because $\zeta = \eta = \beta > 0$ and $(1 - \alpha)\zeta/\eta = 1 - \alpha$, this game is type +I if $-1/2 \leq \alpha < 1$, type +II if $-1 \leq \alpha < -1/2$, and type +III if $\alpha < -1$ by Lemma 3, which corresponds to the result in Section 4.1. For example,

[18] Strong externalities induce players' overreaction to public information relative to the efficiency benchmark, which results in the detrimental effects of public information, as discussed by MS.

consider a large Cournot game with a homogeneous product studied by AP. The inverse demand function is $\theta - \rho \int a_j dj$, where $\rho > 0$ and θ is normally distributed. The cost function is ca_i^2 with $c > 0$. Then, player i's profit is

$$\Big(\theta - \rho \int a_j dj\Big) a_i - c a_i^2 = -c a_i^2 - \rho a_i \int a_j dj + \theta a_i.$$

Dividing this by c, we have (10) with $\alpha = -\rho/c$, $\beta = 1/c$, and $\kappa = \lambda = \mu = 0$. Thus, this game is type +I if $c \geq 2\rho$, type +II if $\rho \leq c < 2\rho$, and type +III if $c < \rho$. Note that if the slope of the demand curve is sufficiently large or the quadratic cost is sufficiently small, then this game is type +III.

Bergemann and Morris (2012a) also study this game and show the following: if $\alpha \geq -1$, then the optimal information structure is complete information, and if $\alpha < -1$, then it is $(-\tau_\theta/(\alpha+1), 0)$. This is a special case of Corollary 2 because this game is type either +I or +II if $\alpha \geq -1$ and type +III if $\alpha < -1$. Bergemann and Morris (2012a) obtain these optimal information structures from optimal Bayesian correlated equilibria. See Section 5.6.

AP's Corollary 10 states that expected total profits necessarily increase with the precision of private information, but can decrease with that of public information. This implies that a large Cournot game cannot be type +III, which is inconsistent with the above result. See Appendix G for an error in AP.

5.4 Bertrand competition

Consider a large Bertrand game with differentiated products studied by AP. Player i produces good i and chooses its price a_i. The demand function is $\theta - a_i + \rho \int a_j dj$, where $\rho > 0$ and θ is normally distributed. The cost function is cq^2 with $c > 0$. Then, player i's profit is

$$\Big(\theta - a_i + \rho \int a_j dj\Big) a_i - c\Big(\theta - a_i + \rho \int a_j dj\Big)^2 =$$
$$-(c+1)a_i^2 + \rho(2c+1)a_i \int a_j dj + (2c+1)\theta a_i - \rho^2 c \Big(\int a_j dj\Big)^2 - 2\rho c\theta \int a_j dj - c\theta^2.$$

Dividing this by $c+1$, we have (10) with $\alpha = \rho(2c+1)/(2(c+1))$, $\beta = (2c+1)/(2(c+1))$, $\kappa = 0$, $\lambda = -\rho^2 c/(c+1)$, $\mu = -2\rho c/(c+1)$, $\nu = 0$, and $f(\theta) = -c\theta^2$. Assume $\alpha < 1$ to guarantee the uniqueness of equilibrium, i.e., $\rho = 2(c+1)\alpha/(2c+1) < 2(c+1)/(2c+1)$. Then, we have $\zeta = (2(1-2\rho)c+1)/(2(c+1))$ and $\eta = (c(2c+1)\rho^2 - 4c(c+1)\rho + 2c^2 + 3c + 1)/(2(c+1)^2)$.

Note that $\zeta < 0$ if and only if $\rho > (2c+1)/(4c)$. Thus, if $\zeta < 0$, then we must have $(2c+1)/(4c) < 2(c+1)/(2c+1)$ because $\alpha < 1$, which is rewritten as $c > (-1+\sqrt{2})/2$. Note also that $\eta > 0$ because the discriminant of the numerator of η as a quadratic function of ρ is $-4c^2 - 4c < 0$.

Therefore, this game is type $-$IV if $c > (-1 + \sqrt{2})/2$ and $(2c + 1)/(4c) < \rho < 2(c + 1)/(2c + 1)$. For example, if we set $\rho = 4/5$ and $c = 1$, then $\zeta = -1/20 < 0$, $\eta = 19/100 > 0$, and $X = 8$. Thus, expected total profits decrease with the precision of private information if and only if $\tau_x < (\tau_y + \tau_\theta)/8$.

Otherwise, this game is type +I. To see this, suppose that $\rho \leq \min\{(2c + 1)/(4c), 2(c + 1)/(2c + 1)\}$. Then, $\zeta, \eta > 0$ and

$$(1 - \alpha)\zeta - 3\eta/2 = ((2c^2 + c)\rho^2 - \rho - 2c^2 - 3c - 1)/(4(c + 1)^2).$$

The numerator of the fraction above is strictly negative for all $\rho \in (0, 2(c+1)/(2c+1))$ because it is so at the endpoints of the interval, which implies that $(1-\alpha)\zeta/\eta < 3/2$. Therefore, this game is type +I by Lemma 3.

AP's Corollary 11 states that expected total profits necessarily increase with the precision of both public and private information. This implies that a large Bertrand game is type +I for all $\rho, c > 0$, which is inconsistent with the above result. See Appendix H for an error in AP.

5.5 Games that are efficient under complete information

Consider a class of games in which the equilibrium is inefficient under incomplete information but efficient under complete information. AP study this class of games and state that welfare can decrease with either public or private information, but not with both. We show that this is true if the socially optimal strategy profiles exist, which is a crucial assumption in AP, but without this assumption, there exists a game in which welfare can decrease with both public and private information.

To identify this class of games, assume that players directly observe θ. The equilibrium strategy is to choose $\beta\theta/(1 - \alpha)$. When each player chooses $x \in \mathbb{R}$, the payoff is

$$-x^2+2\alpha x^2+2\beta\theta x+\kappa x^2+\lambda x^2+\mu\theta x+\nu x+f(\theta) = -(1-2\alpha-\kappa-\lambda)x^2+(2\beta+\mu)\theta x+\nu x+f(\theta),$$

which is maximized at $x^*(\theta) = (2\beta+\mu)\theta/(2(1-2\alpha-\kappa-\lambda))$ if $1-2\alpha-\kappa-\lambda > 0$. Thus, a unique efficient strategy is to choose $x^*(\theta)$ if $1-2\alpha-\kappa-\lambda > 0$, and the equilibrium is efficient under complete information if and only if $x^*(\theta) = \beta\theta/(1-\alpha)$ for all $\theta \in \mathbb{R}$, i.e., $\mu = -2\beta(\alpha+\kappa+\lambda)/(1-\alpha)$ and $1-2\alpha-\kappa-\lambda > 0$. Plugging this into ζ and η, we have $\zeta = \beta(1-3\alpha-\alpha\kappa-\kappa-2\lambda)/(1-\alpha)$ and $\eta = \beta(1-2\alpha-\kappa-\lambda) > 0$. Because $\eta > 0$, possible types are +I, +II, +III, and −IV. Only in type +III can welfare decrease with both public and private information. Because $(1-\alpha)\zeta - 2\eta = (1-\alpha)\beta(\kappa-1)$, $(1-\alpha)\zeta/\eta > 2$ if and only if $\kappa > 1$. Thus, this game is type +III if and only if $\kappa > 1$ by Lemma 3.

If $\kappa > 1$, then the expected payoff is unbounded above because it equals $(\kappa - 1)\mathrm{var}[\sigma_i] + (2\alpha+\lambda)\mathrm{cov}[\sigma_i, \sigma_j] + (2\beta+\mu)\mathrm{cov}[\theta, \sigma_i]$ plus a constant by (11). Therefore, welfare can decrease with both public and private information in this class of games only if there is no socially optimal strategy profile maximizing the expected payoff. AP require the existence, whereas we do not, which enables us to study the social value of information in a broader class of games than AP's class.

5.6 Optimal Bayesian correlated equilibrium

Consider a mediator who knows the true state and makes private, perhaps correlated, action recommendations to players who have no information about the state. If each player has an incentive to follow the mediator's recommendation, we say that the resulting action distribution is a Bayesian correlated equilibrium. We are interested in the optimal Bayesian correlated equilibrium that achieves the highest welfare.

Bergemann and Morris (2012a) study a game with (10) and characterize the set of all Bayesian correlated equilibria with normally distributed action recommendations. It is known that the set of all Bayesian correlated equilibria coincides with the set of all action distributions of Bayesian Nash equilibria generated by all possible signal structures. Clearly, action distributions of Bayesian Nash equilibria generated by the bivariate (public and private) signal structures are Bayesian correlated equilibria. Bergemann and Morris (2012a) show that the converse is also true in games with a continuum of players; that is, the set of all Bayesian correlated equilibria coincides with the set of all action distributions of Bayesian Nash equilibria generated by the bivariate signal structures. Thus, the bivariate signal structures are rich enough to generate every Bayesian correlated equilibrium.

The finding of Bergemann and Morris (2012a) implies that the action distribution of the Bayesian Nash equilibrium under the optimal information structure in Corollary 2 is the optimal Bayesian correlated equilibrium. Therefore, in the optimal

Bayesian correlated equilibrium of types +I, +II, and −IV, actions are completely correlated with θ; in that of types −I, −II, and −III, actions are constant; in that of types +III and +IV, actions are conditionally independent given θ. To summarize, we obtain the following characterization of the optimal Bayesian correlated equilibrium.

Corollary 5. *Consider a Bayesian correlated equilibrium that achieves the highest welfare. If $\zeta \leq 2\eta/(1-\alpha)$ and $\eta \geq 0$, then the recommended action for all players is $\beta\theta/(1-\alpha)$. If $\zeta \leq 0$ and $\eta < 0$, then the recommended action for all players is $\beta\bar{\theta}/(1-\alpha)$. If $\zeta > \max\{0, 2\eta/(1-\alpha)\}$, then the recommended action for player i is $\beta(\theta + \varepsilon_i - \bar{\theta})/(1-\alpha+X) + \beta\bar{\theta}/(1-\alpha)$, where ε_i is an i.i.d. normally distributed random variable with mean zero and variance τ_θ/X.*

Proof. Plugging the optimal information structure in Corollary 2 into the equilibrium strategy (4), we obtain the action in the equilibrium, which is the recommended action in the optimal Bayesian correlated equilibrium. □

This corollary complements the result of Bergemann and Morris (2012a), who go in the opposite direction. Studying large Cournot games with $\kappa = \lambda = \mu = 0$, they first obtain optimal Bayesian correlated equilibria, which correspond to the first case in Corollary 5 if $\alpha \geq -1$ and the third case if $\alpha < -1$; they then obtain the corresponding optimal information structures mentioned in Section 5.3 using the equivalence of Bayesian correlated and Nash equilibria.[19] Because $\overline{W}(\tau) = \mathrm{var}[\sigma_i]$ when $\kappa = \lambda = \mu = 0$ by (12), large Cournot games are a special class of games and only two types of optimal Bayesian correlated equilibria arise. Corollary 5 shows that there are three types of optimal Bayesian correlated equilibria in total and provides a necessary and sufficient condition for each of them.

The difference between Bergemann and Morris (2012a) and this paper is that the focus of the former is equilibria determined by (α, β),[20] whereas that of the latter is welfare determined by (ζ, η) given (correlated or Nash) equilibria. Combining the results of both papers, we can characterize all the optimal Bayesian correlated equilibria in terms of $(\alpha, \beta, \zeta, \eta)$ as in Corollary 5. In games with a finite number of players, however, the equivalence of Bayesian correlated and Nash equilibria does not hold, and the optimal Bayesian correlated equilibria in the finite case do not correspond to the optimal information structures in Corollary 2. Even in the finite case, by combining the results of both papers, we can characterize all the optimal

[19] They also discuss its implication for the optimal sharing of information among firms.

[20] Their welfare analysis is also determined by (α, β) because they assume $\kappa = \lambda = \mu = 0$.

Bayesian correlated equilibria in terms of $(\alpha_n, \beta, \zeta_n, \eta_n)$. In Appendix I, we provide a counterpart of Corollary 5 for the finite case. This result complements that of Bergemann and Morris (2012a,b), who obtain the optimal Bayesian correlated equilibria in the finite case with $\kappa = \lambda = \mu = 0$.

6 Concluding remarks

The analysis of quadratic Bayesian games with normally distributed signals originates in the work of Radner (1962). The symmetric models studied by AP are especially useful and have a number of applications. This paper classifies the symmetric models into eight types, in each of which increased precision of public or private information has different effects on welfare. This paper also characterizes the optimal precision of information and the optimal Bayesian correlated equilibrium in each type.

Methodologically, this paper differs from the previous literature in the following way. AP, MS, and Hellwig (2005) adopt the socially optimal strategy profile as an efficiency benchmark, which is a useful instrument for understanding the welfare effects of information in the equilibrium. Especially in AP, the efficiency benchmark plays a crucial role, its use being the core methodological contribution. AP's method is effective not only in quadratic Bayesian games with normally distributed signals but also in all Bayesian games. When we focus on quadratic Bayesian games, however, AP's method has the limitation that a full characterization of the social value of information is not straightforward.

In contrast, this paper exploits the property of quadratic Bayesian games that the expected payoff is affine in the second moments of actions. That is, we write welfare as a linear combination of the covariance and the difference between the variance and covariance of actions. In the continuum case, they are equal to the volatility and the dispersion in the sense of Bergemann and Morris (2012a), respectively. The former increases with both public and private information, whereas the latter can decrease, so their relative weights together with the cross-derivative of the payoff function determine the types of games. This representation of welfare is also useful in studying Bayesian correlated equilibria. In addition, insofar as payoff functions are quadratic, expected payoffs are affine in the second moments of actions even if signals are not normally distributed, and thus a similar analysis is possible.

Bayesian potential games have unique equilibria if potential functions are strictly concave (Radner, 1962; Ui, 2009), including quadratic Bayesian games in this paper.

The social value of information in these games is worth studying, where our classification of games serves as a benchmark. As discussed in Remark 2 to Lemma 3, the classification depends upon how far a game is from a team with the same equilibrium. Thus, when we study Bayesian potential games with strictly concave potential functions, a similar comparison to such a team could be useful.

In recent years, a growing number of researchers have studied endogenous information structures in quadratic Bayesian potential games with normally distributed signals. Examples include Colombo and Femminis (2008), Dewan and Myatt (2008), Hellwig and Veldkamp (2009), Hagenbach and Koessler (2010), and Myatt and Wallace (2012), in which underlying games are type +I or +II. However, private collection of information and aggregation of information may have different effects on the expected payoff depending upon the types. Thus, comparing endogenous information structures and their welfare properties in different types would be an interesting topic for future research.

Appendix

A Proof of Lemma 1

The first order condition (2) has a unique solution if $-(n-1) < \alpha_n < 1$ by Theorem 5 of Radner (1962). Let $(\sigma_i)_{i \in N}$ be the unique solution. Because the joint probability distribution of $(\mathbf{s}_1, \ldots, \mathbf{s}_n)$ is symmetric, for any permutation $\pi : N \to N$, (2) is equivalent to

$$-\sigma_i(\mathbf{s}_{\pi(i)}) + \alpha \sum_{j \neq i} E[\sigma_j(\mathbf{s}_{\pi(j)}) | \mathbf{s}_{\pi(i)}] + \beta E[\theta | \mathbf{s}_{\pi(i)}] = 0,$$

which implies that a strategy profile $(\sigma'_i)_{i \in N}$ with $\sigma'_{\pi(i)} = \sigma_i$ is also a unique solution of (2). Hence, we must have $\sigma_i(\cdot) = \sigma_j(\cdot)$ for all i, j. Then, (2) is reduced to

$$-\sigma_i(\mathbf{s}_i) + \alpha \sum_{j \neq i} E[\sigma_i(\mathbf{s}_j) | \mathbf{s}_i] + \beta E[\theta | \mathbf{s}_i] = 0.$$

Because $E[\sigma_i(\mathbf{s}_j)|\mathbf{s}_i] = E[\sigma_i(\mathbf{s}_k)|\mathbf{s}_i]$ for all $j, k \neq i$, this is rewritten as

$$-\sigma_i(\mathbf{s}_i) + \gamma E[\sigma_i(\mathbf{s}_j) | \mathbf{s}_i] + \beta E[\theta | \mathbf{s}_i] = 0, \tag{A1}$$

where $\gamma = (n-1)\alpha$. Equation (A1) has a unique solution if $-(n-1) < \gamma < 1$ for all n, i.e., $\gamma < 1$. Therefore, $\gamma = \alpha_n < 1$ guarantees the uniqueness of an equilibrium.

For completeness, we obtain $\mathbf{b}$ and c. Plugging (4) into (A1),

$$-(\mathbf{b}^\top(\mathbf{s}_i - \bar{\mathbf{s}}) + c) + \alpha_n(\mathbf{b}^\top(E[\mathbf{s}_j|\mathbf{s}_i] - \bar{\mathbf{s}}) + c) + \beta E[\theta|\mathbf{s}_i] = 0. \tag{A2}$$

By the property of multivariate normal distributions,[21] $E[\mathbf{s}_j|\mathbf{s}_i] = \bar{\mathbf{s}} + DC^{-1}(\mathbf{s}_i - \bar{\mathbf{s}})$ and $E[\theta|\mathbf{s}_i] = \bar{\theta} + \mathbf{g}^\top C^{-1}(\mathbf{s}_i - \bar{\mathbf{s}})$ hold. Plugging these into (A2),

$$-\Big(\mathbf{b}^\top(I - \alpha_n DC^{-1}) - \beta\mathbf{g}^\top C^{-1}\Big)(\mathbf{s}_i - \bar{\mathbf{s}}) - (1-\alpha_n)c + \beta\bar{\theta} = 0$$

for all $\mathbf{s}_i \in \mathbb{R}^2$. This implies that $\mathbf{b}^\top = \beta\mathbf{g}^\top(C - \alpha_n D)^{-1}$ and $c = \beta\bar{\theta}/(1-\alpha_n)$.

B Proof of Lemma 2

To evaluate $W(\tau) = \left(\zeta_n \mathbf{b}^\top(C-D)\mathbf{b} + \eta_n \mathbf{b}^\top D\mathbf{b}\right)/\beta$, it is more convenient to use the variances rather than the precision. We write $x = \tau_x^{-1}$, $y = \tau_y^{-1}$, $z = \tau_\theta^{-1}$, and $V(x,y) = W(\tau)$. Then,

$$C = \begin{pmatrix} x+z & z \\ z & y+z \end{pmatrix},\ D = \begin{pmatrix} z & z \\ z & y+z \end{pmatrix},\ \mathbf{g} = \begin{pmatrix} z \\ z \end{pmatrix},$$

and thus

$$V(x,y) = \frac{\beta z^2\left((1-\alpha_n)^2 xy^2\zeta_n + \left(x^2y + (x + (1-\alpha_n)\,y)^2 z\right)\eta_n\right)}{(1-\alpha_n)^2(xy + (1-\alpha_n)\,yz + zx)^2}. \tag{B1}$$

Therefore, $W(0,0) = \lim_{y\to\infty}\lim_{x\to\infty} V(x,y) = \lim_{x\to\infty}\lim_{y\to\infty} V(x,y) = 0$.

C Proof of Proposition 1

First, note that $\partial W/\partial\tau_x > 0$ is equivalent to $\partial V/\partial x < 0$. By differentiating (B1), we get

$$\frac{\partial V}{\partial x} \equiv -\frac{\beta y^2 z^2\left(2yz\eta_n + (xz - (1-\alpha_n)\,yz + yx)\,\zeta_n\right)}{(xz + (1-\alpha_n)\,yz + yx)^3}. \tag{C1}$$

Because the denominator is positive, $\partial V/\partial x < 0$ if and only if

$$2yz\eta_n + (xz - (1-\alpha_n)\,yz + yx)\,\zeta_n = x\,(y+z)\,\zeta_n - ((1-\alpha_n)\zeta_n - 2\eta_n)yz > 0. \tag{C2}$$

[21] Let $X = (X_1, X_2)$ be a random vector whose distribution is multivariate normal with $\mu_i = EX_i$ and $C_{ij} = \mathrm{cov}(X_i, X_j)$ for $i,j = 1,2$. Then, $E[X_2|X_1] = \mu_2 + C_{21}C_{11}^{-1}(X_1 - \mu_1)$.

If $\zeta_n = 0$, (C2) is rewritten as $2yz\eta_n > 0$, and thus $\partial V/\partial x < 0$ if and only if $\eta_n > 0$, which establishes the signs of $\partial W/\partial \tau_x$ in (i) and (iii) with $\zeta_n = 0$ because we set $X = -\infty$. If $\zeta_n \gtrless 0$, (C2) is rewritten as

$$X = ((1-\alpha_n)\zeta_n - 2\eta_n)/\zeta_n \lesseqgtr x(y+z)/(yz) = (\tau_y + \tau_\theta)/\tau_x.$$

Thus, $\partial V/\partial x < 0$ if and only if either $X < (\tau_y + \tau_\theta)/\tau_x$ and $\zeta_n > 0$ or $X > (\tau_y + \tau_\theta)/\tau_x$ and $\zeta_n < 0$, which establishes the signs of $\partial W/\partial \tau_x$ in (i), (ii), (iii), and (iv) with $\zeta_n \neq 0$.

Next, note that $\partial W/\partial \tau_y > 0$ is equivalent to $\partial V/\partial y < 0$. By differentiating (B1), we get

$$\frac{\partial V}{\partial y} \equiv -\frac{\beta x^2 z^2 \left((xy + 3(1-\alpha_n)yz + zx)\eta_n - 2(1-\alpha_n)^2 yz\zeta_n\right)}{(1-\alpha_n)^2 (xy + (1-\alpha_n)yz + zx)^3}. \tag{C3}$$

Because the denominator is positive, $\partial V/\partial y < 0$ if and only if

$$\begin{aligned}(xy + 3(1-\alpha_n)yz + zx)\eta_n - 2(1-\alpha_n)^2 yz\zeta_n& \\ = x(y+z)\eta_n - (1-\alpha_n)(2(1-\alpha_n)\zeta_n - 3\eta_n)yz > 0.&\end{aligned} \tag{C4}$$

If $\eta_n = 0$, (C4) is rewritten as $-2(1-\alpha_n)^2 yz\zeta_n > 0$, and thus $\partial V/\partial y < 0$ if and only if $\zeta_n < 0$, which establishes the signs of $\partial W/\partial \tau_y$ in (ii) and (iv) with $\eta_n = 0$. If $\eta_n \gtrless 0$, (C4) is rewritten as

$$Y = (1-\alpha_n)(2(1-\alpha_n)\zeta_n - 3\eta_n)/\eta_n \lesseqgtr x(y+z)/(yz) = (\tau_y + \tau_\theta)/\tau_x.$$

Thus, $\partial V/\partial y < 0$ if and only if either $Y < (\tau_y + \tau_\theta)/\tau_x$ and $\eta_n > 0$ or $Y > (\tau_y + \tau_\theta)/\tau_x$ and $\eta_n < 0$, which establishes the signs of $\partial W/\partial \tau_y$ in (i) and (iii) with $\eta_n \neq 0$. This also establishes the signs of $\partial W/\partial \tau_y$ in (ii) and (iv) with $\eta_n \neq 0$ because $\zeta_n \eta_n < 0$ implies $Y = (1-\alpha_n)(2(1-\alpha_n)\zeta_n/\eta_n - 3) < 0$.

Finally, if $\zeta_n \eta_n > 0$, then

$$Y - X = 2((1-\alpha_n)\zeta_n - \eta_n)^2/(\zeta_n \eta_n) \geq 0,$$

which establishes $X \leq Y$ in (i) and (iii). If $X = Y$, then $(1-\alpha_n)\zeta_n = \eta_n$, and thus $X = Y = \alpha_n - 1 < 0$.

D Proof of Corollary 2

In types +I and +II, $\partial W(\tau)/\partial \tau_x > 0$ for all τ, and thus $W(\tau_x, \tau_y) < W(\infty, \tau_y) = W(\tau_x, \infty)$. In type −IV, $\partial W(\tau)/\partial \tau_y > 0$ for all τ, and thus $W(\tau_x, \tau_y) < W(\tau_x, \infty) = W(\infty, \tau_y)$.

In types +III and +IV, if $\tau_x \geq (\tau_y + \tau_\theta)/X \geq \tau_\theta/X$, then $\partial W(\tau)/\partial \tau_x \leq 0$ and $\partial W(\tau)/\partial \tau_y < 0$, and thus $W(\tau_x, \tau_y) \leq W(\tau_x, 0) \leq W(\tau_\theta/X, 0)$. If $\tau_x < (\tau_y + \tau_\theta)/X$, then $\partial W(\tau)/\partial \tau_x > 0$, and thus $W(\tau_x, \tau_y) < W((\tau_y + \tau_\theta)/X, \tau_y) \leq W(\tau_\theta/X, 0)$, where the last inequality holds by the case with $\tau_x \geq (\tau_y + \tau_\theta)/X$.

In type $-$I, $\partial W(\tau)/\partial \tau_x < 0$ and $\partial W(\tau)/\partial \tau_y < 0$ for all τ, and thus $W(\tau_x, \tau_y) \leq W(0, 0)$.

In types $-$II and $-$III, if $\tau_x \leq (\tau_y + \tau_\theta)/Y$, then $\partial W(\tau)/\partial \tau_x < 0$ and $\partial W(\tau)/\partial \tau_y \leq 0$, and thus $W(\tau_x, \tau_y) \leq W(0, \tau_y) \leq W(0, 0)$. If $\tau_x > (\tau_y + \tau_\theta)/Y$, then $\partial W(\tau)/\partial \tau_y > 0$, and thus $W(\tau_x, \tau_y) < W(\tau_x, Y\tau_x - \tau_\theta) \leq W(0, 0)$, where the last inequality holds by the case with $\tau_x \leq (\tau_y + \tau_\theta)/Y$.

E Proof of Corollary 3

In types +I, +II, and $-$IV, $W(\tau_x, \tau_y) < W(\tau_x, \infty)$ by Corollary 2.

In type +III, if $\tau_y < Y\tau_x - \tau_\theta$, then $\partial W(\tau)/\partial \tau_y < 0$, and if $\tau_y > Y\tau_x - \tau_\theta$, then $\partial W(\tau)/\partial \tau_y > 0$. Thus,

$$\sup_{\tau_y} W(\tau) = \max\{W(\tau_x, 0), W(\tau_x, \infty)\}.$$

Because $W(\tau_x, \infty) = W(\infty, 0)$, we compare $W(\tau_x, 0)$ and $W(\infty, 0)$. Note that if $\tau_x > \tau_\theta/X$, then $\partial W(\tau_x, 0)/\partial \tau_x < 0$,[22] and thus $W(0, 0) < W(\infty, 0) < W(\tau_\theta/X, 0)$. Hence, there exists a unique $\tau_x^* < \tau_\theta/X$ such that $W(\tau_x^*, 0) = W(\infty, 0)$ since $\partial W(\tau_x, 0)/\partial \tau_x > 0$ for $\tau_x < \tau_\theta/X$. Note that $W(\tau_x, 0) < W(\infty, 0)$ if $\tau_x < \tau_x^*$ and $W(\tau_x, 0) \geq W(\infty, 0)$ if $\tau_x \geq \tau_x^*$. Therefore,

$$\sup_{\tau_y} W(\tau) = \begin{cases} W(\tau_x, \infty) & \text{if } \tau_x < \tau_x^*, \\ W(\tau_x, 0) & \text{if } \tau_x \geq \tau_x^*. \end{cases}$$

To find τ_x^*, we solve

$$\begin{aligned} W(\tau_x^*, 0) - W(\infty, 0) &= \frac{\beta\, {\tau_\theta}^{-2} \left(\tau_\theta^{-1}\, \eta_n + \tau_x^{*-1}\, \zeta_n\right)}{\left((1-\alpha_n)\ \tau_\theta^{-1} + \tau_x^{*-1}\right)^2} - \frac{\beta\, \tau_\theta^{-1}\, \eta_n}{(1-\alpha_n)^2} \\ &= \frac{\beta\, \tau_x^{*-1}\, \tau_\theta^{-1} \left(-2\ (1-\alpha_n)\ \tau_\theta^{-1}\, \eta_n - \tau_x^{*-1}\, \eta_n + (1-\alpha_n)^2\, \tau_\theta^{-1}\, \zeta_n\right)}{(1-\alpha_n)^2 \left((1-\alpha_n)\ \tau_\theta^{-1} + \tau_x^{*-1}\right)^2} = 0, \end{aligned}$$

and obtain $\tau_x^* = \eta_n \tau_\theta / ((1-\alpha_n) X \zeta_n)$.

[22] We can verify this directly by (C1) rather than Proposition 1 because $\tau_y = 0$.

In types +IV and −I, $\partial W(\tau)/\partial\tau_y < 0$ for all τ, and thus $W(\tau_x, \tau_y) \le W(\tau_x, 0)$.

In types −II and −III, $\partial W(\tau)/\partial\tau_y > 0$ if $\tau_y < Y\tau_x - \tau_\theta$ and $\partial W(\tau)/\partial\tau_y < 0$ if $\tau_y > Y\tau_x - \tau_\theta$. Thus, $W(\tau_x, \tau_y) \le W(\tau_x, \max\{Y\tau_x - \tau_\theta, 0\})$.

F Proof of Corollary 4

In types +I, +II, and −IV, $W(\tau_x, \tau_y) < W(\infty, \tau_y)$ by Corollary 2.

In types +III and +IV, $\partial W(\tau)/\partial\tau_x > 0$ if $\tau_x < (\tau_y + \tau_\theta)/X$ and $\partial W(\tau)/\partial\tau_x < 0$ if $\tau_x > (\tau_y + \tau_\theta)/X$. Thus, $W(\tau_x, \tau_y) \le W((\tau_y + \tau_\theta)/X, \tau_y)$.

In types −I and −II, $\partial W(\tau)/\partial\tau_x < 0$ for all τ, and thus $W(\tau_x, \tau_y) \le W(0, \tau_y)$.

In type −III, $\partial W(\tau)/\partial\tau_x < 0$ if $\tau_x < (\tau_y + \tau_\theta)/X$ and $\partial W(\tau)/\partial\tau_x > 0$ if $\tau_x > (\tau_y + \tau_\theta)/X$. Thus,

$$\sup_{\tau_x} W(\tau) = \max\{W(0, \tau_y), W(\infty, \tau_y)\} = W(0, \tau_y)$$

because $W(\infty, \tau_y) = W(0, \infty) < W(0, \tau_y)$ by $\partial W(0, \tau_y)/\partial\tau_y < 0$.[23]

G On the proof of AP's Corollary 10

AP consider the following payoff function in p. 1128:

$$U = (a_0 - c_1 + a_1\theta - a_3K)k - (a_2 + c_2)k^2,$$

where $a_0, a_1, a_2, a_3, c_1, c_2 > 0$ are constants, $k \in \mathbb{R}$ is an action, and $K \in \mathbb{R}$ is its mean over all the players. This payoff function is the same as that in Section 5.3.[24] AP define

$$\alpha \equiv -(\partial^2 U/\partial k\partial K)/(\partial^2 U/\partial k^2) = -a_3/(2(a_2 + c_2)),$$

which is the same as α in this paper and assumed to be strictly less than 1.

AP's Corollary 10 states that expected total profits necessarily increase with the precision of private information, but can decrease with that of public information, which is possible if $\alpha < -1$. To prove the former, they use their main result. To prove the latter, they directly calculate the partial derivative of the welfare loss $\mathcal{L}$ due to incomplete information given by (36) in AP. In both proofs, their parameter ϕ plays a key role. They obtain $\phi = \alpha/(2(1-\alpha))$, but this includes an error. By correcting it, we obtain $\phi = \alpha/(1 - 2\alpha)$.

[23] We can verify this directly by (C3) rather than Proposition 1 because $\tau_x = 0$.

[24] Replace $a_2 + c_2$, $a_0 - c_1 + a_1\theta$, and a_3 with c, θ, and ρ, respectively.

Using the correct value of ϕ, we calculate $\mathcal{L}$ and its partial derivatives based upon (36) in AP. We write $\sigma_x^2 \equiv 1/\tau_x$ and $\sigma_z^2 \equiv 1/(\tau_y + \tau_\theta)$, following AP. Then, we have

$$\mathcal{L} = a_1^2/(a_2 + c_2) \times \sigma_x^2\sigma_z^2(\sigma_x^2 + (1-\alpha^2)\sigma_z^2)/(4(1-\alpha)^2(\sigma_x^2 + (1-\alpha)\sigma_z^2)^2),$$
$$\partial\mathcal{L}/\partial\sigma_x^2 = a_1^2/(a_2 + c_2) \times \sigma_z^4(\sigma_x^2 + (1+\alpha)\sigma_z^2)/(4(\sigma_x^2 + (1-\alpha)\sigma_z^2)^3),$$
$$\partial\mathcal{L}/\partial\sigma_z^2 = a_1^2/(a_2 + c_2) \times \sigma_x^4(\sigma_x^2 + (1-\alpha)(2\alpha+1)\sigma_z^2)/(4(1-\alpha)^2(\sigma_x^2 + (1-\alpha)\sigma_z^2)^3).$$

Thus, $\partial\mathcal{L}/\partial\sigma_x^2 < 0$ if and only if $\sigma_x^2 + (1+\alpha)\sigma_z^2 < 0$, which is rewritten as $\sigma_x^2/\sigma_z^2 = (\tau_y + \tau_\theta)/\tau_x < -(\alpha+1)$, and $\partial\mathcal{L}/\partial\sigma_z^2 < 0$ if and only if $\sigma_x^2 + (1-\alpha)(2\alpha+1)\sigma_z^2 < 0$, which is rewritten as $\sigma_x^2/\sigma_z^2 = (\tau_y + \tau_\theta)/\tau_x < -(1-\alpha)(2\alpha+1)$. That is, expected total profits decrease with the precision of private information if and only if $\alpha < -1$ and $\tau_x > -(\tau_y + \tau_x)/(\alpha+1)$, and decrease with that of public information if and only if $\alpha < -1/2$ and $\tau_x > -(\tau_y + \tau_\theta)/((1-\alpha)(2\alpha+1))$. This result is consistent with our result that the large Cournot game is type +I if $-1/2 \leq \alpha < 1$, type +II if $-1 \leq \alpha < -1/2$, and type +III if $\alpha < -1$.

H On the proof of AP's Corollary 11

AP consider the following payoff function in p. 1129:

$$U = (\theta - k + bK)k - c(\theta - k + bK)^2,$$

where $b, c \in \mathbb{R}$ are constants with $0 < b < 1$, k is an action, and $K \in \mathbb{R}$ is its mean over all the players. This payoff function is the same as that in Section 5.4.

AP's Corollary 11 states that expected total profits necessarily increase with the precision of both public and private information. To prove the "private information" part, which is inconsistent with our result, they directly calculate the partial derivative of the welfare loss $\mathcal{L}$ due to incomplete information given by (36) in AP. They then show that $\partial\mathcal{L}/\partial\sigma_x^2 > 0$, where $\sigma_x^2 \equiv 1/\tau_x$, but this includes an error. To see this, we calculate $\mathcal{L}$ and $\partial\mathcal{L}/\partial\sigma_x^2$ based upon (36) in AP assuming that $b = 4/5$ and $c = 1$. We write $\sigma_z^2 \equiv 1/(\tau_y + \tau_\theta)$, following AP. Then, we have

$$\mathcal{L} = 75\sigma_x^2\sigma_z^2(19\sigma_x^2 + 16\sigma_z^2)/(32(5\sigma_x^2 + 2\sigma_z^2)^2),$$
$$\partial\mathcal{L}/\partial\sigma_x^2 = 75\sigma_z^4(8\sigma_z^2 - \sigma_x^2)/(8(5\sigma_x^2 + 2\sigma_z^2)^3).$$

Thus, $\partial\mathcal{L}/\partial\sigma_x^2 < 0$ if and only if $8\sigma_z^2 - \sigma_x^2 < 0$, which is rewritten as $1/\sigma_x^2 = \tau_x < 1/(8\sigma_z^2) = (\tau_y + \tau_\theta)/8$. That is, expected total profits decrease with the precision of private information if and only if $\tau_x < (\tau_y + \tau_\theta)/8$. This result is the same as that in Section 5.4.

I Optimal Bayesian correlated equilibrium in the finite case

Consider a game with n players and a payoff function (1). We obtain the optimal Bayesian correlated equilibrium that achieves the highest welfare. It is different from that in Corollary 5 because the equivalence of Bayesian correlated and Nash equilibria does not hold in the finite case.

Assume that $v(a,\theta)$ in (3) is strictly concave in a, i.e., $-(n-1) < \alpha_n < 1$. Let $\theta \in \mathbb{R}$ be normally distributed with mean $\bar{\theta}$ and variance σ_θ^2. Players have no information about θ, but a mediator knows θ and recommends player $i \in N$ to choose $a_i \in \mathbb{R}$, where $(a_1,\dots,a_n,\theta)$ is normally distributed with

$$E[a_i] = \bar{a},\ \operatorname{var}[a_i] = \sigma_a^2,\ \operatorname{cov}[a_i,a_j] = \rho_a\sigma_a^2,\ \operatorname{cov}[\theta,a_i] = \rho_{a\theta}\sigma_a\sigma_\theta$$

for $i \neq j$. A joint probability distribution of $(a_1,\dots,a_n,\theta)$ is a Bayesian correlated equilibrium if $a_i = \arg\max_{a_i'} E[u_i((a_i',a_{-i}),\theta)|a_i]$ for all $a_i \in \mathbb{R}$ and $i \in N$, where $a_{-i} = (a_j)_{j\neq i}$. Bergemann and Morris (2012a,b) show that a necessary and sufficient condition for a Bayesian correlated equilibrium is

$$\rho_{a\theta}^2 \leq \frac{n-1}{n}\rho_a + \frac{1}{n}, \tag{I1}$$

$$\sigma_a = \frac{\beta\rho_{a\theta}\sigma_\theta}{1-\alpha_n\rho_a}, \tag{I2}$$

$$\bar{a} = \beta\bar{\theta}/(1-\alpha_n). \tag{I3}$$

The condition (I1) guarantees that the covariance matrix of the joint probability distribution of $(a_1,\dots,a_n,\theta)$ is non-negative definite. Note that $\rho_a \geq n\rho_{a\theta}^2/(n-1) - 1/(n-1) \to \rho_{a\theta}^2$ as $n \to \infty$. Thus, ρ_a can be negative when $n < \infty$, but it must be positive when $n \to \infty$, which results in differences between the finite and continuum cases.

The conditions (I2) and (I3) are equivalent to the following first order condition for a Bayesian correlated equilibrium:

$$-a_i + \alpha\sum_{j\neq i} E[a_j|a_i] + \beta E[\theta|a_i] = -a_i + \alpha_n E[a_j|a_i] + \beta E[\theta|a_i] = 0 \tag{I4}$$

for $j \neq i$. In fact, by the property of multivariate normal distributions, $E[a_j|a_i] = \bar{a} + \rho_a(a_i - \bar{a})$ and $E[\theta|a_i] = \bar{\theta} + \rho_{a\theta}\sigma_\theta\sigma_a^{-1}(a_i - \bar{a})$, and by plugging these into (I4), we have $(\beta\rho_{a\theta}\sigma_\theta\sigma_a^{-1} - (1-\alpha_n\rho_a))(a_i - \bar{a}) - (1-\alpha_n)\bar{a} + \beta\bar{\theta} = 0$ for all $a_i \in \mathbb{R}$, which is equivalent to (I2) and (I3).

Given a Bayesian correlated equilibrium, the expected payoff is written as

$$E[u_i(a,\theta)] = \beta^{-1}\Big(\zeta_n(\mathrm{var}[a_i] - \mathrm{cov}[a_i,a_j]) + \eta_n \mathrm{cov}[a_i,a_j]\Big) + \text{const.}$$

by the same argument as that of Lemma 2, where the constant term is independent of $(\sigma_a, \rho_a, \rho_{a\theta})$. We adopt

$$F(\sigma_a,\rho_a) = \beta^{-1}\Big(\zeta_n(\mathrm{var}[a_i] - \mathrm{cov}[a_i,a_j]) + \eta_n \mathrm{cov}[a_i,a_j]\Big) = \beta^{-1}(\zeta_n + (\eta_n - \zeta_n)\rho_a)\sigma_a^2$$

as a measure of welfare. The optimal Bayesian correlated equilibrium is the Bayesian correlated equilibrium that maximizes $F(\sigma_a, \rho_a)$ subject to (I1) and (I2). The following proposition characterizes the optimal Bayesian correlated equilibrium. The case of $n \to \infty$ is Corollary 5.

Proposition A. *Let $(\sigma_a^*, \rho_a^*, \rho_{a\theta}^*)$ be parameters of the optimal Bayesian correlated equilibrium. Then, the following holds.*

- *Suppose that $\zeta_n \leq \max\{\eta_n/n, (2n-1+\alpha_n)\eta_n/(n(1-\alpha_n))\}$. If $\eta_n > 0$, then $\sigma_a^* = \beta\sigma_\theta/(1-\alpha_n)$ and $\rho_a^* = \rho_{a\theta}^* = 1$; that is, the recommended action is $a_i = \beta\theta/(1-\alpha_n)$. If $\eta_n \leq 0$, then $\sigma_a^* = 0$ and $\rho_{a\theta}^* = 0$; that is, the recommended action is $a_i = \beta\bar{\theta}/(1-\alpha_n)$.*

- *Suppose that $\zeta_n > \max\{\eta_n/n, (2n-1+\alpha_n)\eta_n/(n(1-\alpha_n))\}$. Then,*

$$\sigma_a^* = \frac{\beta\rho_{a\theta}^*\sigma_\theta}{1-\alpha_n\rho_a^*}, \tag{I5}$$

$$\rho_a^* = -\frac{(2\alpha_n + n - 2)\zeta_n + \eta_n}{((n-2)\alpha_n - 2(n-1))\zeta_n + (\alpha_n + 2(n-1))\eta_n}, \tag{I6}$$

$$\rho_{a\theta}^* = \sqrt{\frac{n-1}{n}\rho_a^* + \frac{1}{n}}. \tag{I7}$$

Proof. By (I1) and (I2), $-1/(n-1) \leq \rho_a \leq 1$ and

$$\sigma_a^2 = \left(\frac{\beta\rho_{a\theta}\sigma_\theta}{1-\alpha_n\rho_a}\right)^2 \leq \frac{\beta^2\sigma_\theta^2}{(1-\alpha_n\rho_a)^2}\left(\frac{n-1}{n}\rho_a + \frac{1}{n}\right).$$

Thus, by setting

$$f(x) \equiv \beta\sigma_\theta^2 \frac{\zeta_n + (\eta_n - \zeta_n)x}{(1-\alpha_n x)^2}\left(\frac{n-1}{n}x + \frac{1}{n}\right),$$

we have $F(\sigma_a, \rho_a) \leq f(\rho_a)$ for $\rho_a \in [-1/(n-1), 1]$, where the equality holds if

$$\rho_{a\theta}^2 = \frac{n-1}{n}\rho_a + \frac{1}{n}. \tag{I8}$$

Therefore, to maximize $F(\sigma_a, \rho_a)$ subject to (I1) and (I2), it is enough to solve

$$\max_{-1/(n-1)\leq x\leq 1} f(x). \tag{I9}$$

We first calculate

$$f'(x) = \frac{(((n-2)\,\alpha_n - 2\,(n-1))\,\zeta_n + (\alpha_n + 2\,(n-1))\,\eta_n)\,x + (2\alpha_n + n - 2)\,\zeta_n + \eta_n}{n(1-\alpha_n x)^3}.$$

The denominator is positive since $-(n-1) < \alpha_n < 1$ and $-1/(n-1) \leq x \leq 1$. The numerator is a linear function of x.

Suppose that $f'(-1/(n-1)) > 0$ and $f'(1) < 0$. It can be verified that this is true if and only if $\zeta_n > \max\{\eta_n/n, (2n-1+\alpha_n)/(n(1-\alpha_n))\}$. In this case, (I9) has an interior solution ρ_a^* with $f'(\rho_a^*) = 0$, and ρ_a^* in (I6) is the unique solution. (I7) is implied by (I8) and (I5) is implied by (I2).

Suppose otherwise. Then, (I9) has a corner solution. Thus,

$$\max_{-1/(n-1)\leq x\leq 1} f(x) = \max\{f(-1/(n-1)), f(1)\} = \max\left\{0, \frac{\beta\sigma_\theta^2\eta_n}{(1-\alpha_n)^2}\right\}.$$

This implies that if $\eta_n > 0$, then $\rho_a^* = \rho_{a\theta}^* = 1$, and if $\eta_n \leq 0$, then $\rho_{a\theta}^* = 0$. In each case, σ_a^* is given by (I2). □

Bergemann and Morris (2012a,b) obtain the following special case with $\kappa = \lambda = \mu = 0$, i.e., $\zeta_n = \eta_n = \beta > 0$.

Corollary B. *Suppose that $\kappa = \lambda = \mu = 0$. Let $(\sigma_a^*, \rho_a^*, \rho_{a\theta}^*)$ be parameters of the optimal Bayesian correlated equilibrium. Then, the following holds.*

- *Suppose that $\alpha_n \geq -(n-1)/(n+1)$. Then, $\sigma_a^* = \beta\sigma_\theta/(1-\alpha_n)$ and $\rho_a^* = \rho_{a\theta}^* = 1$; that is, the recommended action is $a_i = \beta\theta/(1-\alpha_n)$.*

- *Suppose that $\alpha_n < -(n-1)/(n+1)$. Then,*

$$\sigma_a^* = \frac{\beta\rho_{a\theta}^*\sigma_\theta}{1-\alpha_n\rho_a^*},\ \ \rho_a^* = -\frac{2\alpha_n + n - 1}{(n-1)\alpha_n},\ \ \rho_{a\theta}^* = \sqrt{\frac{n-1}{n}\rho_a^* + \frac{1}{n}}.$$

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