Derivative Pricing Models with Inter-commodity Price Relations

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Contents

Chapter 1

Introduction

Commodities, such as energy, metals, and agricultural crops, constitute fundamental parts of economics. Energy policy affects the future of nations. At one time, OPEC had an enormous influence on the global economy through the determination of oil prices. Now, new energy sources such as shale gas and oil are expected to make new industries. Metals are used everywhere especially in transportations and electronic devices. For example, iron, copper, zinc, and aluminum are base metals used in automobiles, batteries, and electronic materials. Agricultural crops are the cause of dispute in international trade. The 8th General Agreement on Tariffs and Trade (GATT) round, also known as the Uruguay Round, ended in a failure because countries did not agree with agriculture trade policies. Furthermore, one of the difficulties in the ongoing negotiation on Trans-Pacific Strategic Economic Partnership (TPP) is agriculture. Hence, commodity-pricing models are an important issue, not just in finance, but also in the international economy.

Our motivation of this paper is to utilize equilibrium or long-term economic relationships in the area of commodity derivative pricing. We are interested in how the relation between commodities affects derivative prices of a commodity theoretically and empirically. During the last few years, it has been observed that commodity prices, e.g., crude oil, coal, and natural gas prices rise and fall simultaneously. Another example is that sugar cane is becoming more correlated with oil prices through production of ethanol. Commodity prices are related to each other. Thus, it is natural to construct a pricing model for commodity prices to incorporate these relationships and analyze their effect.

In this paper, we investigate commodity price models with relations between commodity prices. More specifically, we focus on two subjects of interest: emission allowance price and its relation with other commodity prices, and a commodity price model that incorporates a long-term relationship, i.e. cointegration among prices.

The first theme of our analysis is on emission allowance price in relation to other commodity prices. An emission allowance is the right to emit greenhouse gases, such as CO_2. As these gases are emitted when energy is consumed, this right is a contingent claim on the commodity. Therefore, emission allowance is a commodity-related asset.

Research on emission allowance prices and their derivatives is already in progress. Cronshaw and Kruse (1996), Rubin (1996), Schennach (2000), Carmona, Fehr, and Hinz (2009), Seifert, Uhrig-Homberg, and Wagner (2008) have theoretically investigated emission allowances, as discussed later. However, this body of research does not explicitly model the relation between emission allowances and other commodities, such as electricity, coal, and natural gas. In contrast, some empirical papers, such as Fezzi and Bunn (2009) and Mansanet-Bataller, Pardo, and Valor (2007), have analyzed the futures prices of emission allowances and found a relation with the price of electricity, natural gas and temperature. These relations and their implications for valuation of derivatives are not yet fully understood in existing works. Thus, we focus on (linear) relations of commodity spot and emission allowance prices and analyze the valuation of emission allowances and their derivatives within this framework.

In Chapter 2, we relate emission allowance prices with other commodity prices through profit maximization of a firm; e.g., a power-generating company that produces electricity by burning coal and/or natural gas while emitting CO_2. Under a trading system similar to the EU-ETS, we derive the inter- and intratemporal relations among the emission allowance prices and commodity prices. These relations are necessary conditions for equilibrium. The intertemporal relation shows that the emission allowance price at time t is the present value of the emission allowance price at the end of period T. This is simply a no-arbitrage condition for a financial claim. From the intratemporal relations, we have two equations. The first equation implies that the emission allowance price is the spread between output (e.g. electricity) and input (e.g. coal or natural gas) prices adjusted by the production/emission ratio. The second equation implies that the emission allowance price is the price difference between two input (e.g. coal and natural gas) prices adjusted by the production/emission ratio, i.e., the fuel switching cost. Emission allowance price and commodity price satisfy these equations because marginal revenue is equal to marginal costs of fuel and emission allowance per unit of production. In addition, we find that emission allowance at the end of period T can not be greater than the penalty. We also analyze how change in the prices of the other commodities affect the coefficients of the spread relation to emission prices for the Cobb-Douglas and the constant elasticity of substitution (CES) production functions. The implications of the results lead to an empirical model for the emission allowance price which is the issue of next chapter.

In Chapter 3, we characterize the price of emission allowances by incorporating the interrelations between emission allowances and other commodities. We assume three conditions. The first is the intertemporal condition that emission allowances at time t are the present value of the emission allowances at the end of period T. The second is the intratemporal condition that the emission allowance spot price at the end of period T should be positive and equal to the minimum of the spread of the two commodities prices and the penalty. To make the model tractable, we also assume that commodity prices follow the Gibson-Schwartz model which is the third assumption. We derive the valuation formula of emission allowance as a commodity spread option. The emission allowance price is not just the price of commodity spread, because it embeds options such that the price has to be lower than the penalty and positive. We also analyze the option values embedded in emission allowances, derive valuation formulae for futures and options on emission allowances, and characterize a hedging strategy of emission allowances using commodity futures. We emphasize the interrelation between prices of emission allowances and commodities, which had not been incorporated in preceding papers on the valuation of emission allowances. We calibrate the model to real market data and use the parameter to analyze emission allowance with its embedded option value and the behavior of the hedge ratios of emission allowance futures by commodity futures. We find that the electricity and natural gas price explain emission allowance price to some extent. From the numerical analysis using the calibrated model, we find that the option values for the penalty embedded in emission allowances are relatively large, which imply that the penalty is an important component in evaluating emission allowances.

The second theme of this paper focuses on commodity prices with a long-term relationship, i.e. cointegration. In the academic literature, the benchmark model for commodity price is the Gibson and Schwartz (1990) commodity pricing model. Other models which generalize this model include Schwartz (1997), Miltersen and Schwartz (1998), Nielsen and Schwartz (2004) and Casassus and Collin-Dufresne (2005). Although these models have their own characteristics, we argue that these models ignore the relations among commodity prices, which should be a significant element. Indeed, empirical research (Malliaris and Urrutia, 1996; Girma and Paulson, 1999) suggests the existence of cointegration between commodity prices.

The concept of cointegration was first established by Engle and Granger (1987) and is interpreted as a long-term relationship or equilibrium between variables. Although papers on cointegration are plentiful in economic issues, they are limited in the area of finance and commodity pricing. Of these, a few apply cointegration to financial derivatives (see Duan and Pliska, 2004; and Dempster, Medova, and Tang, 2008). Duan and Pliska's (2004) model is for stocks and does not readily apply to commodity pricing since it ignores the key factor for commodity pricing, i.e., convenience yield.

Therefore, we generalize the Gibson-Schwartz model by explicitly incorporating cointegration in Chapter 4. More specifically, we formulate a commodity pricing model in which the temporary deviation of drift terms from the risk-free rate under a risk-neutral probability is described by convenience yields and linear relations among logarithms of commodity prices, which correspond to error terms under an appropriate condition. We derive futures and call option pricing formulae and show that, in contrast to Duan and Pliska (2004), the linear relations among log commodity prices, or the error term under appropriate conditions, should affect these derivative prices in the standard setup of commodity pricing. Cortazar, Milla, and Severino (2008), Paschke and Prokopczuk (2009), and Casassus, Liu, and Tang (2011) have also independently studied commodity price models which is similar to ours. In contrast to the papers mentioned above, we provide a sufficient condition for the model to be cointegrated. We estimate the model by using crude and heating oil prices. Using the estimated parameter, we apply this model to a hedging strategy of long-term futures contracts using short-term futures.

In Chapter 5, we develop a model of commodity spread option with cointegration based on Chapter 4. Commodity spread options are options on the spread of two commodity prices. As energy companies' profits are the spread of commodity prices, they can be used as risk-hedging tools and these derivatives are traded on the NYMEX. There are many papers considering the valuation of spread options and commodity spread options such as Margrabe (1978), Wilcox (1990), Shimko (1994), Kirk (1995), Pearson (1995), Poitras (1998), Zhang (1998), Carmona and Durrleman (2003), and Nakajima and Maeda (2007). Yet again, these studies do not consider the relation among commodity prices that we analyze in this chapter. Casassus, Liu, and Tang (2011) also provide a commodity spread option model which consider the relation of commodity prices. They conduct a numerical analysis by Monte Carlo simulation using estimated parameters. In contrast to their paper, we derive European commodity spread option formulae analytically and conduct sensitivity analysis against parameters such as volatilities. Also, we present an analytical approximation formula for American call commodity spread options using the framework of Bjerksund and Stensland (1994). Using the estimated parameters of Chapter 4, we compare our model with the Shimko (1994) model which applied the Gibson-Schwartz model to commodity spread options. From numerical analysis, we show that the price of commodity spread options for long maturity given by the Gibson-Schwartz spread option model is much higher than that of our spread option model. This is because the cointegration binds commodity price spread, forces the commodity prices back into the long-term relation, and prevents the spread from diverging. This implies that the Gibson-Schwartz spread option model might overprice option values when pricing long-term maturity commodity spread options.

Chapter 2

Relations among Emission Allowance and Other Commodity Prices

2.1 Introduction

Global warming is one of the hottest topics in the world at present. In December 1997, Australia, Canada, China, EU, Japan, Norway, Russia, and many other countries adopted the Kyoto Protocol to take action against global warming. There were two features of this protocol. First, it provided targets for 2012 of greenhouse gases (GHG) emissions for each country.[1] Second, it adopted an economic approach called "flexible mechanism" to meet the GHG emissions targets. Emission trading is one of the mechanism of this economic approach. The Chicago Climate Exchange started trading six major GHG in 2003, and there was state level activity, for example in New York and California, in order to reduce emissions. The EU started the EU Emission Trading Scheme (EU ETS) in 2005. In Japan, the host country of the Kyoto Protocol, the government examined the effects of an emission trading market.

An emission allowance trading system is controlled by a central authority. The central authority sets the limit of CO_2 emissions, the period for implementation, defines which pollutants, such as energy companies, should

[1] Greenhouse gases are CO_2, CH_4, N_2O, HFCs, PFCs, SF_6. In this chapter and the next, we mainly deal with CO_2. However, other gases can be treated same as well.

be in the emission allowance market, allocates the initial allowance to them, and sets penalties that the polluters should pay when their CO_2 emissions exceed their allowance. Then the polluters trade their emission allowances during the normal course of business and emit CO_2. At the end of each period, polluters whose emissions exceed their allowance must pay the penalty. There are many economic issues associated with this trading system, such as how to set the initial allocation and/or penalty and nature of the emission price dynamics. In this paper, we investigate a firm's optimal policy in such emission trading system.

Several studies in both environmental and financial economics investigated emission allowance trading. Dale (1968) and Montgomery (1972) were among the pioneers who predicted and analyzed emission trading system. Cronshaw and Kruse (1996) extended Montgomery's model with banking in a multiperiod discrete-time model without uncertainty. Rubin (1996) studied emission trading with banking and borrowing in a continuous-time model. Schennach (2000) explored the effect of nonborrowing and uncertainty. Stevens and Rose (2002), based on Rubin's model with constraints on emission trading, indicated that cost effectiveness would be achieved by allowing trading across countries. Carmona, Fehr, and Hinz (2009) studied the market equilibrium of emission allowances based on the EU ETS framework and fuel switching technology in a discrete-time setting. Seifert, Uhrig-Homberg, and Wagner (2008) proposed a continuous stochastic equilibrium model based on Rubin's model with a penalty as in the EU ETS and analyzed the emission allowance spot price process. Cetin and Verschuere (2009) priced emission allowances under complete and incomplete information when the banking of permits is not allowed. Taschini (2009) surveyed theoretical models of environmental economics including emission trading. These papers investigated and clarified important properties and effects of emission allowance trading. However, they treated emission allowances alone, and did not explicitly model the relation between emission allowances and other commodities such as electricity, coal, and natural gas.

However, several papers found that emission allowances were related to electricity, natural gas, and temperature. Fezzi (2006) and Fezzi and Bunn (2009) empirically analyzed the relationships among electricity, gas, and carbon prices in Germany and the United Kingdom using a vector error correction model (VECM). They showed that the prices of carbon and natural gas jointly affect the equilibrium price of electricity and derived the dynamic pass-through of carbon into electricity prices. Mansanet-Bataller, Pardo, and Valor (2007) found that the CO_2 price level was determined by energy sources and that only extreme temperatures influence the price. These studies suggest that an emission allowance price model should be related to energy prices, which has not been fully investigated theoretically yet.

Hence, this chapter investigates theoretically the relation among commodity spot prices and the emission allowance price. In Section 2, we characterize the spot price of emission allowances through profit maximization model of a firm. We show that both inter- and intratemporal relations should be satisfied. That is, the emission allowance price at time t is the present value of the emission allowance price at the end of period T, and the emission allowance price is expressed as the spread between other commodity spot prices. Section 3 analyzes how changes in the prices of the other commodities affect the coefficients of the spread relation to emission prices using the Cobb-Douglas and the constant elasticity of substitution (CES) production functions. Section 4 discusses the implications of the results to empirical analysis, and suggests empirical models for the emission allowance price that have not yet been studied. Section 5 concludes.

2.2 A Model of Emission Allowance Price with Production

Let us consider the profit maximization problem of a risk-neutral firm. This firm uses two types of inputs q_2 and q_3 to produce $Q_1(q_2, q_3)$ units of output. However, in the course of production it emits $Q_e(q_1, q_2)$ units of CO_2, where $Q_1 : \mathbb{R}^2_+ \to \mathbb{R}$ and $Q_e : \mathbb{R}^2_+ \to \mathbb{R}$ are the production function and the emission function, respectively.[2][3] Furthermore, we assume the production function Q_1 to be twice continuously differentiable on $\mathbb{R}^2_+$ and concave, and the emission function Q_e to be twice continuously differentiable on $\mathbb{R}^2_{++}$ and convex. We will use fuel switching in electricity generation between coal and natural gas as an example of this model and assume a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. Furthermore, $S_1(t)$, $S_2(t)$, $S_3(t)$, and $S_e(t)$, are the

[2] In this paper, $\mathbb{R}^n$ is n-dimensional Euclidean space, $\mathbb{R}^n_+ = \{x = (x_i)^n_{i=1} | x_i \geq 0, i = 1, \cdots, n\}$ and $\mathbb{R}^n_{++} = \{x = (x_i)^n_{i=1} | x_i > 0, i = 1, \cdots, n\}$. We omit the superscript when $n = 1$.

[3] In this chapter and the next, we assume two inputs in order to produce one output. We can extend this to more than three inputs or specialize to the one input case. However, the arguments are the same as in the two inputs case.

prices of output, two types of inputs, and emission allowances, respectively, where they are positive adapted processes. The firm can trade emission allowances of amount $q_{et}(t)$ at time t. At the end of period T the firm pays the penalty Z if the amount of emissions exceeds the sum of the initial endowment of emission allowances A and the amount of emission allowances purchased $(\sum_{t=1}^{T} q_{et}(t))$. The penalty Z and the initial endowment of emission allowances A are both constant. We denote the interest rate by r, which is constant.

Because the firm is assumed to be risk-neutral, its objective is to maximize the expected profit as follows.

$$\max_{q(\cdot)\in\mathcal{U}[0,T]} E_0\left[\sum_{t=1}^{T} e^{-rt}(Q_1(q_2(t),q_3(t))S_1(t) - q_2(t)S_2(t) - q_3(t)S_3(t) - q_{et}(t)S_e(t)) \right.$$
$$\left. -e^{-rT} Z\left(\sum_{t=1}^{T} Q_e(q_2(t),q_3(t)) - \sum_{t=1}^{T} q_{et}(t) - A\right)^{+}\right], \tag{1}$$

where $q(\cdot) = [q_2(\cdot), q_3(\cdot), q_{et}(\cdot)]$, $\mathcal{U}[0,T] = \{u : \{0,\dots,T\} \times \Omega \to \mathbb{R}^3 : \{\mathcal{F}_t\}_{t\geq 0} - \text{adapted}\}$, and $E_0[\cdot]$ is the expectation operator. Note that the objective function is concave.

Before solving this problem, first note that if there is no arbitrage opportunity, then $S_e(T) \leq Z$. Because if $S_e(T) > Z$ is true, then the firm sells arbitrarily large amounts of emission allowances, accepts the penalty, and obtains arbitrage profit.

Because the object function of problem (1) is not differentiable, we resort to the following problem whose optimal solution is the same as the optimal solution of problem (1).[4]

$$\max_{q(\cdot)\in\mathcal{U}[0,T]} E_0\left[\sum_{t=1}^{T} e^{-rt}(Q_1(q_2(t),q_3(t))S_1(t) - q_2(t)S_2(t) - q_3(t)S_3(t) - q_{et}(t)S_e(t))\right],$$
$$\text{subject to} \quad \sum_{t=1}^{T} Q_e(q_2(t),q_3(t)) - \sum_{t=1}^{T} q_{et}(t) - A \leq 0. \tag{2}$$

Problem (2) is easier than (1) because its Lagrangian is differentiable. In addition, because of the uniqueness of the optimal solution $[q_2^*, q_3^*]$ (proved

[4]See the Appendix for the proof.

in the Appendix), we can identify the optimal solution and its properties through solving (2).

Let us assume that $q_2^*(t)$ and $q_3^*(t)$ are positive. As the Appendix shows, the first order conditions for $q_2^*(t)$, $q_3^*(t)$, and $q_{et}^*(t)$ are

$$\begin{aligned} 0 &= -S_e(t) + E_t\left[e^{-r(T-t)}S_e(T)\right], \\ 0 &= Q_{1,q_i}S_1(t) - S_i(t) - Q_{e,q_i}E_t\left[e^{-r(T-t)}S_e(T)\right], \quad i = 2, 3. \end{aligned}$$

The first equation implies

$$S_e(t) = E_t\left[e^{-r(T-t)}S_e(T)\right].$$

Furthermore, the second equation implies

$$\begin{aligned} S_1(t) &= \frac{S_2(t) + Q_{e,q_2}S_e(t)}{Q_{1,q_2}} \\ &= \frac{S_3(t) + Q_{e,q_3}S_e(t)}{Q_{1,q_3}}, \end{aligned}$$

where $Q_{1,q_3}, Q_{1,q_2}, Q_{e,q_2}$, and Q_{e,q_3} are partial derivatives of the production and emission functions with respect to variables q_2 and q_3. These equations imply that the marginal revenues are equal to the marginal costs of inputs and emission allowances. By solving this equation for $S_e(t)$, we obtain

$$S_e(t) = \frac{S_2(t)}{Q_{1,q_2}\left(\frac{Q_{e,q_3}}{Q_{1,q_3}} - \frac{Q_{e,q_2}}{Q_{1,q_2}}\right)} - \frac{S_3(t)}{Q_{1,q_3}\left(\frac{Q_{e,q_3}}{Q_{1,q_3}} - \frac{Q_{e,q_2}}{Q_{1,q_2}}\right)}.$$

Proposition 2.2.1. Suppose that there is no arbitrage opportunity, i.e., $S_e(T) \leq Z$. If $q_2^*(t)$ and $q_3^*(t)$ are positive, $Q_{1,q_i} \neq 0, Q_{e,q_i} \neq 0, i = 2, 3$, and $\frac{Q_{e,q_3}}{Q_{1,q_3}} - \frac{Q_{e,q_2}}{Q_{1,q_2}} \neq 0$, we have

$$\begin{aligned} S_e(t) &= E_t\left[e^{-r(T-t)}S_e(T)\right], \\ S_e(t) &= \frac{Q_{1,q_i}}{Q_{e,q_i}}S_1(t) - \frac{1}{Q_{e,q_i}}S_i(t), \quad i = 2, 3, \\ S_e(t) &= \left(\frac{Q_{e,q_3}}{Q_{1,q_3}} - \frac{Q_{e,q_2}}{Q_{1,q_2}}\right)^{-1}\left(\frac{S_2(t)}{Q_{1,q_2}} - \frac{S_3(t)}{Q_{1,q_3}}\right), \end{aligned}$$

where the first subscripts indicate the production and emission functions with "1" and "e," respectively, and the second subscripts indicate the partial derivatives.

The first equation is the intertemporal relation of the emission allowance price. It asserts that the emission allowance price at time t is the present value of the emission allowance price at the end of period T.

The second equation is the intratemporal relation among the emission allowance, output, and input prices. While the relation is simply from the equality between marginal revenue and marginal cost under profit maximization, it suggests that the emission allowance price should be the spread of output and input prices adjusted by the production and emission rates. If we consider a power company, emission allowances should be equal to the spread between electricity (output) and coal (input) prices.[5]

The third equation is another intratemporal relation among the emission allowance price and two inputs prices. Again, this relation is satisfied when the marginal costs of inputs with emission allowances meet. It implies that the emission allowance price is the price difference between the two input

[5] Although this model considers only one firm, the spread relations can also be obtained when there are multiple firms. Suppose that there are J firms in the emission allowance trading market. By the same argument, we have

$$Q_{e,j,q_i} S_e(t) = Q_{1,j,q_i} S_1(t) - S_i(t), \quad i = 2, 3,$$
$$\left(\frac{Q_{e,j,q_3}}{Q_{1,j,q_3}} - \frac{Q_{e,j,q_2}}{Q_{1,j,q_2}} \right) S_e(t) = \left(\frac{S_2(t)}{Q_{1,j,q_2}} - \frac{S_3(t)}{Q_{1,j,q_3}} \right),$$

where we denote $Q_{1,j}$ and $Q_{e,j}$ as the production and emission functions, respectively, and the third subscripts indicate the partial derivative. Summing these equations

$$\sum_{j=1}^{J} Q_{e,j,q_i} S_e(t) = \sum_{j=1}^{J} (Q_{1,j,q_i} S_1(t) - S_i(t)), \quad i = 2, 3,$$
$$\sum_{j=1}^{J} \left(\frac{Q_{e,j,q_3}}{Q_{1,j,q_3}} - \frac{Q_{e,j,q_2}}{Q_{1,j,q_2}} \right) S_e(t) = \sum_{j=1}^{J} \left(\frac{S_2(t)}{Q_{1,j,q_2}} - \frac{S_3(t)}{Q_{1,j,q_3}} \right).$$

Thus,

$$S_e(t) = \left(\sum_{j=1}^{J} Q_{e,j,q_i} \right)^{-1} \sum_{j=1}^{J} (Q_{1,j,q_i} S_1(t) - S_i(t)), \quad i = 2, 3,$$
$$S_e(t) = \left(\sum_{j=1}^{J} \frac{Q_{e,j,q_3}}{Q_{1,j,q_3}} - \frac{Q_{e,j,q_2}}{Q_{1,j,q_2}} \right)^{-1} \sum_{j=1}^{J} \left(\frac{S_2(t)}{Q_{1,j,q_2}} - \frac{S_3(t)}{Q_{1,j,q_3}} \right).$$

These are the spread relations for multiple firms.

prices adjusted by the emission/production ratio. For instance, the emission allowance price is the spread of coal and natural gas prices that are modified by the heat and emission rates, or in other words the fuel switching cost. This fundamental relation was implicitly utilized in Carmona, Fehr, and Hinz (2009) and confirmed empirically by Fezzi and Bunn (2009).[6]

We emphasize that these are necessary conditions for a firm's profit maximization. Although we do not discuss the equilibrium, if there is an equilibrium, these intra- and intertemporal conditions should also be satisfied.

2.3 Sensitive Analysis on Hedge Ratio

As we showed in Proposition 2.2.1, the spread relation was

$$S_e(t) = H_2(t)S_2(t) - H_3(t)S_3(t),$$

where

$$H_i(t) \triangleq \left\{ Q_{1,q_i} \left(\frac{Q_{e,q_3}}{Q_{1,q_3}} - \frac{Q_{e,q_2}}{Q_{1,q_2}} \right) \right\}^{-1}, \quad i = 2, 3.$$

In other words,

$$S_e(t) - H_2(t)S_2(t) + H_3(t)S_3(t) = 0,$$

which suggests that one unit of emission allowance can be hedged by selling $H_2(t)$ amount of $S_2(t)$ and buying $H_3(t)$ amount of $S_3(t)$. In this sense, we call the ratio $\frac{H_2(t)}{H_3(t)}$ of spread coefficients the relative hedge ratio in the following.

By assuming the form of the production and emission functions, we can analyze the behavior of the spread relation. We study how the relation changes depending on the technologies, production levels, and prices for commonly used production functions. We assume that the emission function is linear, i.e.,

$$Q_e(q_2, q_3) = Q_{e2}q_2 + Q_{e3}q_3,$$

[6] See also Delarue, Lamberts, and D'haeseleer (2007), Delarue and D'haeseleer (2008), and Delarue, Voorspools, and D'haeseleer (2008), who use fuel switching models to simulate GHG emission reduction potentials.

where Q_{e2} and Q_{e3} are constants. We consider two cases which are the Cobb-Douglas and the CES production functions.

Case1

Assume the Cobb-Douglas production function

$$Q_1(q_2, q_3) = Q_{1a} q_2^{\gamma_2} q_3^{\gamma_3}, \\ \gamma_2, \gamma_3 \geq 0.$$

Then, the spread coefficients are given by

$$H_2(t) = \left(\frac{\gamma_2 Q_{e3} q_3^*(t)}{\gamma_3 q_2^*(t)} - Q_{e2} \right)^{-1}, \\ H_3(t) = \left(Q_{e3} - \frac{\gamma_3 Q_{e2} q_2^*(t)}{\gamma_2 q_3^*(t)} \right)^{-1}. \qquad (3)$$

The relative hedge ratio $H_2(t)/H_3(t)$ becomes

$$\frac{H_2(t)}{H_3(t)} = \frac{Q_{e3} - \frac{\gamma_3 Q_{e2} q_2^*(t)}{\gamma_2 q_3^*(t)}}{\frac{\gamma_2 Q_{e3} q_3^*(t)}{\gamma_3 q_2^*(t)} - Q_{e2}}.$$

We can calculate the sensitivity of the relative hedge ratio to optimal input ratio $q_2^*(t)/q_3^*(t)$ to obtain

$$\frac{\partial \left(\frac{H_2(t)}{H_3(t)} \right)}{\partial \left(\frac{q_2^*(t)}{q_3^*(t)} \right)} = \frac{\left(\frac{\sqrt{\gamma_3}}{\sqrt{\gamma_2}} Q_{e2} - \frac{\sqrt{\gamma_2} Q_{e3} q_3^*(t)}{\sqrt{\gamma_3} q_2^*(t)} \right)^2}{\left(\frac{\gamma_2 Q_e 3 q_3^*(t)}{\gamma_3 q_2^*(t)} - Q_{e2} \right)^2} \geq 0.$$

This means that the relative hedge ratio increases when the optimal input ratio increases.

Meanwhile, the first order conditions for $q_2^*(t), q_3^*(t)$ are

$$0 = \gamma_2 Q_{1a} q_2^*(t)^{\gamma_2 - 1} q_3^*(t)^{\gamma_3} S_1(t) - S_2(t) - Q_{e2} S_e(t), \\ 0 = \gamma_3 Q_{1a} q_2^*(t)^{\gamma_2} q_3^*(t)^{\gamma_3 - 1} S_1(t) - S_3(t) - Q_{e3} S_e(t).$$

Thus, the optimal solutions are given by

$$q_2^*(t) = \left\{ \frac{(\gamma_2 Q_{1a} S_2(t))^{\frac{\gamma_3-1}{\gamma_3}} (S_3(t) + Q_{e3} S_e(t))}{\gamma_3 Q_1 S_1(t)(S_2(t) + Q_{e2} S_e(t))^{\frac{\gamma_3-1}{\gamma_3}}} \right\}^{\frac{\gamma_3}{\gamma_2+\gamma_3-1}},$$

$$q_3^*(t) = \left\{ \frac{(\gamma_3 Q_{1a} S_3(t))^{\frac{\gamma_2-1}{\gamma_2}} (S_2(t) + Q_{e2} S_e(t))}{\gamma_2 Q_1 S_1(t)(S_3(t) + Q_{e3} S_e(t))^{\frac{\gamma_2-1}{\gamma_2}}} \right\}^{\frac{\gamma_2}{\gamma_2+\gamma_3-1}}.$$

Therefore, the optimal input ratio $q_2^*(t)/q_3^*(t)$ is

$$\frac{q_2^*(t)}{q_3^*(t)} = \frac{\gamma_2(S_3(t) + Q_{e3}S_e(t))}{\gamma_3(S_2(t) + Q_{e2}S_e(t))}.$$

Then, we can calculate the sensitivity of the optimal input ratio to the relative price $S_2(t)/S_3(t)$ to satisfy

$$\frac{\partial\left(\frac{q_2^*(t)}{q_3^*(t)}\right)}{\partial\left(\frac{S_2(t)}{S_3(t)}\right)} = \frac{-\gamma_2 S_3(t)^2 - \gamma_2 Q_{e3} S_3(t) S_e(t) - \gamma_2 S_3(t)^2 - \gamma_2 Q_{e2} \frac{S_3(t)^2 S_e(t)}{S_2(t)}}{\gamma_3(S_2(t) + Q_{e2}S_e(t))^2} \leq 0,$$

which implies that the optimal input ratio decreases when the relative price increases.

Finally, we can calculate the sensitivity of the relative hedge ratio $H_2(t)/H_3(t)$ to the relative price $S_2(t)/S_3(t)$ to obtain

$$\begin{aligned} \frac{\partial\left(\frac{H_2(t)}{H_3(t)}\right)}{\partial\left(\frac{S_2(t)}{S_3(t)}\right)} &= \frac{\partial\left(\frac{H_2(t)}{H_3(t)}\right)}{\partial\left(\frac{q_2^*(t)}{q_3^*(t)}\right)} \cdot \frac{\partial\left(\frac{q_2^*(t)}{q_3^*(t)}\right)}{\partial\left(\frac{S_2(t)}{S_3(t)}\right)} \\ &= \left(\frac{\frac{\sqrt{\gamma_3}}{\sqrt{\gamma_2}} Q_{e2} - \frac{\sqrt{\gamma_2} Q_{e3} q_3^*(t)}{\sqrt{\gamma_3} q_2^*(t)}}{\frac{\gamma_2 Q_{e3} q_3^*(t)}{\gamma_3 q_2^*(t)} - Q_{e2}} \right)^2 \\ &\quad \times \frac{-\gamma_2 S_3(t)^2 - \gamma_2 Q_{e3} S_3(t) S_e(t) - \gamma_2 S_3(t)^2 - \gamma_2 Q_{e2} \frac{S_3(t)^2 S_e(t)}{S_2(t)}}{\gamma_3(S_2(t) + Q_{e2}S_e(t))^2} \leq 0. \end{aligned}$$

Thus, the relative hedge ratio decreases as the relative price increases. That is, for the Cobb-Douglas production function, when an input price increases relative to the other, the coefficient of the input in the spread relation decreases relative to the other. This is a testable implication for empirical research on the spread relation among emission allowances and other commodity prices.

Case2

Assume the CES production function

$$Q_1(q_2, q_3) = Q_{1b}\left(Q_{12}q_2^{\gamma} + Q_{13}q_3^{\gamma}\right)^{\frac{1}{\gamma}},$$
$$Q_{1b}, Q_{12}, Q_{13} \geq 0.$$

Then, the spread coefficients are given by

$$H_2(t) = \left(\frac{Q_{12}Q_{e3}q_2^*(t)^{\gamma-1}}{Q_{13}q_3^*(t)^{\gamma-1}} - Q_{e2}\right)^{-1},$$
$$H_3(t) = \left(Q_{e3} - \frac{Q_{13}Q_{e2}q_3^*(t)^{\gamma-1}}{Q_{12}q_2^*(t)^{\gamma-1}}\right)^{-1}.$$

The relative hedge ratio $H_2(t)/H_3(t)$ becomes

$$\frac{H_2(t)}{H_3(t)} = \frac{Q_{e3} - \frac{Q_{13}Q_{e2}q_3^*(t)^{\gamma-1}}{Q_{12}q_2^*(t)^{\gamma-1}}}{\frac{Q_{12}Q_{e3}q_2^*(t)^{\gamma-1}}{Q_{13}q_3^*(t)^{\gamma-1}} - Q_{e2}}.$$

We can calculate the sensitivity of the relative hedge ratio to the optimal input ratio $q_2^*(t)/q_3^*(t)$ to obtain

$$\frac{\partial\left(\frac{H_2(t)}{H_3(t)}\right)}{\partial\left(\frac{q_2^*(t)}{q_3^*(t)}\right)} = \frac{\left\{\frac{\sqrt{Q_{13}(1-\gamma)}Q_{e2}}{\sqrt{Q_{12}}}\left(\frac{q_2^*(t)}{q_3^*(t)}\right)^{-\frac{\gamma}{2}} - \frac{\sqrt{Q_{12}(1-\gamma)}Q_{e3}}{\sqrt{Q_{13}}}\left(\frac{q_2^*(t)}{q_3^*(t)}\right)^{\frac{\gamma}{2}-1}\right\}^2}{\left(\frac{Q_{12}Q_{e3}q_2^*(t)^{\gamma-1}}{Q_{13}q_3^*(t)^{\gamma-1}} - Q_{e2}\right)^2} \geq 0,$$

which means that the relative hedge ratio increases when the optimal input ratio increases.

The first order conditions for $q_2^*(t), q_3^*(t)$ are

$$0 = Q_{1b}Q_{1i}(Q_{12}q_2^*(t)^{\gamma} + Q_{13}q_3^*(t)^{\gamma})^{\frac{1-\gamma}{\gamma}}q_i^*(t)^{\gamma-1}S_1(t) - S_i(t) - Q_{e2}S_e(t) \quad i = 2, 3,$$

and the optimal input ratio $q_2^*(t)/q_3^*(t)$ is[7]

$$\frac{q_2^*(t)}{q_3^*(t)} = \left\{\frac{(S_3(t) + Q_{e3}S_e(t))Q_{12}}{(S_2(t) + Q_{e2}S_e(t))Q_{13}}\right\}^{\frac{1}{1-\gamma}}.$$

[7]Because the first order condition is a nonlinear equation, we cannot solve the optimal solution as in the case of the Cobb-Douglas production function.

The sensitivity of the optimal input ratio to the relative price $S_2(t)/S_3(t)$ can be calculated as

$$
\begin{aligned}
\frac{\partial\left(\frac{q_2^*(t)}{q_3^*(t)}\right)}{\partial\left(\frac{S_2(t)}{S_3(t)}\right)} \quad = \quad & \frac{Q_{13}S_3(t)}{(1-\gamma)Q_{12}(S_3(t)+Q_{e3}S_e(t))}\left\{\frac{Q_{13}(S_2(t)+Q_{e2}S_e(t))}{Q_{12}(S_3(t)+Q_{e3}S_e(t))}\right\}^{\frac{\gamma}{1-\gamma}} \\
& \times\left\{1+\frac{(S_2(t)+Q_{e2}S_e(t))S_3(t)}{(S_3(t)+Q_{e3}S_e(t))S_2(t)}\right\}\left\{\begin{array}{ll} \geq 0, & 1-\gamma>0, \\ \leq 0, & 1-\gamma<0. \end{array}\right.
\end{aligned}
$$

Thus, the sensitivity of the relative hedge ratio $H_2(t)/H_3(t)$ to the relative price $S_2(t)/S_3(t)$ is

$$
\begin{aligned}
\frac{\partial\left(\frac{H_2(t)}{H_3(t)}\right)}{\partial\left(\frac{S_2(t)}{S_3(t)}\right)} \quad = \quad & \left\{\frac{\frac{\sqrt{Q_{13}(1-\gamma)}Q_{e2}}{\sqrt{Q_{12}}}\left(\frac{q_2^*(t)}{q_3^*(t)}\right)^{-\frac{\gamma}{2}}-\frac{\sqrt{Q_{12}(1-\gamma)}Q_{e3}}{\sqrt{Q_{13}}}\left(\frac{q_2^*(t)}{q_3^*(t)}\right)^{\frac{\gamma}{2}-1}}{\frac{Q_{12}Q_{e3}q_2^*(t)^{\gamma-1}}{Q_{13}q_3^*(t)^{\gamma-1}}-Q_{e2}}\right\}^2 \\
& \times\frac{Q_{13}S_3(t)}{(1-\gamma)Q_{12}(S_3(t)+Q_{e3}S_e(t))}\left\{\frac{Q_{13}(S_2(t)+Q_{e2}S_e(t))}{Q_{12}(S_3(t)+Q_{e3}S_e(t))}\right\}^{\frac{\gamma}{1-\gamma}} \\
& \times\left\{1+\frac{(S_2(t)+Q_{e2}S_e(t))S_3(t)}{(S_3(t)+Q_{e3}S_e(t))S_2(t)}\right\}\left\{\begin{array}{ll} \geq 0, & 1-\gamma>0, \\ \leq 0, & 1-\gamma<0. \end{array}\right.
\end{aligned}
$$

This equation implies that the relative hedge ratio decreases as the relative price increases if $1-\gamma<0$, but decreases if $1-\gamma>0$. Though it depends on the value of γ, this result is also a testable implication for empirical study.

2.4 Implications for Empirical Analysis

We saw that the hedge ratio $H_2(t)/H_3(t)$ depends on the price ratio $S_2(t)/S_3(t)$ through the production function and relative optimal solution $q_2^*(t)/q_3^*(t)$. If the production function is the Cobb-Douglas functional form, the hedge ratio decreases as the price ratio increases. On the other hand, if the production function is the CES functional form, the sensitivity of the hedge ratio depends on the parameter γ. These imply that an empirical model for the prices of emission allowances, natural gas, and coal should include time-varying coefficients and not constant coefficients.

This implication can be tested empirically by the following model.

$$\begin{aligned} S_e(t) &= (\eta_{2,1} + \eta_{2,2} 1_{S_2(t)/S_3(t) < R_S}) S_2(t) \\ &\quad + (\eta_{3,1} + \eta_{3,2} 1_{S_2(t)/S_3(t) < R_S}) S_3(t) + z(t). \end{aligned} \tag{4}$$

If the price ratio is greater than the threshold R_S, $\eta_{i,1}$ $(i = 2, 3)$ are the coefficients for $S_i(t)$; if not, $\eta_{i,1} + \eta_{i,2}$ $(i = 2, 3)$ are the coefficients. In other words, the hedge ratio changes depending on the level of the price ratio.

If there are autocorrelated disturbances in the above model, we can interpret (4) as a long run relationship with a vector autoregression structure, i.e., cointegration. That is,

$$\Delta \boldsymbol{S}(t) = \boldsymbol{\alpha} + \boldsymbol{\Gamma}_1 \Delta \boldsymbol{S}(t-1) + \cdots + \boldsymbol{\Gamma}_p \Delta \boldsymbol{S}(t-p+1) + \boldsymbol{b} z(t-1) + \boldsymbol{e}(t), \tag{5}$$

where $\boldsymbol{b}$ is the adjustment coefficient. While (4) incorporates the dependence of the hedge ratio on the level of the price ratio, it maintains the relationship by $z(t)$ and $\boldsymbol{b}$ through (5). Indeed, Fezzi and Bunn (2009) analyzed the cointegration among emission allowances, electricity, and natural gas prices. However, they modeled the long run relationship using constant coefficients. Our results suggest an extension of the model to one with state-dependent coefficients.

Other empirical models of interest are models with time-varying parameter coefficients and regime-switching coefficients. We emphasize that state-dependent coefficients are more realistic for modeling the relation of prices among emission allowances and other energy commodities. Thus, we need empirical models with coefficients that change with prices, price ratio, and/or other factors.

2.5 Conclusion

We characterized the inter- and intratemporal relations among the emission allowance price and commodity prices under a firm's profit maximization. These relations are necessary conditions for equilibrium. The first equation showed that the emission allowance price at time t is the present value of the emission allowance price at the end of period T. The second equation implied the emission allowance price was the spread between output and input prices adjusted by production/emission ratio. The third point was that the emission allowance price was the price difference between two input prices adjusted by the production/emission ratio, i.e., the fuel switching cost.

We analyzed how the relative hedge ratio changes as the input commodity prices change. Based on our results, we suggested empirical models of emission allowances and commodity prices. The coefficients of the spread relation should not be constant and depend on prices, price ratios, and/or other factors.

There are several other issues left for future research. First, the theoretical model could be extended to investigate the general equilibrium of emission allowances and commodities. Furthermore, the central authority that controls the penalty and initial allocation should be included so as to allow an analysis of the welfare of emission trading. Also, it would be interesting to analyze derivative pricing of emission allowances using these relations which is the issue of the next chapter.

2.6 Appendix

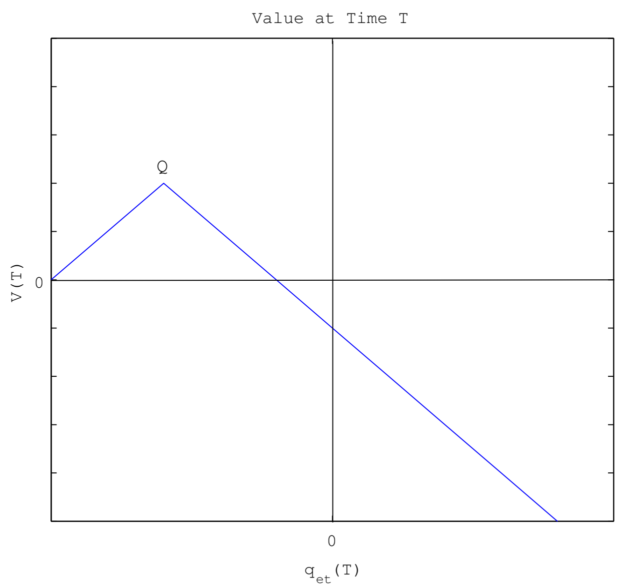
In this section, we prove that the optimal solution of problems (1) and (2) are the same. Note that there are some difficulties in obtaining the optimal solution for (1) because of a point at which the objective function cannot be differentiated. This difficulty can be overcome by proving the equivalence between the optimal solutions of (2) and that of (1).

Let us consider the end of period T of this problem, which will be adequate for describing our point. Suppose $[q_2(t), q_3(t), q_{et}(t)]_{t=1}^{T-1} \in \mathbb{R}^{3(T-1)}$ and $q_2(T), q_3(T) \in \mathbb{R}$ are fixed.

$$\max_{q_{et}(T)} -e^{-rT} q_{et}(T) S_e(T) - e^{-rT} Z \left(\sum_{t=1}^{T} Q_e(q_2(t), q_3(t)) - \sum_{t=1}^{T} q_{et}(t) - A \right)^+ .$$

This is the problem for $q_{et}(T)$ that can be solved by examining the following Figure 2.1. This figure is a graph of

Figure 2.1: The value at time T. Q is $\sum_{t=1}^{T} Q_{e,2}(q_2(t), q_3(t)) - \sum_{t=1}^{T-1} q_{et}(t) - A$, which gives the maximum value for $q_{et}(T)$.



$$
\begin{aligned}
V' &= \max_{q_{et}(T)} -e^{-rT} q_{et}(T) S_e(T) - e^{-rT} Z \left(\sum_{t=1}^{T} Q_e(q_2(t), q_3(t)) - \sum_{t=1}^{T} q_{et}(t) - A \right)^{+} \\
&= \begin{cases} -e^{-rT} q_{et}(T) S_e(T), & \text{for } \; Q(T) \le q_{et}(T), \\ -e^{-rT} Z Q(T) + e^{-rT}(Z - S_e(T)) q_{et}(T), & \text{for } \; Q(T) > q_{et}(T), \end{cases}
\end{aligned}
$$

where $Q(T) = \sum_{t=1}^{T} Q_e(q_2(t), q_3(t)) - \sum_{t=1}^{T-1} q_{et}(t) - A$. We can ignore the case $S_e(T) > Z$, because if this is true, then the firm sells arbitrarily large amounts of emission allowances, accepts the penalty, and obtains arbitrage profit. Hence, it is clear from Figure 2.1 that the optimal solution is[8]

$$
q_{et}^{*}(T) = \sum_{t=1}^{T} Q_e(q_2(t), q_3(t)) - \sum_{t=1}^{T-1} q_{et}(t) - A.
$$

However, this is the point at which the objective function cannot be differentiated, which implies that we may not obtain a simple first order condition. In order to handle this difficulty, we need to incorporate another constraint. Thus, we need problem (2). We now prove that the optimal solution of these problems is the same.

Lemma 2.6.1. Let us assume that there is no arbitrage or in other words $S_e(T) \le Z$.

1. If there exists an optimal solution to (1), then the solution is an optimal solution to problem (2).

2. If there exists an optimal solution to (2), then the solution is an optimal solution to problem (1).

3. If there exists an optimal solution to any one of the problems, then problems (1) and (2) have the same optimal value.

[8] To be more precise, for the case $S_e(T) = Z$, optimality can be obtained by any $q_{et}(T)$ such that $q_{et}(T) \le \sum_{t=1}^{T} Q_e(q_2(t), q_3(t)) - \sum_{t=1}^{T-1} q_{et}(t) - A$. However, we consider only $q_{et}(T) = \sum_{t=1}^{T} Q_e(q_2(t), q_3(t)) - \sum_{t=1}^{T-1} q_{et}(t) - A$ as the optimal solution. This can be achieved by defining the optimal solution to be the nearest to the origin if they are not unique. Economically, q_{et} is the number of carbon allowance contracts that are traded with transaction costs. This suggests that optimality should be near the origin where the transaction costs are the lowest.

4. Suppose there exists an optimal solution to any one of the problems. Furthermore, assume that the optimal solutions are interior points and that $\partial^2 Q_1/\partial q_i^2 < 0$ or $\partial^2 Q_e/\partial q_i^2 > 0$ for $i = 2, 3$. Then, the optimal solution for inputs $[q_2^*, q_3^*]$ to problem (2) is unique and hence the uniqueness of the optimal solution of inputs $[q_2^*, q_3^*]$ to problem (1) is satisfied.

Proof. We first prove 1 and 2. 3 is obvious from 1 and 2. Let us construct another problem.

$$\max_{q(\cdot)\in\mathcal{U}[0,T]} E_0 \left[\sum_{t=1}^{T} e^{-rt}(Q_1(q_2(t), q_3(t))S_1(t) - q_2(t)S_2(t) - q_3(t)S_3(t) - q_{et}(t)S_e(t)) \right.$$
$$\left. -e^{-rT} Z \left(\sum_{t=1}^{T} Q_e(q_2(t), q_3(t)) - \sum_{t=1}^{T} q_{et}(t) - A\right)^{+}\right],$$

$$\text{subject to} \quad \sum_{t=1}^{T} Q_e(q_2(t), q_3(t)) - \sum_{t=1}^{T} q_{et}(t) - A \leq 0. \tag{6}$$

Denote $x^* = [x_2^*(t), x_3^*(t), x_e^*(t)]_{t=1}^T$, $y^* = [y_2^*(t), y_3^*(t), y_e^*(t)]_{t=1}^T$, and $z^* = [z_2^*(t), z_3^*(t), z_e^*(t)]_{t=1}^T$ as optimal solutions to (1), (2), and (6), respectively. Define

$$D = \left\{ (q_2(t), q_3(t), q_{et}(t))_{t=1}^T \middle| \sum_{t=1}^{T} Q_e(q_2(t), q_3(t)) - \sum_{t=1}^{T} q_{et}(t) - A \leq 0 \right\}.$$

Obviously, y^*, $z^* \in D$ and from the discussion above $x^* \in D$ also. We have

$$\sum_{t=1}^{T} Q_e(y_2^*(t), y_3^*(t)) - \sum_{t=1}^{T} y_e^*(t) - A = 0 = \sum_{t=1}^{T} Q_e(z_2^*(t), z_3^*(t)) - \sum_{t=1}^{T} z_e^*(t) - A.$$

Therefore, if there exists an optimal solution to (2), then it is an optimal solution to (6), and vice versa with the same optimal value.

Now, we show that problems (1) and (6) have the same optimal solution. Suppose there exists an optimal solution to (6). Let $[x_2(t), x_3(t), x_e(t)]_{t=1}^T$ be any feasible solution to (1). Define solution $[z_2(t), z_3(t), z_e(t)]_{t=1}^T$, where $[z_2(t), z_3(t), z_e(t)] = [x_2(t), x_3(t), x_e(t)]$ for $t < T$ and

$$z_i(T) = x_i(T), i = 2, 3,$$
$$z_e(T) = \sum_{t=1}^{T} Q_e(x_2(t), x_3(t)) - \sum_{t=1}^{T-1} x_e(t) - A.$$

We have already seen that the value of the objective function of problem (1) for $[z_2(t), z_3(t), z_e(t)]_{t=1}^T$ is equal to or larger than the value corresponding to $[x_2(t), x_3(t), x_e(t)]_{t=1}^T$. Furthermore, by definition, $[z_2(t), z_3(t), z_e(t)]_{t=1}^T \in D$, i.e., feasible for problem (1) and problem (6). Thus, $[z_2^*(t), z_3^*(t), z_e^*(t)]_{t=1}^T$ is an optimal solution to (1), since it is an optimal solution to (6).

It is easy to see that the optimal solution for problem (1) is indeed optimal for (6). We have already seen that $[x_2^*(t), x_3^*(t), x_e^*(t)]_{t=1}^T$ satisfy $\sum_{t=1}^T Q_e(x_2^*(t), x_3^*(t)) - \sum_{t=1}^T x_e^*(t) - A = 0$, so this solution must be optimal for (6).

We now prove 4, the uniqueness of the optimal solution. For this, we define several terms. Let us assume

$$\begin{aligned} J(0, x_0; u(\cdot)) &= E_0\left[\sum_{t=1}^T p_t(x(t), u_t(x(t)))\right], \\ x(t+1) &= f_t(x(t), q(t), \omega(t)), \end{aligned}$$

where

$$\begin{aligned} p_t(x(t), u_t(x(t))) &= e^{-rt}(Q_1(q_2(t), q_3(t))S_1(t) - q_2(t)S_2(t) \\ &\quad - q_3(t)S_3(t) - q_{et}(t)S_e(t)), \\ f_t(x(t), q(t), \omega(t)) &= \begin{bmatrix} Q_{ce}(t-1) + Q_e(q_2(t), q_3(t)) \\ Q_{cet}(t-1) + q_{et}(t) \\ f_{1t}(S_1(t), \omega(t)) \\ f_{2t}(S_2(t), \omega(t)) \\ f_{3t}(S_3(t), \omega(t)) \\ f_{et}(S_e(t), \omega(t)), \end{bmatrix}, \end{aligned}$$

$x(t) = [Q_{ce}(t), Q_{cet}(t), S_1(t), S_2(t), S_3(t), S_e(t)]^\top$, $x_0 = [0, 0, s_1, s_2, s_3, s_e]^\top$, $\omega(t) \in \Omega$ is a disturbance term, $f_{it}(\cdot, \cdot) : \mathbb{R} \times \Omega \to \mathbb{R}$ are measurable functions, and $u(\cdot)$ is a sequence of functions $\{u_1, \ldots, u_T\}$ where u_t maps states $x(t)$ to controls $u_t(x(t)) = [\ q_2(t) \quad q_3(t) \quad q_{et}(t)\ ]$ and is such that $u_t(x(t)) \in \mathcal{U}_t(x(t)) = \{q_i \leq \bar{q}_i, i = 2, 3, Q_{ce}(T) - Q_{cet}(T) - A \leq 0\}$ for all $x(t) \in \mathbb{R}^6$ and call these policies admissible.

The value function is

$$V(0, x_0) = \max_{u(\cdot) \in \mathcal{U}[0,T]} J(0, x_0; u(\cdot)).$$

Proceeding backwards, the Bellman equation of (2) at time t is[9]

$$\begin{aligned} V(t,x(t)) &= \max_{q_2(t),q_3(t),q_{et}(t)} Q_1(q_2(t),q_3(t))S_1(t) - q_2(t)S_2(t) - q_3(t)S_3(t) - q_{et}(t)S_e(t) \\ &\quad + E_t[V(t+1,x(t+1))], \end{aligned}$$

where

$$\begin{aligned} V(t+1,x(t+1)) &= \sum_{s=t+1}^{T-1} p_s(x(s),u_s(x(s))) + p_T(x(T),u_T(x(T))), \\ u_T(x(T)) &= \begin{bmatrix} q_2^*(T) \\ q_3^*(T) \\ \sum_{s=1}^{T} Q_e(q_2^{**}(s),q_3^{**}(s)) - \sum_{s=1}^{T-1} q_{et}^{**}(s) - A \end{bmatrix}. \end{aligned}$$

We have used the fact that $q_{et}(T) = \sum_{t=1}^{T} Q_e(q_2(t),q_3(t)) - \sum_{t=1}^{T-1} q_{et}(t) - A$ is the optimal solution.

The first order conditions at t are

$$\begin{aligned} 0 &= \frac{\partial Q_1(q_2^*(t),q_3^*(t))}{\partial q_i} S_1(t) - S_i(t) \\ &\quad - \frac{\partial Q_e(q_2^*(t),q_3^*(t))}{\partial q_i} E_t\left[e^{-r(T-t)}S_e(T)\right], i = 2,3, \quad (7) \\ 0 &= -S_e(t) + E_t\left[e^{-r(T-t)}S_e(T)\right]. \quad (8) \end{aligned}$$

By assumption, we have

$$\frac{\partial^2 Q_1(q_2(t),q_3(t))}{\partial q_i^2} S_1(t) - S_e(t)\frac{\partial^2 Q_e(q_2(t),q_3(t))}{\partial q_i^2} < 0, i = 2,3.$$

This implies that

$$\frac{\partial Q_1(q_2(t),q_3(t))}{\partial q_i} S_1(t) - S_i(t) - S_e(t)\frac{\partial Q_e(q_2(t),q_3(t))}{\partial q_i}, i = 2,3,$$

is a strictly decreasing function and there should be only one $(q_2^*(t),q_3^*(t))$. Thus, the uniqueness of the optimal solution of inputs (q_2,q_3) to problem

[9] See Bertsekas and Shreve (1978) and Bertsekas (2005) for dynamic programming and the Bellman equation.

(2) has been proved. If $(q_2^*(t), q_3^*(t), q_{et}^*(t))_{t=1}^T$ and $(q_2'(t), q_3'(t), q_{et}^*(t))_{t=1}^T$ are optimal to (1), these solutions are optimal to (2). We have $(q_i^*(t))_{t=1}^T = (q_i'(t))_{t=1}^T, i = 2, 3$ by the uniqueness of the optimal solution of (2). Thus, it also implies the uniqueness of the optimal solution of inputs (q_2, q_3) to problem (1). The same argument applies to (6).[10] □

Unfortunately, we cannot determine $q_{et}^*(t)$ uniquely. From the previous discussion, we can only say that it must satisfy

$$\sum_{t=1}^T q_{et}^*(t) = \sum_{t=1}^T Q_e(q_2^*(t), q_3^*(t)) - A.$$

One optimal solution is

$$q_{et}^*(t) = 0, t = 1, \ldots, T-1,$$
$$q_{et}^*(T) = \sum_{t=1}^T Q_e(q_2^*(t), q_3^*(t)) - A.$$

Although our focus is on commodity prices and their relations, it is also important to know the behavior of the optimal price solutions. We show some results of sensitivity analysis. Let us assume $\partial^2 Q_1/\partial q_i^2 < 0$ or $\partial^2 C/\partial q_2^2 > 0$. From the first order conditions for problem (2), we obtained

$$0 = \frac{\partial Q_1(q_2^*(t), q_3^*(t))}{\partial q_i} S_1(t) - S_i(t) - S_e(t) \frac{\partial Q_e(q_2^*(t), q_3^*(t))}{\partial q_i}. \tag{9}$$

We use this equation for sensitivity analysis.

From the implicit function theorem, we have the following result.

Proposition 2.6.1. If $\partial^2 Q_1(q_2^*(t), q_3^*(t))/\partial q_i^2 < 0$ and $0 < q_i^*(t)$ (which implies $(q_2^*(t), q_3^*(t))$ is an interior point of (9)), then

$$\begin{aligned}
\frac{\partial q_i^*(t)}{\partial S_1(t)} &= \frac{-\frac{\partial Q_1(q_2^*(t), q_3^*(t))}{\partial q_i}}{\frac{\partial^2 Q_1(q_2^*(t), q_3^*(t))}{\partial q_i^2} S_1(t) - \frac{\partial Q_e(q_2^*(t), q_3^*(t))}{\partial q_i} S_e(t)} \geq 0, \\
\frac{\partial q_i^*(t)}{\partial S_i(t)} &= \frac{1}{\frac{\partial^2 Q_1(q_2^*(t), q_3^*(t))}{\partial q_i^2} S_1(t) - \frac{\partial Q_e(q_2^*(t), q_3^*(t))}{\partial q_2} S_e(t)} < 0, \\
\frac{\partial q_i^*(t)}{\partial S_e(t)} &= \frac{\frac{\partial Q_e(q_2^*(t), q_3^*(t))}{\partial q_i}}{\frac{\partial^2 Q_1(q_2^*(t), q_3^*(t))}{\partial q_i^2} S_1(t) - \frac{\partial Q_e(q_2^*(t), q_3^*(t))}{\partial q_i} S_e(t)} \leq 0, i = 2, 3.
\end{aligned}$$

[10] In the proof, we have used the fact that the penalty is nonpositive, so the method cannot be generalized to any concave function that has a nondifferentiable point.

The first inequality implies that the optimal solution $q_i^*(t)$ increases as $S_1(t)$ increases. The second equation implies that the optimal solution $q_i^*(t)$ decreases as $S_i(t)$ increases. The last equation implies that the optimal solution $q_i^*(t)$ decreases as $S_e(t)$ increases.

Chapter 3

Emission Allowance as a Derivative on Commodity-Spread

3.1 Introduction

Since the EU Emission Trading System (EU ETS) was launched in October 2003, the European Energy Exchange (EEX), European Climate Exchange (ECX), Powernext, Chicago Climate Exchange (CCX), and New York Mercantile Exchange (NYMEX) began trading EU allowance (EUA), certified emission reduction (CER), and their derivatives. The trading volumes in these markets are increasing as the EU ETS expands; therefore, the pricing of emission allowances is becoming an important issue.

In the academic literature, Cronshaw and Kruse (1996), Rubin (1996), Schennach (2000), Fehr and Hinz (2007), and Seifert, Uhrig-Homberg, and Wagner (2008) investigated theoretically the price of emission allowance. These studies focused on describing the univariate properties of emission allowance price in terms of abatement costs, but did not examine the relations between prices of emission allowances and other commodities. In contrast, some empirical papers, such as Fezzi and Bunn (2009) and Mansanet-Bataller, Pardo, and Valor (2007), found relations between futures prices of commodities such as emission allowance, electricity, natural gas, and temperature. These results suggest the need for a model that incorporates price relations between emission allowance and other commodities.

Studies also exist on derivatives of emission allowance. Chao and Wilson (1993) assumed a fixed supply and stochastic demand for allowance, and derived an explicit valuation formula for options on emission allowance. Maeda (2001) presented a forward pricing model of emission allowance with and without banking. Kijima, Maeda, and Nishide (2010) built a pricing model of emission allowance in a general equilibrium framework. Chesney and Taschini (2009) proposed a model under asymmetric information that allowed intertemporal banking and borrowing, and derived a closed-form pricing formula for a European option. Daskalakis, Psychoyois, and Markellos (2009) studied the spot and futures markets of emission allowance with several price processes including a mean reverting square-root process and jump process, and addressed the difference between inter- and intraphase markets. Again, these studies only considered the emission allowance price, and did not model explicitly the relations between commodities prices.

In this chapter, we characterize the price of emission allowance by incorporating the interrelations between emission allowance and other commodities. More precisely, we focus on a input-output relation of energy and assume that it is the main source of emission allowance price. Since marginal revenue (e.g., electricity price) is equal to marginal cost (e.g., natural gas price adjusted by heat/emission rate) under competitive markets, this relation is natural. We also assume that emission allowance is traded in a system similar to the EU ETS. That is, emission allowance is traded in a certain predetermined period, and compliance with the reductions is required at the end of period with a penalty for violation.

In this situation, the profit maximization of producers leads to inter- and intratemporal conditions on prices of emission allowance and commodities. The former requires that the emission allowance price at any time should be equal to the present value of the emission allowance price at the end of the trading period. The latter requires that marginal revenue is equal to marginal cost of fuel and emission allowance per unit of production. Imposing these conditions, taking account of the penalty, and assuming that commodities prices follow the Gibson-Schwartz type (1990) stochastic processes, we provide a valuation formula for the emission allowance price in terms of a spread between commodities prices with the penalty.

It is worth noting that many other factors may affect the emission allowance price but are not incorporated in our analysis; such factors include demand for emission allowance from other industries, investment in abatement technologies, asymmetric information on emission reduction, uncertainty of institutional change, and so on. In reality, these factors can also be important determinants of emission allowance price. However, by focusing on the relation between input-output of production and the emission allowance price, we are able to characterize explicitly at least a part of the emission allowance price in terms of a spread between commodities prices with the penalty. In other words, we obtain an approximation of the emission allowance price that can be described by observable variables, i.e., prices of other commodities. This allows us to value emission allowance in part but in a tractable way.

This chapter is organized as follows. In Section 2, we characterize the spot price of emission allowance as a derivative of a spread between commodity prices with the penalty. We also analyze the option values embedded in emission allowance and derive valuation formulae for futures and options on emission allowance. Using these valuation formulae, in Section 3, we characterize a hedging strategy of emission allowance using commodity futures. In Section 4, empirical and numerical analyses are provided. Section 5 concludes.

3.2 Prices of Emission Allowance and Their Derivatives

3.2.1 The Setup

Let us consider an economy that has a CO_2 emission trading system similar to the EU ETS. That is, the emission allowance of CO_2 is traded and its cumulative amount of emissions in period $[0, T]$ is required to be less than a certain limit. Otherwise the excess emissions over the limit are penalized at the end of period T and emission allowance is traded throughout the period $[0, T]$. In this economy, we are interested in characterizing the price of emission allowance. For this purpose, we focus on the relation between the emission allowance and input-output prices in electricity generation.

To be more concrete, assume that there are competitive power companies that generate electricity by burning fuel while emitting CO_2 as a by-product. The power companies generate electricity depending on the prices of electricity, fuel, and emission allowance. Assume for simplicity that these companies are the dominant CO_2 emitters in the economy and that their power-generating activities determine the relative prices of emission allowance, electricity, and fuel.

In this situation, it is well known that profit-maximization of the power companies requires prices of electricity, fuel, and emission allowance to satisfy both intertemporal and intratemporal conditions. The former requires that the emission allowance price at time $t(\leq T)$ should be equal to the present value of the emission allowance price at the end of period T.[1] The latter requires equality between marginal revenue of output (electricity) and marginal cost of input (fuel and emission allowance), which leads to an expression of the allowance price as a spread between commodities prices. In the following, we utilize these inter- and intratemporal conditions and characterize the price of emission allowance in terms of commodities prices.

Denote by $S_e(t)$ the spot price of emission allowance at $t(\leq T)$ and by $S_e(T)$ the price at T. The intertemporal condition that the emission allowance price should satisfy is

$$S_e(t) \quad = \quad E_t[e^{-r(T-t)}S_e(T)], \tag{1}$$

where $E_t[\cdot]$ is expectation under risk-neutral probability P given $\mathcal{F}_t$.[2] Thus, to derive the emission allowance price $S_e(t)$, we need to know its value $S_e(T)$ at T, which we obtain from the intratemporal condition.

Let us denote by Z the per-unit penalty for excess emissions over the limit at time T. We assume that Z is constant. Let us also denote by $S_1(t)$ and $S_2(t)$ as the price of output (e.g. electricity) and input (e.g. natural gas) at time t, respectively. Then, if Z is sufficiently large so that it is not binding, the intratemporal condition, or the equality of marginal costs for fuel and emission allowance, leads to the equality of the allowance price and a spread between commodities prices at T.[3] Denote this spread by $H_1S_1(T)-H_2S_2(T)$,

[1]This is because emission allowance is needed only at the end of period T when the central authority checks the companies in order to penalize any offenders. Thus, if the emission allowance price at $t(\leq T)$ is lower (resp. higher) than the present value of the price at T, the companies can increase their profits by adopting a trading strategy to buy (resp. sell) allowance at t and to sell (resp. buy) them back at T, which contradicts their profit maximization. See also Chapter 2.

[2]Here we assume a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$.

[3]This can be understood intuitively as follows. Let h_2 be the amount of fuel necessary for producing one marginal unit of electricity. Let k_e be the amount of CO_2 emissions associated with burning one marginal unit of electricity. Then, the equality of marginal revenue and marginal costs implies $S_1(T) = h_2S_2(T) + k_eS_e(T)$, which leads to $S_e(T) = \frac{1}{k_e}(S_1(T) - h_2S_2(T))$. See Chapter 2 for more general cases.

where we assume H_1 and H_2 are constant for simplicity. On the other hand, if the spread is larger than the penalty Z, i.e., $H_1S_1(T) - H_2S_2(T) > Z$, the emission allowance price cannot be equal to the spread; otherwise, power companies will short sell emission allowance and pay the penalty Z that is less than the spread or the emission allowance price which implies arbitrage opportunity. Hence, the penalty Z sets the price ceiling, or the upper bound, of the emission allowance price at T.[4] Furthermore, the emission allowance price cannot be negative. Thus, the emission allowance price at T is given by

$$S_e(T) = [\{H_1S_1(T) - H_2S_2(T)\} \wedge Z] \vee 0, \tag{2}$$

where $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.[5]

Finally, to describe the spot commodity prices, we assume that commodity prices follow the Gibson-Schwartz (1990) model (here after the GS model).[6] That is, for a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, we assume that the commodities prices $S_i(t)$ and convenience yields $\delta_i(t)$ satisfy the following stochastic differential equations.

$$\begin{aligned} d\ln S_i(t) &= \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t)\right)dt + \sigma_{S_i} dW_{S_i}(t), && (3) \\ d\delta_i(t) &= \kappa_i(\hat{\alpha}_i - \delta_i(t))dt + \sigma_{\delta_i} dW_{\delta_i}(t), && (4) \end{aligned}$$

where $\boldsymbol{W}(t) = [W_{S_1}(t), W_{S_2}(t), W_{\delta_1}(t), W_{\delta_2}(t)]^\top$ is a four-dimensional standard Brownian motion under the risk-neutral probability.[7]

To summarize, we assume three conditions. The first is the intertemporal condition that emission allowance price at time t is the present value of the emission allowance price at the end of period T. The second is the

[4]This is indicated in Chapter 2.

[5]If power companies generate electricity by using several kinds of fuels, say coal and natural gas, a similar spread relation is obtained among the prices of emission allowance and different fuels through fuel-switching by the power companies. Naturally, the same theoretical results in the following hold for this relation among emission allowance and fuel prices.

[6]Although Gibson and Schwartz (1990) constructed their model as a set of price and convenience yield process for a single commodity, here we slightly generalize their model to two sets of commodity price and convenience yield processes with correlations between each process.

[7]The volatility structure is given in the Appendix.

intratemporal condition that the emission allowance price at the end of period T should be positive and equal to the minimum of the spread of the two commodities prices and the penalty. The last is that commodity prices follow the GS model with correlations.

3.2.2 Spot Price of Emission Allowance

Under the assumptions above, we can derive the spot price of emission allowance as follows.

Proposition 3.2.1.

Under assumptions (1)–(4), the spot price of emission allowance is given by

$$S_e(t) = \hat{H}_1(t,T)S_1(t) - \hat{H}_2(t,T)S_2(t) + \hat{H}_3(t,T)Z, \tag{5}$$

where $\hat{H}_i(t,T)$ are defined in the Appendix.

Proof. The proof is in the Appendix. □

Thus, if the input-output production relation is the main factor of the emission allowance price, and if we can regard other factors as negligible, the emission allowance price is expressed as a spread of commodities prices with the penalty appropriately discounted.

Observe, however, that the spread relation is not simple: even if the coefficients (H_1 and H_2) on commodities prices in the spread are constant at the end of trading period T, the corresponding coefficients ($\hat{H}_1(t,T)$ and $\hat{H}_2(t,T)$) in the spread that determines the emission allowance price in (5) are not constant and depend on the stochastic properties of commodities prices as well as time to maturity. Indeed, $\hat{H}_1(t,T)$ (resp. $\hat{H}_2(t,T)$) can be interpreted as H_1 (resp. H_2) multiplied by the discount factor and the risk-adjusted probability of the spread $H_1S_1(T) - H_2S_2(T)$ of commodities prices between 0 and Z at the end of period T.

One implication of this expression is that the emission allowance spot price may inherit stochastic properties from the commodities prices. For example, while the coefficients ($\hat{H}_1(t,T)$ and $\hat{H}_2(t,T)$) are changing stochastically over time, if the commodities prices exhibit convenience yields, the emission allowance price may also exhibit a convenience yield through the spread relation in (5).

The emission allowance price also depends on the penalty Z multiplied by $\hat{H}_3(t,T)$ because the producers have the option of emitting any amount of CO_2 by paying the penalty Z per unit of emission at the end of period T. Here, the coefficient $\hat{H}_3(t,T)$ can be interpreted as the adjusted discount factor that consists of the risk-free discount rate multiplied by the risk-neutral probability that the spread $H_1S_1(T) - H_2S_2(T)$ of commodities prices exceeds Z at the end of period T.

3.2.3 The Option Value Embedded in the Emission Allowance Spot Price

From equations (1) and (2), we can see that the emission allowance is a bull call spread of European call options on the spread $H_1S_1(T) - H_2S_2(T)$ at the end of period T. That is, the value of the emission allowance can be replicated by a portfolio of buying 1 unit of European call option with exercise price 0 on the spread $H_1S_1(T) - H_2S_2(T)$ at the end of period T and selling 1 unit of European call option with exercise price Z on the spread.

In this section, we analyze the values of these embedded options in more detail. For this purpose, we first derive the valuation formula for the emission allowance spot price when the option to emit CO_2 by paying penalty Z is ignored, or when Z is taken to be infinite, as follows.

Proposition 3.2.2. Define

$$S_e'(t) \equiv E_t[e^{-r(T-t)}((H_1S_1(T) - H_2S_2(T)) \vee 0)].$$

Then

$$S_e'(t) = H_1'(t,T)S_1(t) - H_2'(t,T)S_2(t), \tag{6}$$

where

$$H_1'(t,T) = H_1 \exp\left\{ -r(T-t) + \mu_{X_1}(t,T) + \frac{1}{2}\sigma_{X_1}^2(t,T) \right\}(1 - \Phi(\hat{\mu}_1(t,T))),$$

$$H_2'(t,T) = H_2 \exp\left\{ -r(T-t) + \mu_{X_2}(t,T) + \frac{1}{2}\sigma_{X_2}^2(t,T) \right\}(1 - \Phi(\hat{\mu}_2(t,T))),$$

and $\Phi(\cdot)$ is the standard normal distribution function.

Proof. The derivation is similar to that of Proposition 3.2.1 and hence we omit. □

Notice that equation (6) can be regarded as the present value of the maximum of the spread of two commodity prices and 0 at the end of period T. Thus, the option value of price ceiling by penalty Z embedded in the emission allowance spot price is expressed by the difference between $S'_e(t)$ and $S_e(t)$.

Corollary 3.2.1. The option value of price ceiling by penalty Z embedded in the emission allowance is given by

$$\begin{aligned} S_e(t) - S'_e(t) &= \exp\left\{-r(T-t) + \mu_{X_1}(t,T) + \frac{1}{2}\sigma^2_{X_1}(t,T)\right\} H_1 S_1(t) E_1 \\ &\quad - \exp\left\{-r(T-t) + \mu_{X_2}(t,T) + \frac{1}{2}\sigma^2_{X_2}(t,T)\right\} H_2 S_2(t) E_2 \\ &\quad + \hat{H}_3(t,T)Z, \end{aligned}$$

where

$$\begin{aligned} E_1 &= -\int_{-\infty}^{\infty} \Phi(-d_1(x_2,Z)) n(x_2 | \mu_{X_2}(t,T) + \sigma_{X_1X_2}(t,T), \sigma^2_{X_2}(t,T)) dx_2, \\ E_2 &= -\int_{-\infty}^{\infty} \Phi(-d_2(x_2,Z)) n(x_2 | \mu_{X_2}(t,T) + \sigma^2_{X_2}(t,T), \sigma^2_{X_2}(t,T)) dx_2. \end{aligned}$$

and $n(\cdot|\mu,\sigma^2)$ is the normal density function with μ and σ^2 as mean and variance, respectively.

This is the value of the option to emit any amount of CO_2 by paying penalty Z when the spread $H_1S_1(T) - H_2S_2(T)$ exceeds Z. From this corollary, we can see that the option value of price ceiling by penalty Z embedded in the emission allowance spot price is affected not only by penalty Z, but also by $S_1(t)$ and $S_2(t)$ through E_i $(=1,2)$.

Similarly, if the investor (mis-)values the emission allowance spot price just as the spread of two commodity prices, he/she will be subject to the next equation.

Proposition 3.2.3. Define

$$S_e''(t) \;\equiv\; E_t[e^{-r(T-t)}(H_1S_1(T) - H_2S_2(T))]. \tag{7}$$

Then

$$\begin{aligned} S_e''(t) \;=\; & \exp\left\{ -r(T-t) + \mu_{X_1}(t,T) + \frac{1}{2}\sigma^2_{X_1}(t,T) \right\} H_1S_1(t) \\ & - \exp\left\{ -r(T-t) + \mu_{X_2}(t,T) + \frac{1}{2}\sigma^2_{X_2}(t,T) \right\} H_2S_2(t). \end{aligned}$$

Proof. The formula for $S_e''(t)$ is obvious by the linearity of expectations. □

Equation (7) is merely the present value of the spread of two commodity prices. Comparing equation (7) with $S_e'(t)$, we have the following result.

Corollary 3.2.2.

$$\begin{aligned} & S_e'(t) - S_e''(t) \\ = \;& -\exp\left\{ -r(T-t) + \mu_{X_1}(t,T) + \frac{1}{2}\sigma^2_{X_1}(t,T) \right\} H_1S_1(t)\Phi(\hat{\mu}_1(t,T)) \\ & + \exp\left\{ -r(T-t) + \mu_{X_2}(t,T) + \frac{1}{2}\sigma^2_{X_2}(t,T) \right\} H_2S_2(t)\Phi(\hat{\mu}_2(t,T)). \end{aligned}$$

Thus, the option value embedded in emission allowance against the spread of the two commodity prices can be decomposed into two components. The first component $S_e(t) - S_e'(t)$ is the option value of price ceiling by penalty Z and the second component $S_e'(t) - S_e''(t)$ is the option value of price floor at 0.[8] Grüll and Taschini (2011) also pointed out that emission allowance under hybrid scheme can be decomposed to ordinary cap-and-trade scheme with European or American style call and put options. In this chapter, we emphasize the relation between emission allowance price with other commodity prices and derive the valuation formula which have embedded option value.

[8] From another point of view, $S_e(t) - S_e'(t)$ and $S_e(t) - S_e''(t) = S_e(t) - S_e'(t) + S_e'(t) - S_e''(t)$ can be interpreted as pricing errors for the emission allowance price when $((H_1S_1(T) - H_2S_2(T)) \wedge Z) \vee 0$ is replaced by $(H_1S_1(T) - H_2S_2(T)) \vee 0$ and $H_1S_1(T) - H_2S_2(T)$, respectively, i.e., when the investor misprices the emission allowance spot price at the end of period T.

3.2.4 Derivatives of Emission Allowance

Given the spot price, we can derive the prices of emission allowance derivatives. Note that as the penalty is paid at the end of period T, a power company that needs to hedge the penalty will naturally focus on the payment at time T. Thus, derivatives of the emission allowance that mature at T should be adequate for risk hedging.

First, we calculate the emission allowance futures price in the following proposition.

Proposition 3.2.4. The futures price of the emission allowance that matures at T is

$$\begin{aligned} G_e(t,T) &= E_t[S_e(T)] \\ &= E_t[((H_1S_1(T) - H_2S_2(T)) \wedge Z) \vee 0] \\ &= e^{r(T-t)}\hat{H}_1(t,T)S_1(t) - e^{r(T-t)}\hat{H}_2(t,T)S_2(t) + e^{r(T-t)}\hat{H}_3(t,T)Z. \end{aligned}$$

Proof. The first equation is from Cox, Ingersoll, and Ross (1981). The proof for the third equation is the same as Proposition 3.2.1. □

Next, we obtain the valuation formula for a European call option of emission allowance.

Proposition 3.2.5. Suppose $Z > K > 0$. The European call option price on the emission allowance that matures at T is

$$\begin{aligned} C_e(t,T) &= E_t[e^{-r(T-t)}(S_e(T) - K)^+] \\ &= E_t[e^{-r(T-t)}(((H_1S_1(T) - H_2S_2(T) - K) \wedge (Z - K)) \vee 0)] \\ &= \bar{H}_1(t,T)S_1(t) - \bar{H}_2(t,T)S_2(t) - \bar{H}_3(t,T)K + \bar{H}_4(t,T)Z, \end{aligned}$$

where

$$\begin{aligned}
\bar{H}_1(t,T) &= H_1 \exp\left\{ -r(T-t) + \mu_{X_1}(t,T) + \frac{1}{2}\sigma^2_{X_1}(t,T) \right\} \\
&\quad \times \int_{-\infty}^{\infty} (\Phi(d_1(x_2,Z)) - \Phi(d_1(x_2,K))) \\
&\quad \times n(x_2|\mu_{X_2}(t,T) + \sigma_{X_1X_2}(t,T), \sigma^2_{X_2}(t,T))dx_2, \\
\bar{H}_2(t,T) &= H_2 \exp\left\{ -r(T-t) + \mu_{X_2}(t,T) + \frac{1}{2}\sigma^2_{X_2}(t,T) \right\} \\
&\quad \times \int_{-\infty}^{\infty} (\Phi(d_2(x_2,Z)) - \Phi(d_2(x_2,K))) \\
&\quad \times n(x_2|\mu_{X_2}(t,T) + \sigma^2_{X_2}(t,T), \sigma^2_{X_2}(t,T))dx_2, \\
\bar{H}_3(t,T) &= \exp(-r(T-t)) \int_{-\infty}^{\infty} (\Phi(d_2(x_2,Z) - \Phi(d_2(x_2,K))) \\
&\quad \times n(x_2|\mu_{X_2}(t,T), \sigma^2_{X_2}(t,T))dx_2, \\
\bar{H}_4(t,T) &= \exp(-r(T-t)) \int_{-\infty}^{\infty} (1 - \Phi(d_2(x_2,Z))) \\
&\quad \times n(x_2|\mu_{X_2}(t,T), \sigma^2_{X_2}(t,T))dx_2.
\end{aligned}$$

Proof. Again, the derivation is similar to that of Proposition 3.2.1 and hence we omit. □

Trivially, if $Z \leq K$ then the option price is 0, and if $K \leq 0$ then the option price is the sum of the emission allowance spot price and $-e^{-r(T-t)}K$.[9]

[9] Since the payoff at maturity T is

$$\begin{aligned}
(S_e(T) - K)^+ &= ((((H_1S_1(T) - H_2S_2(T)) \wedge Z) \vee 0) - K) \vee 0 \\
&= (((H_1S_1(T) - H_2S_2(T)) \wedge Z) \vee 0) - K,
\end{aligned}$$

when $K \leq 0$, the emission allowance spot price is $S_e(t) - e^{-r(T-t)}K$.

3.3 Hedging Emission Allowances Using Commodity Futures

Equation (5) seems to suggest that the value of emission allowances at time t could be replicated by holding $\hat{H}_1(t,T)$ units of commodity 1, short-selling $\hat{H}_2(t,T)$ units of commodity 2, and holding $\hat{H}_3(t,T)Z$ units in risk-free assets at time t. However, since commodity 1 is assumed to be electricity, which is not storable, it is not possible to implement this trading strategy in the current setting.

On the other hand, because the compliance of emission reductions is checked and the penalty is paid only at the end of period T, a firm that needs to hedge the penalty only has to manage the payment to emission allowance at time T. Thus, the derivatives of emission allowance that mature at T should be enough for its risk hedging. Moreover, because the futures price of emission allowance is equal to the spot price at maturity, a firm that wishes to hedge the allowance price at maturity can satisfy its need by hedging the futures. Hence, we investigate the hedging strategy to replicate the emission allowance futures with maturity T by trading the commodity futures with the same maturity.

We can derive the hedging strategy for the emission allowance futures as follows.

Proposition 3.3.1. Assume (1)–(4). The hedging equation for emission allowances using commodity futures is

$$dG_e(t,T) \;=\; \varphi_B(t)dB(t) + \varphi_{G_1}(t)dG_1(t,T) + \varphi_{G_2}(t)dG_2(t,T),$$

where the hedging strategies are

$$\begin{aligned}\varphi_B(t) \;=\; & \left\{ \frac{\partial \hat{\hat{H}}_1(t,T)}{\partial t} G_1(t,T) - \frac{\partial \hat{\hat{H}}_2(t,T)}{\partial t} G_2(t,T) + Z \frac{\partial \hat{\hat{H}}_3(t,T)}{\partial t} \right. \\ & + \sum_{j,k=1}^{2} \left(\frac{G_1(t,T)}{2} \frac{\partial^2 \hat{\hat{H}}_1(t,T)}{\partial G_j(t,T)\partial G_k(t,T)} - \frac{G_2(t,T)}{2} \frac{\partial^2 \hat{\hat{H}}_2(t,T)}{\partial G_j(t,T)\partial G_k(t,T)} \right.\end{aligned}$$

$$
\begin{aligned}
& +\frac{Z}{2}\frac{\partial^2\hat{\hat{H}}_3(t,T)}{\partial G_j(t,T)\partial G_k(t,T)}\Bigg)G_j(t,T)G_k(t,T)\Bigg(\sigma_{S_jS_k}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{j,j}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{k,k} \\
& -\sigma_{S_j\delta_k}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{j,j}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{k,2+k}-\sigma_{S_k\delta_j}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{k,k}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{j,2+j} \\
& +\sigma_{\delta_j\delta_k}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{j,2+j}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{k,2+k}\Bigg) \\
& +\frac{\partial\hat{\hat{H}}_1(t,T)}{\partial G_1(t,T)}G_1(t,T)^2\Bigg(\sigma_{S_1}^2\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}-2\sigma_{S_1\delta_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3} \\
& +\sigma_{\delta_1}^2\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3}\Bigg) \\
& +\frac{\partial\hat{\hat{H}}_1(t,T)}{\partial G_2(t,T)}G_1(t,T)G_2(t,T)\Bigg(\sigma_{S_1S_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2} \\
& -\sigma_{S_1\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}-\sigma_{S_2\delta_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3} \\
& +\sigma_{\delta_1\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}\Bigg) \\
& -\frac{\partial\hat{\hat{H}}_2(t,T)}{\partial G_2(t,T)}G_2(t,T)^2\Bigg(\sigma_{S_2}^2\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}-2\sigma_{S_2\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4} \\
& +\sigma_{\delta_2}^2\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}^2\Bigg) \\
& -\frac{\partial\hat{\hat{H}}_2(t,T)}{\partial G_1(t,T)}\Bigg(\sigma_{S_1S_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}-\sigma_{S_1\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4} \\
& -\sigma_{S_2\delta_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3}+\sigma_{\delta_1\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}\Bigg)\Bigg\}(rB(t))^{-1}dt, \\
\varphi_{G_i}(t) \quad = \quad & \sum_{j=1}^{2}\frac{\partial\hat{\hat{H}}_j(t,T)}{\partial G_i(t,T)}G_j(t,T)dG_i(t,T)+\hat{\hat{H}}_i(t,T)dG_i(t,T) \\
& -\sum_{j=1}^{2}\frac{\partial\hat{\hat{H}}_j(t,T)}{\partial G_i(t,T)}G_j(t,T)dG_i(t,T)+\frac{\partial\hat{\hat{H}}_3(t,T)}{\partial G_i(t,T)}ZdG_i(t,T), i=1,2.
\end{aligned}
$$

Proof. The proof is in the Appendix. □

The advantage of hedging the futures is that we need only two futures of the commodities with the same maturity as the emission allowance futures. With the same maturity, there is no convenience yield and we do not need to control it using other commodity futures. This greatly simplifies the calculation.

We emphasize that we use tradable commodity futures instead of spots since electricity spots are neither tradable nor even storable. This hedging strategy should be useful for financial institutions and power companies. For example, financial institutions can hedge risks for emission allowance trading, and power companies can replicate emission allowance by maintaining their power portfolios.

3.4 Empirical and Numerical Analysis

In this section, we empirically estimate the emission allowance price model using market data in two steps. First, we estimate the parameters (i.e., $\sigma_{S_i}, \sigma_{\delta_i}$, and so on) for equations (3) and (4) using the Kalman filter. Second, using the estimated parameters, we calibrate the model to emission allowance price.

We use emission allowance OTC prices at EU-ETS, baseload electricity futures prices traded at European Energy Exchange and natural gas futures prices traded at ICE Futures from April 3, 2008 to August 31, 2011 which are plotted in Figure 3.1. All prices are daily closing prices in terms of euros. For parameter estimation, six futures contracts of electricity and natural gas labeled Maturity 1, 2, 3, 4, 5, and 6 are used. Maturity 1 stands for the contract closest to maturity, 2 stands for the second closest maturity, and so on. Time to maturity corresponding to these prices are also used. We set the risk-free rate to be 3% and the penalty Z is 100 euros.

The basic statistics for these data are described in Table 3.1. Comparing emission allowance, electricity, and natural gas, we can see that the mean price return of emission allowance is negative where the other commodities are positive.

Figure 3.1: Emission allowance OTC price, electricity futures price and natural gas futures prices from April 3, 2008 to August 31, 2011. The blue solid line, the red dashed line and the green chained line are the price of emission allowance, electricity and natural gas, respectively.

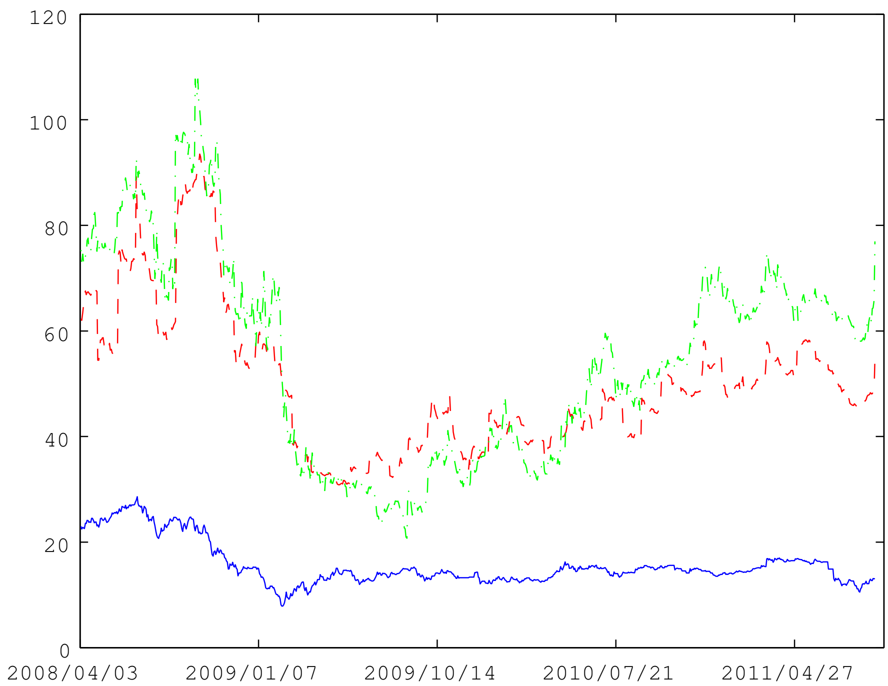


Table 3.1: Statistics of Data.

Futures Contract	Mean price (Standard deviation)	Mean price return (Standard deviation)	Mean maturity (Standard deviation)
Emission allowance	15.71 (4.11)	-0.0342 % (2.4117 %)	na
Electricity			
Maturity 1	49.83 (13.51)	0.0280 % (2.9813 %)	0.04 (0.02)
Maturity 2	51.51 (14.21)	0.0491 % (2.6179 %)	0.12 (0.02)
Maturity 3	52.76 (14.59)	0.0406 % (2.4756 %)	0.21 (0.01)
Maturity 4	53.75 (15.39)	0.0247 % (2.2996 %)	0.29 (0.02)
Maturity 5	54.42 (15.68)	0.0418 % (2.4077 %)	0.37 (0.02)
Maturity 6	55.26 (15.56)	0.0329 % (2.3488 %)	0.46 (0.02)
Natural gas			
Maturity 1	54.93 (19.49)	0.0710 % (3.8420 %)	0.05 (0.02)
Maturity 2	58.02 (21.99)	0.0718 % (3.4782 %)	0.13 (0.02)
Maturity 3	60.55 (24.17)	0.0616 % (3.0605 %)	0.21 (0.02)
Maturity 4	62.66 (25.96)	0.0595 % (2.9400 %)	0.30 (0.02)
Maturity 5	64.47 (27.19)	0.0579 % (2.8941 %)	0.38 (0.02)
Maturity 6	65.82 (27.13)	0.0454 % (2.9360 %)	0.46 (0.02)

3.4.1 Estimation Results

As mentioned before, we first estimate the GS model (3) and (4) using the Kalman filter. We assume that the market price of risks θ are constants. Thus, we estimate the following equations under natural probability.

$$
\begin{aligned}
d\ln S_i(t) &= \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t) + \theta_{S_i}\right) dt + \sigma_{S_i} dW_{S_i}(t), \\
d\delta_i(t) &= \{\kappa_i(\hat{\alpha}_i - \delta_i(t)) + \theta_{\delta_i}\} dt + \sigma_{\delta_i} dW_{\delta_i}(t).
\end{aligned}
$$

Table 3.2 reports the estimated parameters with standard errors. Electricity and natural gas spot prices have positive correlation ($\rho_{S_1 S_2} = 0.44$). The other correlations among spot prices and convenience yields are also positive and not small. Market prices of risks are all negative but not significant.

Figure 3.2: Figure on the left hand side shows electricity futures price of maturity 1 and its theoretical price. Figure on the right hand side shows natural gas futures price of maturity 1 and its theoretical price. The blue solid line and the red dashed line represent the futures market prices and theoretical prices, respectively.

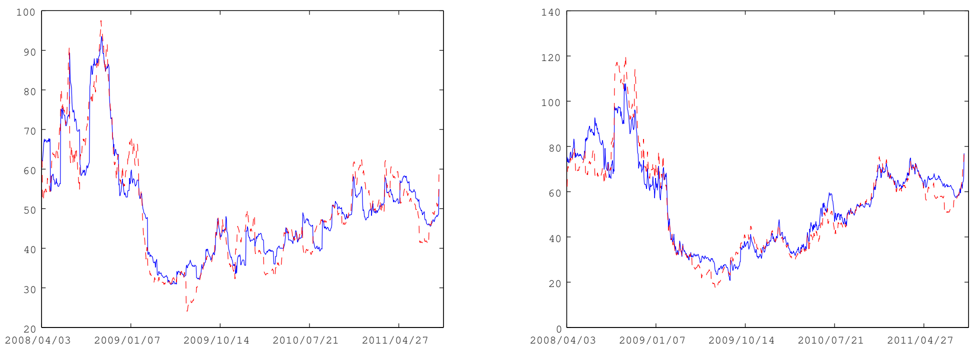


Table 3.3 shows root mean square error (RMSE) and mean error (ME) of the model. In Figure 3.2, we show the futures market prices and their theoretical prices. We see that the model is well fitted.

We now proceed to the second step which is the calibration of emission allowance using the estimated parameters. Specifically, we calculate the fol-

Table 3.2: Parameters estimates and standard errors in parenthesis. The data used are electricity futures price and natural gas futures prices from April 3, 2008 to August 31, 2011.

Volatility Prameters	
σ_{S_1}	0.497136 (0.017781)
σ_{S_2}	0.648273 (0.028053)
σ_{δ_1}	1.793224 (0.071091)
σ_{δ_2}	2.021003 (0.074879)
$\rho_{S_1 S_2}$	0.436194 (0.032242)
$\rho_{S_1 \delta_1}$	0.811057 (0.014139)
$\rho_{S_1 \delta_2}$	0.296837 (0.041366)
$\rho_{S_2 \delta_1}$	0.410148 (0.032402)
$\rho_{S_2 \delta_2}$	0.777341 (0.024868)
$\rho_{\delta_1 \delta_2}$	0.469754 (0.031623)
Covenience yield parameters	
κ_1	0.722765 (0.141487)
κ_2	0.727691 (0.062573)
$\hat{\alpha}_1$	0.368187 (0.120499)
$\hat{\alpha}_2$	0.512544 (0.112107)
Market price of risk parameters	
$\theta_{S_1 0}$	-0.573257 (0.827919)
$\theta_{S_2 0}$	-0.777342 (0.901076)
$\theta_{\delta_1 0}$	-0.371621 (0.969878)
$\theta_{\delta_2 0}$	-0.535114 (1.104714)
Variances of observation equation	
$R(1,1)$	0.009118 (0.000489)
$R(2,2)$	0.000071 (0.000010)
$R(3,3)$	0.004160 (0.000316)
$R(4,4)$	0.004613 (0.000334)
$R(5,5)$	0.000062 (0.000013)
$R(6,6)$	0.007776 (0.000539)
$R(7,7)$	0.008047 (0.000425)
$R(8,8)$	0.000101 (0.000019)
$R(9,9)$	0.002428 (0.000125)
$R(10,10)$	0.002047 (0.000124)
$R(11,12)$	0.000035 (0.000018)
$R(12,12)$	0.005877 (0.000330)
Log-likelihood	17035.085477
AIC	-34010.170954
sample size	890

Table 3.3: RMSE (root mean square error) and ME (mean error) for each futures.

Contracts	RMSE	ME
Electricity		
Maturity 1	0.092625	-0.003160
Maturity 2	0.022860	0.000100
Maturity 3	0.066663	0.001453
Maturity 4	0.069359	0.001353
Maturity 5	0.025235	-0.000041
Maturity 6	0.091555	0.004394
Natural gas		
Maturity 1	0.090430	-0.003256
Maturity 2	0.029190	0.000088
Maturity 3	0.053379	0.000460
Maturity 4	0.047981	0.000111
Maturity 5	0.028412	0.000158
Maturity 6	0.087340	0.001753

lowing minimization problem.

$$\min_{H_1,H_2} \sum_{t=1}^{T} \left(\frac{S_e(t) - S_{market}(t)}{S_{market}(t)} \right)^2,$$

where $S_e(t)$ is the theoretical price of emission allowance obtained by Equation (5) and $S_{market}(t)$ is the market price of emission allowance at t.

The results are $H_1 = 0.290234$, $H_2 = 0.171998$ and the value of object function is 92.688944. We plot theoretical emission allowance price $S_e(t)$ with the market price $S_{market}(t)$ in Figure 3.3. Although the market price and theoretical price of emission allowance have some differences, they do not seem too far off even with the simplicity of the model. The discrepancy between the prices can be due to the effects of other commodity prices, political factor such as the result of COP 17, and the economic downturn in 2008 (the bankruptcy of Lehman Brothers) and 2011 (European sovereign-debt crisis).

3.4.2 Numerical Analysis

Based on the estimation above, we now conduct a numerical analysis of emission allowance. The spot prices of electricity and natural gas are set to be 50 and 80 euros, respectively, with initial convenience yields as zero. The maturity is $T = 1733/365$ (1733 is the number of days starting from April 3rd, 2008 to December 31th, 2012.) and t is 2 years before maturity.

Figure 3.4 illustrates theoretical emission allowance spot prices with penalty $S_e(t)$ and without penalty $S_e'(t)$. (We only compare $S_e(t)$ and $S_e'(t)$.) Commodity spot price $S_1(t)$ affects the emission allowance spot price more than commodity spot price $S_2(t)$ does. This is because H_1 is larger than H_2. Comparing $S_e(t)$ with $S_e'(t)$, $S_e(t)$ is consistently lower than $S_e'(t)$ because the emission allowance spot price at T has the upper bound Z. The difference between $S_e'(t)$ and $S_e(t)$ is the value of price-ceiling option by penalty Z, the option to emit CO_2 as much as one wants by paying the penalty Z. As Figure 3.4 shows the value of price-ceiling option by penalty Z becomes large as $S_1(t)$ increases.

The sensitivity of the emission allowance spot price to σ_{S_1} and σ_{S_2} is shown in Figures 3.5. It is interesting to observe that as σ_{S_1} decreases, $S_e(t)$, $S_e'(t)$, and the spread $S_e(t) - S_e'(t)$ increase. This can be partly attributed to their complicated dependence on the volatilities and correlations of the two

Figure 3.3: Emission allowance market prices and theoretical prices from April 3, 2008 to August 31, 2011. The blue solid line and the red dashed line are the price of emission allowance market prices and theoretical prices, respectively.

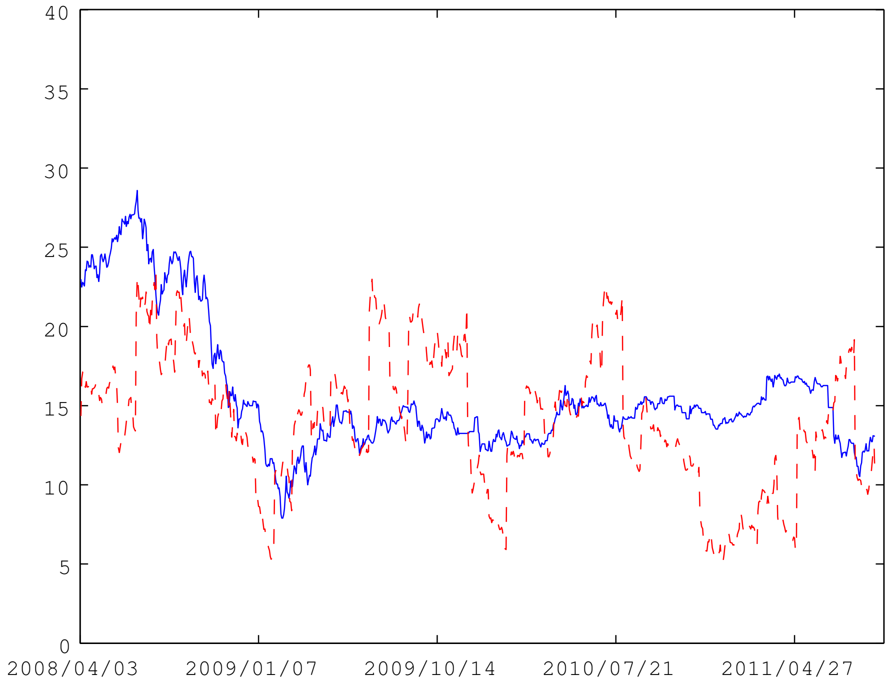
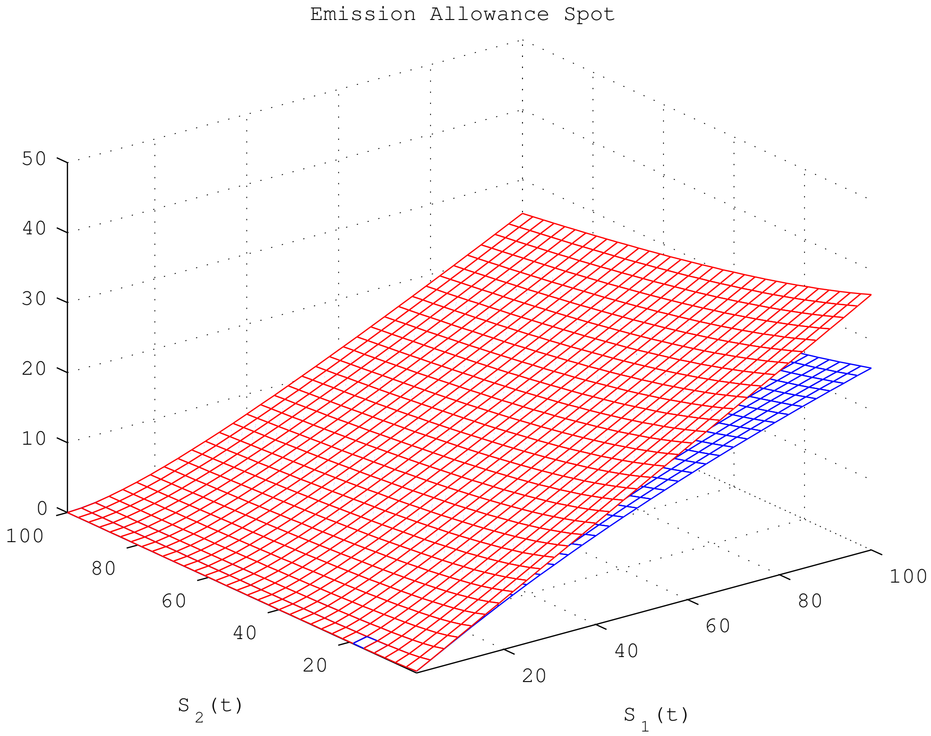
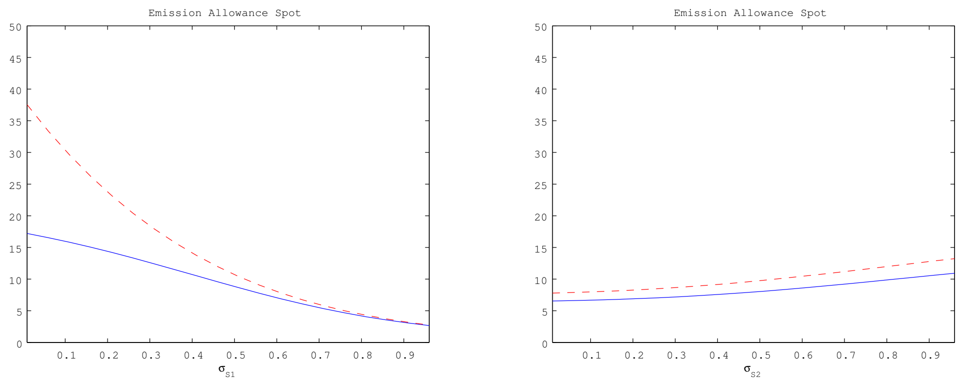


Figure 3.4: Sensitivity of the emission allowance spot price to commodity spot prices. The lower blue and upper red surfaces represent the theoretical emission allowance spot prices $S_e(t)$ and $S_e'(t)$, respectively.



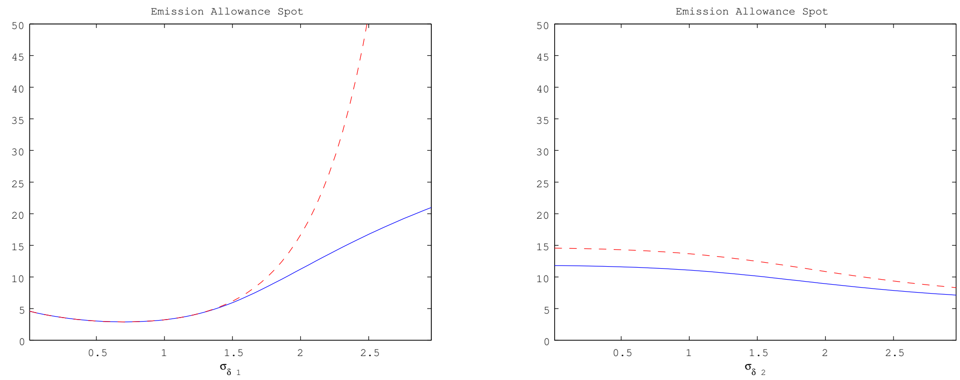
commodity prices and the convenience yields. As Proposition 3.2.1, 3.2.2, and Corollary 3.2.1 show, $S_e(t)$ and $S'_e(t)$ depend on $\sigma^2_{X_1}(t,T)$ and $\sigma^2_{X_2}(t,T)$ that are complex functions of the volatilities and correlations. Thus, the effect of volatilities on $S_e(t)$, $S'_e(t)$, and $S_e(t) - S'_e(t)$ can be different from that on a plain vanilla option. In addition, this may occur because the underlying asset of $S_e(t)$ and $S'_e(t)$ is the spread $S_1(t) - S_2(t)$ of the two commodity prices. Recall that $S_e(t)$ is the price of a bull call spread and $S'_e(t)$ is that of a call option on the underlying spread. Thus, if the mean of payoff of the underlying is above the penalty, and if the volatility of $S_1(t)$ decreases, the price of the bull call spread, whose price is capped by the penalty Z, is likely to be lower than that of the call option, whose price is not capped. Finally, notice that H_1 is larger than H_2. Hence, the sensitivity of $S_e(t)$ and $S'_e(t)$ to σ_{S_1} is much larger than that to σ_{S_2}, which yields the smaller change of $S_e(t)$ and $S'_e(t)$ in response to the change in σ_{S_2}.

Figure 3.5: Sensitivity of the emission allowance spot price to σ_{S_1} and σ_{S_2}. The blue solid line and the red dashed line represent the theoretical emission allowance spot prices $S_e(t)$ and $S'_e(t)$, respectively.



Figures 3.6 plot the price of the commodity spread option with σ_{δ_1} and σ_{δ_2}. As the result of sensitivity to σ_{S_1} and σ_{S_2} in Figures 3.5, the effect of volatility in convenience yields on $S_e(t)$, $S'_e(t)$, and $S_e(t) - S'_e(t)$ can be different from that suggested by a plain vanilla option. In this case, $S_e(t)$, $S'_e(t)$, and $S_e(t) - S'_e(t)$ mostly increase as σ_{δ_1} increases (though $S_e(t)$ and $S'_e(t)$ are slightly U-shaped), while they decrease as σ_{δ_2} increases. Again, the change of $S_e(t) - S'_e(t)$ in response to the change in σ_{δ_1} is larger than that in σ_{δ_2} probably because H_1 is larger than H_2.

Figure 3.6: Sensitivity of the emission allowance spot price to σ_{δ_1} and σ_{δ_2}. The blue solid line and the red dashed line represent the theoretical emission allowance spot prices $S_e(t)$ and $S'_e(t)$, respectively.



We now turn to the numerical analysis of hedge ratios. Figure 3.7 shows the sensitivity of hedge ratios of commodity futures to commodity futures prices. The upper surface depicts the hedge ratio of G_1 and the lower surface depicts that of G_2. The former is positive and larger than the latter that is slightly negative in the most depicted area. Although this hedging strategy is for emission allowance which have the penalty Z as for price-ceiling, we can try to imagine the difference between the hedging strategy for $S'_e(t)$ which does not have the price ceiling. We conjecture that if this hedging strategy for emission allowance is compared with that for $S'_e(t)$, the hedging strategy for emission allowance will have more bond within its strategy since we need to replicate the penalty Z for the state $H_1S_1(T) - H_2S_2(T) > Z$.

Figure 3.8 shows the sensitivity of the hedge ratios to the volatilities of commodity prices. The hedge ratios of G_1 decrease sharply as σ_{S_1} increases, but does not change much as σ_{S_2} varies. On the other hand, the hedge ratios of G_2 increase sharply as σ_{S_2} increases, but does not change much as σ_{S_1} varies.

Figure 3.9 shows the sensitivity of the hedge ratios to σ_{δ_1} and σ_{δ_2}. The hedge ratios of G_1 moves in a complicated way as σ_{δ_1} increases. On the other hand, the hedge ratios of G_2 decrease sharply as σ_{δ_2} increases.

Figure 3.7: Sensitivity of hedge ratios of commodity futures to $G_1(t,T)$ and $G_2(t,T)$. The upper blue and lower red surfaces represent the hedge ratio of commodity futures 1 and the hedge ratio of commodity futures 2, respectively.

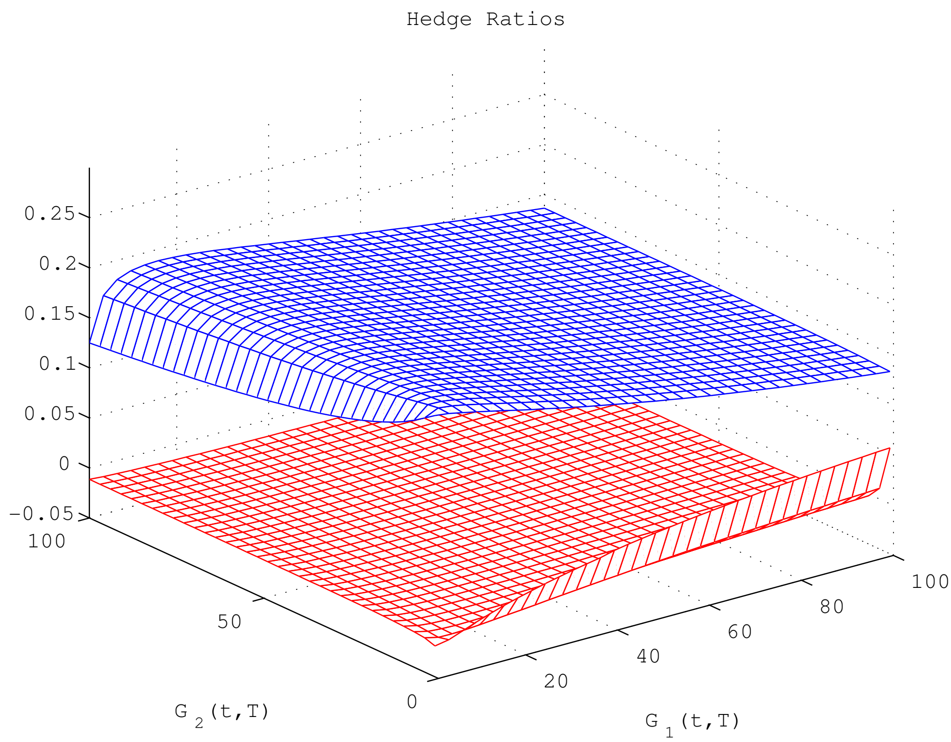


Figure 3.8: Sensitivity of hedge ratios of commodity futures to σ_{S_1} and σ_{S_2}. The blue solid line and the red dashed line represent the hedge ratio of commodity futures 1 and the hedge ratio of commodity futures 2, respectively.

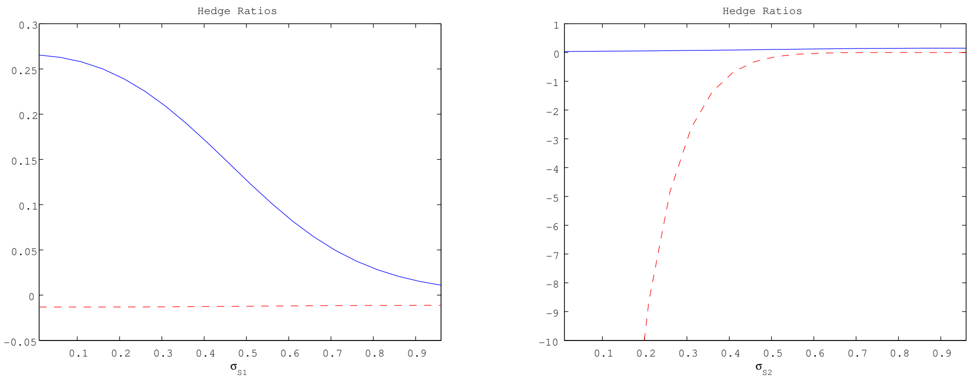
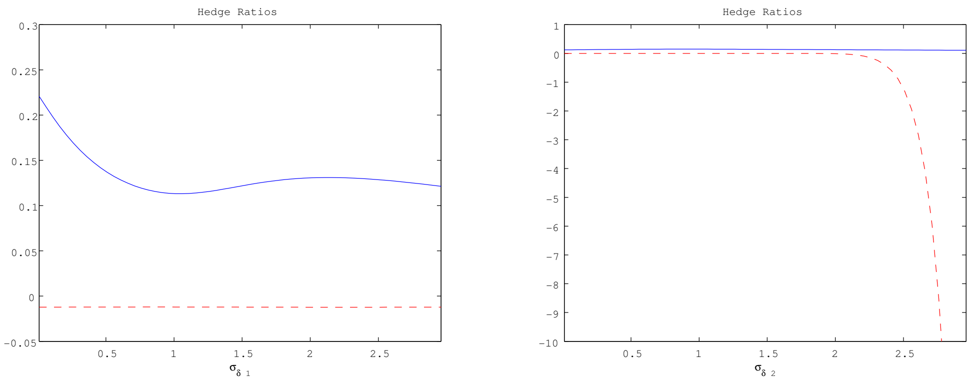


Figure 3.9: Sensitivity of hedge ratios of commodity futures to σ_{δ_1} and σ_{δ_2}. The blue solid line and the red dashed line represent the hedge ratio of commodity futures 1 and the hedge ratio of commodity futures 2, respectively.



3.5 Conclusion

In this chapter, we proposed a model of emission allowance spot price as a derivative on the commodity spread. We assumed that the emission allowance spot price at the end of the trading period was equal to the minimum of the spread of the two commodity prices and the penalty when it was positive, or equal to zero otherwise. We focused on the interrelation among emission allowance and commodities prices, which had not been incorporated in preceding papers on the valuation of emission allowance.

We characterized the emission allowance spot price in terms of the value of a portfolio of commodities and a risk-free asset. We also characterized the values of options embedded in emission allowance. In addition, we derived the formulae for emission allowance futures and options. We calibrated the model to real market data. From the numerical analysis with certain parameter values, we found that the option value of price ceiling by the penalty Z embedded in emission allowances was relatively large, which implied that the price-ceiling option by penalty was an important component in evaluating the emission allowance.

For future research, it would be interesting to explore the model using alternative assumptions. For example, we could investigate a model in which the emission allowance price at the end of the period is determined by a spread of fuel prices (e.g. coal and natural gas prices), which arises from the fuel-switching of profit-maximizing power companies. We could also analyze a model whose underlying commodity prices follow stochastic processes that are different from the Gibson-Schwartz process and may include seasonality, jumps, or stochastic volatility. As we emphasized, the interrelation between the prices of emission allowances and commodities should be the key to understanding the properties of emission allowance price. With this point in mind, empirical analyses on the prices of natural gas, electricity and other commodities should form the foundation for the study of emission allowance price.

3.6 Appendices

3.6.1 The Solutions of Spot Prices

The closed formulae for (3) and (4) are derived as follows. Let

$$\begin{aligned} d\ln S_i(t) &\equiv (\beta_{S_i0}(t) + \beta_{S_i\delta_i}\delta_i(t))dt + \sigma_{S_i}dW_{S_i}(t), \\ d\delta_i(t) &\equiv (\beta_{\delta_i0} + \beta_{\delta_i\delta_i}\delta_i(t))dt + \sigma_{\delta_i}dW_{\delta_i}(t), \end{aligned}$$

where

$$\begin{aligned} &\beta_{S_i0}(t) = r - \frac{\sigma_{S_i}^2}{2}, \\ &\beta_{S_i\delta_i} = -1, \\ &\beta_{\delta_i0} = \kappa_i\hat{\alpha}_i, \\ &\beta_{\delta_i\delta_i} = -\kappa_i. \end{aligned}$$

This equation can be solved as follows.[10]

$$\boldsymbol{X}(T) = e^{T\boldsymbol{\beta}}\left\{e^{-t\boldsymbol{\beta}}\boldsymbol{X}(t) + \int_t^T e^{-s\boldsymbol{\beta}}\boldsymbol{\beta}_0(s)ds + \int_t^T e^{-s\boldsymbol{\beta}}d\boldsymbol{W}_0(s)\right\}, \quad (8)$$

where

$$\begin{aligned} &\boldsymbol{X}(t) = [\ln S_1(t), \ln S_2(t), \delta_1(t), \delta_2(t)]^\top, \\ &\boldsymbol{\beta}_0(t) = [\beta_{S_10}(t), \beta_{S_20}(t), \beta_{\delta_10}, \beta_{\delta_20}]^\top, \\ &\boldsymbol{\beta} = \begin{bmatrix} 0 & 0 & \beta_{S_1\delta_1} & 0 \\ 0 & 0 & 0 & \beta_{S_2\delta_2} \\ 0 & 0 & \beta_{\delta_1\delta_1} & 0 \\ 0 & 0 & 0 & \beta_{\delta_2\delta_2} \end{bmatrix}, \end{aligned}$$

and $\boldsymbol{W}_0(t) = [\sigma_{S_1}W_{S_1}(t), \cdots, \sigma_{S_n}W_{S_n}(t), \sigma_{\delta_1}W_{\delta_1}(t), \cdots, \sigma_{\delta_n}W_{\delta_n}(t)]^\top$ is a scaled Brownian motion vector.

[10]Cf. Karatzas and Shreve (1991), Section 5.6 or Liptser and Shiryaev (2001), p.151, Thm. 4.10.

The mean and covariances of $\ln S_i(T)$ are

$$\begin{aligned}
\mu_{X_i}(t,T) &= E_t[\ln S_i(T)] \\
&= \left[e^{T\boldsymbol{\beta}} \left\{ e^{-t\boldsymbol{\beta}} \boldsymbol{X}(t) + \int_t^T e^{-s\boldsymbol{\beta}} \boldsymbol{\beta}_0(s) ds \right\} \right]_i, \\
\sigma_{X_i X_j}(t,T) &= E_t[(\ln S_i(T) - \mu_{X_i}(t,T))(\ln S_j(T) - \mu_{X_i}(t,T))] \\
&= \left[\int_t^T (e^{(T-s)\boldsymbol{\beta}}) \boldsymbol{\Sigma} (e^{(T-s)\boldsymbol{\beta}})^\top ds \right]_{ij},
\end{aligned}$$

where $[\cdot]_i$ and $[\cdot]_{ij}$ are i th element of vector and $[i,j]$ th element of matrix, respectively, and the covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{S_1}^2 & \rho_{S_1 S_2}\sigma_{S_1}\sigma_{S_2} & \rho_{S_1}\sigma_{S_1}\sigma_{\delta_1} & \rho_{S_1\delta_2}\sigma_{S_1}\sigma_{\delta_2} \\ \rho_{S_2 S_1}\sigma_{S_2}\sigma_{S_1} & \sigma_{S_2}^2 & \rho_{S_2\delta_1}\sigma_{S_2}\sigma_{\delta_1} & \rho_{S_2\delta_2}\sigma_{S_2}\sigma_{\delta_2} \\ \rho_{\delta_1 S_1}\sigma_{\delta_1}\sigma_{S_1} & \rho_{\delta_1 S_2}\sigma_{\delta_1}\sigma_{S_2} & \sigma_{\delta_1}^2 & \rho_{\delta_1\delta_2}\sigma_{\delta_1}\sigma_{\delta_2} \\ \rho_{\delta_2 S_1}\sigma_{\delta_2}\sigma_{S_1} & \rho_{\delta_2 S_2}\sigma_{\delta_2}\sigma_{S_2} & \rho_{\delta_2\delta_1}\sigma_{\delta_2}\sigma_{\delta_1} & \sigma_{\delta_2}^2 \end{bmatrix}.$$

We use notations

$$\begin{aligned}
\boldsymbol{\mu}_X(t,T) &\triangleq \begin{bmatrix} \mu_{X_1}(t,T) \\ \mu_{X_2}(t,T) \end{bmatrix}, \\
\boldsymbol{\Sigma}_X(t,T) &\triangleq \begin{bmatrix} \sigma_{X_1X_1}(t,T) & \sigma_{X_1X_2}(t,T) \\ \sigma_{X_2X_1}(t,T) & \sigma_{X_2X_2}(t,T) \end{bmatrix}.
\end{aligned}$$

3.6.2 Proof of Proposition 3.2.1

Let us use the notation $\Phi(\cdot)$ and $\phi(\cdot)$ as the standard normal distribution and density function, respetively, and also $N(\cdot|\mu,\sigma^2)$ and $n(\cdot|\mu,\sigma^2)$ as normal distribution and density function with μ and σ^2 as mean and variance, respectively. We calculate the following equation in this subsection.

$$S_e(t) = e^{-r(T-t)}E_t[((H_1S_1(T) - H_2S_2(T)) \wedge Z) \vee 0].$$

For notational convenience, we will omit the time parameters such as $\mu_{X_i} = \mu_{X_i}(t,T)$. The expectation is

$$\begin{aligned}
& E_t[((H_1S_1(T) - H_2S_2(T)) \wedge Z) \vee 0] \\
= \;& H_1 \int_{D_1} \exp\{x_1\} n(\boldsymbol{x}|\boldsymbol{\mu}_X, \Sigma_X) d\boldsymbol{x} - H_2 \int_{D_1} \exp\{x_2\} n(\boldsymbol{x}|\boldsymbol{\mu}_X, \Sigma_X) d\boldsymbol{x} \\
& + Z \int_{D_2} n(\boldsymbol{x}|\boldsymbol{\mu}_X, \Sigma_X) d\boldsymbol{x},
\end{aligned}$$

where

$$\begin{aligned}
d(x_2, Z) &= \ln(H_2 \exp\{x_2\} + Z) - \ln H_1, \\
D_1 &= \{\boldsymbol{x} = [x_1, x_2]^\top | d(x_2, 0) \le x_1 \le d(x_2, Z)\}, \\
D_2 &= \{\boldsymbol{x} = [x_1, x_2]^\top | x_1 > d(x_2, Z)\}.
\end{aligned}$$

We calculate each integral. Let us use $\boldsymbol{e}_i$ to be the unit vector which i-th element is one. For the integrals of the first and second term, we have

$$\begin{aligned}
& \int_{D_1} \exp\{x_i\} n(\boldsymbol{x}|\boldsymbol{\mu}_X, \Sigma_X) d\boldsymbol{x} \\
= \;& \exp\left\{\mu_{X_i} + \frac{1}{2}\sigma_{X_i}^2\right\} \int_{D_1} (2\pi)^{-1} |\Sigma_X|^{-\frac{1}{2}} \\
& \times \exp\left\{ -\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu}_X - \Sigma_X \boldsymbol{e}_i)^\top \Sigma_X^{-1} (\boldsymbol{x} - \boldsymbol{\mu}_X - \Sigma_X \boldsymbol{e}_i) \right\} d\boldsymbol{x},
\end{aligned}$$

where we completed the squares. Furthermore, the integral can be expanded by changing the variables.

$$\begin{aligned}
&\int_{D_1} (2\pi)^{-1}|\Sigma_X|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_X-\Sigma_X\boldsymbol{e}_1)^\top\Sigma_X^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_X-\Sigma_X\boldsymbol{e}_1)\right\} d\boldsymbol{x} \\
= &\int_{-\infty}^{\infty}\int_{d_1(x_2,0)}^{d_1(x_2,Z)} (2\pi(1-\rho_{X_1X_2}^2))^{-\frac{1}{2}}\sigma_{X_1}^{-1}\exp\left\{-\frac{y^2}{2}\right\}(1-\rho_{X_1X_2}^2)^{\frac{1}{2}}\sigma_{X_1}dy \\
&\times(2\pi)^{-\frac{1}{2}}\sigma_{X_2}^{-1}\exp\left\{-\frac{1}{2}\left(\frac{x_2-\mu_{X_2}-\sigma_{X_1X_2}}{\sigma_{X_2}}\right)^2\right\}dx_2 \\
= &\int_{-\infty}^{\infty}(\Phi(d_1(x_2,Z))-\Phi(d_1(x_2,0)))n(x_2|\mu_{X_2}+\sigma_{X_1X_2},\sigma_{X_2}^2)dx_2,
\end{aligned}$$

where

$$\begin{aligned}
d_1(x,z) &= \frac{\ln(H_2\exp\{x\}+z)-\ln H_1-\mu_{X_1}-\sigma_{X_1}^2}{\sigma_{X_1}\sqrt{1-\rho_{X_1X_2}^2}} \\
&\quad -\frac{\rho_{X_1X_2}\sigma_{X_1}\frac{x-\mu_{X_2}-\sigma_{X_1X_2}}{\sigma_{X_2}}}{\sigma_{X_1}\sqrt{1-\rho_{X_1X_2}^2}}.
\end{aligned}$$

In addition, we can simplify the second part of the integration. Generally it is known that,

$$\begin{aligned}
\Phi(d_1) &= P(X_1\le d_1) = P(X_1\le d_1, X_2\le\infty) \\
&= \int_{-\infty}^{\infty}\Phi\left(\frac{d_1-\rho_{12}x_2}{\sqrt{1-\rho_{12}^2}}\right)\phi(x_2)dx_2,
\end{aligned} \tag{9}$$

where

$$[X_1,X_2]\sim N(\mathbf{0},\boldsymbol{\Sigma}),$$
$$\boldsymbol{\Sigma}=\begin{bmatrix}1 & \rho_{12}\\ \rho_{12} & 1\end{bmatrix}.$$

Notice that,

$$\begin{aligned} d_1(x_2,0) &= \frac{\ln(H_2/H_1) - \mu_{X_1} - \sigma_{X_1}^2 + \rho_{X_1X_2}\sigma_{X_1}\frac{\mu_{X_2}+\sigma_{X_1X_2}}{\sigma_{X_2}}}{\sigma_{X_1}\sqrt{1-\rho_{X_1X_2}^2}} \\ &\quad -\frac{\left(\frac{\rho_{X_1X_2}\sigma_{X_1}}{\sigma_{X_2}} - 1\right)(\sigma_{X_2}\hat{x}_1 + \mu_{X_2} + \sigma_{X_1X_2})}{\sigma_{X_1}\sqrt{1-\rho_{X_1X_2}^2}} \\ &= \frac{\hat{\mu}_1 - \hat{\rho}\hat{x}_1}{\sqrt{1-\hat{\rho}^2}}, \end{aligned}$$

where we defined $\hat{x}_1 = (x_2 - \mu_{X_2} - \sigma_{X_1X_2})/\sigma_{X_2}$ and used the following facts.

$$\begin{aligned} \frac{\hat{\rho}}{\sqrt{1-\hat{\rho}^2}} &\equiv \frac{\left(\frac{\rho_{X_1X_2}\sigma_{X_1}}{\sigma_{X_2}} - 1\right)\sigma_{X_2}}{\sigma_{X_1}\sqrt{1-\rho_{X_1X_2}^2}} = \frac{\rho_{X_1X_2}\sigma_{X_1} - \sigma_{X_2}}{\sigma_{X_1}\sqrt{1-\rho_{X_1X_2}^2}} \\ \Rightarrow \hat{\rho} &= \frac{\rho_{X_1X_2}\sigma_{X_1} - \sigma_{X_2}}{\sqrt{\sigma_{X_1}^2 - 2\rho_{X_1X_2}\sigma_{X_1}\sigma_{X_2} + \sigma_{X_2}^2}}, \\ \hat{\mu}_1 &\equiv \sqrt{1-\hat{\rho}^2}\Bigg(\frac{\ln(H_2/H_1) - \mu_{X_1} - \sigma_{X_1}^2 + \rho_{X_1X_2}\sigma_{X_1}\frac{\mu_{X_2}+\sigma_{X_1X_2}}{\sigma_{X_2}}}{\sigma_{X_1}\sqrt{1-\rho_{X_1X_2}^2}} \\ &\quad -\frac{\left(\frac{\rho_{X_1X_2}\sigma_{X_1}}{\sigma_{X_2}} - 1\right)(\mu_{X_2} + \sigma_{X_1X_2})}{\sigma_{X_1}\sqrt{1-\rho_{X_1X_2}^2}}\Bigg) \\ &= \frac{\ln(H_2/H_1) - \mu_{X_1} + \mu_{X_2} - \sigma_{X_1}^2 + \sigma_{X_1X_2}}{\sqrt{\sigma_{X_1}^2 - 2\sigma_{X_1X_2} + \sigma_{X_2}^2}}. \end{aligned}$$

Now, we have

$$\begin{aligned} & -\int_{-\infty}^{\infty} \Phi(d_1(x_2,0))n(x_2|\mu_{X_2} + \sigma_{X_1X_2}, \sigma_{X_2}^2)dx_2 \\ = & -\int_{-\infty}^{\infty} \Phi\left(\frac{\hat{\mu}_1 - \hat{\rho}\hat{x}_1}{\sqrt{1-\hat{\rho}^2}}\right)\phi(\hat{x}_1)d\hat{x}_1 = -\Phi(\hat{\mu}_1), \end{aligned}$$

where we used $n(x|\mu_{X_2} + \sigma_{X_1X_2}, \sigma_{X_2}^2) = \frac{1}{\sigma_{X_2}}\phi(\frac{x-\mu_{X_2}-\sigma_{X_1X_2}}{\sigma_{X_2}})$, changed the variables and (9).

The other integrals are calculated in similar manner.

$$\begin{aligned}
&\int_{D_1} (2\pi)^{-1} |\Sigma_X|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} (\boldsymbol{x} - \boldsymbol{\mu}_X - \Sigma_X \boldsymbol{e}_2)^\top \Sigma_X^{-1} (\boldsymbol{x} - \boldsymbol{\mu}_X - \Sigma_X \boldsymbol{e}_2) \right\} d\boldsymbol{x} \\
= &\int_{-\infty}^{\infty} \Phi(d_2(x_2, Z)) n(x_2 | \mu_{X_2} + \sigma_{X_2}^2, \sigma_{X_2}^2) dx_2 - \Phi(\hat{\mu}_2),
\end{aligned}$$

where

$$\begin{aligned}
d_2(x, z) &= \frac{\ln(H_2 \exp\{x\} + z) - \ln H_1 - \mu_{X_1} - \rho_{X_1 X_2} \sigma_{X_1} \frac{x - \mu_{X_2}}{\sigma_{X_2}}}{\sigma_{X_1} \sqrt{1 - \rho_{X_1 X_2}^2}}, \\
\hat{\mu}_2 &= \frac{\ln(H_2/H_1) - \mu_{X_1} + \mu_{X_2} - \sigma_{X_1 X_2} + \sigma_{X_2}^2}{\sqrt{\sigma_{X_1}^2 - 2\sigma_{X_1 X_2} + \sigma_{X_2}^2}}.
\end{aligned}$$

The integral of the last term is

$$\begin{aligned}
&\int_{D_2} (2\pi)^{-1} |\Sigma_X|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} (\boldsymbol{x} - \boldsymbol{\mu}_X)^\top \Sigma_X^{-1} (\boldsymbol{x} - \boldsymbol{\mu}_X) \right\} d\boldsymbol{x} \\
= &\int_{-\infty}^{\infty} \int_{d(x_2, Z)}^{\infty} (2\pi)^{-1} \left(\sigma_{X_1} \sigma_{X_2} \sqrt{1 - \rho_{X_1 X_2}^2} \right)^{-1} \exp\Bigg\{ -\frac{1}{2(1 - \rho_{X_1 X_2}^2)} \\
&\times \left(\left(\frac{x_1 - \mu_{X_1}}{\sigma_{X_2}} \right)^2 - 2\rho_{X_1 X_2} \left(\frac{x_1 - \mu_{X_1}}{\sigma_{X_1}} \right) \left(\frac{x_2 - \mu_{X_2}}{\sigma_{X_2}} \right) \right. \\
&\left. + \left(\frac{x_2 - \mu_{X_2}}{\sigma_{X_2}} \right)^2 \right) \Bigg\} dx_2 dx_1 \\
= &\int_{-\infty}^{\infty} \int_{d(x_2, Z)}^{\infty} (2\pi(1 - \rho_{X_1 X_2}^2))^{-\frac{1}{2}} \sigma_{X_1}^{-1} \\
&\times \exp\left\{ -\frac{\left(x_1 - \mu_{X_1} - \rho_{X_1 X_2} \sigma_{X_1} \frac{x_2 - \mu_{X_2}}{\sigma_{X_2}} \right)^2}{2(1 - \rho_{X_1 X_2}^2) \sigma_{X_1}^2} \right\} dx_1 \\
&\times (2\pi)^{-\frac{1}{2}} \sigma_{X_2}^{-1} \exp\left\{ -\frac{1}{2} \left(\frac{x_2 - \mu_{X_2}}{\sigma_{X_2}} \right)^2 \right\} dx_2
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{d_2(x_2,Z)}^{\infty} (2\pi(1-\rho_{X_1X_2}^2))^{-\frac{1}{2}}\sigma_{X_1}^{-1}\exp\left\{-\frac{y^2}{2}\right\}(1-\rho_{X_1X_2}^2)^{\frac{1}{2}}\sigma_{X_1}dy \\
&\quad \times(2\pi)^{-\frac{1}{2}}\sigma_{X_2}^{-1}\exp\left\{-\frac{1}{2}\left(\frac{x_2-\mu_{X_2}}{\sigma_{X_2}}\right)^2\right\}dx_2 \\
&= \int_{-\infty}^{\infty}(1-\Phi(d_2(x_2,Z)))n(x_2|\mu_{X_2},\sigma_{X_2}^2)dx_2 \\
&= \int_{-\infty}^{\infty}\Phi(-d_2(x_2,Z))n(x_2|\mu_{X_2},\sigma_{X_2}^2)dx_2.
\end{aligned}
$$

Collecting all terms, we have the valuation formula.

$$
S_e(t) = \hat{H}_1(t,T) - \hat{H}_2(t,T) + \hat{H}_3(t,T)Z, \tag{10}
$$

where

$$
\begin{aligned}
\hat{H}_1(t,T) &= H_1\exp\left\{-r(T-t)+\mu_{X_1}(t,T)+\frac{1}{2}\sigma_{X_1}^2(t,T)\right\} \\
&\quad\times\left(\int_{-\infty}^{\infty}\Phi(d_1(x_2,Z))n(x_2|\mu_{X_2}(t,T)+\sigma_{X_1X_2}(t,T),\sigma_{X_2}^2(t,T))dx_2\right. \\
&\quad\left.-\Phi(\hat{\mu}_1(t,T))\right), \\
\hat{H}_2(t,T) &= H_2\exp\left\{-r(T-t)+\mu_{X_2}(t,T)+\frac{1}{2}\sigma_{X_2}^2(t,T)\right\} \\
&\quad\times\left(\int_{-\infty}^{\infty}\Phi(d_2(x_2,Z))n(x_2|\mu_{X_2}(t,T)+\sigma_{X_2}^2(t,T),\sigma_{X_2}^2(t,T))dx_2\right. \\
&\quad\left.-\Phi(\hat{\mu}_2(t,T))\right), \\
\hat{H}_3(t,T) &= \exp(-r(T-t))\int_{-\infty}^{\infty}\Phi(-d_2(x_2,Z))n(x_2|\mu_{X_2}(t,T),\sigma_{X_2}^2(t,T))dx_2,
\end{aligned}
$$

$$
\begin{aligned}
d_1(x,z) &= d_2(x,z) - \sigma_{X_1}(t,T)\sqrt{1-\rho^2_{X_1X_2}(t,T)}, \\
d_2(x,z) &= \frac{\ln(H_2\exp\{x\}+z) - \ln H_1 - \mu_{X_1}(t,T)}{\sigma_{X_1}(t,T)\sqrt{1-\rho^2_{X_1X_2}(t,T)}} \\
&\quad - \frac{\rho_{X_1X_2}(t,T)\sigma_{X_1}(t,T)\frac{x-\mu_{X_2}(t,T)}{\sigma_{X_2}(t,T)}}{\sigma_{X_1}(t,T)\sqrt{1-\rho^2_{X_1X_2}(t,T)}}, \\
\hat{\mu}_1(t,T) &= \frac{\ln(H_2/H_1) - \mu_{X_1}(t,T) + \mu_{X_2}(t,T) - \sigma^2_{X_1}(t,T) + \sigma_{X_1X_2}(t,T)}{\sqrt{\sigma^2_{X_1}(t,T) - 2\sigma_{X_1X_2}(t,T) + \sigma^2_{X_2}(t,T)}}, \\
\hat{\mu}_2(t,T) &= \hat{\mu}_1(t,T) + \sqrt{\sigma^2_{X_1}(t,T) - 2\sigma_{X_1X_2}(t,T) + \sigma^2_{X_2}(t,T)}.
\end{aligned}
$$

3.6.3 Proof of Proposition 3.3.1

In this subsection, we derive the hedging strategy for emission allowance using futures commodities. First, we use the future commodity prices equation written in terms of spot commodity prices and derive the future price process using Ito's lemma. This price process can be explicitly written in terms of futures price levels. Then, we calculate the expectation and covariance of stochastic terms of futures price using properties of stochastic calculus.

First, notice that for each commodity i

$$G_i(t,T) = e^{\mu_{X_i}(t,T)+\frac{\sigma^2_{X_i}(t,T)}{2}}.$$

The partial derivatives are

$$\begin{aligned}
\frac{\partial G_i(t,T)}{\partial S_i(t)} &= \frac{\left[e^{(T-t)\boldsymbol{\beta}}\right]_{i,i}}{S_i(t)} G_i(t,T), \\
\frac{\partial G_i(t,T)}{\partial \delta_i(t)} &= \left[e^{(T-t)\boldsymbol{\beta}}\right]_{i,n+i} G_i(t,T),
\end{aligned}$$

where we denote $[A]_{i,j}$ as $[i,j]$ th entry of matrix A.

Since the futures price $G_i(t,T)$ is a function of $S_i(t), \delta_i(t)$ and twice differentiable, we can use the Ito's lemma and the dynamics of future price is

$$dG_i(t,T) = \sigma_{S_i} S_i(t) \frac{\partial G_i}{\partial S_i} dW_{S_i}(t) + \sigma_{\delta_i} \frac{\partial G_i}{\partial \delta_i} dW_{\delta_i}(t),$$

where the drift term is 0 since $G_i(t,T)$ is martingale under the risk-neutral probability.

Again, using Ito's lemma we have,

$$\begin{aligned}
& d\log G_i(t,T) \\
= & -\frac{1}{2}\left\{\sigma^2_{S_i}\left[e^{(T-t)\boldsymbol{\beta}}\right]^2_{i,i} + 2\sigma_{S_i\delta_i}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{i,i}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{i,n+i} + \sigma^2_{\delta_i}\left[e^{(T-t)\boldsymbol{\beta}}\right]^2_{i,n+i}\right\}dt \\
& +\sigma_{S_i}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{i,i} dW_{S_i}(t) + \sigma_{\delta_i}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{i,n+i} dW_{\delta_i}(t).
\end{aligned}$$

The futures price can be expressed as follows.

$$G_i(T_0,T_i) = G_i(t,T_i)e^{\hat{X}_{G_i}(t,T_0,T_i)}, \quad t \le T_0 \le T_i,$$

where

$$\begin{aligned}\hat{X}_{G_i}(t,T_0,T_i) &\equiv \mu_{\hat{X}_{G_i}}(t,T_0,T_i) \\ &\quad + \int_t^{T_0} \sigma_{S_i} \left[e^{(T_i-t)\boldsymbol{\beta}}\right]_{i,i} dW_{S_i}(u) + \int_t^{T_0} \sigma_{\delta_i} \left[e^{(T_i-t)\boldsymbol{\beta}}\right]_{i,n+i} dW_{\delta_i}(u).\end{aligned}$$

The expectation value for each commodity i is

$$\begin{aligned}&\mu_{\hat{X}_{G_i}}(t,T_0,T_i) \equiv E_t[\hat{X}_{G_i}(t,T_0,T_i)] \\ =\ & -\frac{1}{2}\int_t^{T_0} \sigma_{S_i}^2 \left[e^{(T_i-u)\boldsymbol{\beta}}\right]_{i,i}^2 du - \int_t^{T_0} \sigma_{S_i\delta_i} \left[e^{(T_i-u)\boldsymbol{\beta}}\right]_{i,i} \left[e^{(T_i-u)\boldsymbol{\beta}}\right]_{i,n+i} du \\ & -\frac{1}{2}\int_t^{T_0} \sigma_{\delta_i}^2 \left[e^{(T_i-u)\boldsymbol{\beta}}\right]_{i,n+i}^2 du.\end{aligned}$$

The covariance of $\hat{X}_{G_i}(t,T_0,T_i)$ and $\hat{X}_{G_j}(t,T_0,T_j)$ is

$$\begin{aligned}&\sigma_{\hat{X}_{G_i}\hat{X}_{G_j}}(t,T_0,T_i,T_j) \equiv \mathrm{cov}_t[\hat{X}_{G_i}(t,T_0,T_i), \hat{X}_{G_i}(t,T_0,T_j)] \\ =\ & \int_t^{T_0} \sigma_{S_iS_j} \left[e^{(T_i-u)\boldsymbol{\beta}}\right]_{i,i} \left[e^{(T_j-u)\boldsymbol{\beta}}\right]_{j,j} du \\ & + \int_t^{T_0} \sigma_{S_i\delta_j} \left[e^{(T_i-u)\boldsymbol{\beta}}\right]_{i,i} \left[e^{(T_j-u)\boldsymbol{\beta}}\right]_{j,n+j} du \\ & + \int_t^{T_0} \sigma_{S_j\delta_i} \left[e^{(T_j-u)\boldsymbol{\beta}}\right]_{j,j} \left[e^{(T_i-u)\boldsymbol{\beta}}\right]_{i,n+i} du \\ & + \int_t^{T_0} \sigma_{\delta_i\delta_j} \left[e^{(T_i-u)\boldsymbol{\beta}}\right]_{i,n+j} \left[e^{(T_j-u)\boldsymbol{\beta}}\right]_{j,n+j} du.\end{aligned}$$

Now we derive the emission allowance futures price using commodity future prices.

$$\begin{aligned}G_e(t,T) &= E_t[S_e(T)] = E_t[((H_1S_1(T) - H_2S_2(T)) \wedge Z) \vee 0] \\ &= E_t[((H_1G_1(t,T)e^{\hat{X}_{G_1}(t,T,T)} - H_2G_2(t,T)e^{\hat{X}_{G_2}(t,T,T)}) \wedge Z) \vee 0].\end{aligned}$$

With the same argument as in the proof of Proposition 3.2.1, we have

$$G_e(t,T) = \hat{\hat{H}}_1(t,T)G_1(t,T) - \hat{\hat{H}}_2(t,T)G_2(t,T) + \hat{\hat{H}}_3(t,T)Z,$$

where

$$
\begin{aligned}
\hat{\hat{H}}_1(t,T) &= H_1 \exp\left\{\mu_{\hat{X}_{G_1}}(t,T,T) + \frac{1}{2}\sigma^2_{\hat{X}_{G_1}}(t,T,T,T)\right\} \\
&\quad \times \left\{ \int_{-\infty}^{\infty} \Phi(d_{G_1}(x_2,Z)) \right. \\
&\quad \times n(x_2|\mu_{\hat{X}_{G_2}}(t,T,T) + \sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T), \sigma^2_{\hat{X}_{G_2}}(t,T,T,T))dx_2 \\
&\quad \left. -\Phi(\hat{\mu}_{G_1}(t,T)) \right\}, \\
\hat{\hat{H}}_2(t,T) &= H_2 \exp\left\{\mu_{\hat{X}_{G_2}}(t,T,T) + \frac{1}{2}\sigma^2_{\hat{X}_{G_2}}(t,T,T,T)\right\} \\
&\quad \times \left\{ \int_{-\infty}^{\infty} \Phi(d_{G_2}(x_2,Z)) \right. \\
&\quad \times n(x_2|\mu_{\hat{X}_{G_2}}(t,T,T) + \sigma^2_{\hat{X}_{G_2}}(t,T,T,T), \sigma^2_{\hat{X}_{G_2}}(t,T,T,T))dx_2 \\
&\quad \left. -\Phi(\hat{\mu}_{G_2}(t,T)) \right\}, \\
\hat{\hat{H}}_3(t,T) &= \int_{-\infty}^{\infty} \Phi(-d_{G_2}(x_2,Z))n(x_2|\mu_{\hat{X}_{G_2}}(t,T,T), \sigma^2_{\hat{X}_{G_2}}(t,T,T,T))dx_2, \\
d_{G_1}(x,Z) &= d_{G_2}(x,Z) - \sigma_{\hat{X}_{G_1}}(t,T,T,T)\sqrt{1-\rho^2_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T)}, \\
d_{G_2}(x,Z) &= \frac{\ln(H_2G_2(t,T)\exp(x)+Z) - \ln(H_1G_1(t,T)) - \mu_{\hat{X}_{G_1}}(t,T,T)}{\sigma_{\hat{X}_{G_1}}(t,T,T,T)\sqrt{1-\rho^2_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T)}} \\
&\quad - \frac{\rho_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T)\sigma_{\hat{X}_{G_1}}(t,T,T,T)\frac{x-\mu_{\hat{X}_{G_2}}(t,T,T)}{\sigma_{\hat{X}_{G_2}}(t,T,T,T)}}{\sigma_{\hat{X}_{G_1}}(t,T,T,T)\sqrt{1-\rho^2_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T)}}, \\
\hat{\mu}_{G_1}(t,T) &= \frac{\ln(H_2G_2(t,T)/H_1G_1(t,T)) - \mu_{\hat{X}_{G_1}}(t,T,T) + \mu_{\hat{X}_{G_2}}(t,T,T)}{\sqrt{\sigma^2_{\hat{X}_{G_1}}(t,T,T,T) - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T) + \sigma^2_{\hat{X}_{G_2}}(t,T,T,T)}} \\
&\quad + \frac{-\sigma^2_{\hat{X}_{G_1}}(t,T,T,T) + \sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T)}{\sqrt{\sigma^2_{\hat{X}_{G_1}}(t,T,T,T) - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T) + \sigma^2_{\hat{X}_{G_2}}(t,T,T,T)}},
\end{aligned}
$$

$$\hat{\mu}_{G_2}(t,T) = \hat{\mu}_{G_1}(t,T) + \sqrt{\sigma^2_{\hat{X}_{G_1}}(t,T,T,T) - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T) + \sigma^2_{\hat{X}_{G_2}}(t,T,T,T)}.$$

We now derive the hedging equation for emission allowance futures price using commodity futures prices. Using Ito's lemma, the dynamics of emission allowance futures price $dG_e(t,T)$ is

$$\begin{aligned} dG_e(t,T) &= G_1(t,T)d\hat{\hat{H}}_1(t,T) + \hat{\hat{H}}_1(t,T)dG_1(t,T) + d\hat{\hat{H}}_1(t,T)dG_1(t,T) \\ &\quad -G_2(t,T)d\hat{\hat{H}}_2(t,T) - \hat{\hat{H}}_2(t,T)dG_2(t,T) - d\hat{\hat{H}}_2(t,T)dG_2(t,T) \\ &\quad +Zd\hat{\hat{H}}_3(t,T), \end{aligned}$$

and $d\hat{\hat{H}}_i(t,T)$ is

$$\begin{aligned} d\hat{\hat{H}}_i(t,T) &= \frac{\partial \hat{\hat{H}}_i(t,T)}{\partial t}dt + \sum_{j=1}^{2} \frac{\partial \hat{\hat{H}}_i(t,T)}{\partial G_j(t,T)} dG_j(t,T) \\ &\quad + \frac{1}{2}\frac{\partial^2 \hat{\hat{H}}_i(t,T)}{\partial G_1(t,T)\partial G_1(t,T)} G_1(t,T)^2 \Bigg(\sigma_{S_1S_1} \left[e^{(T-t)\boldsymbol{\beta}}\right]^2_{1,1} \\ &\quad -2\sigma_{S_1\delta_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3} + \sigma_{\delta_1\delta_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]^2_{1,3}\Bigg)dt \\ &\quad + \frac{1}{2}\frac{\partial^2 \hat{\hat{H}}_i(t,T)}{\partial G_1(t,T)\partial G_2(t,T)} G_1(t,T)G_2(t,T) \Bigg(\sigma_{S_1S_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2} \\ &\quad -\sigma_{S_1\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4} - \sigma_{S_2\delta_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3} \\ &\quad +\sigma_{\delta_1\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}\Bigg)dt \\ &\quad + \frac{1}{2}\frac{\partial^2 \hat{\hat{H}}_i(t,T)}{\partial G_2(t,T)\partial G_1(t,T)} G_1(t,T)G_2(t,T) \Bigg(\sigma_{S_2S_1} \\ &\quad -\sigma_{S_2\delta_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3} - \sigma_{S_1\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4} \\ &\quad +\sigma_{\delta_2\delta_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}\Bigg)dt \end{aligned}$$

$$+\frac{1}{2}\frac{\partial^2\hat{\hat{H}}_i(t,T)}{\partial G_2(t,T)\partial G_2(t,T)}G_2(t,T)^2\Bigg(\sigma_{S_2S_2}$$
$$-2\sigma_{S_2\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}+\sigma_{\delta_2\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}^2\Bigg)dt.$$

Substituting $d\hat{\hat{H}}_i(t,T)$ to $dG_e(t,T)$, we have

$$\begin{aligned}
& dG_e(t,T) \\
= \quad & \Bigg\{\frac{\partial\hat{\hat{H}}_1(t,T)}{\partial t}G_1(t,T)-\frac{\partial\hat{\hat{H}}_2(t,T)}{\partial t}G_2(t,T)+Z\frac{\partial\hat{\hat{H}}_3(t,T)}{\partial t} \\
& +\Bigg(\frac{G_1(t,T)}{2}\frac{\partial^2\hat{\hat{H}}_1(t,T)}{\partial G_1(t,T)\partial G_1(t,T)}-\frac{G_2(t,T)}{2}\frac{\partial^2\hat{\hat{H}}_2(t,T)}{\partial G_1(t,T)\partial G_1(t,T)} \\
& +\frac{Z}{2}\frac{\partial^2\hat{\hat{H}}_3(t,T)}{\partial G_1(t,T)\partial G_1(t,T)}\Bigg)G_1(t,T)^2\Bigg(\sigma_{S_1S_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}^2 \\
& -2\sigma_{S_1\delta_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3} \\
& +\sigma_{\delta_1\delta_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3}^2\Bigg) \\
& +\Bigg(\frac{G_1(t,T)}{2}\frac{\partial^2\hat{\hat{H}}_1(t,T)}{\partial G_1(t,T)\partial G_2(t,T)}-\frac{G_2(t,T)}{2}\frac{\partial^2\hat{\hat{H}}_2(t,T)}{\partial G_1(t,T)\partial G_2(t,T)} \\
& +\frac{Z}{2}\frac{\partial^2\hat{\hat{H}}_3(t,T)}{\partial G_1(t,T)\partial G_2(t,T)}\Bigg)G_1(t,T)G_2(t,T)\Bigg(\sigma_{S_1S_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2} \\
& -\sigma_{S_1\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}-\sigma_{S_2\delta_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3} \\
& +\sigma_{\delta_1\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}\Bigg)
\end{aligned}$$

$$+\left(\frac{G_1(t,T)}{2}\frac{\partial^2\hat{\hat{H}}_1(t,T)}{\partial G_2(t,T)\partial G_1(t,T)}-\frac{G_2(t,T)}{2}\frac{\partial^2\hat{\hat{H}}_2(t,T)}{\partial G_2(t,T)\partial G_1(t,T)}\right.$$

$$\left.+\frac{Z}{2}\frac{\partial^2\hat{\hat{H}}_3(t,T)}{\partial G_2(t,T)\partial G_1(t,T)}\right)G_1(t,T)G_2(t,T)\left(\sigma_{S_2S_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}\right.$$

$$-\sigma_{S_2\delta_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3}-\sigma_{S_1\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}$$

$$\left.+\sigma_{\delta_2\delta_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}\right)$$

$$+\left(\frac{G_1(t,T)}{2}\frac{\partial^2\hat{\hat{H}}_1(t,T)}{\partial G_2(t,T)\partial G_2(t,T)}-\frac{G_2(t,T)}{2}\frac{\partial^2\hat{\hat{H}}_2(t,T)}{\partial G_2(t,T)\partial G_2(t,T)}\right.$$

$$\left.+\frac{Z}{2}\frac{\partial^2\hat{\hat{H}}_3(t,T)}{\partial G_2(t,T)\partial G_2(t,T)}\right)G_2(t,T)^2\left(\sigma_{S_2S_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}^2\right.$$

$$\left.-2\sigma_{S_2\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}+\sigma_{\delta_2\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}^2\right)$$

$$+\frac{\partial\hat{\hat{H}}_1(t,T)}{\partial G_1(t,T)}G_1(t,T)^2\left(\sigma_{S_1}^2\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}^2-2\sigma_{S_1\delta_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3}\right.$$

$$\left.+\sigma_{\delta_1}^2\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3}^2\right)$$

$$+\frac{\partial\hat{\hat{H}}_1(t,T)}{\partial G_2(t,T)}G_1(t,T)G_2(t,T)\left(\sigma_{S_1S_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}\right.$$

$$-\sigma_{S_1\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}-\sigma_{S_2\delta_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3}$$

$$\left.+\sigma_{\delta_1\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}\right)$$

$$-\frac{\partial\hat{\hat{H}}_2(t,T)}{\partial G_2(t,T)}G_2(t,T)\left(\sigma_{S_2}^2\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}^2-2\sigma_{S_2\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}\right.$$

$$\left.+\sigma_{\delta_2}^2\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}^2\right)$$

$$
\begin{aligned}
&-\frac{\partial\hat{\hat{H}}_2(t,T)}{\partial G_1(t,T)}G_1(t,T)G_2(t,T)\Bigg(\sigma_{S_1S_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}\\
&-\sigma_{S_1\delta_2}(T-t)\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}-\sigma_{S_2\delta_1}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3}\\
&+\sigma_{\delta_1\delta_2}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{1,3}\left[e^{(T-t)\boldsymbol{\beta}}\right]_{2,4}\Bigg)\Bigg\}dt\\
&+\sum_{j=1}^{2}\frac{\partial\hat{\hat{H}}_1(t,T)}{\partial G_j(t,T)}G_1(t,T)dG_j(t,T)+\hat{\hat{H}}_1(t,T)dG_1(t,T)\\
&-\sum_{j=1}^{2}\frac{\partial\hat{\hat{H}}_2(t,T)}{\partial G_j(t,T)}G_2(t,T)dG_j(t,T)-\hat{\hat{H}}_2(t,T)dG_2(t,T)\\
&+\sum_{j=1}^{2}\frac{\partial\hat{\hat{H}}_3(t,T)}{\partial G_j(t,T)}ZdG_j(t,T).
\end{aligned}
$$

Here after, we omit the time parameters. The partial derivatives are calculated as follows.

$$
\begin{aligned}
\frac{\partial\hat{\hat{H}}_i}{\partial G_j} &= H_i\exp\left\{\mu_{\hat{X}_{G_i}}+\frac{1}{2}\sigma^2_{\hat{X}_{G_i}}\right\}\\
&\times\Bigg\{\int_{-\infty}^{\infty}\phi(d_{G_i}(x_2,Z))\frac{\partial d_{G_i}(x_2,Z)}{\partial G_j}n(x_2|\mu_{\hat{X}_{G_2}}+\sigma_{\hat{X}_{G_2}\hat{X}_{G_i}},\sigma^2_{\hat{X}_{G_2}})dx_2\\
&-\phi(\hat{\mu}_{G_i})\frac{\partial\hat{\mu}_{G_i}}{\partial G_j}\Bigg\}, i,j=1,2,\\
\frac{\partial^2\hat{\hat{H}}_i}{\partial G_j\partial G_k} &= H_i\exp\left\{\mu_{\hat{X}_{G_i}}+\frac{1}{2}\sigma^2_{\hat{X}_{G_i}}\right\}\\
&\times\Bigg\{\int_{-\infty}^{\infty}\phi'(d_{G_i}(x_2,Z))\frac{\partial d_{G_i}(x_2,Z)}{\partial G_k}\frac{\partial d_{G_i}(x_2,Z)}{\partial G_j}\\
&\times n(x_2|\mu_{\hat{X}_{G_2}}+\sigma_{\hat{X}_{G_2}\hat{X}_{G_i}},\sigma^2_{\hat{X}_{G_2}})dx_2
\end{aligned}
$$

$$
\begin{aligned}
& +\int_{-\infty}^{\infty}\phi(d_{G_i}(x_2,Z))\frac{\partial^2 d_{G_i}(x_2,Z)}{\partial G_j\partial G_k}n(x_2|\mu_{\hat{X}_{G_2}}+\sigma_{\hat{X}_{G_2}\hat{X}_{G_i}},\sigma^2_{\hat{X}_{G_2}})dx_2 \\
& \left.-\phi'(\hat{\mu}_{G_i})\frac{\partial\hat{\mu}_{G_i}}{\partial G_k}\frac{\partial\hat{\mu}_{G_i}}{\partial G_j}-\phi(\hat{\mu}_{G_i})\frac{\partial^2\hat{\mu}_{G_i}}{\partial G_j\partial G_k}\right\}, i,j,k=1,2, \\
\frac{\partial\hat{\hat{H}}_i}{\partial t} = {} & H_i\left(\frac{\partial\mu_{\hat{X}_{G_i}}}{\partial t}+\frac{1}{2}\frac{\partial\sigma^2_{\hat{X}_{G_i}}}{\partial t}\right)\exp\left\{\mu_{\hat{X}_{G_i}}+\frac{1}{2}\sigma^2_{\hat{X}_{G_i}}\right\} \\
& \times\left\{\int_{-\infty}^{\infty}\Phi(d_{G_i}(x_2,Z))n(x_2|\mu_{\hat{X}_{G_2}}+\sigma_{\hat{X}_{G_2}\hat{X}_{G_i}},\sigma^2_{\hat{X}_{G_2}})dx_2-\Phi(\hat{\mu}_{G_i})\right\} \\
& +H_i\exp\left\{\mu_{\hat{X}_{G_i}}+\frac{1}{2}\sigma^2_{\hat{X}_{G_i}}\right\} \\
& \times\left\{\int_{-\infty}^{\infty}\phi(d_{G_i}(x_2,Z))\frac{\partial d_{G_i}(x_2,Z)}{\partial t}n(x_2|\mu_{\hat{X}_{G_2}}+\sigma_{\hat{X}_{G_2}\hat{X}_{G_i}},\sigma^2_{\hat{X}_{G_2}})dx_2\right. \\
& +\int_{-\infty}^{\infty}\Phi(d_{G_i}(x_2,Z))\frac{\partial n(x_2|\mu_{\hat{X}_{G_2}}+\sigma_{\hat{X}_{G_2}\hat{X}_{G_i}},\sigma^2_{\hat{X}_{G_2}})}{\partial t}dx_2 \\
& \left.-\phi(\hat{\mu}_{G_i})\frac{\partial\hat{\mu}_{G_i}}{\partial t}\right\}, i=1,2, \\
\frac{\partial\hat{\hat{H}}_3}{\partial t} = {} & -\int_{-\infty}^{\infty}\phi(-d_{G_2}(x_2,Z))\frac{\partial d_{G_2}(x_2,Z)}{\partial t}n(x_2|\mu_{\hat{X}_{G_2}},\sigma^2_{\hat{X}_{G_2}})dx_2 \\
& +\int_{-\infty}^{\infty}\Phi(-d_{G_2}(x_2,Z))\frac{\partial n(x_2|\mu_{\hat{X}_{G_2}},\sigma^2_{\hat{X}_{G_2}})}{\partial t}dx_2, \\
\frac{\partial\hat{\hat{H}}_3}{\partial G_j} = {} & -\int_{-\infty}^{\infty}\phi(-d_{G_2}(x_2,Z))\frac{\partial d_{G_2}(x_2,Z)}{\partial G_j}n(x_2|\mu_{\hat{X}_{G_2}},\sigma^2_{\hat{X}_{G_2}})dx_2, j=1,2, \\
\frac{\partial\hat{\hat{H}}_3}{\partial G_j\partial G_k} = {} & \int_{-\infty}^{\infty}\phi'(-d_{G_2}(x_2,Z))\frac{\partial d_{G_2}(x_2,Z)}{\partial G_k}\frac{\partial d_{G_2}(x_2,Z)}{\partial G_j}n(x_2|\mu_{\hat{X}_{G_2}},\sigma^2_{\hat{X}_{G_2}})dx_2 \\
& -\int_{-\infty}^{\infty}\phi(-d_{G_2}(x_2,Z))\frac{\partial^2 d_{G_2}(x_2,Z)}{\partial G_j\partial G_k}n(x_2|\mu_{\hat{X}_{G_2}},\sigma^2_{\hat{X}_{G_2}})dx_2, j,k=1,2,
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial \mu_{\hat{X}_{G_i}}}{\partial t} &= \frac{1}{2}\sigma_{S_i}^2\left[e^{(T_i-t)\boldsymbol{\beta}}\right]_{i,i}^2 - \sigma_{S_i\delta_i}\left[e^{(T_i-t)\boldsymbol{\beta}}\right]_{i,i}\left[e^{(T_i-t)\boldsymbol{\beta}}\right]_{i,n+i} \\
&\quad + \frac{1}{2}\sigma_{\delta_i}^2\left[e^{(T_i-t)\boldsymbol{\beta}}\right]_{i,n+i}^2, i=1,2, \\
\frac{\partial \sigma_{\hat{X}_{G_i}\hat{X}_{G_j}}}{\partial t} &= -\sigma_{S_iS_j}[e^{(T_i-t)\boldsymbol{\beta}}]_{i,i}\left[e^{(T_j-t)\boldsymbol{\beta}}\right]_{j,j} - \sigma_{S_i\delta_j}[e^{(T_i-t)\boldsymbol{\beta}}]_{i,i} \\
&\quad \times\left[e^{(T_j-t)\boldsymbol{\beta}}\right]_{j,n+j} - \sigma_{S_j\delta_i}[e^{(T_i-t)\boldsymbol{\beta}}]_{i,n+i}\left[e^{(T_j-t)\boldsymbol{\beta}}\right]_{j,j} \\
&\quad -\sigma_{\delta_i\delta_j}[e^{(T_i-t)\boldsymbol{\beta}}]_{i,n+i}\left[e^{(T_j-t)\boldsymbol{\beta}}\right]_{j,n+j}, i,j=1,2, \\
\frac{\partial \rho_{\hat{X}_{G_i}\hat{X}_{G_j}}}{\partial t} &= \frac{\frac{\partial \sigma_{\hat{X}_{G_i}\hat{X}_{G_j}}}{\partial t} - \rho_{\hat{X}_{G_i}\hat{X}_{G_j}}\left(\frac{\partial \sigma_{\hat{X}_{G_i}}}{\partial t}\sigma_{\hat{X}_{G_j}} + \sigma_{\hat{X}_{G_i}}\frac{\partial \sigma_{\hat{X}_{G_j}}}{\partial t}\right)}{\sigma_{\hat{X}_{G_i}}\sigma_{\hat{X}_{G_j}}}, i,j=1,2, \\
\frac{\partial \sigma_{\hat{X}_{G_i}}}{\partial t} &= \frac{1}{2}\sigma_{\hat{X}_{G_i}}^{-1}\frac{\partial \sigma_{\hat{X}_{G_i}}^2}{\partial t}, i=1,2, \\
\frac{\partial d_{G_2}(x_2,Z)}{\partial t} &= (\sigma_{\hat{X}_{G_1}}^2(1-\rho_{\hat{X}_{G_1}\hat{X}_{G_2}}^2))^{-1}\Bigg\{\Bigg(-\frac{\partial \mu_{\hat{X}_{G_1}}}{\partial t} \\
&\quad + \frac{-\frac{\partial \rho_{\hat{X}_{G_1}\hat{X}_{G_2}}}{\partial t}\sigma_{\hat{X}_{G_1}}\sigma_{\hat{X}_{G_2}}(x_2-\mu_{\hat{X}_{G_2}})}{\sigma_{\hat{X}_{G_2}}^2} \\
&\quad - \frac{\frac{\partial \sigma_{\hat{X}_{G_1}}}{\partial t}\rho_{\hat{X}_{G_1}\hat{X}_{G_2}}\sigma_{\hat{X}_{G_2}}(x_2-\mu_{\hat{X}_{G_2}})}{\sigma_{\hat{X}_{G_2}}^2} + \frac{\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}\frac{\partial \mu_{\hat{X}_{G_2}}}{\partial t}}{\sigma_{\hat{X}_{G_2}}^2} \\
&\quad + \rho_{\hat{X}_{G_1}\hat{X}_{G_2}}\sigma_{\hat{X}_{G_1}}(x_2-\mu_{\hat{X}_{G_2}})\frac{\frac{\partial \sigma_{\hat{X}_{G_2}}}{\partial t}}{\sigma_{\hat{X}_{G_2}}^2}\Bigg)\sqrt{\sigma_{\hat{X}_{G_1}}^2(1-\rho_{\hat{X}_{G_1}\hat{X}_{G_2}}^2)} \\
&\quad - \Bigg(\ln(H_2G_2e^{x_2}+Z) - \ln(H_1G_1) - \mu_{\hat{X}_{G_1}} - \rho_{\hat{X}_{G_1}\hat{X}_{G_2}}\sigma_{\hat{X}_{G_1}} \\
&\quad \times\frac{x_2-\mu_{\hat{X}_{G_2}}}{\sigma_{\hat{X}_{G_2}}}\Bigg)\Bigg(\frac{\partial \sigma_{\hat{X}_{G_1}}}{\partial t}\sqrt{1-\rho_{\hat{X}_{G_1}\hat{X}_{G_2}}^2} - \rho_{\hat{X}_{G_1}\hat{X}_{G_2}}\sigma_{\hat{X}_{G_1}} \\
&\quad \times(1-\rho_{\hat{X}_{G_1}\hat{X}_{G_2}}^2)^{-\frac{1}{2}}\frac{\partial \rho_{\hat{X}_{G_1}\hat{X}_{G_2}}}{\partial t}\Bigg)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial d_{G_1}(x_2,Z)}{\partial t} &= \frac{\partial d_{G_2}(x_2,Z)}{\partial t} - \frac{\partial \sigma_{\hat{X}_{G_1}}}{\partial t}\sqrt{1-\rho^2_{\hat{X}_{G_1}\hat{X}_{G_2}}} \\
&\quad +\rho_{\hat{X}_{G_1}\hat{X}_{G_2}}\sigma_{\hat{X}_{G_1}}(1-\rho^2_{\hat{X}_{G_1}\hat{X}_{G_2}})^{-\frac{1}{2}}\frac{\partial \rho_{\hat{X}_{G_1}\hat{X}_{G_2}}}{\partial t}, \\
\frac{\partial \hat{\mu}_{G_1}}{\partial t} &= (\sigma^2_{\hat{X}_{G_1}} - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}} + \sigma^2_{\hat{X}_{G_2}})^{-1} \\
&\quad \times\Bigg\{\Bigg(-\frac{\partial \mu_{\hat{X}_{G_1}}}{\partial t} + \frac{\partial \mu_{\hat{X}_{G_2}}}{\partial t} - \frac{\partial \sigma^2_{\hat{X}_{G_1}}}{\partial t} + \frac{\partial \sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}}{\partial t}\Bigg) \\
&\quad \times\sqrt{\sigma^2_{\hat{X}_{G_1}} - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}} + \sigma^2_{\hat{X}_{G_2}}} \\
&\quad -(\ln(H_2G_2/H_1G_1) - \mu_{\hat{X}_{G_1}} + \mu_{\hat{X}_{G_2}} - \sigma^2_{\hat{X}_{G_1}} + \sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}) \\
&\quad \times\frac{1}{2}(\sigma^2_{\hat{X}_{G_1}} - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}} + \sigma^2_{\hat{X}_{G_2}})^{-\frac{1}{2}} \\
&\quad \times\Bigg(\frac{\partial \sigma^2_{\hat{X}_{G_1}}}{\partial t} - 2\frac{\partial \sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}}{\partial t} + \frac{\partial \sigma^2_{\hat{X}_{G_2}}}{\partial t}\Bigg)\Bigg\}, \\
\frac{\partial \hat{\mu}_{G_2}}{\partial t} &= \frac{\partial \hat{\mu}_{G_1}}{\partial t} + \frac{1}{2}(\sigma^2_{\hat{X}_{G_1}} - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}} + \sigma^2_{\hat{X}_{G_2}})^{-\frac{1}{2}} \\
&\quad \times\Bigg(\frac{\partial \sigma^2_{\hat{X}_{G_1}}}{\partial t} - 2\frac{\partial \sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}}{\partial t} + \frac{\partial \sigma^2_{\hat{X}_{G_2}}}{\partial t}\Bigg), \\
\frac{\partial d_{G_i}(x_2,Z)}{\partial G_1} &= -(\sigma_{\hat{X}_{G_1}}\sqrt{1-\rho^2_{\hat{X}_{G_1}\hat{X}_{G_2}}})^{-1}G_1^{-1}, i=1,2, \\
\frac{\partial d_{G_i}(x_2,Z)}{\partial G_2} &= (\sigma_{\hat{X}_{G_1}}\sqrt{1-\rho^2_{\hat{X}_{G_1}\hat{X}_{G_2}}})^{-1}\frac{H_2e^{x_2}}{H_2G_2e^{x_2}+Z}, i=1,2, \\
\frac{\partial \hat{\mu}_{G_i}}{\partial G_1} &= -(\sigma^2_{\hat{X}_{G_1}} - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}} + \sigma^2_{\hat{X}_{G_2}})^{-\frac{1}{2}}G_1^{-1}, i=1,2, \\
\frac{\partial \hat{\mu}_{G_i}}{\partial G_2} &= (\sigma^2_{\hat{X}_{G_1}} - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}} + \sigma^2_{\hat{X}_{G_2}})^{-\frac{1}{2}}G_2^{-1}, i=1,2, \\
\phi'(x) &= \frac{-x}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}, \\
\frac{\partial^2 d_{G_i}(x_2,Z)}{\partial G_1^2} &= (\sigma_{\hat{X}_{G_1}}\sqrt{1-\rho^2_{\hat{X}_{G_1}\hat{X}_{G_2}}})^{-1}G_1^{-2}, i=1,2, \\
\frac{\partial^2 d_{G_i}(x_2,Z)}{\partial G_2^2} &= -(\sigma_{\hat{X}_{G_1}}\sqrt{1-\rho^2_{\hat{X}_{G_1}\hat{X}_{G_2}}})^{-1}\left(\frac{H_2e^{x_2}}{H_2G_2e^{x_2}+Z}\right)^2, i=1,2,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 d_{G_i}(x_2, Z)}{\partial G_1 \partial G_2} &= 0, i = 1, 2, \\
\frac{\partial^2 \hat{\mu}_{G_i}}{\partial G_1^2} &= (\sigma^2_{\hat{X}_{G_1}} - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}} + \sigma^2_{\hat{X}_{G_2}})^{-\frac{1}{2}} G_1^{-2}, i = 1, 2, \\
\frac{\partial^2 \hat{\mu}_{G_i}}{\partial G_2^2} &= -(\sigma^2_{\hat{X}_{G_1}} - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}} + \sigma^2_{\hat{X}_{G_2}})^{-\frac{1}{2}} G_2^{-2}, i = 1, 2, \\
\frac{\partial^2 \hat{\mu}_{G_i}}{\partial G_1 \partial G_2} &= 0, i = 1, 2, \\
\frac{\partial n(x_2 | \mu_{\hat{X}_{G_2}}, \sigma^2_{\hat{X}_{G_2}})}{\partial t} &= (2\pi)^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2}\left(\frac{x_2 - \mu_{\hat{X}_{G_2}}}{\sigma_{\hat{X}_{G_2}}}\right)^2 \right\} \sigma^{-1}_{\hat{X}_{G_2}} \\
&\quad \times \left(\frac{x_2 - \mu_{\hat{X}_{G_2}}}{\sigma_{\hat{X}_{G_2}}}\right) \frac{\frac{\partial \mu_{\hat{X}_{G_2}}}{\partial t}\sigma_{\hat{X}_{G_2}} + (x_2 - \mu_{\hat{X}_{G_2}})\frac{\partial \sigma_{\hat{X}_{G_2}}}{\partial t}}{\sigma^2_{\hat{X}_{G_2}}} \\
&\quad - (2\pi)^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2}\left(\frac{x_2 - \mu_{\hat{X}_{G_2}}}{\sigma_{\hat{X}_{G_2}}}\right)^2 \right\} \frac{\frac{\partial \sigma_{\hat{X}_{G_2}}}{\partial t}}{\sigma^2_{\hat{X}_{G_2}}}.
\end{aligned}
$$

Chapter 4

A Cointegrated Commodity Pricing Model

4.1 Introduction

Economies are full of equilibrium relations. These include, for example, purchasing power parity, covered or uncovered interest rate parity, spot–forward relations, money demand equations, consumption spending, and relations among commodity prices. Although these relations are widely known, they do not seem to be adequately utilized in finance, especially in the area of derivative valuations.

These relations have been modeled using cointegration techniques, which were first implicitly used by Davidson, Hendry, Srba, and Yeo (1978) and later established by Engle and Granger (1987). Cointegration refers to a property that holds between two or more nonstationary time series variables. That is, if certain linear combinations of several nonstationary variables are stationary, these variables are said to be cointegrated. Cointegration is interpreted as a long-term relationship or an equilibrium between variables. This is because cointegrated variables are tied to each other to keep certain linear combinations stationary, and hence they tend to move together. Thus, it is natural to consider whether and how such relations among cointegrated variables affect the prices of their derivatives.

Although academic papers that analyze cointegration relationships among economic variables are plentiful, research on derivative pricing using cointegration is limited. To the best of our knowledge, Duan and Pliska (2004) are the first to use cointegration in examining derivative pricing. They focused on stocks and priced their options under an assumption termed the local risk-neutral valuation relationship, which by definition implies that the drift terms of stock returns are equal to the risk-free rate under the risk-neutral probability. In this setting, they conclude that cointegration affects option prices only when volatilities are stochastic.

Commodity prices, however, behave differently from stock prices. They are strongly affected by production and inventory conditions, and tend to deviate temporarily from the prices that would exist without those effects. These characteristics are recognized from the theory of storage by Kaldor (1939) and Working (1949). To incorporate such temporary deviations, the concept of convenience yield is introduced, which is a crucial element in commodity pricing models. For example, Gibson and Schwartz (1990, 1993) propose a two-factor model with commodity spot prices and mean reverting convenience yields, and price commodity futures and options. Schwartz (1997) investigate three different (one-, two-, and three-factor) models including the Gibson-Schwartz model, using data for crude oil, gold, and copper prices, and analyze their long-term hedging strategies. Schwartz and Smith (2000) model commodity dynamics in a different setting using long- and short-term factors and find that their model is equivalent to the Gibson-Schwartz model. Many other models generalize the above, including those of Miltersen and Schwartz (1998), Nielsen and Schwartz (2004), and Casassus and Collin-Dufresne (2005).

When a convenience yield exists, the drift in commodity prices may deviate from the risk-free rate even under risk-neutral probability. Thus, in standard commodity pricing models, Duan and Pliska's (2004) risk-neutral valuation framework does not hold, and their results cannot be directly applied to commodity derivative pricing. This is why we need to extend Duan and Pliska's (2004) framework and investigate commodity pricing using cointegration or, more generally, linear relations among the logarithms of commodity prices.

For this purpose, we generalize the Gibson-Schwartz model by explicitly incorporating linear relations among log commodity prices, which includes cointegration under certain conditions. More specifically, we formulate a commodity pricing model in which the temporary deviation of drift terms from the risk-free rate under a risk-neutral probability is described by convenience yields and linear relations among log commodity prices, which correspond to error terms under an appropriate condition. In previous studies, such temporary deviations are modeled using only the convenience yield; therefore, this chapter also can be regarded as proposing a model that specifies a part of the temporary deviation of commodity prices by their cointegrating relationship.

Intuitively, we can expect that relations among commodity prices should characterize part of the deviation for the following reason. As explained, in standard commodity pricing models, drifts in commodity returns may deviate from the risk-free rate. Such deviations are thought to occur because of frictions (e.g. nonnegative constraints and/or transaction costs) in commodity trading. However, if the deviation occurs because of such frictions, then temporary deviations from the long-term relation among commodity prices may not dissolve immediately, either. Consequently, the relations among commodity prices may affect the deviation in addition to "convenience."

It is important to note that several studies on commodities incorporate linear relations among prices, or cointegration, into their pricing models. Dempster, Medova, and Tang (2008) analyze spread options on two commodity prices, assuming that the spread is stationary. However, they do not explicitly model the spot prices, instead directly modeling the spread. This approach simplifies their model, but it does not enable us to value futures and options on each commodity, whose prices are cointegrated.

Cortazar, Milla, and Severino (2008) develop a general multicommodity model in which prices of commodities share a set of common factors, through which movements of different commodity prices are related. Such a relation among commodity prices should then provide useful information for describing the movement of each commodity price more accurately. Using data on WTI oil and Brent oil and data on WTI oil and gasoline, Cortazar et al. (2008) assess multicommodity models and find them to be superior to traditional individual-commodity models.

Based on a similar idea, Paschke and Prokopczuk (2009) also develop a general and tractable multifactor model in which commodity spot prices are characterized by the weighted sum of latent factors. Using NYMEX data for crude oil, heating oil, and gasoline, and unlike Cortazar et al. (2008), they estimate the model simultaneously with three different commodities, and identify the latent factors that jointly characterized those commodity prices.

Adopting a different approach, Casassus, Liu, and Tang (2011) consider an economy in which commodity prices are related through production, and provide a general model of long-term relationships among commodity prices that is similar to the one in this chapter. They estimate their model using NYMEX data for spread options and compared their results with existing models.

While addressing similar questions to those papers, the model in this chapter is different from them in deriving the condition for cointegration, which is a modification of Duan and Pliska's (2004) to commodity-pricing environments. Note that this condition is very important for empirical analysis because without the condition for cointegration among prices, i.e., stationarity of their linear combinations, standard tests may not apply to estimated results. The above mentioned papers, except Dempster et al. (2008), do not deal with this issue. Thus, this paper is the first to provide a sufficient condition for cointegration in this setting of multiple commodities.

In the following, we investigate the effect of linear relations among log spot prices on commodity derivative pricing for which Duan and Pliska's (2004) risk-neutral valuation does not hold. More precisely, based on Duan and Pliska's (2004) framework, we formulate the Gibson-Schwartz model with linear relations among log commodity spot prices, or cointegration under a certain condition. We obtain an analytical formula for commodity futures and options prices, and then investigate empirically the effect of such spot price relationships on derivative prices using crude oil and heating oil data from NYMEX.

The rest of this chapter is organized as follows. In Section 2, we model commodity spot prices and convenience yields using linear relations among the logarithms of commodity prices with an error term in the drift of spot prices. We also derive the closed-form pricing formulae of futures and call options. In Section 3, we show the state equation and observation equation for the Kalman filter, and conduct an empirical analysis using crude oil and heating oil data. We also conduct hedging simulations for long-term futures using short-term futures in Section 4. Section 5 discusses the results and Section 6 concludes.

4.2 The Model

4.2.1 The Gibson-Schwartz with Cointegration (GSC) Model

We propose a model that extends the GS model (Gibson and Schwartz, 1990; Schwartz, 1997) to explicitly incorporate linear relations among commodity prices, or cointegration, under certain conditions. We adopt the continuous-time specification of cointegrated systems shown by Duan and Pliska (2004).[1] As usual in commodity pricing models, we start by describing the behavior of spot prices and convenience yields under the risk-neutral probability.

Assume that there are n commodities whose spot prices and convenience yields under the risk-neutral probability are as follows:[2]

$$\begin{aligned} d\ln S_i(t) &= \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t) + b_i z(t)\right) dt + \sigma_{S_i} dW_{S_i}(t), \quad i = 1, \dots, n, & (1) \\ d\delta_i(t) &= \kappa_i(\hat{\alpha}_i - \delta_i(t))dt + \sigma_{\delta_i} dW_{\delta_i}(t), \quad i = 1, \dots, n. & (2) \end{aligned}$$

Here, r is the risk-free interest rate, which is assumed to be constant. b_i, σ_{S_i}, κ_i, $\hat{\alpha}_i$, and σ_{δ_i} are constant coefficients. $\boldsymbol{W}(t) = [W_{S_1}(t), ..., W_{S_n}(t), W_{\delta_1}(t), ..., W_{\delta_n}(t)]^\top$ is a $2n$-dimensional Brownian motion under the risk-neutral probability with

$$dW_{S_i}(t)dW_{S_j}(t) = \rho_{S_iS_j}dt, dW_{S_i}(t)dW_{\delta_j}(t) = \rho_{S_i\delta_j}dt, dW_{\delta_i}(t)dW_{\delta_j}(t) = \rho_{\delta_i\delta_j}dt, \quad i, j = 1, \dots, n.$$

We assume that the commodity prices are related linearly through

$$z(t) = \mu_z + a_0 t + \sum_{i=1}^{n} a_i \ln S_i(t), \tag{3}$$

where μ_z, a_0, and a_is are constants. If $\ln S_i$ are cointegrated, then by rearranging the equation as $\ln S_1(t) = (-\mu_z - a_0 t - \sum_{i=2}^{n} a_i \ln S_i(t) + z(t))/a_1$,

[1] Duan and Pliska (2004) considered stock prices where $\delta_i(t) \equiv 0$ (no convenience yield for stocks), and showed that the diffusion limit of discrete stock price processes with cointegration among their log prices $lnS_i(t)$ is given by $dS_i(t) = S_i(t)(r+\lambda_i\sigma_{S_i}+b_i z(t))dt+ S_i(t)\sigma_{S_i}dW^P_{S_i}(t)$ under the natural probability where λ_i is the market price of risk.

[2] In Subsection 4.5.2, we discuss how we can enhance our model to incorporate seasonality.

if $a_1 \neq 0$, $z(t)$ can be interpreted as an error term, a_i as cointegration vectors, and b_i as adjustment speeds of the error term. Using Ito's lemma, the dynamics of $z(t)$ is

$$\begin{aligned} dz(t) &= a_0 dt + \sum_{i=1}^{n} a_i d\ln S_i(t) \\ &= \left(a_0 + \sum_{i=1}^{n} a_i r - \frac{1}{2}\sum_{i=1}^{n} a_i \sigma_{S_i}^2 - \sum_{i=1}^{n} a_i \delta_i(t) + \sum_{i=1}^{n} a_i b_i z(t) \right) dt \\ &\quad + \sum_{i=1}^{n} a_i \sigma_{S_i} dW_{S_i}(t). \end{aligned} \tag{4}$$

Define $b = \sum_{i=1}^{n} b_i a_i$. If $b \neq 0$, the above equation can be written as

$$\begin{aligned} dz(t) &= -b(m - z(t))dt - \sum_{i=1}^{n} a_i \delta_i(t) dt + \sum_{i=1}^{n} a_i \sigma_{S_i} dW_{S_i}(t), \quad (5) \\ m &= \frac{-a_0 - \sum_{i=1}^{n} a_i r + \frac{1}{2}\sum_{i=1}^{n} a_i \sigma_{S_i}^2}{b}. \end{aligned}$$

The set of equations (1), (2), and (3) is an extension of the GS model with a linear relation $z(t)$ among the logarithms of commodity prices that affects the drift terms. $z(t)$ represents the error term of the cointegrating relationship among commodity prices.[3] We call this model the Gibson-Schwartz with cointegration model (hereafter the GSC model).

It is worth mentioning that while the GSC model bases its specification on Duan and Pliska (2004), the drift term is different from that of Duan and Pliska (2004), in which the drift is equal to the risk-free rate under the risk-neutral probability. This difference comes from the characteristics of the underlying assets. For stocks, which Duan and Pliska (2004) focused on, it is natural to assume that the drift terms of returns should be equal to the risk-free rate under the risk-neutral probability. On the other hand, for commodities, it is standard to assume that the drift terms may deviate temporarily from the risk-free rate even under the risk-neutral probability by reflecting inventory and production conditions. The GSC model assumes

[3] In Subsection 4.2.3, we show a sufficient condition for cointegration.

that such deviations are described by the convenience yield and the term $z(t)$.[4]

4.2.2 Futures and Option Prices for the GSC Model

We derive the futures and European call option prices on commodity i in matrix form.

$$\begin{aligned}
d\ln S_i(t) &= \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t) + b_i\mu_z + b_i a_0 t + \sum_{j=1}^{n} b_i a_j \ln S_j(t)\right) dt + \sigma_{S_i} dW_{S_i}(t) \\
&\equiv \left(\beta_{S_i0}(t) + \beta_{S_i\delta_i}\delta_i(t) + \sum_{j=1}^{n} \beta_{S_iS_j} \ln S_j(t)\right) dt + \sigma_{S_i} dW_{S_i}(t), \\
d\delta_i(t) &\equiv (\beta_{\delta_i0} + \beta_{\delta_i\delta_i}\delta_i(t))dt + \sigma_{\delta_i} dW_{\delta_i}(t),
\end{aligned}$$

where

$$\begin{aligned}
&\beta_{S_i0}(t) = r - \frac{\sigma_{S_i}^2}{2} + b_i\mu_z + b_i a_0 t, \\
&\beta_{S_iS_j} = b_i a_j, \\
&\beta_{S_i\delta_i} = -1, \\
&\beta_{\delta_i0} = \kappa_i\hat{\alpha}_i, \\
&\beta_{\delta_i\delta_i} = -\kappa_i.
\end{aligned}$$

This equation can be solved as follows.[5]

$$\boldsymbol{X}(T) = e^{(T-t)\boldsymbol{\beta}} \left\{ \boldsymbol{X}(t) + \int_t^T e^{-s\boldsymbol{\beta}} \boldsymbol{\beta}_0(s) ds + \int_t^T e^{-s\boldsymbol{\beta}} d\boldsymbol{W}_0(s) \right\}, \quad (6)$$

[4] See Section 4.1 for the intuition behind including the linear relation $z(t)$ in the drift terms. One may think of generalization of the model to incorporate other economic factors such as foreign exchange and/or interest rates as proposed in Bali, Hume, and Martell (2007).

[5] Cf. Karatzas and Shreve (1991), Section 5.6 or Liptser and Shiryaev (2001), p. 151, Thm. 4.10.

where

$$\boldsymbol{X}(t) = [\ln S_1(t), \cdots, \ln S_n(t), \delta_1(t), \cdots, \delta_n(t)]^\top,$$
$$\boldsymbol{\beta}_0(t) = [\beta_{S_1 0}(t), \cdots, \beta_{S_n 0}(t), \beta_{\delta_1 0}, \cdots, \beta_{\delta_n 0}]^\top,$$
$$\boldsymbol{\beta} = \begin{bmatrix} \beta_{S_1S_1} & \cdots & \beta_{S_1S_n} & \beta_{S_1\delta_1} & & 0 \\ \vdots & \ddots & \vdots & & \ddots & \\ \beta_{S_nS_1} & \cdots & \beta_{S_nS_n} & 0 & & \beta_{S_n\delta_n} \\ & & & \beta_{\delta_1\delta_1} & & 0 \\ & \text{\huge 0} & & & \ddots & \\ & & & 0 & & \beta_{\delta_n\delta_n} \end{bmatrix},$$

and $\boldsymbol{W}_0(t) = [\sigma_{S_1} W_{S_1}(t), \cdots, \sigma_{S_n} W_{S_n}(t), \sigma_{\delta_1} W_{\delta_1}(t), \cdots, \sigma_{\delta_n} W_{\delta_n}(t)]^\top$ is a scaled Brownian motion vector.

Denote by $E_t[\cdot]$ the expectation under the risk-neutral probability given $\mathcal{F}_t$.[6] The mean and covariances of $\ln S_i(T)$ are

$$\begin{aligned} \mu_{X_i}(t,T) &= E_t[\ln S_i(T)] \\ &= \left[e^{(T-t)\boldsymbol{\beta}} \left\{ \boldsymbol{X}(t) + \int_t^T e^{-s\boldsymbol{\beta}} \boldsymbol{\beta}_0(s) ds \right\} \right]_i, \\ \sigma_{X_iX_j}(t,T) &= E_t[(\ln S_i(T) - \mu_{X_i}(t,T))(\ln S_j(T) - \mu_{X_i}(t,T))] \\ &= \left[\int_t^T (e^{(T-t-s)\boldsymbol{\beta}}) \boldsymbol{\Omega} (e^{(T-t-s)\boldsymbol{\beta}})^\top ds \right]_{ij}, \end{aligned}$$

where $[\cdot]_i$ and $[\cdot]_{ij}$ are the ith vector element and the $[i,j]$th matrix element, respectively, and the covariance matrix is

$$\boldsymbol{\Omega} = \begin{bmatrix} \rho_{S_1S_1}\sigma_{S_1}\sigma_{S_1} & \cdots & \rho_{S_1S_n}\sigma_{S_1}\sigma_{S_n} & \rho_{S_1}\sigma_{S_1}\sigma_{\delta_1} & \cdots & \rho_{S_1\delta_n}\sigma_{S_1}\sigma_{\delta_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{S_nS_1}\sigma_{S_n}\sigma_{S_1} & \cdots & \rho_{S_nS_n}\sigma_{S_n}\sigma_{S_n} & \rho_{S_n\delta_1}\sigma_{S_n}\sigma_{\delta_1} & \cdots & \rho_{S_n\delta_n}\sigma_{S_n}\sigma_{\delta_n} \\ \rho_{\delta_1S_1}\sigma_{\delta_1}\sigma_{S_1} & \cdots & \rho_{\delta_1S_n}\sigma_{\delta_1}\sigma_{S_n} & \rho_{\delta_1\delta_1}\sigma_{\delta_1}\sigma_{\delta_1} & \cdots & \rho_{\delta_1\delta_n}\sigma_{\delta_1}\sigma_{\delta_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{\delta_nS_1}\sigma_{\delta_n}\sigma_{S_1} & \cdots & \rho_{\delta_nS_n}\sigma_{\delta_n}\sigma_{S_n} & \rho_{\delta_n\delta_1}\sigma_{\delta_n}\sigma_{\delta_1} & \cdots & \rho_{\delta_n\delta_n}\sigma_{\delta_n}\sigma_{\delta_n} \end{bmatrix}.$$

Using risk-neutrality and property of a moment generating function, we obtain the futures price of commodity i as follows (cf. Cox, Ingersoll, and Ross, 1981).

[6]In this chapter, we assume a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$.

Proposition 4.2.1. Assuming (1), (2), and (3), the futures price of commodity i with maturity T at t is given by

$$\begin{aligned} G_i(t,T) &= E_t[S_i(T)] \\ &= \exp\left\{\mu_{X_i}(t,T) + \frac{\sigma_{X_i}^2(t,T)}{2}\right\}, \end{aligned}$$

where $\sigma_{X_i}^2(t,T) = \sigma_{X_iX_i}(t,T)$.

Note that there is $\ln S_i(t)$ in $\mu_{X_i}(t,T)$ implicitly, so $S_i(t)$ do not appear in the formula.

In the following proposition, we derive the call option pricing formula. This is not addressed by other papers that incorporate multicommodity prices, such as Cortazar et al. (2008), Paschke and Prokopczuk (2009), and Casassus et al. (2011).

Proposition 4.2.2. Assuming (1), (2), and (3), the European call option price of commodity i with maturity T at t is given by

$$\begin{aligned} C_i(t,T) &= e^{-r(T-t)}E_t[(S_i(T)-K)^+] \\ &= e^{-r(T-t)+\mu_{X_i}(t,T)+\frac{\sigma_{X_i}^2(t,T)}{2}}\Phi(d_{i1}(t,T)) - Ke^{-r(T-t)}\Phi(d_{i2(t,T)}), \\ d_{i1}(t,T) &= \frac{-\ln K + \mu_{X_i}(t,T) + \sigma_{X_i}^2(t,T)}{\sigma_{X_i}(t,T)}, \\ d_{i2}(t,T) &= d_{i1}(t,T) - \sigma_{X_i}(t,T). \end{aligned}$$

Proof. See the Appendix for the derivation. □

In the Appendix, we elaborate the formulae of the derivatives without using integrals or matrices. These formulae are more tractable than the matrix formulae in this subsection for calculating the hedge weights and applying them to risk management.

4.2.3 A Sufficient Condition for Cointegration

We now show a sufficient condition for the GSC model to be cointegrated.

Proposition 4.2.3. Let us assume $e^{b\triangle t} < 1(\iff b < 0), e^{-\kappa_i \triangle t} < 1(\iff \kappa_i > 0)$. Then, the GSC model is cointegrated.

Proof. In the Appendix. □

This condition is very important for the estimated parameters in the model to be valid. If the condition of Proposition 4.2.3 does not hold, the estimation of the model may lead to a spurious regression. The estimated coefficients in this case are not consistent and the sample residual of $z(t)$ will be nonstationary.[7] This condition is similar to that of Duan and Pliska (2004). However, because our setting is different from their model, we cannot simply apply their results. In particular, convenience yields are unobservable in our model.[8] Therefore, we need a different condition for cointegration for our model. Note that the papers that deal with relations among commodity prices, including Cortazar et al. (2008), Paschke and Prokopczuk (2009), and Casassus et al. (2011), do not examine this important issue. This paper is the first to show a sufficient condition for cointegration in this setting.

4.3 Empirical Analysis

4.3.1 The Dynamics of Commodity Spot Prices, Convenience Yields, and Error Terms under Natural Probability

As neither commodity spot prices nor convenience yields are observable, we have to estimate their parameters using the Kalman filter[9] with their futures prices. We have already modeled commodity spot prices and convenience yields under the risk-neutral probability, and thus, we can calculate the Kalman filter with only these SDEs. However, because it would be useful to check whether the model performs well under the natural probability, the SDEs of commodity spot prices and convenience yields under the natural probability are needed to estimate the model. For this purpose, we assume the market price of risk that transforms the risk-neutral probability into the natural probability.

Let us assume that a Brownian motion under the risk-neutral probability $\boldsymbol{W}(t)$ and a Brownian motion under the natural probability $\boldsymbol{W}^P(t)$ satisfy

[7] See Hamilton (1994), Section 18.3 for the properties of spurious regressions.

[8] For unobservability, see Section 4.5.1.

[9] For the Kalman filter, see Hamilton (1994), Chapter 13.

$$\boldsymbol{W}(t) = \boldsymbol{W}^P(t) + \int_0^t \boldsymbol{\theta}_0 ds,$$
$$\boldsymbol{W}(t) = [W_{S_1}(t), \cdots, W_{S_n}(t), W_{\delta_1}(t), \cdots, W_{\delta_n}(t)]^\top,$$
$$\boldsymbol{W}^P(t) = [W^P_{S_1}(t), \cdots, W^P_{S_n}(t), W^P_{\delta_1}(t), \cdots, W^P_{\delta_n}(t)]^\top,$$
$$\boldsymbol{\theta}_0 = [\theta_{S_1 0}, \cdots, \theta_{S_n 0}, \theta_{\delta_1 0}, \cdots, \theta_{\delta_n 0}]^\top,$$

where $\boldsymbol{\theta}$ is the market price of risk, which is assumed to be constant. The consequence of this assumption can be seen in the following SDEs under the natural probability.

$$\begin{aligned} d\ln S_i(t) &= \left(\gamma_{S_i 0}(t) + \sum_{j=1}^n \gamma_{S_i S_j} \ln S_j(t) + \gamma_{S_i \delta_i} \delta_i(t)\right) dt \\ &\quad + \sigma_{S_i} dW^P_{S_i}(t), \qquad (7) \\ d\delta_i(t) &= \left(\gamma_{\delta_i 0} + \gamma_{\delta_i \delta_i} \delta_i(t)\right) dt + \sigma_{\delta_i} dW^P_{\delta_i}(t), \qquad (8) \end{aligned}$$

where

$$\begin{aligned} \gamma_{S_i 0}(t) &= r - \frac{\sigma^2_{S_i}}{2} + b_i \mu_z + b_i a_0 t + \sigma_{S_i} \theta_{S_i 0}, \\ \gamma_{S_i S_j} &= b_i a_j, \\ \gamma_{S_i \delta_i} &= -1, \\ \gamma_{\delta_i 0} &= \kappa_i \hat{\alpha}_i + \sigma_{\delta_i} \theta_{\delta_i 0}, \\ \gamma_{\delta_i \delta_i} &= -\kappa_i. \end{aligned}$$

To implement the empirical analysis, for ease of calculation, we classify the model into five cases, estimate each of them separately, and compare them. The cases are: (i) $b \neq 0$ and $a_1 = 1$, (ii) $b \neq 0$ and $a_2 = 1$, (iii) $b = 0, a_1 = 1, b_1 = -a_2 b_2$, (iv) $b = 0, a_1 = 0, a_2 = 0$, and (v) $b = 0, a_1 = 0, b_2 = 0$.[10] This enables us to calculate the log-likelihood using the scalar

[10] These cases are collectively exhaustive. This can be shown as follows. There are only two cases: $b \neq 0$ and $b = 0$. For $b \neq 0$, there are only two cases, $a_1 \neq 0$ or $a_2 \neq 0$, because otherwise we have $b = 0$. Furthermore, $a_1 \neq 0$ or $a_2 \neq 0$ can be rescaled to 1, case (i) or (ii), respectively. For $b = 0$, we have $a_1 \neq 0$, which can be rescaled to $a_1 = 1$, which is case (iii), or $a_1 = 0$. If $a_1 = 0$, then there are two possibilities, which are $a_2 = 0$ or $b_2 = 0$, case (iv) or (v), respectively; otherwise $b = a_2 b_2 \neq 0$. Furthermore, note that cases (i) and (ii) may satisfy the cointegration condition ($b < 0$), but the other cases do not because $b = 0$.

forms and to avoid the time-consuming calculation of the general case using the matrix formula. In this subsection, we show the result when $b \neq 0$ and $a_2 = 1$.[11] For the other results, see the Appendix. We compare the model with the GS model. The GS model is enhanced to have two commodity price processes and two convenience yield processes, with correlations between each process.

4.3.2 Data

We use WTI and heating oil daily closing prices traded on the NYMEX from January 2, 1990, to July 30, 2010. Five futures contracts, labeled Maturity 1, Maturity 3, Maturity 5, Maturity 7, Maturity 9, are used in the estimation. Maturity 1 stands for the contract closest to maturity, Maturity 3 stands for the third closest maturity, and so on. The time to maturity corresponding to these prices is also used. We set the risk-free rate equal to 4%.

The basic statistics for these data are described in Table 4.1. As the maturity dates are fixed, the time to maturity changes over time. Comparing WTI crude oil with heating oil, we can see that the standard deviation of heating oil is higher because the average price of heating oil is higher than that of crude oil. The mean maturity and its standard deviation are quite close to each other. Furthermore, note that the correlation between the futures prices of WTI and heating oil is 0.995.

4.3.3 Estimation Results

We estimate the model using the Kalman filter. In Table 4.2, we report the estimated parameters with standard errors. The subscripts describe the commodities where 1 and 2 indicate for crude oil and heating oil, respectively. Note that the AIC for the GSC model is lower than that for the GS model, which implies that the GSC model fits the data better. As we can see, the estimated linear relation vectors are $[a_1, a_2] = [-1.19, 1.00]$ and the adjustment speeds are $[b_1, b_2] = [-0.05, -0.36]$. A comparison of these values with the standard errors suggests that the linear relation among commodity prices empirically affects the derivative prices.

As the values of b_1, b_2 measure how much the linear relation affects the spot prices, it also suggests that the heating oil price is much more affected

[11] In our empirical analysis, this case is found to have the smallest AIC among the cases that satisfy the cointegration condition in Proposition 4.2.3.

Figure 4.1: WTI and heating oil daily closing prices from January 2, 1990, to July 30, 2010. The black solid line and the blue dashed line denote the prices of crude oil and heating oil, respectively.

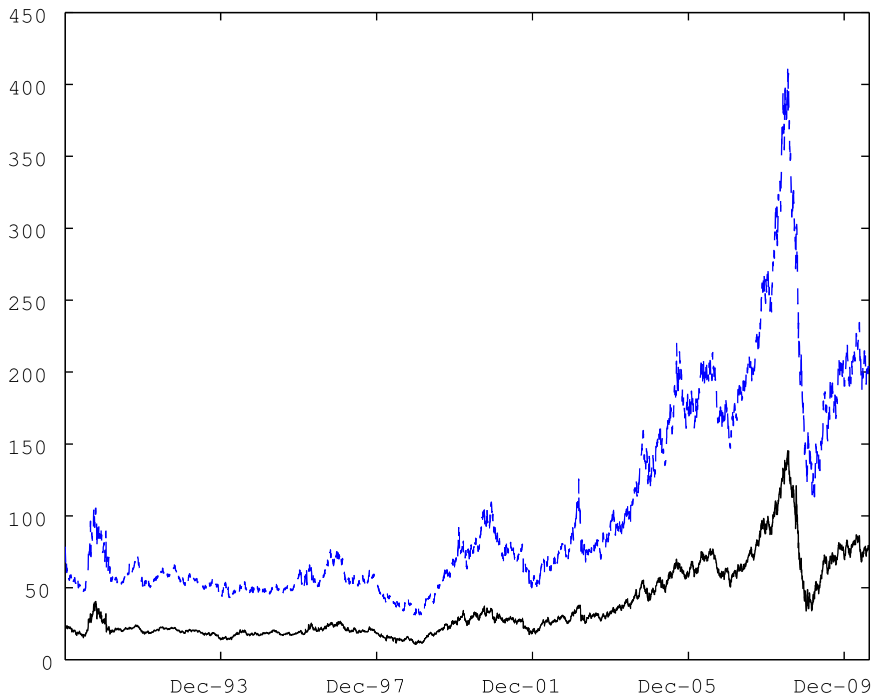


Table 4.1: Data statistics.

Futures contract	Mean price (Standard deviation)	Mean log return (Standard deviation)	Mean maturity (Standard deviation)
WTI crude oil			
Maturity 1	36.69 (25.02)	0.0240 % (2.5417 %)	0.10 (0.04)
Maturity 3	36.79 (25.43)	0.0250 % (2.0547 %)	0.35 (0.04)
Maturity 5	36.71 (25.72)	0.0259 % (1.8571 %)	0.59 (0.04)
Maturity 7	36.60 (25.93)	0.0267 % (1.7382 %)	0.83 (0.04)
Maturity 9	36.48 (26.08)	0.0272 % (1.6528 %)	1.08 (0.04)
Heating oil			
Maturity 1	101.67 (70.14)	0.0188 % (2.4679 %)	0.09 (0.04)
Maturity 3	102.23 (71.50)	0.0240 % (2.0456 %)	0.34 (0.04)
Maturity 5	102.40 (72.58)	0.0260 % (1.8614 %)	0.58 (0.04)
Maturity 7	102.36 (73.35)	0.0264 % (1.7512 %)	0.82 (0.04)
Maturity 9	102.17 (73.68)	0.0260 % (1.6702 %)	1.07 (0.04)

by the linear relation, or the error term, than the crude oil price is. Note that a_0 is -0.000072 and its standard error is 0.000004, which implies that the relation term $z(t)$ includes time drift, and μ_z is large compared with the standard error. Furthermore, b is -0.29. With both κ_i positive, the cointegration conditions are satisfied. Therefore, we can compare the coefficients with their standard deviations to check the significance of the coefficients.

Except $\hat{\alpha}_1$ in the GS model, $\hat{\alpha}_i$ are significant. However, they are different between the two models. This is a result of the relation term $z(t)$. As mentioned above, the GSC model assumes that the deviation of the drift terms from the risk-free rate is described by the convenience yield and the term $z(t)$. Thus, it is only a matter of which factor explains most of the deviation, and $\hat{\alpha}_i$ depends on these factors. Both κ_i exceed 1, which is the same as in the GS model.

Let us turn now to the volatility parameters. In the GSC model, crude oil and heating oil spot prices have a positive correlation ($\rho_{S_1 S_2} = 0.75$). The corresponding spot prices and convenience yields have relatively high positive correlations $\rho_{S_1\delta_1} = 0.77$ and $\rho_{S_2\delta_2} = 0.62$, respectively, which is consistent with the GS model. Moreover, crude oil spot prices and heating oil convenience yields have no correlation ($\rho_{S_1\delta_2} = 0.00$). However, we see that the correlation for heating oil spot prices and crude oil convenience yields $\rho_{S_2\delta_1}$

Table 4.2: Estimated parameters, with standard errors in parentheses. Data are WTI and heating oil daily closing prices traded on the NYMEX from January 2, 1990, to July 30, 2010.

	GS	GSC
Volatility parameters		
σ_{S_1}	0.414476 (0.003512)	0.381896 (0.002241)
σ_{S_2}	0.377914 (0.002247)	0.406307 (0.002546)
σ_{δ_1}	0.320532 (0.002552)	0.287109 (0.001652)
σ_{δ_2}	0.507958 (0.005171)	0.699693 (0.007743)
$\rho_{S_1S_2}$	0.698858 (0.005467)	0.748660 (0.004673)
$\rho_{S_1\delta_1}$	0.793308 (0.004015)	0.767305 (0.004128)
$\rho_{S_1\delta_2}$	0.000058 (0.012618)	0.000072 (0.012604)
$\rho_{S_2\delta_1}$	0.505952 (0.005636)	0.628424 (0.005645)
$\rho_{S_2\delta_2}$	0.600362 (0.009234)	0.620154 (0.008083)
$\rho_{\delta_1\delta_2}$	0.108853 (0.011792)	0.165843 (0.014062)
Convenience yield parameters		
κ_1	1.070822 (0.005328)	1.140883 (0.006597)
κ_2	1.294663 (0.014874)	1.085038 (0.015096)
$\hat{\alpha}_1$	0.001375 (0.001417)	0.006611 (0.003161)
$\hat{\alpha}_2$	0.038074 (0.002409)	-0.037714 (0.020791)
Linear relation parameters		
μ_z		1.144262 (0.046325)
a_0		-0.000072 (0.000004)
a_1		-1.187431 (0.006754)
a_2		1.000000 (n.a.)
b_1		-0.052615 (0.001626)
b_2		-0.356252 (0.005272)
Market price of risk parameters		
$\theta_{S_1 0}$	0.083425 (0.267456)	0.478595 (0.235855)
$\theta_{S_2 0}$	-0.357933 (0.212986)	0.817002 (0.231929)
$\theta_{\delta_1 0}$	0.074827 (0.247218)	-0.002131 (0.241285)
$\theta_{\delta_2 0}$	-0.281003 (0.282557)	-0.351462 (0.269419)
$R(1,1)$	0.000509 (0.000005)	0.000520 (0.000006)
$R(2,2)$	0.000000 (0.000000)	0.000002 (0.000000)
$R(3,3)$	0.000009 (0.000000)	0.000008 (0.000000)
$R(4,4)$	0.000000 (0.000000)	0.000000 (0.000000)
$R(5,5)$	0.000023 (0.000001)	0.000020 (0.000001)
$R(6,6)$	0.000002 (0.000001)	0.000003 (0.000001)
$R(7,7)$	0.001043 (0.000029)	0.001019 (0.000030)
$R(8,8)$	0.000742 (0.000024)	0.000700 (0.000022)
$R(9,9)$	0.000002 (0.000000)	0.000007 (0.000000)
$R(10,10)$	0.001138 (0.000029)	0.000999 (0.000027)
Log-likelihood	153030.832494	154335.136395
AIC	-306005.664988	-308604.272790
Sample size	51590	51590

is relatively high (0.63). This is the same as in the GS model. It is intuitive that spot prices and convenience yields among different commodities should not be strongly correlated; however, heating oil prices are affected by crude oil convenience yields. Volatilities of spot prices σ_{S_1}, σ_{S_2} and convenience yields σ_{δ_1}, σ_{δ_2} seem to be similar in the two models.

Table 4.3: Root mean square error (RMSE) and Mean error (ME) for each futures contract.

Contracts	RMSE		ME	
Models	GS	GSC	GS	GSC
Crude oil				
Maturity 1	0.032872	0.032943	-0.002727	-0.002741
Maturity 3	0.020227	0.020206	0.000462	-0.000003
Maturity 5	0.018617	0.018556	0.000565	0.000026
Maturity 7	0.017313	0.017296	0.000434	-0.000030
Maturity 9	0.017211	0.017152	0.000625	0.000170
Heating oil				
Maturity 1	0.024144	0.024064	0.000889	0.000023
Maturity 3	0.037593	0.037136	0.000502	-0.001603
Maturity 5	0.032656	0.031699	0.000677	-0.000226
Maturity 7	0.018059	0.017767	0.000717	-0.000012
Maturity 9	0.038907	0.036419	0.000424	-0.003012

Table 4.3 shows the root mean square error (RMSE) and mean error (ME) of the model. Although both models have small values, which indicates that the models are well fitted, the RMSE and ME both favor the GSC model with few exceptions.

4.4 Hedging Futures

In this section, we implement the GSC and GS models for hedging long-term futures contracts,[12] which we call the target futures, using short-term

[12]Recall that we are assuming the risk-free rate is constant, which implies that futures and forwards are equally valued.

futures. We empirically analyze the logarithms of the crude oil and heating oil prices.

As the GS model has two stochastic variables, we need two futures that have different maturities to hedge, and the weights can be calculated by solving the following system of equations.[13]

$$\boldsymbol{\Phi w} = \boldsymbol{\varphi}, \tag{9}$$
$$\boldsymbol{\Phi} = \begin{bmatrix} \frac{\partial G_i(t,T_1)}{\partial S_i} & \frac{\partial G_i(t,T_2)}{\partial S_i} \\ \frac{\partial G_i(t,T_1)}{\partial \delta_i} & \frac{\partial G_i(t,T_2)}{\partial \delta_i} \end{bmatrix},$$
$$\boldsymbol{w} = [w_1, w_2]^\top, \tag{10}$$
$$\boldsymbol{\varphi} = \left[\frac{\partial G_i(t,T)}{\partial S_i}, \frac{\partial G_i(t,T)}{\partial \delta_i}\right]^\top, \tag{11}$$

where w_i are weights for futures with maturity T_i and T is the maturity of the target futures.

On the other hand, for the GSC model, which has S_i, δ_i, δ_j, and z as stochastic variables,[14] we need four futures to hedge when there are two commodities to consider. Now, the system of equations for (9) is

$$\boldsymbol{\Phi} = \begin{bmatrix} \frac{\partial G_i(t,T_1)}{\partial S_i} & \frac{\partial G_i(t,T_2)}{\partial S_i} & \frac{\partial G_i(t,T_3)}{\partial S_i} & \frac{\partial G_i(t,T_4)}{\partial S_i} \\ \frac{\partial G_i(t,T_1)}{\partial \delta_i} & \frac{\partial G_i(t,T_2)}{\partial \delta_i} & \frac{\partial G_i(t,T_3)}{\partial \delta_i} & \frac{\partial G_i(t,T_4)}{\partial \delta_i} \\ \frac{\partial G_i(t,T_1)}{\partial \delta_j} & \frac{\partial G_i(t,T_2)}{\partial \delta_j} & \frac{\partial G_i(t,T_3)}{\partial \delta_j} & \frac{\partial G_i(t,T_4)}{\partial \delta_j} \\ \frac{\partial G_i(t,T_1)}{\partial z} & \frac{\partial G_i(t,T_2)}{\partial z} & \frac{\partial G_i(t,T_3)}{\partial z} & \frac{\partial G_i(t,T_4)}{\partial z} \end{bmatrix},$$
$$\boldsymbol{w} = [w_1, w_2, w_3, w_4]^\top,$$
$$\boldsymbol{\varphi} = \left[\frac{\partial G_i(t,T)}{\partial S_i}, \frac{\partial G_i(t,T)}{\partial \delta_i}, \frac{\partial G_i(t,T)}{\partial \delta_j}, \frac{\partial G_i(t,T)}{\partial z}\right]^\top.$$

We emphasize that we use the futures formula in Proposition 4.7.1 in the Appendix to derive the hedging weights.

[13] See Brennan and Crew (1997) and Schwartz (1997) for hedging long-term forwards using short-term futures.

[14] For calculating the hedge weights, we can use S_i, S_j, δ_i, and δ_j instead of S_i, δ_i, δ_j, and z. If we are considering more than three commodities, for example four commodities, then using state variable z is more convenient for calculating hedge weights as we only need $n+2$ futures to hedge, whereas we need $2n$ when using S_i, δ_i.

To calculate the hedging portfolio, we need the values of state variables $S_i(t)$, $\delta_i(t), \delta_j(t)$, and $z(t)$. There are two methods for calculating the values of the state variables. One method is to use a Kalman filter, which we call the Kalman filter method. The other method is to calculate the values of the state variables by solving the observation equation, which only requires futures prices and the estimated parameters. We call this method the simultaneous equation method. We implement both methods. The hedging error ratio is calculated by dividing the hedging error value by the target futures price of each hedging start period. The hedging error value is the difference between the target futures price and the value of the hedge portfolio.

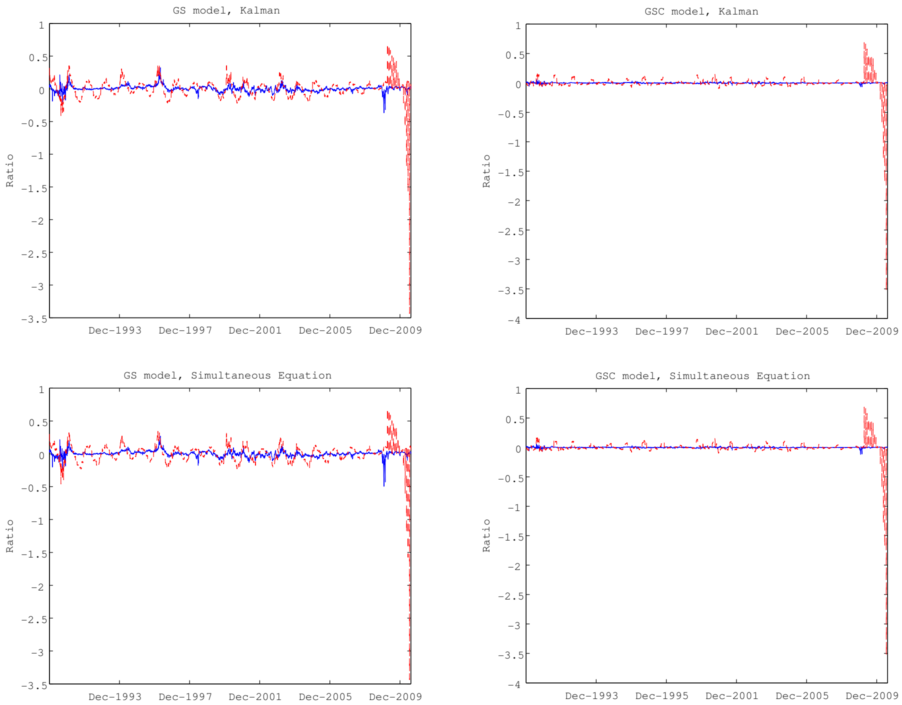
We hedge the futures that mature in 1 year and in 10 years with the futures that mature in 1, 3, 5, and 7 months for the GSC model. For the GS model, we use the futures that mature in 1 and 3 months. As long-term futures, e.g. 10-year futures, are not traded in the market, we cannot calculate their hedging error precisely. Hence, to evaluate the hedging error, we also hedge 1-year futures. We calculate the hedging error for 10-year futures by using their theoretical price. The total hedging period is from January 2, 1990, to July 30, 2010. We roll the futures 3 business days before they mature, and each hedging period is roughly 1 month. The hedging weight and the hedging error are calculated daily.

Table 4.4: Performance of hedging 1-year futures. "Kalman filter" indicates that the state variables are calculated using the Kalman filters. "Simultaneous" indicates that the state variables are calculated by solving the observation equation.

Contracts	Mean of hedging error ratio	
Method	GS	GSC
Crude oil		
Kalman filter	0.042603	-0.019405
Simultaneous	0.023230	-0.027480
Heating oil		
Kalman filter	0.010182	0.004493
Simultaneous	-0.026031	-0.012187

The performance of the hedging simulation for the 1-year futures is indicated in Table 4.4 and Figure 4.2. For both commodities, the results indicate

Figure 4.2: Performance of hedging 1-year futures. The two graphs on the left show the results of the GS model, and the two graphs on the right show the results of the GSC model. The blue solid line and the red dashed line indicate hedge performance of WTI crude oil and heating oil, respectively. The upper two graphs show the result of the Kalman filter method and the bottom two graph show the result of the simultaneous equation method.



that the hedging error ratios are relatively small. This is true for both the GSC and the GS models. Comparing the two models, we see that the GSC model has a relatively good performance using the simultaneous equation method, except for the case of crude oil.

Figure 4.3 shows the weights of the futures in the hedging portfolio whose state variables are calculated by the Kalman filters. For the GS model, the hedge weights for 3-month futures are positive and those for 1-month futures are negative. For the GSC model, the hedge weights for 7-month and 3-month futures are positive and the others are negative.

Table 4.5: Performance of hedging 10-year futures. “Kalman filter” indicates that the state variables are calculated using the Kalman filters. “Simultaneous” indicates that the state variables are calculated by solving the observation equation.

Contracts	Mean of hedging error ratio	
Method	GS	GSC
Crude oil		
Kalman filter	-0.333988	-12.315692
Simultaneous	0.009467	-13.809328
Heating oil		
Kalman filter	-0.043942	-35.425612
Simultaneous	-0.023776	-42.889075

In Table 4.5 and Figure 4.4, we show the performance of the hedging simulation for 10-year futures. Obviously, the hedging error ratio is poorer than that for the 1-year futures. However, note that this hedging error ratio is calculated by using the theoretical price and, therefore, we cannot estimate the hedge errors exactly. Note also that for both commodities, the GS model performs significantly better than the GSC model. For the GSC model, this is because the absolute hedge weight is very large, as indicated in Figure 4.5. Recall that the hedge weight is calculated as $\boldsymbol{w} = \boldsymbol{\Phi}^{-1}\boldsymbol{\varphi}$. Some of the values in $\boldsymbol{\Phi}$, especially the partial derivatives of the other convenience yield δ_j and z, are too small and hence $\boldsymbol{\Phi}^{-1}$ and the hedge weight are very large. If we erase the partial derivatives for δ_j and z and calculate the hedge weight, the performance of the hedging simulation for 10-year futures improves, but the performance for 1-year futures will not be as good as described above. This

Figure 4.3: Weights of futures for hedging 1-year futures, for which the state variables are calculated by the Kalman filters. The upper two figures show the results for the GS model. The blue solid line and the red dashed line indicate the hedging weights of the 1-month futures and 3-month futures, respectively. The lower two figures show the results for the GSC model. The blue solid line, the red dashed line, the green dotted line, and the black chained line indicate the hedging weights of the 1-month futures, 3-month futures, 5-month futures, and 7-month futures, respectively.

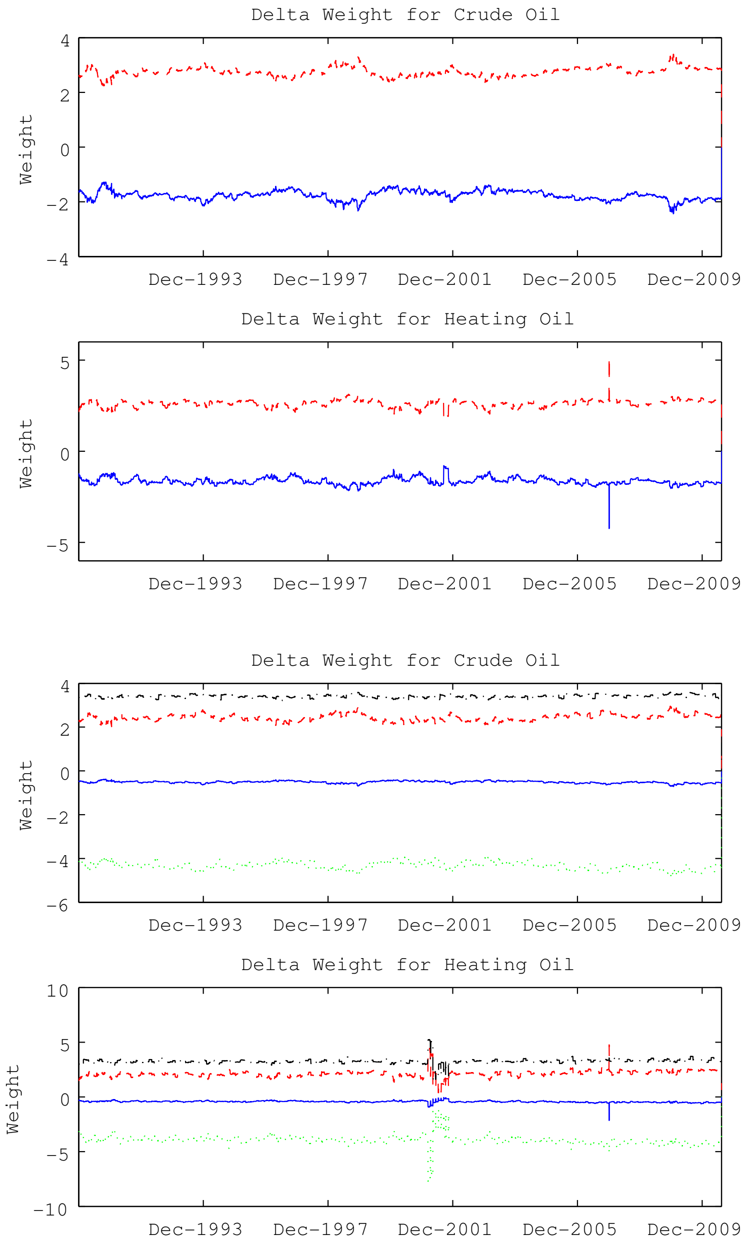
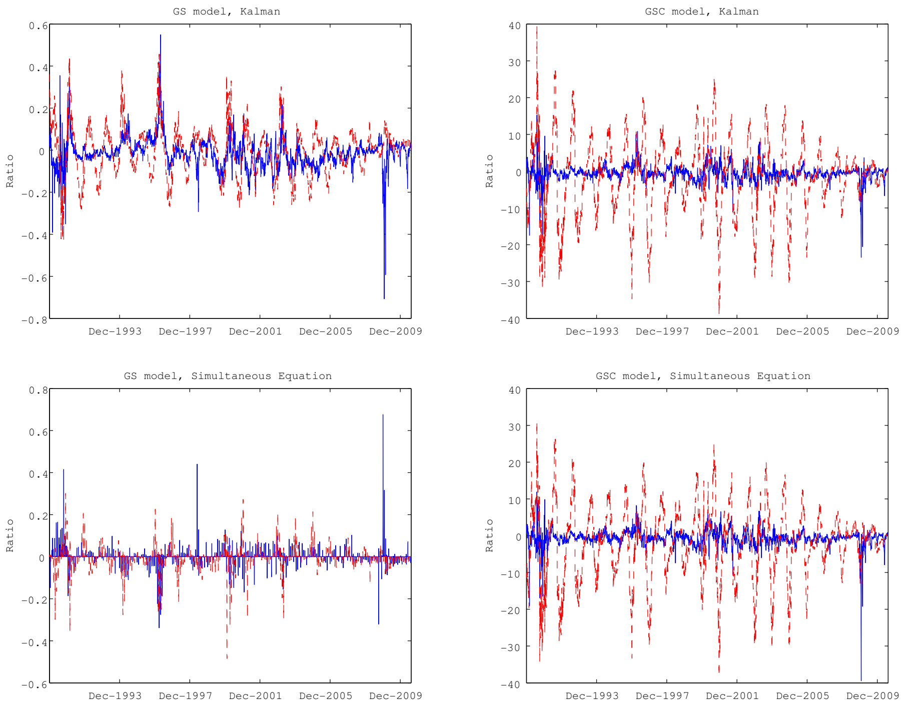


Figure 4.4: Performance of hedging 10-year futures. The two graphs on the left present the results of the GS model and the two graphs on the right present the results of the GSC model. The blue solid line and the red dashed line indicate the hedge performance of WTI crude oil and heating oil, respectively. The upper two graphs show the result of the Kalman filter method and the bottom two graph show the result of the simultaneous equation method.



implies that the GS model is good enough for hedging long-term commodity futures.

4.5 Discussion

4.5.1 Relations among Futures Prices

It should be noted that, in our setting, the linear relations among commodity spot prices do not automatically apply to the linear relations among their futures prices. Let us look at the dynamics of the logarithms of futures prices. From Proposition 4.2.1, it is easy to see that the logarithms of futures prices $\ln G_i(t,T)$ can be represented by

$$\ln G_i(t,T) = \sum_{j=1}^{n} c_{S_j}(t,T)\ln S_j(t) + \sum_{j=1}^{n} c_{\delta_j}(t,T)\delta_j(t) + X'(t,T),$$

where

$$\frac{\partial G_i(t,T)}{\partial \ln S_j(t)} = c_{S_j}(t,T)G_i(t,T), \qquad \frac{\partial G_i(t,T)}{\partial \delta_j(t)} = c_{\delta_j}(t,T)G_i(t,T),$$

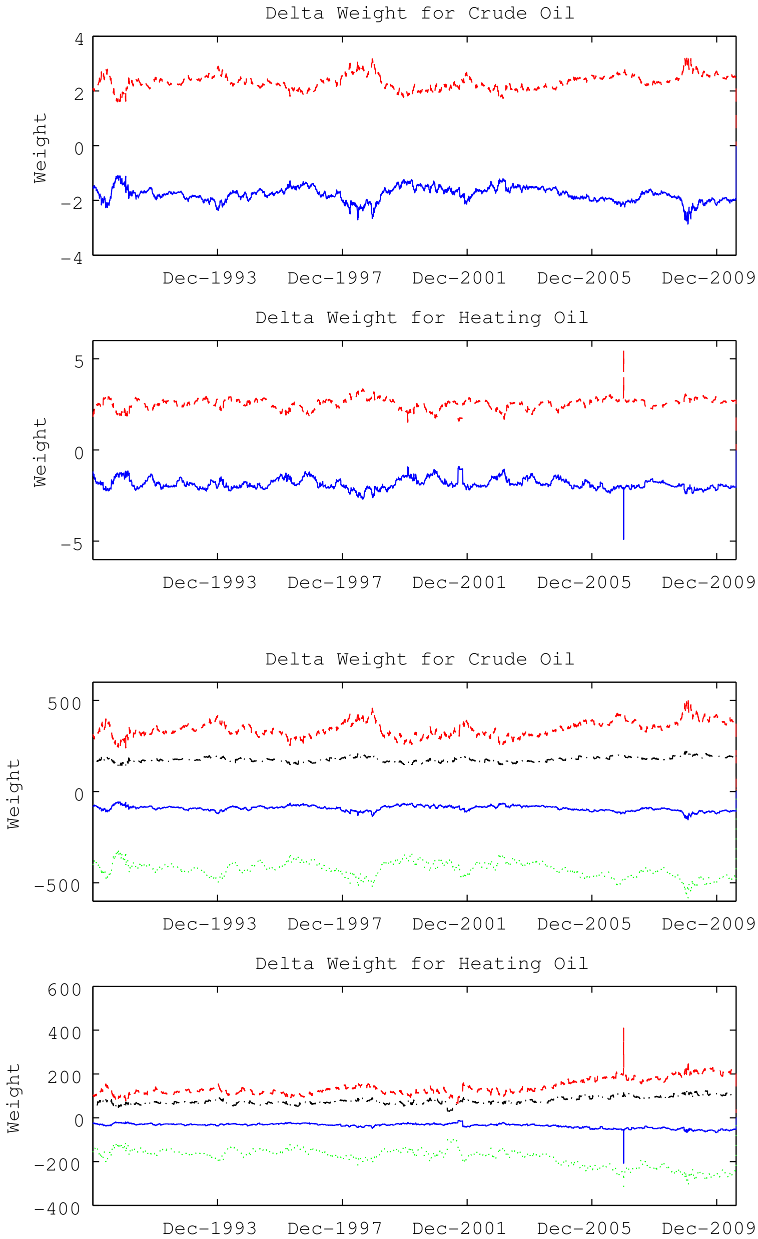
and $X'(t,T)$ represents the residual.

Using Ito's lemma and the martingale property of futures prices, we have

$$\begin{aligned} & dG_i(t,T) \\ = \; & \sum_{j=1}^{n} c_{S_j}(t,T)G_i(t,T)\sigma_{S_j}dW_{S_j}(t) + \sum_{j=1}^{n} c_{\delta_j}(t,T)G_i(t,T)\sigma_{\delta_j}dW_{\delta_j}(t) \\ = \; & \sum_{j=1}^{n}\left(\sigma_{S_j}\theta_{S_j0}c_{S_j}(t,T)G_i(t,T) + \sigma_{\delta_j}\theta_{\delta_j0}c_{\delta_j}(t,T)G_i(t,T)\right)dt \\ & + \sum_{j=1}^{n}\sigma_{S_j}c_{S_j}(t,T)G_i(t,T)dW^P_{S_j}(t) + \sum_{j=1}^{n}\sigma_{\delta_j}c_{\delta_j}(t,T)G_i(t,T)dW^P_{\delta_j}(t). \end{aligned}$$

This equation states two facts. First, the drift term under the risk-neutral probability includes an error term equal to 0. Second, the drift term under the natural probability is nonlinearly affected by futures prices $G_j(t,T)$. This

Figure 4.5: Weights of futures for hedging 10-year futures, for which the state variables are calculated using the Kalman filters. The upper two figures show the results for the GS model. The blue solid line and the red dashed line indicate the hedging weights of 1-month futures and 3-month futures, respectively. The lower two figures show the result for the GSC model. The blue solid line, the red dashed line, the green dotted line, and the black chained line indicate the hedging weights of 1-month futures, 3-month futures, 5-month futures, and 7-month futures, respectively.



means that in either case, the adjustment coefficients for futures prices are different from the coefficients of linear relations a_i and adjustment coefficients b_i for spot prices.

We emphasize that the linear relation is not observable in the GSC model. There are two aspects of this unobservability. First, it is modeled as spot prices, which are not observable. If we model the linear relation using futures prices, the advantage of our model will be the observability of the price, which allows us to use simple regression analysis and avoid using the more technical Kalman filter. Second, we modeled the linear relation under the risk-neutral probability, which is not observable from the historical data. Thus, it may be interesting to model the linear relations among observable futures prices under the natural probability instead of unobservable spot prices under the risk-neutral probability, and analyze the effects on spot prices and other derivatives.

4.5.2 Multidimensional $z(t)$ and Seasonality

In this chapter, we have assumed that there is only one linear relation, which is represented by the term $z(t)$. This can be relaxed to $h(< n)$ different linear relations $[z_1(t) \dots z_h(t)]^\top$ that can be formalized as

$$\begin{aligned}
d \ln S_i(t) &= \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t) + \sum_{j=1}^{h} b_{ij} z_j(t) \right) dt + \sigma_{S_i} dW_{S_i}(t), \quad i = 1, \dots, n, \\
d\delta_i(t) &= \kappa_i(\hat{\alpha}_i - \delta_i(t))dt + \sigma_{\delta_i} dW_{\delta_i}(t), \quad i = 1, \dots, n, \\
z_j(t) &= \mu_{zj} + a_{0j} t + \sum_{i=1}^{n} a_{ij} \ln S_i(t), \quad j = 1, \dots, h.
\end{aligned}$$

It is then simple to derive the futures and call option formulae. We can also extend the assumption on market price of risk and formalize the state and observation equations for the Kalman filters. The difficulty of this model stems from the number of parameters to consider when estimating the model. The parameters to be estimated are $n(1+2n)$ parameters for volatilities and correlations, $2n$ parameters for convenience yields $(\hat{\alpha}, \kappa)$, $2h(n+1)$ parameters for linear relations $(\mu_{zj}, a_{0j}, a_{ij}, b_{ij})$, $2n$ parameters for the market price of risks $(\boldsymbol{\theta})$, and other parameters that depend on the number of commodities and futures maturity data used for covariance matrix $\boldsymbol{R}$ in the observation equation. If we assume three commodities and two linear relations for the model using three maturities of futures for each commodity, there will be 55 parameters to be estimated. To conduct a realistic empirical analysis, the numbers of commodities and linear relations used have to be much smaller.

Furthermore, we can incorporate seasonality into the model. There are various ways of modeling seasonality.[15] One suggestion is the following.

$$\begin{aligned}
d\ln S_i(t) &= \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t) + b_i z(t)\right) dt + \sigma_{S_i} dW_{S_i}(t) \\
&\quad + \left(\sum_{m_i=1}^{M_i} \phi_{i,m_i,1}\cos(2\pi m_i t) + \phi_{i,m_i,2}\sin(2\pi m_i t)\right) dt + \sigma_{S_i} dW_{S_i}(t), \\
d\delta_i(t) &= \kappa_i(\hat{\alpha}_i - \delta_i(t))dt + \sigma_{\delta_i} dW_{\delta_i}(t), \quad i = 1, \ldots, n, \\
z(t) &= \mu_z + a_0 t + \sum_{i=1}^{n} a_i \ln S_i(t).
\end{aligned}$$

In this model, the seasonality is in the drift term of the dynamics of the log commodity prices $d\ln S_i(t)$. This can be interpreted as demeaned seasonality in the log commodity prices. Consider the dynamics of the log commodity prices without the $z(t)$ term. Integrating

$$\begin{aligned}
d\ln S_i(t) &= \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t)\right) dt \\
&\quad + \left(\sum_{m_i=1}^{M_i} \phi_{i,m_i,1}\cos(2\pi m_i t) + \phi_{i,m_i,2}\sin(2\pi m_i t)\right) dt + \sigma_{S_i} dW_{S_i}(t),
\end{aligned}$$

from 0 to t, we have

$$\begin{aligned}
\ln S_i(t) &= \ln S_i(0) + \int_0^t \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t)\right) dt \\
&\quad + \sum_{m_i=1}^{M_i} \frac{\phi_{i,m_i,1}}{2\pi m_i}\sin(2\pi m_i t) - \frac{\phi_{i,m_i,2}}{2\pi m_i}\sin(2\pi m_i t) - \sum_{m_i=1}^{M_i} \frac{\phi_{i,m_i,2}}{2\pi m_i} \\
&\quad + \sigma_{S_i} W_{S_i}(t).
\end{aligned}$$

[15] Other models that include seasonality in commodity spot prices are Hannan, Terrell, and Tuckwell (1970), Manoliu and Tompaidis (2002), Richter and Sorensen (2002), Sorensen (2002), Geman and Nguyen (2005), Cortazar et al. (2008), Paschke and Prokopczuk (2009), and Casassus et al. (2011).

This implies that the above model includes the demeaned seasonality in the log commodity prices.

4.6 Conclusion

In this chapter, we formulate a commodity pricing model that incorporates the effect of linear relations among log commodity prices, which includes cointegration under a certain condition. We derive futures and call option pricing formulae and show that, in contrast to Duan and Pliska (2004), the linear relations among log commodity prices, or the error term under appropriate conditions, should affect these derivative prices in the standard setup of commodity pricing. Furthermore, we derive the condition for the model to be cointegrated.

We emphasize that the proposed model can be interpreted as a generalization of standard commodity model, the Gibson-Schwartz model. This is because we decompose the deviation of the drift in commodity returns from the risk-free rate under the risk-neutral probability into two components; convenience yield and the linear relation term $z(t)$. The proposed model can thus describe not only the usual storage effects captured by the convenience yield, but also other causes such as impacts from other commodity prices and transaction costs.

In the empirical analysis, we assume that the market price of risk is linear in the convenience yield and the term $z(t)$, and utilize the Kalman filter technique. Using crude oil and heating oil market data, we estimate the proposed model. We also implement the model to examine the hedging of long-term futures.

Finally, it should be noted that while the linear relations among log spot prices play an important role, such spot prices are assumed to be unobservable in standard commodity pricing models, including ours. Thus, it would be interesting to model the linear relations among observable log futures prices instead of unobservable spot prices, and analyze the effects of the linear relation, or cointegration under certain conditions, on derivatives.

It should also be noted that, as Duan and Pliska (2004) showed, if the volatilities of commodity returns are stochastic, then cointegration affects derivative prices. Although they do not investigate the effect of linear relations among log spot prices on derivative prices, Trolle and Schwartz (2009) developed a commodity derivative pricing model with stochastic volatility.

Hence, it would also be interesting to advance a commodity derivative pricing model to incorporate linear relations among log spot prices under stochastic volatility of their returns.

These questions are left for future study.

4.7 Appendices

4.7.1 Proof of Proposition 4.2.2

We prove the call option pricing formula. From Harrison and Kreps (1979) or Harrison and Pliska (1981), we have

$$\begin{aligned} C_i(t,T) &= e^{-r(T-t)}E_t[(S_i(T)-K)^+] \\ &= e^{-r(T-t)}\int_D (e^{x_i}-K)n(x_i|\mu_{X_i}(t,T),\sigma^2_{X_i}(t,T))dx_i, \end{aligned}$$

where $n(x|\mu,\sigma^2)$ is the density function of the normal distribution with mean μ and variance σ^2, and

$$D = \{x_i | x_i \geq \ln K\}.$$

The integral can be calculated as

$$\int_D \exp\{x_i\}n(x_i|\mu_{X_i},\sigma^2_{X_i})dx_i = \exp\left\{\mu_{X_i}+\frac{\sigma^2_{X_i}}{2}\right\}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_{i1}}\exp\left\{-\frac{y^2}{2}\right\}dy,$$

where

$$d_{i1} = \frac{-\ln K + \mu_{X_i} + \sigma^2_{X_i}}{\sigma_{X_i}},$$

and we omit the time parameters such as $\mu_{X_i} = \mu_{X_i}(t,T)$ for notational convenience. Furthermore,

$$\int_D \frac{1}{\sqrt{2\pi}\sigma_{X_i}}\exp\left\{-\frac{(x_i-\mu_{X_i})^2}{2\sigma^2_{X_i}}\right\}dx_i = \int_{-\infty}^{d_{i2}}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{y^2}{2}\right\}dy,$$

where

$$d_{i2} = \frac{-\ln K + \mu_{X_i}}{\sigma_{X_i}},$$

and again we omit the time parameters. Collecting all terms, we have

$$C_i(t,T) = e^{-r(T-t)+\mu_{X_i}(t,T)+\frac{\sigma^2_{X_i}(t,T)}{2}}\Phi(d_{i1}(t,T)) - Ke^{-r(T-t)}\Phi(d_{i2}(t,T)).$$

4.7.2 Derivation of Spot and Futures Commodity Prices

In this subsection, we derive the closed formula for $S_i(T)$ and futures price $G_i(t,T)$ without integrals and matrix forms for the case when $b \neq 0$.[16] We assume that $\kappa_i \neq 0, \forall i$. Furthermore, we assume that $b-\kappa_i \neq 0$, $b+\kappa_i \neq 0, \forall i$ and $\kappa_i + \kappa_j \neq 0, \forall i,j$.

First, note that equation (1) is equivalent to

$$\begin{aligned} S_i(T) &= S_i(t)\exp\{\hat{X}_i(t,T)\}, \\ \hat{X}_i(t,T) &= \int_t^T \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(s) + b_i z(s)\right) ds + \int_t^T \sigma_{S_i} dW_{S_i}(s). \end{aligned}$$

The key point of the derivation is the calculation of the term $z(t)$ as follows:

$$\begin{aligned} \int_t^T z(s)ds &= \frac{1}{b}(z(T)-z(t)) + m(T-t) + \sum_{j=1}^n \int_t^T \frac{a_j}{b}\delta_j(s)ds \\ &\quad - \sum_{i=1}^n \frac{a_i}{b}\sigma_{S_i}(W_{S_i}(T) - W_{S_i}(t)), \end{aligned}$$

and

$$\begin{aligned} z(T) &= e^{b(T-t)}z(t) + \int_t^T e^{b(T-s)}(-bm - \sum_{i=1}^n a_i\delta_i(s))ds \\ &\quad + \sum_{i=1}^n \int_t^T e^{b(T-s)} a_j \sigma_{S_j} dW_{S_i}(s) \\ &= e^{b(T-t)}z(t) + m(1-e^{b(T-t)}) - \int_t^T e^{b(T-s)} \sum_{i=1}^n a_i\delta_i(s))ds \\ &\quad + \sum_{j=1}^n \int_t^T e^{b(T-s)} a_j \sigma_{S_j} dW_{S_i}(s). \end{aligned}$$

Hence, we have

$$\hat{X}_i(t,T) \triangleq \int_t^T \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(s) + b_i z(s)\right) ds + \int_t^T \sigma_{S_i} dW_{S_i}(s)$$

[16]For other cases, including the case when $b = 0$, the proofs are similar and we omit.

$$\begin{aligned}
= \;& \left(r + b_i m - \frac{\sigma_i^2}{2} - \hat{\alpha}_i + \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b}\right)(T-t) \\
& + \frac{b_i(m - z(t))}{b}(1 - e^{b(T-t)}) - \sum_{j=1}^{n} \frac{b_i a_j \delta_j(t)}{b(b+\kappa_j)}(e^{b(T-t)} - e^{-\kappa_j(T-t)}) \\
& - \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b^2}(e^{b(T-t)} - 1) + \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b(b+\kappa_j)}(e^{b(T-t)} - e^{-\kappa_j(T-t)}) \\
& + \frac{(\hat{\alpha}_i - \delta_i(t))}{\kappa_i}(1 - e^{-\kappa_i(T-t)}) - \sum_{j=1}^{n} \frac{b_i a_j(\hat{\alpha}_j - \delta_j(t))}{b\kappa_j}(1 - e^{-\kappa_j(T-t)}) \\
& + \sigma_{S_i}(W_{S_i}(T) - W_{S_i}(t)) - \frac{1}{\kappa_i}\sigma_{\delta_i}(W_{\delta_i}(T) - W_{\delta_i}(t)) \\
& - \sum_{j=1}^{n} \frac{b_i a_j}{b}\sigma_{S_j}(W_{S_j}(T) - W_{S_j}(t)) + \sum_{j=1}^{n} \frac{b_i a_j}{b\kappa_j}\sigma_{\delta_j}(W_{S_i}(T) - W_{S_i}(t)) \\
& + \sum_{j=1}^{n} \frac{b_i a_j}{b} \int_t^T e^{b(T-s)} \sigma_{S_j} dW_{S_j}(s) \\
& + \int_t^T \frac{e^{-\kappa_i(T-s)}}{\kappa_i} \sigma_{\delta_i} dW_{\delta_i}(s) - \sum_{j=1}^{n} \int_t^T \frac{b_i a_j e^{-\kappa_j(T-s)}}{b\kappa_j} \sigma_{\delta_j} dW_{\delta_j}(s) \\
& - \sum_{j=1}^{n} \int_t^T \frac{b_i a_j}{b(b+\kappa_j)}(e^{b(T-s)} - e^{-\kappa_j(T-s)})\sigma_{\delta_j} dW_{\delta_j}(s).
\end{aligned}$$

The $\mu_{\hat{X}_i}(t,T)$ and $\sigma_{\hat{X}_i}(t,T)$ are

$$\begin{aligned}
\mu_{\hat{X}_i}(t,T) \;=\;& E_t[\hat{X}_i(t,T)] \\
=\;& \left(r + b_i m - \frac{\sigma_{S_i}^2}{2} - \hat{\alpha}_i + \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b}\right)(T-t) \\
& + \frac{b_i(m - z(t))}{b}(1 - e^{b(T-t)}) - \sum_{j=1}^{n} \frac{b_i a_j \delta_j(t)}{b(b+\kappa_j)}(e^{b(T-t)} - e^{-\kappa_j(T-t)}) \\
& - \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b^2}(e^{b(T-t)} - 1) + \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b(b+\kappa_i)}(e^{b(T-t)} - e^{-\kappa_j(T-t)})
\end{aligned}$$

$$+\frac{(\hat{\alpha}_i-\delta_i(t))}{\kappa_i}(1-e^{-\kappa_i(T-t)})-\sum_{j=1}^{n}\frac{b_i a_j(\hat{\alpha}_j-\delta_j(t))}{b\kappa_j}(1-e^{-\kappa_j(T-t)}),$$

and

$$\begin{aligned}
\sigma^2_{\hat{X}_i}(t,T) &= E_t[(\hat{X}_i(t,T)-\mu_{\hat{X}_i}(t,T))^2] \\
&= \Bigg(\sigma^2_{S_i}+\frac{\sigma^2_{\delta_i}}{\kappa_i^2}-\frac{2\sigma_{S_i\delta_i}}{\kappa_i}-\sum_{j=1}^{n}\frac{2b_i a_j}{b\kappa_i\kappa_j}\sigma_{\delta_i\delta_j} \\
&\quad +\sum_{j=1}^{n}\frac{2b_i a_j\sigma_{S_j\delta_i}}{b\kappa_i}+\sum_{j,k=1}^{n}\frac{b_i^2 a_j a_k\sigma_{\delta_j\delta_k}}{b^2\kappa_i\kappa_k}+\sum_{j=1}^{n}\frac{2b_i a_j\sigma_{S_i\delta_j}}{b\kappa_j} \\
&\quad -\sum_{j,k=1}^{n}\frac{2b_i^2 a_j a_k\sigma_{S_j\delta_k}}{b^2\kappa_k}-\sum_{j=1}^{n}\frac{2b_i a_j\sigma_{S_iS_j}}{b}+\sum_{j,k=1}^{n}\frac{b_i^2 a_j a_k\sigma_{S_jS_k}}{b^2}\Bigg)(T-t) \\
&\quad +\frac{\sigma^2_{\delta_i}}{2\kappa_i^3}(1-e^{-2\kappa_i(T-t)})-\sum_{j=1}^{n}\frac{2b_i a_j\sigma_{\delta_i\delta_j}}{b\kappa_i\kappa_j(\kappa_i+\kappa_j)}(1-e^{-(\kappa_i+\kappa_j)(T-t)}) \\
&\quad +\sum_{j,k=1}^{n}\frac{b_i^2 a_j a_k\sigma_{\delta_j\delta_k}}{b^2\kappa_j\kappa_k(\kappa_j+\kappa_k)}(1-e^{-(\kappa_j+\kappa_k)(T-t)}) \\
&\quad -\frac{1}{\kappa_j-b}(1-e^{-(\kappa_j-b)(T-t)})-\frac{1}{\kappa_k-b}(1-e^{-(\kappa_k-b)(T-t)}) \\
&\quad \left.+\frac{1}{\kappa_j+\kappa_k}(1-e^{-(\kappa_j+\kappa_k)(T-t)})\right\} \\
&\quad -\sum_{j,k=1}^{n}\frac{b_i^2 a_j a_k\sigma_{S_jS_k}}{2b^3}(1-e^{2b(T-t)}) \\
&\quad -\sum_{j,k=1}^{n}\frac{2b_i^2 a_j a_k\sigma_{S_k\delta_j}}{b^2(b+\kappa_j)}\left\{\frac{-1}{2b}(1-e^{2b(T-t)})-\frac{1}{\kappa_j-b}(1-e^{-(\kappa_j-b)(T-t)})\right\} \\
&\quad +2\left(-\frac{\sigma^2_{\delta_i}}{\kappa_i^3}+\frac{\sigma_{S_i\delta_i}}{\kappa_i^2}+\sum_{j=1}^{n}\frac{b_i a_j\sigma_{\delta_i\delta_j}}{b\kappa_i^2\kappa_j}-\sum_{j=1}^{n}\frac{b_i a_j\sigma_{S_j\delta_i}}{b\kappa_i^2}\right)(1-e^{-\kappa_i(T-t)}) \\
&\quad +\sum_{j=1}^{n}2\left(\frac{b_i a_j\sigma_{\delta_i\delta_j}}{b\kappa_i\kappa_j^2}-\frac{b_i a_j\sigma_{S_i\delta_j}}{b\kappa_j^2}-\sum_{k=1}^{n}\frac{b_i^2 a_j a_k\sigma_{\delta_j\delta_k}}{b^2\kappa_j^2\kappa_k}+\sum_{k=1}^{n}\frac{b_i^2 a_j a_k\sigma_{S_k\delta_j}}{b^2\kappa_j^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\times(1-e^{-\kappa_j(T-t)})\\
&+\sum_{j=1}^{n}2\left(\frac{b_i a_j\sigma_{\delta_i\delta_j}}{b\kappa_i(b+\kappa_j)}-\frac{b_i a_j\sigma_{S_i\delta_j}}{b(b+\kappa_j)}-\sum_{k=1}^{n}\frac{b_i^2 a_j a_k\sigma_{\delta_j\delta_k}}{b^2\kappa_k(b+\kappa_j)}\right.\\
&\left.+\sum_{k=1}^{n}\frac{b_i^2 a_j a_k\sigma_{S_k\delta_j}}{b^2(b+\kappa_j)}\right)\left\{\frac{-1}{b}(1-e^{b(T-t)})-\frac{1}{\kappa_j}(1-e^{-\kappa_j(T-t)})\right\}\\
&+2\left(\sum_{j=1}^{n}\frac{b_i a_j\sigma_{S_j\delta_j}}{\kappa_i b^2}-\sum_{j,k=1}^{n}\frac{b_i^2 a_j a_k\sigma_{S_j\delta_k}}{\kappa_k b^3}-\sum_{j=1}^{n}\frac{b_i a_j\sigma_{S_iS_j}}{b^2}+\sum_{j,k=1}^{n}\frac{b_i^2 a_j a_k\sigma_{S_jS_k}}{b^3}\right)\\
&\times(1-e^{b(T-t)})\\
&-\sum_{j=1}^{n}\frac{2b_i a_j\sigma_{\delta_i\delta_j}}{b\kappa_i(b+\kappa_j)}\left\{\frac{1}{\kappa_i-b}(1-e^{-(\kappa_i-b)(T-t)})-\frac{1}{\kappa_i+\kappa_j}(1-e^{-(\kappa_i+\kappa_j)(T-t)})\right\}\\
&+\sum_{j=1}^{n}\frac{2b_i a_j\sigma_{S_j\delta_i}}{b\kappa_i(\kappa_i-b)}(1-e^{-(\kappa_i-b)(T-t)})\\
&+\sum_{j,k=1}^{n}\frac{2b_i^2 a_j a_k\sigma_{\delta_j\delta_k}}{b^2\kappa_k(b+\kappa_j)}\left\{\frac{1}{\kappa_k-b}(1-e^{-(\kappa_k-b)(T-t)})-\frac{1}{\kappa_j+\kappa_k}(1-e^{-(\kappa_j+\kappa_k)(T-t)})\right\}\\
&-\sum_{j,k=1}^{n}\frac{2b_i^2 a_j a_k\sigma_{S_j\delta_k}}{b^2\kappa_k(\kappa_k-b)}(1-e^{-(\kappa_k-b)(T-t)}).
\end{aligned}
$$

We have the following proposition, which shows the price formula for futures.

Proposition 4.7.1. Assuming (1), (2), and (3), the futures price of commodity i with maturity T at t is given by

$$
\begin{aligned}
G_i(t,T) &= E_t[S_i(T)]\\
&= S_i(t)\exp\left\{\mu_{\hat{X}_i}(t,T)+\frac{\sigma^2_{\hat{X}_i}(t,T)}{2}\right\},
\end{aligned}
$$

where $\mu_{\hat{X}_i}(t,T)=E_t[\hat{X}_i(t,T)]$ and $\sigma^2_{\hat{X}_i}(t,T)=E_t[(\hat{X}_i(t,T)-\mu_{\hat{X}_i}(t,T))^2]$.

Proof. Using risk-neutrality and the property of the moment generating function, we obtain the futures price of commodity i. $\square$

4.7.3 Cointegration Condition for the GSC Model

In this subsection, we provide the cointegration condition for the GSC model. Recall that the definition of cointegration is that $\ln S_i(t) - \ln S_i(t - \triangle t)$ is $I(0)$ for every i and $\sum_{i=1}^{n} a_i \ln S_i(t)$ is stationary.

We use the following proposition, which enhances a proposition from Hamilton (1994).[17]

Proposition 4.7.2. Let $\boldsymbol{x}(t)$ be a vector satisfying

$$\boldsymbol{x}(t) = \boldsymbol{\mu}_x + \sum_{s=0}^{\infty} \boldsymbol{\Phi}(s)\boldsymbol{\varepsilon}(t-s),$$

where $\boldsymbol{\varepsilon}(t)$ is a zero mean covariance-stationary process, i.e. $E[\boldsymbol{\varepsilon}(t)] = \mathbf{0}$, $E[\boldsymbol{\varepsilon}(t)\boldsymbol{\varepsilon}(t-s)^\top] = \boldsymbol{\Omega}(s)$ and $\{\boldsymbol{\Phi}(t)\}$ is absolutely summable, i.e. $\sum_{s=0}^{\infty} |\phi(s)_{ij}| < \infty$, where $\phi_{ij}(t)$ is the row i, column j element of $\boldsymbol{\Phi}(t)$. Then the autocovariance is

$$E[(\boldsymbol{x}(t) - \boldsymbol{\mu}_x)(\boldsymbol{x}(t-s) - \boldsymbol{\mu}_x)^\top] = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \boldsymbol{\Phi}(u)\boldsymbol{\Omega}(-u+s+v)\boldsymbol{\Phi}^\top(v),$$

which implies that the autocovariance is only a function of lag s.

Proof. The proof follows the proposition shown in Hamilton (1994). First, define

$$y_{il}(t) = \sum_{u=0}^{\infty} \phi_{il}(u)\varepsilon_l(t-u),$$

where $\phi_{il}(u)$ is the $[i,l]$ element of matrix $\boldsymbol{\Phi}(u)$. Note that

$$x_i(t) = \mu_{xi} + \sum_{l=1}^{n} y_{il}(t),$$

where $x_i(t)$ and μ_{xi} are the ith elements of $\boldsymbol{x}(t)$ and $\boldsymbol{\mu}_x$, respectively. Let us calculate the autocovariance of $y_{il}(t)$.

$$E[y_{il}(t)y_{jm}(t-s)] = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \phi_{il}(u)\phi_{jm}(v)\omega_{lm}(-u+s+v),$$

[17] Cf. Hamilton(1994), Proposition 10.2, p. 263.

where we denote $\omega_{lm}(t)$ as the $[l, m]$ element of $\Omega(t)$ and interchange the expectation operator and summation operator because

$$\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}|\phi_{il}(u)\phi_{jm}(v)| = \sum_{u=0}^{\infty}|\phi_{il}(u)| \cdot \sum_{v=0}^{\infty}|\phi_{jm}(v)| < \infty.$$

We calculate the autocovariance of $\boldsymbol{x}(t)$.

$$E[(x_i(t) - \mu_{xi})(x_j(t-s) - \mu_{xj})] = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\sum_{l=1}^{n}\sum_{m=1}^{n}\phi_{il}(u)\phi_{jm}(v)\omega_{lm}(-u+s+v).$$

As $\sum_{l=1}^{n}\sum_{m=1}^{n}\phi_{il}(u)\phi_{jm}(v)\omega_{lm}(-u+s+v)$ is the $[i, j]$ element of $\boldsymbol{\Phi}(u)\boldsymbol{\Omega}(-u+s+v)\boldsymbol{\Phi}(v)$, the proposition is proved. $\square$

Let us assume that $e^{b\triangle t} < 1(\iff b < 0), e^{-\kappa_i\triangle t} < 1(\iff \kappa_i > 0)$.

We now prove that $\ln S_i(t)$ is cointegrated. First, we see that $z(t)$ is stationary. Note that

$$\delta_i(t) = e^{-\kappa_i\triangle t}\delta_i(t-\triangle t) + \int_{t-\triangle t}^{t} e^{-\kappa_i(t-s)}\kappa_i\hat{\alpha}_i ds + \int_{t-\triangle t}^{t} e^{-\kappa_i(t-s)}\sigma_{\delta_i} dW_{\delta_i}(s).$$

As $e^{-\kappa_i\triangle t} < 1$, this yields

$$\delta_i(t) = \sum_{s=0}^{\infty} e^{-s\kappa_i\triangle t}(d_{\delta_i}(t-s\triangle t) + \varepsilon_{\delta_i}(t-s\triangle t)),$$

where

$$\begin{aligned} d_{\delta_i}(t) &= \int_{t-\triangle t}^{t} e^{-\kappa_i(t-s)}\kappa_i\hat{\alpha}_i ds, \\ \varepsilon_{\delta_i}(t) &= \int_{t-\triangle t}^{t} e^{-\kappa_i(t-s)}\sigma_{\delta_i} dW_{\delta_i}(s). \end{aligned}$$

Note that $d_{\delta_i}(t)$ does not depend on t, which can be easily confirmed by integrating or using changes of variables.

The $z(t)$ can be expanded as follows.

$$\begin{aligned}
z(t) &= e^{b\triangle t} z(t-\triangle t) + \int_{t-\triangle t}^{t} e^{b(t-s)} \left(\sum_{i=1}^{n} a_i r - \frac{1}{2} \sum_{i=1}^{n} a_i \sigma_{S_i}^2 \right) ds \\
&\quad - \sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)} \delta_i(s) ds \\
&\quad + \sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)} \sigma_{S_i} dW_{S_i}(s) \\
&= e^{b\triangle t} z(t-\triangle t) + \int_{t-\triangle t}^{t} e^{b(t-s)} \left(a_0 + \sum_{i=1}^{n} a_i r - \frac{1}{2} \sum_{i=1}^{n} a_i \sigma_{S_i}^2 \right) ds \\
&\quad - \sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)-\kappa_i(s-t+\triangle t)} ds \\
&\quad \times \left(\sum_{v=1}^{\infty} e^{-(v-1)\kappa_i \triangle t} \left(d_{\delta_i}(t-v\triangle t) + \varepsilon_{\delta_i}(t-v\triangle t) \right) \right) \\
&\quad - \sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)} \hat{\alpha}_i - e^{b(t-s)-\kappa_i(s-t+\triangle t)} \hat{\alpha}_i ds \\
&\quad + \int_{t-\triangle t}^{t} \int_{u}^{t} e^{b(t-s)-\kappa_i(s-u)} \sigma_{\delta_i} ds dW_{\delta_i}(u) \\
&\quad + \sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)} \sigma_{S_i} dW_{S_i}(s),
\end{aligned}$$

where we use Fubini's theorem for stochastic integrals and

$$\begin{aligned}
& \int_{t-\triangle t}^{t} e^{b(t-s)} \delta_i(s) ds \\
= & \int_{t-\triangle t}^{t} e^{b(t-s)-\kappa_i(s-t+\triangle t)} \delta_i(t-\triangle t) + e^{b(t-s)} \hat{\alpha}_i - e^{b(t-s)-\kappa_i(s-t+\triangle t)} \hat{\alpha}_i ds \\
& + \int_{t-\triangle t}^{t} \int_{u}^{t} e^{b(t-s)-\kappa_i(s-u)} \sigma_{\delta_i} ds dW_{\delta_i}(u).
\end{aligned}$$

Thus, from $e^{b\triangle t} < 1$,

$$\begin{aligned} z(t) &= \sum_{s=0}^{\infty} e^{sb\triangle t}(d_z(t-s\triangle t) + \varepsilon_{z1}(t-s\triangle t)) \\ &\quad + \sum_{s=0}^{\infty}\sum_{v=0}^{\infty} e^{sb\triangle t} e^{-v\kappa_i \triangle t}\varepsilon_{z2}(t-(s+v+1)\triangle t), \end{aligned}$$

where

$$\begin{aligned} d_z(t) &= \int_{t-\triangle t}^{t} e^{b(t-s)}\left(\sum_{i=1}^{n} a_i r - \frac{1}{2}\sum_{i=1}^{n} a_i \sigma_{S_i}^2\right) ds \\ &\quad - \sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)-\kappa_i(s-t+\triangle t)} ds \sum_{v=1}^{\infty} e^{-(v-1)\kappa_i\triangle t} d_{\delta_i}(t-v\triangle t) \\ &\quad - \sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)}\hat{\alpha}_i - e^{b(t-s)-\kappa_i(s-t+\triangle t)}\hat{\alpha}_i ds, \\ \varepsilon_{z1}(t) &= \int_{t-\triangle t}^{t}\int_{u}^{t} e^{b(t-s)-\kappa_i(s-u)}\sigma_{\delta_i} ds dW_{\delta_i}(u) \\ &\quad + \sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)}\sigma_{S_i} dW_{S_i}(s), \\ \varepsilon_{z2}(t) &= -\sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)-\kappa_i(s-t+\triangle t)} ds \varepsilon_{\delta_i}(t). \end{aligned}$$

Again, $d_z(t)$ does not depend on t and $\varepsilon_{z1}(t)$ and $\varepsilon_{z2}(t)$ are white noise. From Proposition 4.7.2, we can see that $z(t)$ is stationary.

Next, we check that $\ln S_i(t) - \ln S_i(t-\triangle t)$ is $I(0)$ for every i.

$$
\begin{aligned}
& \ln S_i(t) - \ln S_i(t-\triangle t) \\
= \; & \int_{t-\triangle t}^{t} (r - \frac{\sigma_{S_i}^2}{2} - \delta_i(s) + b_i z(s))ds + \int_{t-\triangle t}^{t} \sigma_{S_i} dW_{S_i}(s) \\
= \; & \mu_{\ln S_i} + \sum_{s=0}^{\infty} \left(- \int_{t-\triangle t}^{t} e^{-\kappa_i(s-t+\triangle t)} ds \right) e^{-s\kappa_i \triangle t} \varepsilon_{\delta_i}(t-(s+1)\triangle t) \\
& + b_i e^{b\triangle t} \triangle t \left(\sum_{s=0}^{\infty} e^{sb\triangle t} \varepsilon_{z_1}(t-(s+1)\triangle t) \right. \\
& \left. + \sum_{s=0}^{\infty} \sum_{v=0}^{\infty} e^{sb\triangle t - v\kappa_i \triangle t} \varepsilon_{z_2}(t-(s+v+2)\triangle t) \right) \\
& - b_i \sum_{j=1}^{n} a_j \int_{t-\triangle t} \int_{t-\triangle t}^{s} e^{b(s-u)-\kappa_j(u-t+\triangle t)} duds \\
& \times \left(\sum_{v=1}^{\infty} e^{-(v-1)\kappa_j \triangle t} \varepsilon_{\delta_j}(t-v\triangle t) \right) + \varepsilon_{\ln S_i}(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_{\ln S_i} \; = \; & \left(r - \frac{\sigma_{S_i}^2}{2} \right) \triangle t - \hat{\alpha}_i \triangle t + \hat{\alpha}_i \int_{t-\triangle t}^{t} e^{-\kappa_i(s-t+\triangle t)} ds \\
& + b_i \left(\sum_{j=1}^{n} a_j r - \frac{\sum_{j=1}^{n} a_j \sigma_{S_j}^2}{2} \right) \int_{t-\triangle t}^{t} \int_{t-\triangle t}^{s} e^{b(s-u)} duds \\
& - \sum_{j=1}^{n} a_j \hat{\alpha}_j \int_{t-\triangle t}^{t} \int_{t-\triangle t}^{s} e^{b(s-u)} (1 - e^{-\kappa_j(u-t+\triangle t)}) duds \\
& - \int_{t-\triangle t}^{t} e^{-\kappa_i(s-t+\triangle t)} ds \sum_{s=0}^{\infty} e^{-s\kappa_i \triangle t} (d_{\delta_i}(t-(s+1)\triangle t))
\end{aligned}
$$

$$
\begin{aligned}
& +b_i e^{b\triangle t}\triangle t\left(\sum_{s=0}^{\infty} e^{sb\triangle t} d_z((s+1)\triangle t)\right) \\
& -b_i \sum_{j=1}^{n} a_j \int_{t-\triangle t}^{t}\int_{t-\triangle t}^{s} e^{b(s-u)-\kappa_j(u-t+\triangle t)} duds \\
& \times\left(\sum_{v=1}^{\infty} e^{-(v-1)\kappa_j\triangle t} d_{\delta_j}(t-v\triangle t)\right), \\
\varepsilon_{\ln S_i} = & \int_{t-\triangle t}^{t}\int_{v}^{t}\int_{u}^{s} e^{b(s-u)-\kappa_i(u-v)}\sigma_{\delta_i} dudsdW_{\delta_i}(v) \\
& +\sum_{j=1}^{n} a_j \int_{t-\triangle t}^{t}\int_{u}^{t} e^{b(s-u)}\sigma_{S_i} dsdW_{S_j}(u) \\
& +\int_{t-\triangle t}^{t}\sigma_{S_i} dW_{S_i}(s).
\end{aligned}
$$

$\mu_{\ln S_i}$ does not depend on time t by using changes of variables. Furthermore, note that the stochastic terms are white noise. Therefore, we can use Proposition 4.7.2, which concludes that $\ln S_i(t) - \ln S_i(t-\triangle t)$ are $I(0)$. This completes the proof.

4.7.4 Other Empirical Results for the GSC Model

In this subsection, we show the empirical results for the other cases: (i) $b \neq 0$ and $a_1 = 1$, (iii) $b = 0, a_1 = 1, b_1 = -a_2 b_2$, (iv) $b = 0, a_1 = 0, a_2 = 0$, and (v) $b = 0, a_1 = 0, b_2 = 0$. The estimation results are presented in Tables 4.6 and 4.7. Except for case (iii), we see that the AIC is lower than that in the GS model. Furthermore, we analyze the result of case (iv), because this case has the lowest AIC, including case (i) $b \neq 0$ and $a_1 = 1$ and the GS model. Recall that this case does not satisfy the condition of cointegration and, thus, the estimated parameters are not valid, which means these are not comparable to standard deviations.

For case (iv), in which the linear relation vectors a_i are both 0, the adjustment speeds are $[b_1, b_2] = [-0.109099, 0.095277]$, respectively. The standard deviations for these parameters are very large. The time drift parameters a_0 and μ_z are 0.000001 and 0.012542, respectively.

Let us turn to the convenience yields. Note that κ_2 is negative, which means that the convenience yield is not stationary. Both long-term means $\hat{\alpha}_i$ are small compared with the standard deviations, but the adjustment parameters κ_i are large compared with the standard deviations.

A comparison of case (i) and the GS model does not reveal any significant differences between the volatility parameters. The differences we indicate are between the volatility parameters of the heating oil convenience yield σ_{δ_2}, correlations of heating oil price and crude oil convenience yield $\rho_{S_2\delta_1}$, and correlations of the convenience yields $\rho_{\delta_1\delta_2}$. σ_{δ_2} for case (iv) is much less volatile than the two previous models. The correlations $\rho_{S_2\delta_1}$ and $\rho_{\delta_1\delta_2}$ are lower.

Tables 4.8 and 4.9 show the root mean square error (RMSE) and mean error (ME) of the four cases. A comparison of the GS model and case (i) indicates that the result of heating oil for maturity 1 is not good, although maturity 5 is somewhat improved; there is no significant improvement or depreciation in the other parts.

Table 4.6: Estimated parameters, with standard errors in parentheses. Data are WTI and heating oil daily closing prices traded on the NYMEX from January 2, 1990, to July 30, 2010.

	(i)	(iii)
Volatility parameters		
σ_{S_1}	0.366667 (0.001942)	0.368996 (0.004328)
σ_{S_2}	0.403866 (0.002648)	0.356059 (0.005757)
σ_{δ_1}	0.292311 (0.001850)	0.198031 (0.003283)
σ_{δ_2}	0.689807 (0.007995)	0.211847 (0.005608)
$\rho_{S_1 S_2}$	0.714434 (0.005201)	0.858236 (0.007080)
$\rho_{S_1 \delta_1}$	0.737688 (0.004359)	0.002200 (0.020520)
$\rho_{S_1 \delta_2}$	0.000051 (0.012178)	0.000382 (0.056260)
$\rho_{S_2 \delta_1}$	0.528267 (0.006447)	-0.050049 (0.020261)
$\rho_{S_2 \delta_2}$	0.651368 (0.007394)	0.388430 (0.052287)
$\rho_{\delta_1 \delta_2}$	0.109110 (0.013473)	-0.000335 (0.001343)
Convenience yield parameters		
κ_1	1.142947 (0.007264)	0.886730 (0.005096)
κ_2	1.256849 (0.016329)	0.026644 (0.029419)
$\hat{\alpha}_1$	0.019111 (0.002216)	0.076056 (0.003200)
$\hat{\alpha}_2$	0.013057 (0.014289)	0.016721 (1.041906)
Linear relation parameters		
μ_z	0.023941 (0.045201)	-0.022223 (1.966758)
a_0	0.000106 (0.000003)	0.006951 (0.005982)
a_1	1.000000 (n.a.)	1.000000 (n.a.)
a_2	-0.725814 (0.007109)	-0.008302 (0.009216)
b_1	0.052334 (0.002303)	-0.000572 (n.a.)
b_2	0.366034 (0.005675)	-0.068889 (0.035397)
Market price of risk parameters		
$\theta_{S_1 0}$	0.199469 (0.222492)	0.082667 (0.330452)
$\theta_{S_2 0}$	-0.233062 (0.233117)	0.185452 (0.339923)
$\theta_{\delta_1 0}$	0.011876 (0.230454)	-1.799341 (0.332545)
$\theta_{\delta_2 0}$	0.006551 (0.273943)	-0.004155 (0.810891)
$R(1,1)$	0.000515 (0.000005)	0.000919 (0.000013)
$R(2,2)$	0.000000 (0.000000)	0.000030 (0.000001)
$R(3,3)$	0.000009 (0.000000)	0.000001 (0.000000)
$R(4,4)$	0.000000 (0.000000)	0.000002 (0.000000)
$R(5,5)$	0.000021 (0.000001)	0.000007 (0.000000)
$R(6,6)$	0.000001 (0.000001)	0.006362 (0.000168)
$R(7,7)$	0.001022 (0.000030)	0.000895 (0.000023)
$R(8,8)$	0.000696 (0.000022)	0.000015 (0.000001)
$R(9,9)$	0.000008 (0.000000)	0.000000 (0.000000)
$R(10,10)$	0.001017 (0.000028)	0.000766 (0.000018)
Log-likelihood	154178.537078	151160.610135
AIC	-308291.074155	-302257.220269
sample size	51590	51590

Table 4.7: Estimated parameters, with standard errors in parentheses. Data are WTI and heating oil daily closing prices traded on the NYMEX from January 2, 1990, to July 30, 2010.

	(iv)	(v)
Volatility parameters		
σ_{S_1}	0.350206 (0.001866)	0.336697 (0.001782)
σ_{S_2}	0.329622 (0.003536)	0.313910 (0.002938)
σ_{δ_1}	0.265981 (0.002176)	0.242048 (0.002008)
σ_{δ_2}	0.244615 (0.005482)	0.205013 (0.004326)
$\rho_{S_1S_2}$	0.625170 (0.008841)	0.685699 (0.008048)
$\rho_{S_1\delta_1}$	0.758072 (0.004341)	0.715504 (0.005356)
$\rho_{S_1\delta_2}$	-0.000129 (0.011527)	-0.000452 (0.013543)
$\rho_{S_2\delta_1}$	0.347681 (0.009722)	0.351016 (0.010844)
$\rho_{S_2\delta_2}$	0.645149 (0.010296)	0.589559 (0.011140)
$\rho_{\delta_1\delta_2}$	0.017574 (0.011490)	0.002437 (0.013464)
Convenience yield parameters		
κ_1	0.949412 (0.005239)	0.951637 (0.005549)
κ_2	-0.231644 (0.017156)	-0.345426 (0.019500)
$\hat{\alpha}_1$	0.016148 (0.051007)	0.007442 (0.043916)
$\hat{\alpha}_2$	0.014541 (0.191621)	0.149694 (0.009358)
Linear relation parameters		
μ_z	0.012542 (12.485066)	0.024759 (2.826261)
a_0	0.000001 (0.000519)	-0.000066 (0.006340)
a_1	0.000000 (n.a.)	0.000000 (n.a.)
a_2	0.000000 (n.a.)	-0.130021 (12.499299)
b_1	-0.109099 (108.382591)	0.030849 (2.963343)
b_2	0.095277 (94.836457)	0.000000 (n.a.)
Market price of risk parameters		
θ_{S_10}	0.005576 (0.235339)	0.002583 (0.222497)
θ_{S_20}	0.026724 (0.252499)	0.371485 (0.242683)
θ_{δ_10}	-0.093062 (0.248346)	-0.560535 (0.232429)
θ_{δ_20}	-0.000270 (0.380532)	0.000262 (0.383377)
$R(1,1)$	0.001041 (0.000016)	0.001124 (0.000018)
$R(2,2)$	0.000046 (0.000001)	0.000051 (0.000001)
$R(3,3)$	0.000000 (0.000000)	0.000000 (0.000000)
$R(4,4)$	0.000000 (0.000000)	0.000000 (0.000000)
$R(5,5)$	0.000009 (0.000000)	0.000008 (0.000000)
$R(6,6)$	0.006745 (0.000173)	0.006392 (0.000162)
$R(7,7)$	0.001100 (0.000028)	0.001091 (0.000029)
$R(8,8)$	0.000000 (0.000000)	0.000002 (0.000000)
$R(9,9)$	0.000000 (0.000000)	0.000002 (0.000000)
$R(10,10)$	0.001038 (0.000027)	0.001113 (0.000031)
Log-likelihood	154897.735778	154743.996275
AIC	-309731.471557	-309423.992550
sample size	51590	51590

Table 4.8: RMSE (root mean square error) and ME (mean error) for each futures contract.

Contracts	RMSE		ME	
Models	(ii)	(iii)	(ii)	(iii)
Crude oil				
Maturity 1	0.032943	0.038575	-0.002741	-0.002193
Maturity 3	0.020206	0.021061	-0.000003	0.000400
Maturity 5	0.018556	0.018369	0.000026	-0.000083
Maturity 7	0.017296	0.017350	-0.000030	-0.000495
Maturity 9	0.017152	0.016782	0.000170	-0.000185
Heating oil				
Maturity 1	0.024064	0.082284	0.000023	-0.000638
Maturity 3	0.037136	0.035022	-0.001603	0.000271
Maturity 5	0.031699	0.018464	-0.000226	0.000495
Maturity 7	0.017767	0.017380	-0.000012	0.000711
Maturity 9	0.036419	0.035860	-0.003012	0.001615

Table 4.9: RMSE (root mean square error) and ME (mean error) for each futures contract.

Contracts	RMSE		ME	
Models	(iv)	(v)	(iv)	(v)
Crude oil				
Maturity 1	0.039733	0.039573	-0.003794	-0.003158
Maturity 3	0.021257	0.021247	0.000062	0.000170
Maturity 5	0.018388	0.018378	0.000443	0.000248
Maturity 7	0.017312	0.017300	0.000412	0.000159
Maturity 9	0.016819	0.016791	0.000619	0.000544
Heating oil				
Maturity 1	0.084311	0.082893	-0.001628	-0.010059
Maturity 3	0.037614	0.037121	-0.000298	-0.003341
Maturity 5	0.018668	0.018706	0.000332	-0.000079
Maturity 7	0.017853	0.017942	0.000348	-0.000137
Maturity 9	0.037304	0.038042	-0.000148	-0.003425

Chapter 5

Commodity Spread Option with Cointegration

5.1 Introduction

Consider an oil refinery company producing heating oil. The profit of this company is given by the difference between the sales of heating oil and the expenses incurred for crude oil, wages, and maintenance costs. Thus, a risk-averse manager of the company should be interested in hedging the risk of profit fluctuations. For this purpose, he may use spread options between prices of heating oil and crude oil. Therefore, it is important for energy companies to value commodity spread options and indeed, such spread options have been traded on the NYMEX since 1994.

Many previous papers in this field depend on commodity pricing and its derivatives. As a type of exotic options, spread options have been studied under many different settings. Margrabe's (1978) paper on exchange options, which were spread options with a zero exercise price, was the first study on this topic. Shimko (1994) derived a valuation formula for commodity spread options where he assumed two commodity prices that follow geometric Brownian motions and the convenience yields to follow mean reverting stochastic processes. Poitras (1998) proposed pricing formulae for spread options under the assumption that the spread followed an arithmetic Brownian motion. Nakajima and Maeda (2007) extended Shimko's model and applied the framework of the Heath-Jarrow-Morton (1992) interest rates and Miltersen-Schwartz convenience yields (1998). Dempster, Medova, and

Tang (2008) derived a valuation formula by assuming a spot spread process. Other approaches for valuation of spread options can be found in Wilcox (1990), Kirk (1995), and Pearson (1995). Zhang (1998) covers a wide variation of exotic options including spread options. Carmona and Durrleman (2003) not only survey this topic from theoretical aspects but also discuss computational issues.

These models, however, did not consider relationships between multiple commodity prices.[1] On the other hand, several papers find evidence of cointegration in commodity prices (Malliaris and Urrutia, 1996; Girma and Paulson, 1999). Thus, a model for commodity spread options is called for that incorporates cointegration or more generally linear relations between log commodity prices. Casassus, Liu, and Tang (2011) developed commodity pricing models with linear relations between commodity prices which is similar to our model in Chapter 4. To check their validity, they conduct some Monte Carlo simulations to calculate prices of European commodity spread options. In this chapter, based on Chapter 4, we develop a model of commodity spread options with linear relations between commodity prices, derive a semianalytic formula for European commodity spread options, provide an approximation formula for American commodity spread options, and investigate properties of spread option prices by conducting sensitivity analyses.

More precisely, we use two models to analyze commodity spread options. One is the GS model (1990), which is the benchmark of commodity derivative models. The other is the GSC model (the GS with cointegration model) developed in Chapter 4. We derive the valuation formulae for European commodity call spread options from both models. Since the GSC model extends the GS model to incorporate linear relations between commodity prices, the model in this chapter can be regarded as an application of the GSC model to spread options. Furthermore, we present in this chapter an analytical approximation formula for American call commodity spread options using the framework of Bjerksund and Stensland (1994). Finally, using the parameter values estimated in Chapter 4, we conduct a numerical analysis and investigate characteristics of commodity spread options.

This chapter is organized as follows: Section 2 formulates the model and

[1] Dempster, Medova, and Tang (2008) assume that a spread of commodity prices follows a stochastic process, but they do not explicitly formulate the stochastic processes that the commodity prices follow in some relationships.

derives the valuation formula for European call commodity spread options and the analytical approximation formula for American ones. Section 3 provides the numerical analysis. Section 4 concludes.

5.2 A Model for Commodity Spread Options

5.2.1 The Gibson-Schwartz (GS) Model

Assume that there are n commodities whose spot prices and convenience yields follow

$$d\ln S_i(t) = \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t)\right)dt + \sigma_{S_i} dW_{S_i}(t), \quad i = 1, 2, \tag{1}$$

$$d\delta_i(t) = \kappa_i(\hat{\alpha}_i - \delta_i(t))dt + \sigma_{\delta_i} dW_{\delta_i}(t), \quad i = 1, 2, \tag{2}$$

under the risk-neutral probability. Here, r is the risk-free interest rate, which is assumed to be constant. Also, σ_{S_i}, κ_i, $\hat{\alpha}_i$, and σ_{δ_i} are constant coefficients. $W(t) = [W_{S_1}(t), ..., W_{S_n}(t), W_{\delta_1}(t), ..., W_{\delta_n}(t)]^\top$ is four-dimensional Brownian motion under the risk-neutral probability with

$$dW_{S_i}(t)dW_{S_j}(t) = \rho_{S_iS_j}dt, dW_{S_i}(t)dW_{\delta_j}(t) = \rho_{S_i\delta_j}dt, dW_{\delta_i}(t)dW_{\delta_j}(t) = \rho_{\delta_i\delta_j}dt, \quad i, j = 1, 2.$$

We derive the futures and European call option prices on commodity i in closed-forms. Note that under the assumptions above, the spot price of commodity i is calculated as[2]

$$\begin{aligned} S_i(T) &= S_i(t)\exp\{\hat{X}_i(t,T)\}, \qquad (3) \\ \hat{X}_i(t,T) &= \left(r - \frac{\sigma_i^2}{2} - \hat{\alpha}_i\right)(T-t) + \frac{(\hat{\alpha}_i - \delta_i(t))}{\kappa_i}(1 - e^{-\kappa_i(T-t)}) \\ &\quad + \sigma_{S_i}(W_{S_i}(T) - W_{S_i}(t)) - \frac{1}{\kappa_i}\sigma_{\delta_i}(W_{\delta_i}(T) - W_{\delta_i}(t)) \\ &\quad + \int_t^T \frac{e^{-\kappa_i(T-s)}}{\kappa_i}\sigma_{\delta_i} dW_{\delta_i}(s). \end{aligned}$$

[2]See Gibson and Schwartz (1990), Bjerksund (1991), and Schwartz (1997) for derivation.

We denote $E_t[\cdot]$ as expectation under the risk-neutral probability given $\mathcal{F}_t$.[3] Using risk-neutrality and the property of moment generating function, we obtain the futures price of commodity i as follows.

Proposition 5.2.1. Assuming (1) and (2), the futures price of commodity i with maturity T at t is given by

$$\begin{aligned} G_i(t,T) &= E_t[S_i(T)] \\ &= S_i(t)\exp\left\{\mu_{\hat{X}_i}(t,T) + \frac{\sigma^2_{\hat{X}_i}(t,T)}{2}\right\}, \end{aligned}$$

where

$$\begin{aligned} \mu_{\hat{X}_i}(t,T) &= E_t[\hat{X}_i(t,T)] \\ &= \left(r - \frac{\sigma^2_{S_i}}{2} - \hat{\alpha}_i\right)(T-t) + \frac{(\hat{\alpha}_i - \delta_i(t))}{\kappa_i}(1 - e^{-\kappa_i(T-t)}), \end{aligned}$$

and

$$\begin{aligned} \sigma^2_{\hat{X}_i}(t,T) &= E_t[(\hat{X}_i(t,T) - \mu_{\hat{X}_i}(t,T))^2] \\ &= \left(\sigma^2_{S_i} + \frac{\sigma^2_{\delta_i}}{\kappa_i^2} - \frac{2\sigma_{S_i\delta_i}}{\kappa_i}\right)(T-t) + \frac{\sigma^2_{\delta_i}}{2\kappa_i^3}(1 - e^{-2\kappa_i(T-t)}) \\ &\quad + 2\left(-\frac{\sigma^2_{\delta_i}}{\kappa_i^3} + \frac{\sigma_{S_i\delta_i}}{\kappa_i^2}\right)(1 - e^{-\kappa_i(T-t)}). \end{aligned}$$

Proof. See Bjerksund (1991). □

We also obtain the price of spread options between the futures prices of commodities i and j as follows.

Proposition 5.2.2. Suppose that $S_i(t)$ and $\delta_i(t)$ follow (1) and (2), respectively. The prices of European call commodity spread options at t, where the option, commodity i, and commodity j futures maturity are T_0, T_i, and T_j,

[3] As before, we assume a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$.

respectively, are

$$
\begin{aligned}
& C^E(G_i, G_j, t, T_0, T_i, T_j) \\
= \; & h_i G_i(t, T_i) \exp\left\{ -r(T_0 - t) + \mu_{\hat{X}_{G_i}GS}(t, T_0, T_i) + \frac{1}{2}\sigma^2_{\hat{X}_{G_i}GS}(t, T_0, T_i, T_i) \right\} \\
& \times \int_{-\infty}^{\infty} \Phi(d_i(x_j)) \\
& \times n(x_j | \mu_{\hat{X}_{G_j}GS}(t, T_0, T_j) + \sigma_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t, T_0, T_i, T_j), \sigma^2_{\hat{X}_{G_j}GS}(t, T_0, T_j, T_j)) dx_j \\
& -h_j G_j(t, T_j) \exp\left\{ -r(T_0 - t) + \mu_{\hat{X}_{G_j}GS}(t, T_0, T_j) + \frac{1}{2}\sigma^2_{\hat{X}_{G_j}GS}(t, T_0, T_j, T_j) \right\} \\
& \times \int_{-\infty}^{\infty} \Phi(d_K(x_j)) \\
& \times n(x_j | \mu_{\hat{X}_{G_j}GS}(t, T_0, T_j) + \sigma^2_{\hat{X}_{G_j}GS}(t, T_0, T_j, T_j), \sigma^2_{\hat{X}_{G_j}GS}(t, T_0, T_j, T_j)) dx_j \\
& -K e^{-r(T_0 - t)} \int_{-\infty}^{\infty} \Phi(d_K(x_j)) n(x_j | \mu_{\hat{X}_{G_j}GS}(t, T_0, T_j), \sigma^2_{\hat{X}_{G_j}GS}(t, T_0, T_j, T_j)) dx_j,
\end{aligned}
$$

where

$$
\begin{aligned}
d_K(x_j) = & -\frac{\ln(h_j G_j(t, T_j) e^{x_j} + K) - \ln(h_i G_i(t, T_i)) - \mu_{\hat{X}_{G_i}GS}(t, t_0, T_i)}{\sigma_{\hat{X}_{G_i}GS}(t, T_0, T_i, T_i)\sqrt{1 - \rho^2_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t, T_0, T_i, T_j)}} \\
& + \frac{\rho_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t, T_0, T_i, T_j)\sigma_{\hat{X}_{G_i}GS}(t, T_0, T_i, T_i)\frac{x_j - \mu_{\hat{X}_{G_j}GS}(t, T_0, T_j)}{\sigma_{\hat{X}_{G_j}GS}(t, T_0, T_i, T_j)}}{\sigma_{\hat{X}_{G_i}GS}(t, T_0, T_i, T_i)\sqrt{1 - \rho^2_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t, T_0, T_i, T_j)}}, \\
d_i(x_j) = d_K(x_j) + & \frac{\sigma^2_{\hat{X}_{G_i}GS}(t, T_0, T_i, T_i) - \rho^2_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t, T_0, T_i, T_j)\sigma^2_{\hat{X}_{G_i}GS}(t, T_0, T_i, T_i)}{\sigma_{\hat{X}_{G_i}GS}(t, T_0, T_i, T_i)\sqrt{1 - \rho^2_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t, T_0, T_i, T_j)}}, \\
\rho_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t, T_0, T_i, T_j) = & \frac{\sigma_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t, T_0, T_i, T_j)}{\sigma_{\hat{X}_{G_i}GS}(t, T_0, T_i, T_i)\sigma_{\hat{X}_{G_j}GS}(t, T_0, T_j, T_j)},
\end{aligned}
$$

$$\begin{aligned}
\mu_{\hat{X}_{G_i}GS}(t,T) &= E_t[\hat{X}_{G_i}(t,T)] \\
&= -\frac{1}{2}\Bigg[\sigma_{S_i}^2(T_0-t) \\
&\quad -2\frac{\sigma_{S_i\delta_i}}{\kappa_i}\left\{(T_0-t)-\frac{1}{\kappa_i}(e^{-\kappa_i(T_i-T_0)}-e^{-\kappa_i(T_i-t)})\right\} \\
&\quad +\sigma_{\delta_i}^2\Bigg\{\frac{T_0-t}{\kappa_i^2}-\frac{2}{\kappa_i^3}(e^{-\kappa_i(T_i-T_0)}-e^{-\kappa_i(T_i-t)}) \\
&\quad +\frac{1}{2\kappa_i^3}(e^{-2\kappa_i(T_i-T_0)}-e^{-2\kappa_i(T_i-t)})\Bigg\},
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t,T) &= E_t[(\hat{X}_{G_i}(t,T)-\mu_{\hat{X}_{G_i}}(t,T))^2] \\
&= \sigma_{S_iS_j}(T_0-t) \\
&\quad -\frac{\sigma_{S_i\delta_j}}{\kappa_j}\left\{(T_0-t)-\frac{1}{\kappa_j}(e^{-\kappa_j(T_j-T_0)}-e^{-\kappa_j(T_j-t)})\right\} \\
&\quad -\frac{\sigma_{\delta_iS_j}}{\kappa_i}\left\{(T_0-t)-\frac{1}{\kappa_i}(e^{-\kappa_i(T_i-T_0)}-e^{-\kappa_i(T_i-t)})\right\} \\
&\quad +\frac{\sigma_{\delta_i\delta_j}}{\kappa_i\kappa_j}\Bigg\{(T_0-t)-\frac{1}{\kappa_i}(e^{-\kappa_i(T_i-T_0)}-e^{-\kappa_i(T_i-T_0)}) \\
&\quad -\frac{1}{\kappa_j}(e^{-\kappa_j(T_j-T_0)}-e^{-\kappa_j(T_j-t)}) \\
&\quad +\frac{1}{\kappa_i+\kappa_j}(e^{-\kappa_i(T_i-T_0)-\kappa_j(T_j-T_0)}-e^{-\kappa_i(T_i-t)-\kappa_j(T_j-t)})\Bigg\}.
\end{aligned}$$

We abbreviate $\sigma_{\hat{X}_{G_i}\hat{X}_{G_i}GS}(t,T)$ as $\sigma^2_{\hat{X}_{G_i}GS}(t,T)$.

Proof. The proof is basically the same as the proof of proposition 5.2.4 in the Appendix. The only differences in that proof are the drifts $\mu_{\hat{X}_{G_i}}$ and the volatilities $\sigma_{\hat{X}_{G_i}\hat{X}_{G_j}}$ are $\mu_{\hat{X}_{G_i}GS}$ and $\sigma_{\hat{X}_{G_i}\hat{X}_{G_j}GS}$, respectively. □

Note that while Shimko (1994) derives the theoretical price of commodity spread options for futures with the same maturities, we assume the futures to have different maturities.

5.2.2 The Gibson-Schwartz with Cointegration (GSC) Model

We now introduce the GSC model developed in Chapter 4. As mentioned, the GSC model is an extension of the GS model and incorporates linear relations between log commodity prices. More precisely, we assume that there are n commodities whose spot prices and convenience yields follow

$$d \ln S_i(t) = \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t) + b_i z(t) \right) dt + \sigma_{S_i} dW_{S_i}(t), \quad i = 1, \ldots, n, \tag{4}$$

$$d\delta_i(t) = \kappa_i(\hat{\alpha}_i - \delta_i(t))dt + \sigma_{\delta_i} dW_{\delta_i}(t), \quad i = 1, \ldots, n, \tag{5}$$

under the risk-neutral probability.[4] b_i, σ_{S_i}, κ_i, $\hat{\alpha}_i$, and σ_{δ_i} are constant coefficients. $W(t) = [W_{S_1}(t), ..., W_{S_n}(t), W_{\delta_1}(t), ..., W_{\delta_n}(t)]^\top$ is $2n$-dimensional Brownian motion under the risk-neutral probability with

$$dW_{S_i}(t)dW_{S_j}(t) = \rho_{S_i S_j} dt, dW_{S_i}(t)dW_{\delta_j}(t) = \rho_{S_i \delta_j} dt, dW_{\delta_i}(t)dW_{\delta_j}(t) = \rho_{\delta_i \delta_j} dt, \quad i, j = 1, \ldots, n.$$

We assume that the commodity prices are related linearly through

$$z(t) = \mu_z + a_0 t + \sum_{i=1}^{n} a_i \ln S_i(t), \tag{6}$$

where μ_z, a_0, and a_is are constants.[5] Assume that $\ln S_i$ are cointegrated. Then by rearranging the equation as $\ln S_1(t) = (-\mu_z - a_0 t - \sum_{i=2}^{n} a_i \ln S_i(t) + z(t))/a_1$ (if $a_1 \neq 0$), $z(t)$ can be interpreted as an error term, a_i as cointegration vectors, and b_i as the adjustment speed of the error correction term.

We obtain the futures price under the GSC model as follows.

[4] Notice that while the GS model only concerns two sets of commodity prices and convenience yields, the GSC model may incorporate n sets of commodity prices and convenience yields linear relations.

[5] Although we treat the case where there is only one linear relation between prices, i.e., the case with one-dimensional $z(t)$, we can extend the model to include several linear relations.

Proposition 5.2.3. Assuming (4), (5), (6), the futures price of commodity at T_0, which matures at T_i is

$$G_i(t,T) = e^{\mu_{X_i}(t,T)+\frac{\sigma^2_{X_i}(t,T)}{2}},$$

where

$$\begin{aligned}
\beta_{S_i0}(t) &= r - \frac{\sigma^2_{S_i}}{2} + b_i\mu_z + b_i a_0 t,\\
\beta_{S_iS_j} &= b_i a_j,\\
\beta_{S_i\delta_i} &= -1,\\
\beta_{\delta_i0} &= \kappa_i\hat{\alpha}_i,\\
\beta_{\delta_i\delta_i} &= -\kappa_i,\\
\boldsymbol{\beta}_0(t) &= [\beta_{S_10}(t),\cdots,\beta_{S_n0}(t),\beta_{\delta_10},\cdots,\beta_{\delta_n0}]^\top,\\
\boldsymbol{\beta} &= \begin{bmatrix}
\beta_{S_1S_1} & \cdots & \beta_{S_1S_n} & \beta_{S_1\delta_1} & & 0\\
\vdots & \ddots & \vdots & & \ddots & \\
\beta_{S_nS_1} & \cdots & \beta_{S_nS_n} & 0 & & \beta_{S_n\delta_n}\\
 & & & \beta_{\delta_1\delta_1} & & 0\\
 & \mathbf{0} & & & \ddots & \\
 & & & 0 & & \beta_{\delta_n\delta_n}
\end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\mu_{X_i}(t,T) &= E_t[\ln S_i(T)]\\
&= \left[e^{T\boldsymbol{\beta}}\left\{e^{-t\boldsymbol{\beta}}\boldsymbol{X}(t) + \int_t^T e^{-s\boldsymbol{\beta}}\boldsymbol{\beta}_0(s)ds\right\}\right]_i,\\
\sigma_{X_iX_j}(t,T) &= E_t[(\ln S_i(T) - \mu_{X_i}(t,T))(\ln S_j(T) - \mu_{X_i}(t,T))]\\
&= \left[\int_t^T (e^{(T-s)\boldsymbol{\beta}})\boldsymbol{\Sigma}(e^{(T-s)\boldsymbol{\beta}})^\top ds\right]_{ij},
\end{aligned}$$

See Chapter 4 for the proof. We use the notation for expectation $\mu_{X_{G_i}}(t,T_0,T_i) = E_t[X_{G_i}(t,T_0,T_i)]$ and covariance $\sigma_{X_{G_i}X_{G_j}}(t,T_0,T_i,T_j) = E_t[(X_{G_i}(t,T_0,T_i) - \mu_{X_{G_i}}(t,T_0,T_i))(X_{G_j}(t,T_0,T_j) - \mu_{X_{G_j}}(t,T_0,T_j))]$. These are calculated in the Appendix.

Let us now show the price formula for a European commodity spread option.

Proposition 5.2.4. Under the GSC model, the prices of the European call commodity spread option at t, where the option, commodity i, and commodity j futures maturity are T_0, T_i, and T_j, respectively, are given by

$$
\begin{aligned}
& C^E(G_i, G_j, t, T_0, T_i, T_j) \\
= \;& h_i G_i(t, T_i) \exp\left\{ -r(T_0 - t) + \mu_{X_{G_i}}(t, T_0, T_i) + \frac{1}{2}\sigma^2_{X_{G_i}}(t, T_0, T_i, T_i) \right\} \\
& \times \int_{-\infty}^{\infty} \Phi(d_i(x_j)) n(x_j | \mu_{X_{G_j}}(t, T_0, T_j) + \sigma_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j), \sigma^2_{X_{G_j}}(t, T_0, T_j, T_j)) dx_j \\
& - h_j G_j(t, T_j) \exp\left\{ -r(T_0 - t) + \mu_{X_{G_j}}(t, T_0, T_j) + \frac{1}{2}\sigma^2_{X_{G_j}}(t, T_0, T_j, T_j) \right\} \\
& \times \int_{-\infty}^{\infty} \Phi(d_K(x_j)) n(x_j | \mu_{X_{G_j}}(t, T_0, T_j) + \sigma^2_{X_{G_j}}(t, T_0, T_j, T_j), \sigma^2_{X_{G_j}}(t, T_0, T_j, T_j)) dx_j \\
& - K e^{-r(T_0 - t)} \int_{-\infty}^{\infty} \Phi(d_K(x_j)) n(x_j | \mu_{X_{G_j}}(t, T_0, T_j), \sigma^2_{X_{G_j}}(t, T_0, T_j, T_j)) dx_j,
\end{aligned}
$$

where

$$
\begin{aligned}
d_K(x_j) &= -\frac{\ln(h_j e^{x_j} + K) - \ln h_i - \mu_{X_{G_i}}(t, t_0, T_i)}{\sigma_{X_{G_i}}(t, T_0, T_i, T_i)\sqrt{1 - \rho^2_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j)}} \\
& \quad + \frac{\rho_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j) \sigma_{X_{G_i}}(t, T_0, T_i, T_i) \frac{x_j - \mu_{X_{G_j}}(t, T_0, T_j}{\sigma_{X_{G_j}}(t, T_0, T_i, T_j)}}{\sigma_{X_{G_i}}(t, T_0, T_i, T_i)\sqrt{1 - \rho^2_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j)}}, \\
d_i(x_j) &= d_K(x_j) + \frac{\sigma^2_{X_{G_i}}(t, T_0, T_i, T_i) - \rho^2_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j)\sigma^2_{X_{G_i}}(t, T_0, T_i, T_i)}{\sigma_{X_{G_i}}(t, T_0, T_i, T_i)\sqrt{1 - \rho^2_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j)}}, \\
\rho_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j) &= \frac{\sigma_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j)}{\sigma_{X_{G_i}}(t, T_0, T_i, T_i)\sigma_{X_{G_j}}(t, T_0, T_j, T_j)},
\end{aligned}
$$

and $\mu_{X_{G_i}}(t, T)$ and $\sigma_{X_{G_i} X_{G_j}}(t, T)$ are in the Appendix. We abbreviate $\sigma_{X_{G_i} X_{G_i}}(t, T)$ as $\sigma^2_{X_{G_i}}(t, T)$.

Proof. See the Appendix. □

Since the spread options traded in the actual markets are the American type,[6] we also show an approximation formula for American commodity spread options. The derivation is in the Appendix.

Proposition 5.2.5. Under the GSC model, the price of the American call commodity spread option at t where the option maturity, commodity i and commodity j futures maturity, are T_0, T_i, and T_j, respectively, is approximated as follows:

$$\begin{aligned} & C^A(G_i, G_j, t, T_0, T_i, T_j; B^{c,s}) \\ = \; & C^E(G_i, G_j, t, T_0, T_i, T_j) + a^A(G_i, G_j, t, T_0, T_i, T_j; B^{c,s}), \end{aligned}$$

where

$$\begin{aligned} & a^A(G_i, G_j, t, T_0, T_i, T_j; B^{c,s}) \\ \approx \; & r\Bigg[h_i \int_t^{T_0} \exp\left\{-r(u-t) + \mu_{X_{G_i}}(t,u,T_i) + \frac{1}{2}\sigma^2_{X_{G_i}}(t,u,T_i,T_i)\right\} \\ & \times \int_{-\infty}^{\infty} \Phi(d_{eep1}(x_j,u)) \\ & \times n(x_j | \mu_{X_{G_j}}(t,u,T_j) + \sigma_{X_{G_i}X_{G_j}}(t,u,T_i,T_j), \sigma^2_{X_{G_j}}(t,u,T_j,T_j)) dx_j du \\ & -h_j \int_t^{T_0} \exp\left\{-r(u-t) + \mu_{X_{G_j}}(t,u,T_j) + \frac{1}{2}\sigma^2_{X_{G_j}}(t,u,T_j,T_j)\right\} \\ & \times \int_{-\infty}^{\infty} \Phi(d_{eepK}(x_j,u)) \\ & \times n(x_j | \mu_{X_{G_j}}(t,u,T_j) + \sigma^2_{X_{G_j}}(t,u,T_j,T_j), \sigma^2_{X_{G_j}}(t,u,T_j,T_j)) dx_j du \\ & -K \int_t^{T_0} e^{-r(u-t)} \int_{-\infty}^{\infty} \Phi(d_{eepK}(x_j,u)) \\ & \times n(x_j | \mu_{X_{G_j}}(t,u,T_j), \sigma^2_{X_{G_j}}(t,u,T_j,T_j)) dx_j du \Bigg], \end{aligned}$$

[6] A crack spread option such as heating oil/crude oil and RBOB gasoline/crude oil, which are traded on the NYMEX, are both American types.

and

$$
\begin{aligned}
d_{eepK}(x_j,u) &= -\frac{\ln(h_jB_je^{x_j}+B_KK)-\ln h_i-\mu_{X_{G_i}}(t,u,T_i)}{\sigma_{X_{G_i}}(t,u,T_i)\sqrt{1-\rho^2_{X_{G_i}X_{G_j}}(t,u,T_i,T_j)}} \\
&\quad +\frac{\rho_{X_{G_i}X_{G_j}}(t,u,T_i,T_j)\sigma_{X_{G_i}}(t,u,T_i)\frac{x_j-\mu_{X_{G_j}}(t,u,T_j)}{\sigma_{X_{G_j}}(t,u,T_j,T_j)}}{\sigma_{X_{G_i}}(t,u,T_i)\sqrt{1-\rho^2_{X_{G_i}X_{G_j}}(t,u,T_i,T_j)}}, \\
d_{eep1}(x_j,u) &= d_{eepK}(x_j,u)+\frac{(1-\rho^2_{X_{G_i}X_{G_j}}(t,u,T_i,T_j))\sigma^2_{X_{G_i}}(t,u,T_i,T_i)}{\sigma_{X_{G_i}}(t,u,T_i)\sqrt{1-\rho^2_{X_{G_i}X_{G_j}}(t,u,T_i,T_j)}}.
\end{aligned}
$$

Proof. See the Appendix. □

The formula is the same as that for the GS model. The only differences are drifts $\mu_{X_{G_i}}$ and volatilities $\sigma_{X_{G_i}X_{G_j}}$ are changed to $\mu_{X_{G_i}GS}$ and $\sigma_{X_{G_i}X_{G_j}GS}$, respectively.

Note that the preceding models such as Shimko (1994) and Nakajima and Maeda (2007) did not consider the linear relations between prices. However, empirical analyses such as those of Malliaris and Urrutia (1996) and Girma and Paulson (1999) show evidence of cointegration. Therefore, the commodity spread option should be priced by incorporating the cointegration or more generally a linear relation between log commodity prices. The valuation formulae for European and American call options that we derive in this subsection incorporate linear relations between commodity prices. These linear relations are interpreted as equilibrium or long-term relationships. Thus, the valuation formulae derived in this subsection reflect the long-term equilibrium in derivative pricing.

Note that a commodity spread option is an option on two commodity prices such that the spread relation is fixed within the contract. However, this fixed spread relation may be different from the linear relation corresponding to, say, cointegration, which we cannot observe and need to estimate. Furthermore, such differences between the spread relation and the linear relation may affect the spread option price. Thus, in order to price a spread option, it is not appropriate to start by directly assuming a stochastic process that a commodity spread must satisfy. It is important to start by formulating the stochastic processes that the commodity prices satisfy with some linear relation, and then to derive the price of the spread option.

5.3 Numerical Analysis

In this section, we numerically analyze the valuation formula of a European commodity future spread option. We use the following parameter values for crude oil (commodity 1) and heating oil (commodity 2) estimated in Chapter 4 as the benchmark.

$$
\begin{aligned}
&G_1(t, T_1) = 35, G_2(t, T_2) = 100,\\
&h_1 = 1, h_2 = 0.42,\\
&\sigma_{S_1} = 0.381896, \sigma_{S_2} = 0.406307, \sigma_{\delta_1} = 0.287109, \sigma_{\delta_2} = 0.699693,\\
&\rho_{S_1 S_2} = 0.748660, \rho_{S_1 \delta_1} = 0.767305, \rho_{S_1 \delta_2} = 0.000072,\\
&\rho_{S_2 \delta_1} = 0.628424, \rho_{S_2 \delta_2} = 0.620154,\\
&\rho_{\delta_1 \delta_2} = 0.165843,\\
&a_1 = -1.187431, a_2 = 1.000000, b_1 = -0.052615, b_2 = -0.356252,\\
&\kappa_1 = 1.140883, \kappa_2 = 1.085038,\\
&T_0 = 1250/250, T_1 = 1256/250, T_2 = 1266/250,\\
&K = 3, r = 0.04.
\end{aligned}
$$

We examine the valuation of the spread option using the GSC model and GS model. The effect of linear relation, or cointegration under certain conditions, can be seen by comparing the GSC model with the GS model. Although the linear relation may include two or more commodity prices, here we assume that there are only two commodity prices in the linear relation.

Figure 5.1 illustrates the theoretical prices of commodity spread options with futures prices. We can see that the prices of the GSC model are lower than in the GS model. This is because the cointegrated prices tend to revert to satisfy the long-term relationship and hence do not diverge.

Sensitivities of commodity spread option prices to σ_{S_1} and σ_{S_2} are shown in Figure 5.2. The price calculated by the GSC model exhibits a u-shaped curve for both σ_{S_i}. While the price in the GSC model is lower than in the GS model for σ_{S_1}, the situation changes for σ_{S_2} when it is volatile. Around the area of estimated parameters, the prices of spread options obtained by the GSC model are lower than those in the GS model. This implies that the cointegration relation is in effect. However, over a certain level of σ_{S_2}, the price of spread options in the GSC model is higher than that in the GS model.

Figure 5.1: Sensitivity of commodity spread option to future prices. The lower blue surface depicts the prices of commodity spread options obtained by the GSC model and the upper red surface depicts the prices obtained by the GS model.

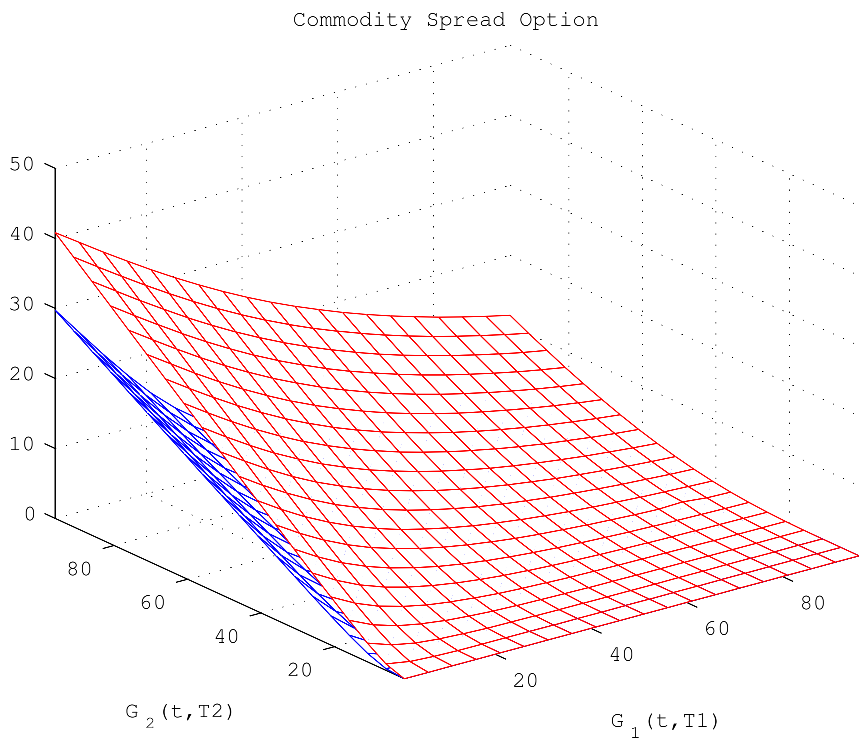
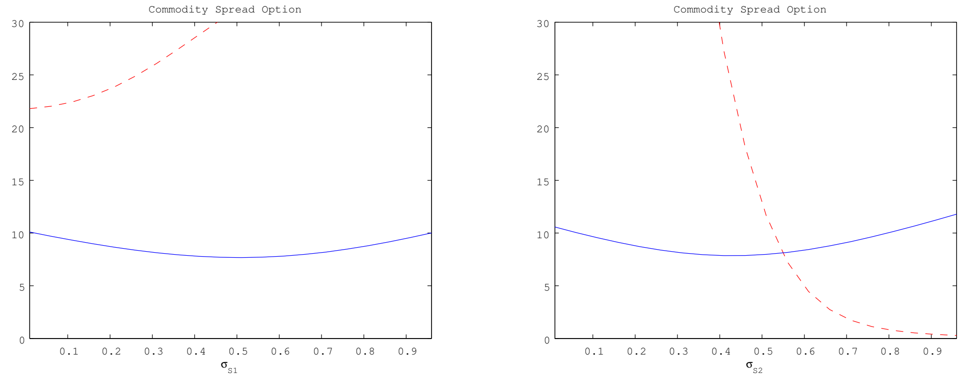
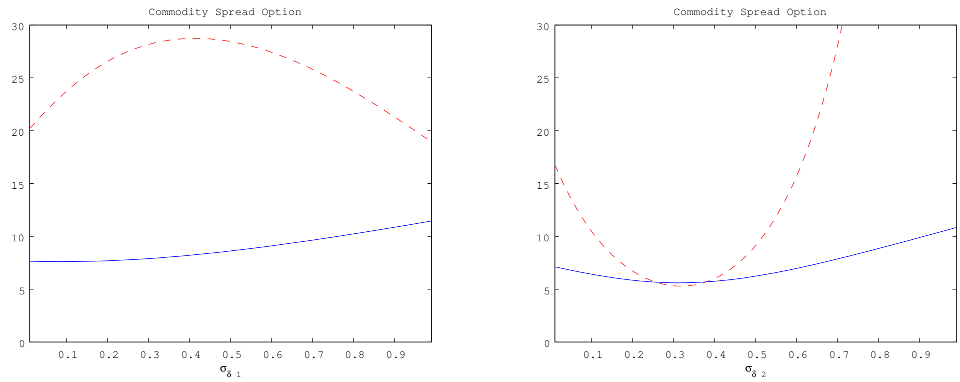


Figure 5.2: Sensitivity of commodity spread option to σ_{S_1} and σ_{S_2}. The blue solid line shows the prices of commodity spread options obtained by the GSC model and the red dashed line shows the prices by the GS model.



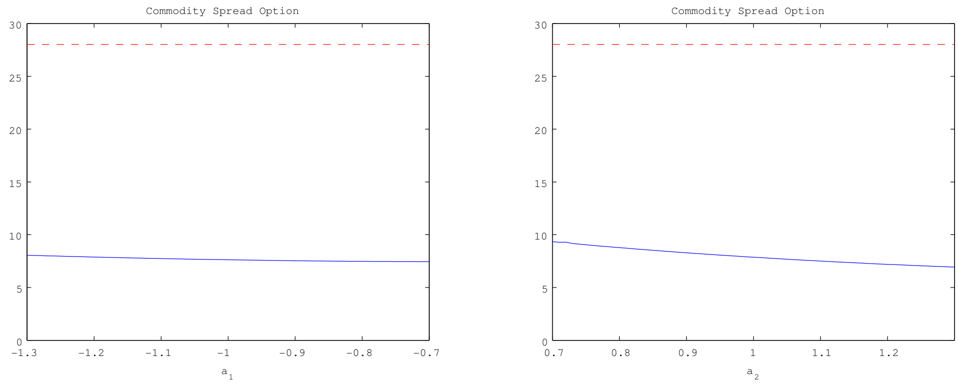
Figures 5.3, plot the price of commodity spread options against σ_{δ_1} and σ_{δ_2}. In most areas, the price obtained by the GSC model is lower than that of the GS model. For σ_{δ_2}, both prices exhibit u-shaped curves. However, the prices of the GSC and GS models are u-shaped and inverted u-shaped, respectively, for σ_{δ_1}. This implies that the cointegration relation changes the effect of volatility on the valuation of spread options.

Figure 5.3: Sensitivity of commodity spread option to σ_{δ_1} and σ_{δ_2}. The blue solid line shows the prices of commodity spread options obtained by the GSC model and the red dashed line shows the prices by the GS model.



The results for the linear relation vector (a_i) are presented in Figure 5.4. Naturally, the price obtained by the GS model, which does not incorporate the linear relation, is insensitive to (a_i). The prices generated by the GSC model are much lower than those of the GS model. In addition, as (a_i) increases, the price generated by the GSC model decreases.

Figure 5.4: Sensitivity of commodity spread option to a_1 and a_2. The blue solid line shows the prices of commodity spread options obtained by the GSC model and the red dashed line shows the price by the GS model.



In Figure 5.5, we report the sensitivity of commodity spread option prices with respect to (b_i). In both figures, the prices obtained by the GS model are higher than those of the GSC model. For b_1, the price exhibits a bell-shaped curve. On the other hand, as $|b_2|$ increases, the commodity spread option price decreases. This indicates that the price under long-term equilibrium is lower, since the higher the absolute values of adjustment parameters are, the more quickly the log spot prices converge to the linear relation.

Figure 5.6 shows the sensitivity to κ_i. Again, the prices calculated by the GSC model are lower than those of the GS model in most areas. The price in sensitivity analysis of κ_2 seems to converge to a certain level.

Finally, Figure 5.7 shows the sensitivity of price to maturity. For both models, the option price rises as maturity becomes longer. The prices obtained by the GSC model are lower than those of the GS model when the maturity is longer than 1.5 years. However, this relation reverses when the maturity becomes short, which is consistent with the result obtained by Casassus, Liu, and Tang (2011).

Notice that the larger the maturity is, the larger is the difference between the prices obtained by the GSC model and those of the GS model. This is because the cointegration relation prevents the commodity prices from diverging and hence makes the value of commodity spread options lower for

Figure 5.5: Sensitivity of commodity spread option to b_1 and b_2. The blue solid line shows the prices of commodity spread options obtained by the GSC model and the red dashed line shows the prices by the GS model.

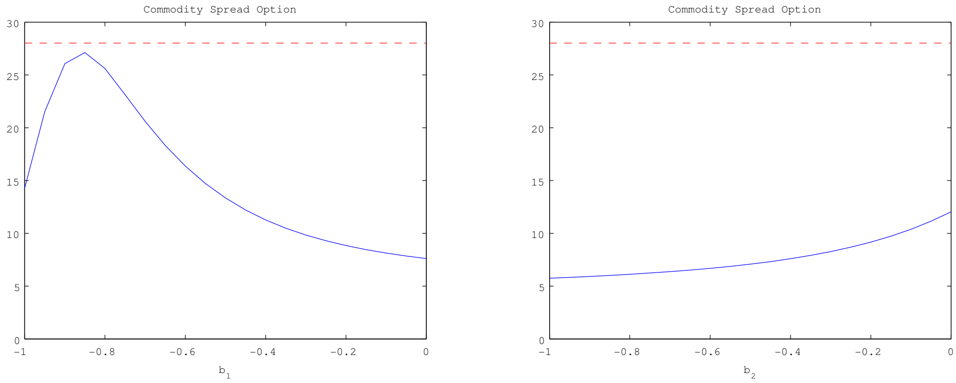
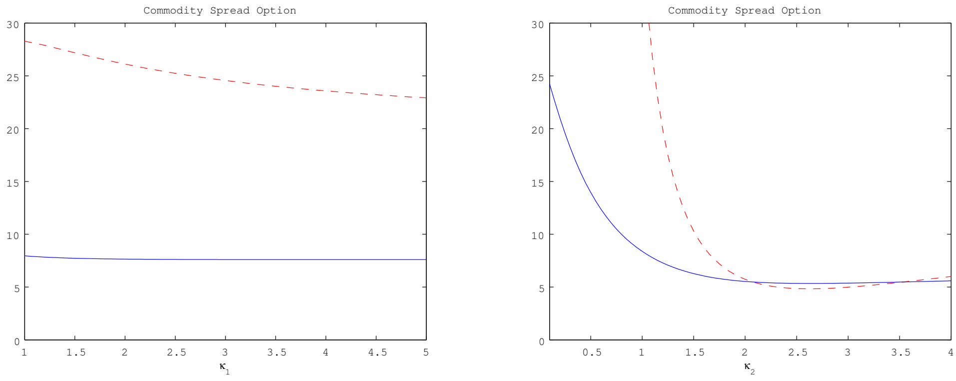
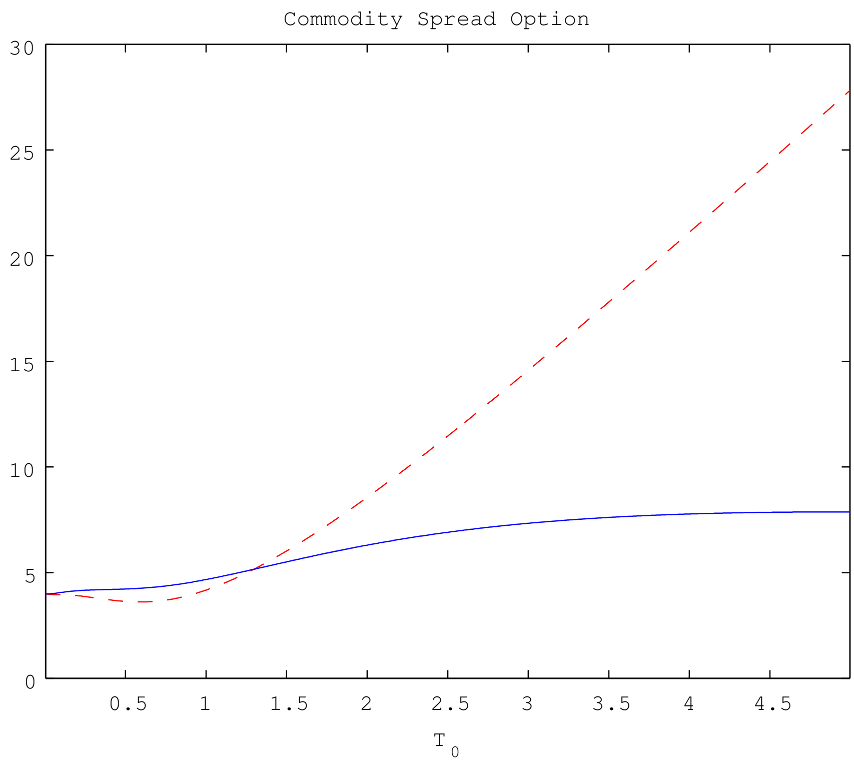


Figure 5.6: Sensitivity of commodity spread option to κ_1 and κ_2. The blue solid line shows the prices of commodity spread options obtained by the GSC model and the red dashed line shows the prices by the GS model.



longer maturities. However, again, this does not work for shorter maturity in both this numerical example and Casassus, Liu, and Tang (2011). This result implies that if cointegration exists, the GS model overprices the commodity spread option especially with longer maturities. Thus, it may be more appropriate to use the GSC model when valuing spread options with long maturity.

Figure 5.7: Sensitivity of commodity spread option to maturity. The blue solid line shows the prices of commodity spread options obtained by the GSC model and the red dashed line shows the prices by the GS model.



5.4 Conclusion

In this chapter, we derive a valuation formula for European call commodity spread options and an analytical approximation formula of American call commodity spread options when commodity prices are cointegrated based on the GSC model developed in Chapter 4. We also derive the valuation formulae of commodity spread options for the GS model, which does not take account of cointegration, and compare the results for the GS and the GSC models.

With numerical analysis, which uses the parameter values estimated in Chapter 4, we show that the prices of the commodity spread options given by the GSC model are lower than those of the GS model in most cases.

This is because cointegrated commodity prices tend to revert to the long-term equilibrium level and hence do not diverge, which lowers the spread and hence the value of the spread option. The GSC model captures this phenomenon.

We also analyze the sensitivities of the commodity spread option price to the change of several parameter values. Among them, we find that as the maturity becomes larger, the difference between the price obtained by the GS model and that by the GSC model becomes larger, where the former is larger than the latter. This is again because the cointegration relation prevents the commodity prices from diverging. However, we also find that for shorter maturity, the spread of price options obtained by the GS model is smaller than that of the GSC model, which is consistent with Casassus, Liu, and Tang (2011). This implies that the GS model may overprice the commodity spread options for the longer maturity without taking account of cointegration. Since the long-term commodity derivatives are often considered as useful tools to hedge risk against long-term projects such as oil mining, it may be more appropriate to use the GSC model that incorporates cointegration rather than the GS model when pricing long-term derivatives.

For future studies, further empirical analysis of derivative pricing that takes account of cointegration seems interesting. In addition, it is also interesting to see how cointegration affects prices of other types of derivatives such as a basket option, which is a generalization of a spread option. Incorporating other economic factors into the model, such as foreign exchange and/or interest rates seems an important direction for future research.

5.5 Appendices

5.5.1 Expectation and Covariances of Log Futures Return

In this subsection, we derive the futures price and expectation value and covariance of log futures return. We use the future price equation written in terms of spot price and derive the future price process using Ito's lemma written in terms of futures price. This price process can be explicitly written in terms of futures price levels. Finally, we calculate the expectation and covariance of stochastic terms of futures price using properties of stochastic calculus.

Note that futures price in terms of spot price is[7]

$$G_i(t,T) = e^{\mu_{X_i}(t,T) + \frac{\sigma^2_{X_i}(t,T)}{2}},$$

where

$$\begin{aligned}
\beta_{S_i0}(t) &= r - \frac{\sigma^2_{S_i}}{2} + b_i\mu_z + b_i a_0 t, \\
\beta_{S_iS_j} &= b_i a_j, \\
\beta_{S_i\delta_i} &= -1, \\
\beta_{\delta_i0} &= \kappa_i\hat{\alpha}_i, \\
\beta_{\delta_i\delta_i} &= -\kappa_i, \\
\boldsymbol{\beta}_0(t) &= [\beta_{S_10}(t), \cdots, \beta_{S_n0}(t), \beta_{\delta_10}, \cdots, \beta_{\delta_n0}]^\top, \\
\boldsymbol{\beta} &= \begin{bmatrix} \beta_{S_1S_1} & \cdots & \beta_{S_1S_n} & \beta_{S_1\delta_1} & & 0 \\ \vdots & \ddots & \vdots & & \ddots & \\ \beta_{S_nS_1} & \cdots & \beta_{S_nS_n} & 0 & & \beta_{S_n\delta_n} \\ & & & \beta_{\delta_1\delta_1} & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & \beta_{\delta_n\delta_n} \end{bmatrix},
\end{aligned}$$

[7]See Chapter 4 for derivation.

$$\begin{aligned}
\mu_{X_i}(t,T) &= E_t[\ln S_i(T)] \\
&= \left[e^{T\boldsymbol{\beta}} \left\{ e^{-t\boldsymbol{\beta}} \boldsymbol{X}(t) + \int_t^T e^{-s\boldsymbol{\beta}} \boldsymbol{\beta}_0(s) ds \right\} \right]_i, \\
\sigma_{X_i X_j}(t,T) &= E_t[(\ln S_i(T) - \mu_{X_i}(t,T))(\ln S_j(T) - \mu_{X_i}(t,T))] \\
&= \left[\int_t^T (e^{(T-s)\boldsymbol{\beta}}) \boldsymbol{\Sigma} (e^{(T-s)\boldsymbol{\beta}})^\top ds \right]_{ij}.
\end{aligned}$$

The partial derivatives are

$$\begin{aligned}
\frac{\partial G_i(t,T)}{\partial S_j(t)} &= \frac{\left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,j}}{S_j(t)} G_i(t,T), \\
\frac{\partial G_i(t,T)}{\partial \delta_j(t)} &= \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,n+j} G_j(t,T),
\end{aligned}$$

where we denote $[A]_{i,j}$ as $[i,j]$ th entry of matrix A.

Since the futures price $G_i(t,T)$ is a function of $S_i(t), \delta_i(t)$ and twice differentiable, we can use the Ito's lemma and the dynamics of future price is

$$dG_i(t,T) = \sum_{k=1}^n \sigma_{S_k} S_k(t) \frac{\partial G_i}{\partial S_k} dW_{S_k}(t) + \sum_{k=1}^n \sigma_{\delta_k} \frac{\partial G_i}{\partial \delta_k} dW_{\delta_k}(t),$$

where the drift term is 0 since $G_i(t,T)$ is martingale under the risk-neutral probability.

Again, using Ito's lemma we have,

$$\begin{aligned}
& d\log G_i(t,T) \\
= & -\frac{1}{2} \left\{ \sum_{k,l=1}^n \sigma^2_{S_k S_l} \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,k} \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,l} \right. \\
& + 2 \sum_{k,l=1}^n \sigma_{S_k \delta_l} \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,k} \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,n+l} \\
& \left. + \sum_{k,l=1}^n \sigma^2_{\delta_k \delta_l} \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,n+k} \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,n+l} \right\} dt \\
& + \sum_{k=1}^n \sigma_{S_k} \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,k} dW_{S_k}(t) + \sum_{k=1}^n \sigma_{\delta_k} \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,n+k} dW_{\delta_k}(t).
\end{aligned}$$

The futures price can be expressed as follows.

$$G_i(T_0, T_i) = G_i(t, T_i) e^{X_{G_i}(t, T_0, T_i)}, \quad t \le T_0 \le T_i,$$

where

$$\begin{aligned} X_{G_i}(t, T_0, T_i) \;\equiv\; & \mu_{X_{G_i}}(t, T_0, T_i) \\ & + \int_t^{T_0} \sum_{k=1}^{n} \sigma_{S_k} \left[e^{(T_i - t)\boldsymbol{\beta}} \right]_{i,k} dW_{S_k}(u) \\ & + \int_t^{T_0} \sum_{k=1}^{n} \sigma_{\delta_k} \left[e^{(T_i - t)\boldsymbol{\beta}} \right]_{i,n+k} dW_{\delta_k}(u). \end{aligned}$$

The expectation value is

$$\begin{aligned} & \mu_{X_{G_i}}(t, T_0, T_i) \equiv E_t[X_{G_i}(t, T_0, T_i)] \\ = \; & -\frac{1}{2} \Bigg\{ \int_t^{T_0} \sum_{k,l=1}^{n} \sigma^2_{S_k S_l} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,k} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,l} du \\ & +2 \int_t^{T_0} \sum_{k,l=1}^{n} \sigma_{S_k \delta_l} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,k} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,n+l} du \\ & + \int_t^{T_0} \sum_{k,l=1}^{n} \sigma_{\delta_k \delta_l} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,n+k} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,n+l} du \Bigg\}. \end{aligned} \tag{7}$$

The covariance of $\hat{X}_{G_i}(t, T_0, T_i)$ and $\hat{X}_{G_j}(t, T_0, T_j)$ is

$$\begin{aligned} & \sigma_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j) \equiv \mathrm{cov}_t[X_{G_i}(t, T_0, T_i), X_{G_i}(t, T_0, T_j)] \\ = \; & \int_t^{T_0} \sum_{k,l=1}^{n} \sigma_{S_k S_l} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,k} \left[e^{(T_j - u)\boldsymbol{\beta}} \right]_{j,l} du \\ & + \int_t^{T_0} \sum_{k,l=1}^{n} \sigma_{S_k \delta_l} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,k} \left[e^{(T_j - u)\boldsymbol{\beta}} \right]_{j,n+l} du \\ & + \int_t^{T_0} \sum_{k,l=1}^{n} \sigma_{S_l \delta_k} \left[e^{(T_j - u)\boldsymbol{\beta}} \right]_{j,l} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,n+k} du \\ & + \int_t^{T_0} \sum_{k,l=1}^{n} \sigma_{\delta_k \delta_l} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,n+k} \left[e^{(T_j - u)\boldsymbol{\beta}} \right]_{j,n+l} du. \end{aligned} \tag{8}$$

5.5.2 Proof of Proposition 5.2.4

In this subsection, we prove Proposition 5.2.4. This is done by the following scheme. What we need to calculate is the expectation which can be expressed in terms of double integrals, since we are only dealing with the bivariate Gaussian processes. We know the expectation and covariance of the stochastic parts as we mentioned previously. The integrals can be calculated using multivariate version of completing squares, decomposing bivariate normal joint distribution in to conditional distribution and marginal distribution, and changing of variables. Finally, collecting all the terms, we have the pricing equation.

From Harrison and Kreps (1979) and Harrison and Pliska (1981), the price of commodity spread option at t, which option maturity is T_0, futures maturity for G_i and G_j are T_i and T_j, respectively, is

$$C^E(G_i, G_j, t, T_0, T_i, T_j) = e^{-r(T_0-t)} E_t[(h_i G_i(T_0, T_i) - h_j G_j(T_0, T_j) - K)^+].$$

The expectation value can be calculated as follows.

$$\begin{aligned} & E_t[(h_i G_i(T_0, T_i) - h_j G_j(T_0, T_j) - K)^+] \\ = & \int_D (h_i G_i(t, T_i) e^{x_i} - h_j G_(t, T_j) e^{x_j} - K) n(\boldsymbol{x} | \boldsymbol{\mu_x}, \boldsymbol{\Sigma_x}) d\boldsymbol{x}, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ \boldsymbol{\mu_x} &= \begin{bmatrix} \mu_{X_{G_i}}(t, T_0, T_i) \\ \mu_{X_{G_j}}(t, T_0, T_j) \end{bmatrix}, \\ \boldsymbol{\Sigma_x} &= \begin{bmatrix} \sigma^2_{X_{G_i}}(t, T_0, T_i, T_i) & \sigma_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j) \\ \sigma_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j) & \sigma^2_{X_{G_j}}(t, T_0, T_j, T_j) \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} D &= \{\boldsymbol{x} | h_i G_i(t, T_i) e^{x_i} - h_j G_j(t, T_j) e^{x_j} - K \geq 0\} \\ &= \{\boldsymbol{x} | \ln(h_j G_j(t, T_j) e^{x_j} + K) - \ln(h_i G_i(t, T_i)) \leq x_i\} \\ &= \{\boldsymbol{x} | d(x_j) \leq x_i\}, \\ d(x_j) &\equiv \ln(h_j G_j(t, T_j) e^{x_j} + K) - \ln(h_i G_i(t, T_i)). \end{aligned}$$

We now calculate the integrals. Suppose that $\boldsymbol{e}_i$ is unit vector which the ith row is 1.

$$
\begin{aligned}
& \int_D e^{x_i} n(\boldsymbol{x}|\boldsymbol{\mu_x}, \boldsymbol{\Sigma_x}) d\boldsymbol{x} \\
= & \int_D (2\pi)^{-1} |\boldsymbol{\Sigma_x}|^{-\frac{1}{2}} \exp\{\boldsymbol{e}_i^\top \boldsymbol{x} - \frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu_x})^\top \boldsymbol{\Sigma_x}^{-1} (\boldsymbol{x} - \boldsymbol{\mu_x})\} d\boldsymbol{x} \\
= & \int_D (2\pi)^{-1} |\boldsymbol{\Sigma_x}|^{-\frac{1}{2}} \exp\{\boldsymbol{e}_i^\top \boldsymbol{\mu_x} + \boldsymbol{e}_i^\top (\boldsymbol{x} - \boldsymbol{\mu_x}) - \frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu_x})^\top \boldsymbol{\Sigma_x}^{-1} (\boldsymbol{x} - \boldsymbol{\mu_x})\} d\boldsymbol{x} \\
= & \int_D (2\pi)^{-1} |\boldsymbol{\Sigma_x}|^{-\frac{1}{2}} \exp\Bigg\{ \boldsymbol{e}_i^\top \boldsymbol{\mu_x} + \frac{1}{2} \boldsymbol{e}_i^\top \boldsymbol{\Sigma_x} \boldsymbol{e}_i \\
& - \frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu_x} - \boldsymbol{\Sigma_x}\boldsymbol{e}_i)^\top \boldsymbol{\Sigma_x}^{-1} (\boldsymbol{x} - \boldsymbol{\mu_x} - \boldsymbol{\Sigma_x}\boldsymbol{e}_i) \Bigg\} d\boldsymbol{x} \\
= & \exp\Bigg\{ \mu_{X_{G_i}}(t, T_0, T_i) + \frac{1}{2}\sigma^2_{X_{G_i}}(t, T_0, T_i, T_i) \Bigg\} \\
& \times \int_D (2\pi)^{-1} |\boldsymbol{\Sigma_x}|^{-\frac{1}{2}} \exp\{-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu_x} - \boldsymbol{\Sigma_x}\boldsymbol{e}_i)^\top \boldsymbol{\Sigma_x}^{-1} (\boldsymbol{x} - \boldsymbol{\mu_x} - \boldsymbol{\Sigma_x}\boldsymbol{e}_i)\} d\boldsymbol{x}.
\end{aligned}
$$

The integral can be expanded as follows. We omit the time prameters for simplicity.

$$
\begin{aligned}
& \int_D (2\pi)^{-1} |\boldsymbol{\Sigma_x}|^{-\frac{1}{2}} \exp\Bigg\{ -\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu_x} - \boldsymbol{\Sigma_x}\boldsymbol{e}_i)^\top \boldsymbol{\Sigma_x}^{-1} (\boldsymbol{x} - \boldsymbol{\mu_x} - \boldsymbol{\Sigma_x}\boldsymbol{e}_i) \Bigg\} d\boldsymbol{x} \\
= & \int_{-\infty}^{\infty} \int_{d(x_j)}^{\infty} (2\pi)^{-1} (\sigma_{X_{G_i}} \sigma_{X_{G_j}} \sqrt{1 - \rho^2_{X_{G_i} X_{G_j}}})^{-1} \exp\Bigg\{ -\frac{1}{2(1 - \rho_{X_{G_i} X_{G_j}})} \\
& \times \Bigg(\left(\frac{x_i - \mu_{X_{G_i}} - \sigma^2_{X_{G_i}})}{\sigma_{X_{G_i}}} \right)^2 - 2\rho_{X_{G_i} X_{G_j}} \left(\frac{x_i - \mu_{X_{G_i}} - \sigma^2_{X_{G_i}}}{\sigma_{X_{G_i}}} \right) \\
& \times \left(\frac{x_j - \mu_{X_{G_j}} - \sigma_{X_{G_i} X_{G_j}}}{\sigma_{X_{G_j}}} \right) + \left(\frac{x_j - \mu_{X_{G_j}} - \sigma_{X_{G_i} X_{G_j}}}{\sigma_{X_{G_j}}} \right)^2 \Bigg)^2 \Bigg\} dx_i dx_j
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{d(x_j)}^{\infty}(2\pi(1-\rho^2_{X_{G_i}X_{G_j}}))^{-\frac{1}{2}}\sigma^{-1}_{X_{G_i}} \\
&\quad\times\exp\left\{-\frac{\left(x_i-\mu_{X_{G_i}}-\sigma^2_{X_{G_i}}-\rho_{X_{G_i}X_{G_j}}\sigma_{X_{G_i}}\frac{x_j-\mu_{X_{G_j}}-\sigma_{X_{G_i}X_{G_j}}}{\sigma_{X_{G_j}}}\right)^2}{2(1-\rho^2_{X_{G_i}X_{G_j}})\sigma^2_{X_{G_i}}}\right\}dx_i \\
&\quad\times(2\pi)^{-\frac{1}{2}}\sigma^{-1}_{X_{G_j}}\exp\left\{-\frac{1}{2}\left(\frac{x_j-\mu_{X_{G_j}}-\sigma_{X_{G_i}X_{G_j}}}{\sigma_{X_{G_j}}}\right)^2\right\}dx_j \\
&= \int_{-\infty}^{\infty}\int_{-d_i(x_j)}^{\infty}(2\pi(1-\rho^2_{X_{G_i}X_{G_j}}))^{-\frac{1}{2}}\sigma^{-1}_{X_{G_i}} \\
&\quad\times\exp\left\{-\frac{y^2}{2}\right\}(1-\rho^2_{X_{G_i}X_{G_j}})^{\frac{1}{2}}\sigma_{X_{G_i}}dy \\
&\quad\times(2\pi)^{-\frac{1}{2}}\sigma^{-1}_{X_{G_j}}\exp\left\{-\frac{1}{2}\left(\frac{x_j-\mu_{X_{G_j}}-\sigma_{X_{G_i}X_{G_j}}}{\sigma_{X_{G_j}}}\right)^2\right\}dx_j \\
&= \int_{-\infty}^{\infty}\Phi(d_i(x_j))n(x_j|\mu_{X_{G_j}}+\sigma_{X_{G_i}X_{G_j}},\sigma^2_{X_{G_j}})dx_j,
\end{aligned}
$$

where

$$
d_i(x_j) = -\frac{d(x_i)-\mu_{X_{G_i}}-\sigma^2_{X_{G_i}}-\rho_{X_{G_i}X_{G_j}}\sigma_{X_{G_i}}\frac{x_j-\mu_{X_{G_j}}-\sigma_{X_{G_i}X_{G_j}}}{\sigma_{X_{G_j}}}}{\sigma_{X_{G_i}}\sqrt(1-\rho^2_{X_{G_i}X_{G_j}})},
$$

and we used change of variables in the third equation. Other integrals can be derived in the same manner. For the second integral,

$$
\begin{aligned}
&\int_D (2\pi)^{-1}|\boldsymbol{\Sigma}_{\boldsymbol{x}}|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_{\boldsymbol{x}}-\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{e}_j)^\top\boldsymbol{\Sigma}^{-1}_{\boldsymbol{x}}(\boldsymbol{x}-\boldsymbol{\mu}_{\boldsymbol{x}}-\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{e}_j)\right\}d\boldsymbol{x} \\
&= \int_{-\infty}^{\infty}\Phi(d_j(x_j))n(x_j|\mu_{X_{G_j}}+\sigma^2_{X_{G_j}},\sigma^2_{X_{G_j}})dx_j,
\end{aligned}
$$

where

$$
d_j(x_j) = -\frac{d(x_i)-\mu_{X_{G_i}}-\sigma_{X_{G_i}X_{G_j}}-\rho_{X_{G_i}X_{G_j}}\sigma_{X_{G_i}}\frac{x_j-\mu_{X_{G_j}}-\sigma^2_{X_{G_j}}}{\sigma_{X_{G_j}}}}{\sigma_{X_{G_i}}\sqrt(1-\rho^2_{X_{G_i}X_{G_j}})}.
$$

And the last integral is

$$\begin{aligned}
&\int_D (2\pi)^{-1}|\boldsymbol{\Sigma}_{\boldsymbol{x}}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_{\boldsymbol{x}})^\top \boldsymbol{\Sigma}_{\boldsymbol{x}}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_{\boldsymbol{x}})\right\} d\boldsymbol{x} \\
= &\int_{-\infty}^{\infty} \Phi(d_j(x_j)) n(x_j|\mu_{X_{G_j}}, \sigma^2_{X_{G_j}}) dx_j,
\end{aligned}$$

where

$$d_i(x_j) = -\frac{d(x_i) - \mu_{X_{G_i}} - \rho_{X_{G_i}X_{G_j}}\sigma_{X_{G_i}}\frac{x_j - \mu_{X_{G_j}}}{\sigma_{X_{G_j}}}}{\sigma_{X_{G_i}}\sqrt{(}1-\rho^2_{X_{G_i}X_{G_j}})}.$$

Collecting all terms, we have

$$\begin{aligned}
&C^E(G_i, G_j, t, T_0, T_i, T_j) \\
= \; & h_i G_i(t,T_i) \exp\left\{-r(T_0-t) + \mu_{X_{G_i}}(t,T_0,T_i) + \frac{1}{2}\sigma^2_{X_{G_i}}(t,T_0,T_i,T_i)\right\} \\
& \times \int_{-\infty}^{\infty} \Phi(d_i(x_j)) n(x_j|\mu_{X_{G_j}}(t,T_0,T_j) + \sigma_{X_{G_i}X_{G_j}}(t,T_0,T_i,T_j), \sigma^2_{X_{G_j}}(t,T_0,T_j,T_j)) dx_j \\
& - h_j G_j(t,T_j) \exp\left\{-r(T_0-t) + \mu_{X_{G_j}}(t,T_0,T_j) + \frac{1}{2}\sigma^2_{X_{G_j}}(t,T_0,T_j,T_j)\right\} \\
& \times \int_{-\infty}^{\infty} \Phi(d_K(x_j)) n(x_j|\mu_{X_{G_j}}(t,T_0,T_j) + \sigma^2_{X_{G_j}}(t,T_0,T_j,T_j), \sigma^2_{X_{G_j}}(t,T_0,T_j,T_j)) dx_j \\
& - K e^{-r(T_0-t)} \int_{-\infty}^{\infty} \Phi(d_K(x_j)) n(x_j|\mu_{X_{G_j}}(t,T_0,T_j), \sigma^2_{X_{G_j}}(t,T_0,T_j,T_j)) dx_j,
\end{aligned}$$

where

$$\begin{aligned}
d_K(x_j) = & -\frac{\ln(h_j G_j(t,T_j)e^{x_j} + K) - \ln(h_i G_i(t,T_i)) - \mu_{X_{G_i}}(t,t_0,T_i)}{\sigma_{X_{G_i}}(t,T_0,T_i,T_i)\sqrt{1-\rho^2_{X_{G_i}X_{G_j}}(t,T_0,T_i,T_j)}} \\
& + \frac{\rho_{X_{G_i}X_{G_j}}(t,T_0,T_i,T_j)\sigma_{X_{G_i}}(t,T_0,T_i,T_i)\frac{x_j-\mu_{X_{G_j}}(t,T_0,T_j}{\sigma_{X_{G_j}}(t,T_0,T_i,T_j)}}{\sigma_{X_{G_i}}(t,T_0,T_i,T_i)\sqrt{1-\rho^2_{X_{G_i}X_{G_j}}(t,T_0,T_i,T_j)}},
\end{aligned}$$

$$d_i(x_j) = d_K(x_j) + \frac{\sigma^2_{X_{G_i}}(t,T_0,T_i,T_i) - \rho^2_{X_{G_i}X_{G_j}}(t,T_0,T_i,T_j)\sigma^2_{X_{G_i}}(t,T_0,T_i,T_i)}{\sigma_{X_{G_i}}(t,T_0,T_i,T_i)\sqrt{1-\rho^2_{X_{G_i}X_{G_j}}(t,T_0,T_i,T_j)}},$$

$$\rho_{X_{G_i}X_{G_j}}(t,T_0,T_i,T_j) = \frac{\sigma_{X_{G_i}X_{G_j}}(t,T_0,T_i,T_j)}{\sigma_{X_{G_i}}(t,T_0,T_i,T_i)\sigma_{X_{G_j}}(t,T_0,T_j,T_j)}.$$

5.5.3 Analytical Approximation for American Commodity Spread Option

In this subsection, we propose an analytical approximation pricing formula for American commodity spread option. The difficulty is the calculation of early exercise premium, more precisely the domain of integration or the condition inequality of exercise which are not analytically tractable. Therefore, we use the scheme of Bjerksund and Stensland (1994) to approximate the condition inequality which split spread option in to two call option that the first option has stochastic exercise price and then use Barone-Adesi and Whaley (1987) framework to approximate the two American option. The formula can now be calculated as we did in European call option which derives the analytical approximated pricing formula for American commodity spread options.

From Broadie and Detemple (1997) the valuation of American spread options are[8]

$$\begin{aligned}
& C^A(G_i, G_j, t, T_0, T_i, T_j; B^{c,s}) \\
= & \sup_{\tau \in \mathcal{S}_{t,T_0}} E_t[e^{-r(\tau-t)}(h_i G_i(\tau, T_i) - h_j G_j(\tau, T_j) - K)^+] \\
= & C^E(G_i, G_j, t, T_0, T_i, T_j) + a^A(G_i, G_j, t, T_0, T_i, T_j; B^{c,s}),
\end{aligned}$$

where $\mathcal{S}_{t,T_0}$ is the class of stopping times of the filtration generated by the underlying the Brownian motion processes, the early exercise premium a^A is defined by

$$\begin{aligned}
& a^A(G_i, G_j, t, T_0, T_i, T_j; B^{c,s}) \\
= & E_t\left[\int_t^{T_0} e^{-r(u-t)}(rh_i G_i(u, T_i) - rh_j G_j(u, T_j) - rK)\right. \\
& \left. \times 1_{\{h_i G_i(u,T_i) \geq B^{c,s}(G_j(u,T_j),u)\}} du\right] \\
= & r\int_t^{T_0} e^{-r(u-t)} E_t[(h_i G_i(u, T_i) - h_j G_j(u, T_j) - K) \\
& \times 1_{\{h_i G_i(u,T_i) \geq B^{c,s}(G_j(u,T_j),u)\}}]du,
\end{aligned}$$

[8]See also Detemple (2006), Section 6.4.

and $B^{c,s}(\cdot,\cdot)$ is a solution to the integral equation

$$B^{c,s}(G_j(t,T_j),t) - K = C^E(G_i, B^{c,s}, t, T_0, T_i, T_j) + a^A(G_i, B^{c,s}, t, T_0, T_i, T_j; B^{c,s}),$$

subject to

$$\begin{aligned}
\lim_{t\to T} B^{c,s}(G_j(t,T_j),t) &= \max(G_i(t,T_i)+K, G_j(t,T_j)+K),\\
B^{c,s}(0,t) &= B^j(t),\\
B^j(t) &= \inf\{G_j(t,T_j) : C^A(G_i, G_j, t, T_0, T_i, T_j)\\
&\qquad = (h_i G_i(\tau,T_i) - h_j G_j(\tau,T_j) - K)^+\}.
\end{aligned}$$

Now, we addopt the framework of Bjerksund and Stensland (1994)[9] to approximate the early exercise premium.

$$\begin{aligned}
& E_t[(h_i G_i(u,T_i) - h_j G_j(u,T_j) - K)1_{\{h_i G_i(u,T_i) \geq B^{c,s}(G_j(u,T_j,u)\}}]\\
\approx\ & E_t[(h_i G_i(u,T_i) - h_j G_j(u,T_j) - K)1_{\{h_i G_i(u,T_i) \geq h_j B_j G_j(u,T_j) + B_K K\}}],
\end{aligned}$$

where

$$\begin{aligned}
B_j &= B(T_0 - u, \sigma^2_{X_{G_i}}(u,T_0) - 2\sigma_{X_{G_i}X_{G_j}}(u,T_0) + \sigma^2_{X_{G_j}}(u,T_0)),\\
B_K &= B(T_0 - u, \sigma^2_{X_{G_i}}(u,T_0)),\\
B(t,\sigma^2) &= e^{h(t,\sigma^2)} + (1 - e^{h(t,\sigma^2)}) B_\infty(\sigma^2),\\
B_\infty(\sigma^2) &= \frac{\beta(\sigma^2)}{\beta(\sigma^2) - 1},\\
\beta(\sigma^2) &= \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2r}{\sigma^2}},\\
h(t,\sigma^2) &= -2\sigma\sqrt{t}(\beta(\sigma^2) - 1).
\end{aligned}$$

The approximation is constructed in two steps. The first step is to split the spread option into an exchange option and vanilla type option. And the second step is due to Barone-Adesi and Whaley (1987) framework of approximating American option.

Note that the exercise region is

$$\begin{aligned}
& h_i G_i(u,T_i) \geq h_j B_j G_j(u,T_j) + B_K K\\
& \Leftrightarrow x_i \leq d_{eep}(x_j,u) \equiv \ln(h_j B_j G_j(t,T_j) e^{x_j} + B_K K) - \ln(h_i G_i(t,T_i)).
\end{aligned}$$

[9]They have also analyzed the performance of their approach.

The integrals of early exercise premium can be calculated just as the integral of European commodity spread option which we now have

$$
\begin{aligned}
& a^A(G_i, G_j, t, T_0, T_i, T_j; B^{c,s}) \\
= \;& r\int_t^{T_0} e^{-r(u-t)} E_t[(h_i G_i(u,T_i) - h_j G_j(u,T_j) - K) \\
& \times 1_{\{h_i G_i(u,T_i) \geq B^{c,s}(G_j(u,T_j,u)\}}] du \\
\approx \;& r\int_t^{T_0} e^{-r(u-t)} E_t[(h_i G_i(u,T_i) - h_j G_j(u,T_j) - K) \\
& \times 1_{\{x_i \leq d_{eep}(x_j,u)\}}] du \\
= \;& r\Bigg[h_i G_i(t,T_i) \int_t^{T_0} \exp\left\{ -r(u-t) + \mu_{X_{G_i}}(t,u,T_i) + \frac{1}{2}\sigma^2_{X_{G_i}}(t,u,T_i,T_i) \right\} \\
& \times \int_{-\infty}^{\infty} \Phi(d_{eep1}(x_j,u)) \\
& \times n(x_j | \mu_{X_{G_j}}(t,u,T_j) + \sigma_{X_{G_i}X_{G_j}}(t,u,T_i,T_j), \sigma^2_{X_{G_j}}(t,u,T_j,T_j)) dx_j du \\
& - h_j G_j(t,T_j) \int_t^{T_0} \exp\left\{ -r(u-t) + \mu_{X_{G_j}}(t,u,T_j) + \frac{1}{2}\sigma^2_{X_{G_j}}(t,u,T_j,T_j) \right\} \\
& \times \int_{-\infty}^{\infty} \Phi(d_{eepK}(x_j,u)) \\
& \times n(x_j | \mu_{X_{G_j}}(t,u,T_j) + \sigma^2_{X_{G_j}}(t,u,T_j,T_j), \sigma^2_{X_{G_j}}(t,u,T_j,T_j)) dx_j du \\
& - K \int_t^{T_0} e^{-r(u-t)} \int_{-\infty}^{\infty} \Phi(d_{eepK}(x_j,u)) \\
& \times n(x_j | \mu_{X_{G_j}}(t,u,T_j), \sigma^2_{X_{G_j}}(t,u,T_j,T_j)) dx_j du \Bigg],
\end{aligned}
$$

where

$$
\begin{aligned}
d_{eepK}(x_j,u) \;=\; & -\frac{\ln(h_j B_j G_j(t,T_j) e^{x_j} + B_K K) - \ln(h_i G_i(t,T_i)) - \mu_{X_{G_i}}(t,u,T_i)}{\sigma_{X_{G_i}}(t,u,T_i)\sqrt{1 - \rho^2_{X_{G_i}X_{G_j}}(t,u,T_i,T_j)}} \\
& + \frac{\rho_{X_{G_i}X_{G_j}}(t,u,T_i,T_j)\sigma_{X_{G_i}}(t,u,T_i)\frac{x_j - \mu_{X_{G_j}}(t,u,T_j)}{\sigma_{X_{G_j}}(t,u,T_j,T_j)}}{\sigma_{X_{G_i}}(t,u,T_i)\sqrt{1 - \rho^2_{X_{G_i}X_{G_j}}(t,u,T_i,T_j)}},
\end{aligned}
$$

$$
\begin{aligned}
d_{eep1}(x_j, u) \quad = \quad & d_{eepK}(x_j, u) \\
& + \frac{\sigma^2_{X_{G_i}}(t, u, T_i, T_i) - \rho^2_{X_{G_i} X_{G_j}}(t, u, T_i, T_j)\sigma^2_{X_{G_i}}(t, u, T_i, T_i)}{\sigma_{X_{G_i}}(t, u, T_i)\sqrt{1 - \rho^2_{X_{G_i} X_{G_j}}(t, u, T_i, T_j)}}.
\end{aligned}
$$

Chapter 6

Conclusion

This thesis studied pricing models that incorporated relations among commodity prices. In Chapter 2, we characterized relations of emission allowance prices and commodity spot prices through profit maximization of a firm. We derived the inter- and intratemporal relations among the emission allowance price and commodity prices. That is, the emission allowance price was expressed as the spread between other commodity spot prices, and the emission allowance price at time t was the present value of the emission allowance price at the end of period T. Moreover, we analyzed how the relative hedge ratio would change as the relative prices of input commodities changed.

In Chapter 3, we derived a valuation formula for emission allowance as a derivative of two commodities. Specifically, we assumed that emission allowance price at time t was the present value of emission allowance at the end of period T, and that at T the emission allowance price was equal to the minimum of spread of two commodity prices and penalty if it was positive or otherwise 0. We characterized the values of options embedded in emission allowances and derived the formulae for emission allowance futures and options. We also calibrated the model using real market data and numerically analyzed the behavior of the hedge ratios of emission allowance futures by commodity (e.g., electricity and natural gas) futures. We found that the electricity and natural gas price could explain emission allowance price to some extent. From the numerical analysis using the calibrated model, we found that the option values for the price ceiling by penalty embedded in emission allowances was relatively large, which implied that the penalty was an important component in evaluating emission allowances.

In Chapter 4, we formulated a commodity pricing model that incorporated the effect of linear relations among logarithms of commodity prices, which included cointegration under a certain condition. More specifically, we formulated a commodity pricing model in which the temporary deviation of drift terms from the risk-free rate under a risk-neutral probability was described by convenience yields and linear relations among log commodity prices, which corresponded to error terms under an appropriate condition. We derived futures and call option pricing formulae and showed that, in contrast to Duan and Pliska (2004), the linear relations among log commodity prices, or the error term under appropriate conditions, should affect these derivative prices in the standard setup of commodity pricing. Furthermore, we provided a sufficient condition for the model to be cointegrated, which had not been proposed by any other related paper. Using crude oil and heating oil market data, we estimated the proposed model. We also implemented the model to examine the hedging of long-term futures using short-term futures.

In Chapter 5, we developed a model of commodity spread option with cointegration based on Chapter 4. We derived the futures price, the valuation formula of European call commodity spread option, and analytical approximation formula of American call commodity spread option with two models; the Gibson-Schwartz model and the model in Chapter 4. We also compared the model numerically with the Shimko (1994) model which applied the Gibson-Schwartz (1990) model to commodity spread options. From numerical analysis, the price of commodity spread option for long maturity given by the Gibson-Schwartz spread option model was much higher than that of our spread option model. Also, we indicated that the Gibson-Schwartz spread option model might overprice option values when pricing long-term maturity commodity spread options.

There are many subjects left for future research. Modifying the model in this thesis will be one reasonable direction for research. Regarding emission allowances, we may incorporate some factors that we did not consider in our model, such as decision on penalty and allocation among countries and effect of economic growth. Also, we may investigate a model in which the emission allowance price at the end of the period is determined in a different way to that assumed in this paper. Moreover, incorporating different characteristics of price processes such as seasonality, jumps, or stochastic volatility, is another area for future research.

As for commodity pricing with cointegration, it would be interesting to formulate a model that incorporates the linear relations among observable futures prices instead of unobservable spot prices, and analyze the effects of the linear relation on derivatives. Note also that as Duan and Pliska (2004) shows, if volatilities of commodity returns are stochastic, then the linear relation affects derivative prices for general assets without convenience yields. Hence, it would be also interesting to enhance a commodity derivative pricing model to incorporate linear relations among logarithms of spot prices under stochastic volatility of their returns. Furthermore, if there are linear relations among commodity prices, a portfolio strategy using these relations should be an interesting issue. Indeed, Girma and Paulson (1999), Simon (1999), and Emery and Liu (2002) have analyzed the possibility of relations and trading strategy with commodities. We may incorporate linear relations among prices or log prices and analyze the profits of trading strategies using these relations.

Our original motivation for this research was to incorporate demand and supply effect into a commodity derivative model. In preceding models, this was the motivation for introducing convenience yield process (Gibson and Schwartz, 1990). For this, it may be natural to develop a general equilibrium model of commodity prices. We may reinterpret convenience yield in the equilibrium model and the valuation of commodity derivative may be reconstructed in terms of supply and demand.

Finally, we assumed the linear relations with constant coefficients in Chapter 3, 4, and 5. However, these coefficients may vary dynamically and/or stochastically as implied by Chapter 2. For example, the parameters may depend on time, and the error term may depend on maturity. In this way, we may describe long-term relation as dynamic equilibrium which should affect derivative pricing. This concept of dynamic equilibrium may be more appropriate to allow for technological changes such as usage of new energy sources. These topics are laid for further studies.

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