CCES Discussion Paper Series
Center for Research on Contemporary Economic Systems

Graduate School of Economics
Hitotsubashi University

CCES Discussion Paper Series, No.48
July 2013

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Securing Basic Well-being for All*

Reiko Gotoh† and Naoki Yoshihara‡

This Version: July, 2013

Abstract

The purpose of this paper is to examine the possibility of a social choice rule to implement a social policy for "securing basic well-being for all." For this purpose, the paper introduces a new scheme of social choice, called a *social relation function* (**SRF),** which associates to each profile of *individual well-being appraisals* and each profile of *group-evaluations* a reflexive and transitive binary relation over the set of social policies. As a part of the domains of **SRF**s, the available class of group evaluations is constrained by the following three conditions: **Basic Well-being Condition**, **Restricted Monotonicity**, and **Refrain Condition**. Furthermore, two axioms, the *non-negative response* (**NR**) and the *weak Pareto* condition (**WP**), are introduced as the two basic condititions of **SRF**s. **NR** demands giving priority to the evaluations of disadvantage groups, while treating them as formally equal relative to each other. **WP** requires treating impartially the well-being appraisals of all individuals. In conclusion, this paper shows that, under some reasonable assumptions, there exists a **SRF** which satisfies **NR** and **WP**.

JEL: D63 (Equity, Justice, Inequality, and Other Normative Criteria and Measurement).

Keywords: basic well-being; individual well-being appraisals; social relation functions.

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*We are extremely grateful to Marc Fleurbaey, James Foster, Peter Hammond, Carmen Herrero, Prasanta Pattanaik and Yongsheng Xu for their fruitful discussions relevant to this paper.

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1 Introduction

The "Convention on the Rights of Persons with Disabilities," which was adopted by the United Nations in 2006, has brought about new insights on human rights and democracy. As is well known, universal human rights have been established since 1948. Yet, persons with disabilities have long been restricted in their effective exercise of human rights. The Convention is innovative in that it requires the effective exercise of human rights for persons with disabilities through, for example, removing discriminatory practices that have built up over time and implementing "reasonable accommodations" in public places.[1] Moreover, what is a remarkable aspect of the procedure used to draft this convention is the role that persons with disabilities have played in assessing alternative articles as experts on their own disabilities. As indicated by the slogan "nothing about us, without us," persons with disabilities have taken the initiative in evaluating alternative articles.

This example urges us to reconsider the appropriateness of the standard framework of social choice theory. For there is little discussion about what types of asymmetrical prior treatments of individual preferences are admissible in what types of social choice problems. In contrast, the above example indicates that the asymmetrical prior treatment of individual preferences could be appropriate when the social choice problem in question is on the effective exercise of universal human rights respective to the particularity of those individuals. The main purpose of this paper is to formulate a social choice procedure which permits asymmetrical prior treatments for disadvantaged groups not as exceptions but as a general rule under some reasonableand socially imposed conditions. More specifically, we focus on a specific type of social choice problem: selecting a public policy in terms of "securing basic well-being for all" and defines the concept of a "group" as a representation of particularity which requires an asymmetrical prior treatment in order to achieve universal aims such as "securing basic well-being for all."

The basic framework of this paper is as follows. Firstly, in this paper we specify no particular type of well-being indicator and allows various types of well-being indicator including "capability"a la Amartya Sen as a typical

[1] Convention on the Rights of Persons with Disabilities (Article 2) defines "reasonable accommodation" as follows: "reasonable accommodation means necessary and appropriate modification and adjustments not imposing a disproportionate or undue burden, where needed in a particular case, to ensure to persons with disabilities the enjoyment or exercise on an equal basis with others of all human rights and fundamental freedoms".

example,[2] while we presume that the contents of individual well-being are multi-dimensional in the space of plural attributes and can be varied by social policies, which typically involve allocations of transferable goods, together with non-transferable personal abilities.[3] The paper also refers to "basic well-being", which means the well-being content that one can legitimately claim to have met by publicly provided goods and services.[4]

Secondly, a social choice rule to select a public policy for "securing basic well-being for all" is introduced and examined. This social choice rule, which we call the *social relation function* (SRF), is defined as having three elements as its informational basis. As argued above, the profile of each individual well-being is identified corresponding to each alternative social policy. Thus, each individual *appraises* her own well-being content based on her own conception of the good, whereas each "group" *appraises* its members' well-beings. Moreover, the group also *evaluates* alternative social policies by taking seriously *its least advantaged members* who are identified based on the appraisal of the group. The individual appraisal is formulated as a binary relation on the space of well-being, whereas the group appraisal is formulated as the intersection of its members' appraisals. Finally, the group evaluation is formulated as a binary relation over social alternatives with the appraisal of that group on the space of well-being. By linking these three elements of information, the role of **SRF**s is to form a social evaluation for an alternative, which is formulated as a binary relation over social alternatives.[5]

[2] The capability of an individual is defined as a set of "functionings" (vectors of various kinds of "doings" and "beings": e.g., nourishing, moving, communicating, having education etc.), which can be realized via various social policies through transforming transferable resources (e.g., by changing social policies) together with non-transferable personal abilities. For example, knowing about universal human rights such as "everyone has the right to recognition everywhere as a person before the law" (Article 6, Universal Declaration of Human Rights) can be counted as one functioning.

[3] For a detailed discussion of evaluative attributes, see Pattanaik and Xu (2007, 2012). Also see Fleurbaey (2007) and Fleurbaey and Hammond (2004) for a discussion of well-being indicators.

[4] For instance, if the well-being indicator is specified by "capability," basic well-being implies "basic capability" (Sen, 1980, p.367). Basic capability refers to lists, scales, and sizes of capability that one can legitimately claim to have met by publicly provided goods and services, preserving the person's freedom to choose functionings from within the opportunity set.

[5] It should be noted that the multi-component structure itself is not original to this paper. In fact, as pointed out by d'Aspremont and Gevers (2002), the framework of Social Welfare Functionals (SWFLs) proposed by Sen enables us to bring a multi-dimensional

Some remarks on the **SRF** framework are necessary. First, because of the multiplicity of attributes for well-being, generally, types of disadvantages may be diversified, which could generate different types of "the least advantaged."[6] Therefore, although these different types of "the least advantaged" share the common feature that they lack access to "basic well-being," the concrete content (the lists, scales, and sizes) of "basic well-being" might be different.[7] Therefore, in this paper, the concept of "group" is operationally defined as a maximum unit that can share a concrete content of "basic well-being" and can identify "the least advantaged" within the group.

Second, due to the multi-dimensionality of well-being contents, the individual and group appraisals could be *incomplete,*[8] which implies that intra-personal full comparability of these appraisals cannot be generally presumed, while inter-personal comparability of these appraisals can be legitimately presumed to some extent at least in a group. In fact, to what extent we should assume inter-personal and intra-personal comparability is one of our research questions.

Given these remarks, a prominent feature of the **SRF** framework is the introduction of group appraisals and group evaluations in addition to individual well-being appraisals. Due to this three component-structure of the informational basis, **SRF**s allow the appropriate asymmetric and prior treatment of specific groups of individuals relevant to the underlying social choice problem in question as well as the symmetric treatment of individual

structure into a social choice rule. According to this viewpoint, there seems to be a correspondence between the framework of our SRFs and that of SWFLs in that, for instance, the individual appraisal of well-being in our SRF can be interpreted as the individual welfare function in a SWFL. Yet, the latter is assumed to be a complete relation and numerically representable, while this not necessarily the case for the former.

[6] Our framework follows John Rawls's difference principle in the sense that securing basic well-being of the least advantaged respective to each policy is necessary and sufficient for achieving the social goal of "securing basic well-being for all." Yet, although Rawls's model assumes inter-personal level-comparability for society as a whole, our model starts from the possibility of different types of "the least advantaged" derived from different types of disadvantages.

[7] For instance, when we consider well-being as capabilities, the lists, scales, and sizes of functionings constituting "basic capability" might be different among different types.

[8] Note that the line of research on ranking opportunity sets initiated by Pattanaik and Xu (1990) also does not presume completeness of binary relations over opportunity sets. However, those relations are typically shown to be complete as a consequence of the combinations of the axioms they regard as proper requirements.

appraisals.[9]

To incorporate this idea formally, we introduce two basic axioms, the *non-negative response* (**NR**) and the *weak Pareto* (**WP**) axioms for **SRF**s within this framework. **NR** requires that **SRF**s should give priority to a disadvantaged group's evaluation whenever any other groups' evaluations are not completely opposite to this group's, while **WP** requires that **SRF**s should treat every individual's appraisal symmetrically.

Given the possibility of a prior treatment of disadvantaged groups, it seems sensible to introduce domain conditions of group evaluations to restrict groups' "decisive-powers" so as not to depart from the general societal goal of "basic well-being for all." We call the domain conditions the *"basic well-being condition,"* *"restricted monotonicity,"* and the *"refrain condition."* These conditions together stipulate that any specific disadvantaged group should evaluate social state x as "more just" than social state y whenever (i) its least advantaged members' well-being contents under x (*resp.* y) are better or at least not worse (*resp.* not better or even worse) than basic well-being; or (ii) its least advantaged members' well-being contents are better under x than under y, given that their respective well-being contents are worse under x and y respectively than basic well-being. Moreover, this group should refrain from comparing x and y whenever its least advantaged members' well-being contents are better under x and y respectively than basic well-being.

If a **SRF** satisfies the above axioms and conditions, it is deemed desirable as a societal goal. However, it is not obvious whether there exists a **SRF** satisfying the three domain conditions as well as the two basic axioms. Indeed, the existence problem of **SRF**s having such properties may have some similarity to the dominance and context-dependence paradox observed by Pattanaik and Xu (2007; 2012). As a typical example of this kind of paradox, recall the Pareto-liberal paradox initiated by Sen (1970) which points out the incompatibility of minimal liberty and the Pareto principle, where the former is formulated as the local decisiveness of some individuals, while the latter is formulated as the global decisiveness of all individuals. Incidentally, in our framework, the three conditions of group evaluations and the non-negative response together imply that a disadvantaged group is given lo-

[9]Indeed, if we formulate analogically to the standard Arrovian framework a social choice rule for social policies to determine well-being assignments, it seems sufficient to collect the information of each individual's well-being appraisal, and then associate a social evaluation over policies to the collected information of individual appraisals. However, such a formulation treats all individuals simply symmetrically.

cally decisive power, in a weak sense,[10] over the specific pairs of alternatives, where at least one of these alternatives does not allow every member of this group access to basic well-being.

Compared to the original Pareto-liberal paradox, the existence issue of **SRF**s is more subtle and non-obvious, since, firstly, the locally decisive power of a disadvantaged group is much weaker than the standard notion of local decisive power discussed in Arrow (1951/1963) and Sen (1970); and secondly, the domain of **SRF**s is not universal but restricted by the three conditions. However, among other things, the key factor of this existence issue is the incompleteness of binary relations as the informational basis of **SRF**s.[11] In fact, our paper shows the extent to which the incompleteness of group appraisals is acceptable so as not to rule out the existence of **SRF**s which are compatible with **NR** and **WP**: if some non-comparable parts of group appraisals remain due to the lack of external scrutiny of their members' well-beings, no **SRF** can satisfy both of **NR** and **WP** simultaneously; otherwise, there exists a **SRF** satisfying **NR** and **WP**. For example, if some non-comparable parts of group appraisals are essentially not "assertive" but "tentative," in that they can be changed to comparable parts via a further scrutiny of disadvantaged members' states, it becomes possible to have a **SRF** satisfying **NR** and **WP**.[12]

In the following discussion, Section 2 provides remarks on the concept of "group" with regard to the aim of "securing basic well-being for all." Section 3 provides the basic **SRF** framework and Section 4 the three conditions for group evaluations and the relevant two axioms. Section 5 discusses the existence problem of **SRF**s satisfying these properties. Finally, Section 6 provides some philosophical implications of the approach of this paper.

[10]The intention of "in a weak sense" here is that this group's 'local decisiveness' over such pairs is conditonal on there being no resistance of any other groups.

[11]Incompleteness of binary relations as the informational basis is not assumed in the context of the Pareto-liberal paradox as well as other types of the dominance and context-dependence paradox discussed by Pattanaik and Xu (2012).

[12]The notions of "tentative incompleteness" and "assertive incompleteness" are due to Sen (2002). See Section 6 for more details on this point.

2 Why group? —Philosophical Inquiry into the Comparability of Capability—

The aim of "securing basic well-being for all" reflects the spirit of universality and equality. To realize this aim substantively, we must particularly take care of differences among individuals in their contents of well-being. It is, however, almost impossible to treat different types of individuals differently, while treating the same type of individuals equally. Given this difficulty, we are faced with the issue of what kind of mechanism can take the difference of individuals into account. Friedrich Hayek gave a clear answer to this question by stating that only markets can do that, because it is each individual who truly knows and can satisfy his diverse needs. Yet, we cannot rely solely on markets, since the market mechanism *per se* does not necessarily guarantee the basic well-being of all participants. To resolve this difficulty, we introduce the notion of "groups." This notion is defined as the representation of any particularity with which society should concern itself. That is, the differences of individuals in the same group can be compared, but the differences of individuals in different groups cannot be compared to one another.

Assuming three types of disadvantages, this section illustrates the difficulties in making trans-group comparisons and in identifying the least advantaged in society as a whole. The three types of disadvantages can be seen as corresponding to three different conceptions of justice that underlie the reasons and the ways that a society should compensate individuals' disadvantages.

The first type of disadvantage is closely related to what Aristotle called "justice as redress." It is based on recognizing the cause of the suffered disadvantage as an injustice that needs to be redressed and the responsibility of society as a whole is seen as engaged in this process. Public repayment represents an idea that it is society's responsibility not to repeat such injustice in the future. Examples are disadvantages that derive from historical injustices such colonial exploitation or the treatment of indigenous populations; or disadvantages suffered by victims of disasters and crime.

The second type of disadvantage is related to the conception of "justice as compensation." This concept implies that some individuals should be recognized as disadvantaged if their vulnerability is due to the failure of social institutions to protect them from social discrimination, such as persons with disabilities, particular diseases, or on the basis of age, nationality, gender, or being a single parent, rather than due to the natural characteristics of individuals as such.

Finally, the third type of disadvantage relates to the concept of "justice as protection." This concept considers it unjust that individuals exist that have less than is necessary for a minimum standard of wholesome and cultured living, even if such individuals are not regarded as disadvantaged in terms of the first or the second type of disadvantage.[13] Redressing this requires a form of outcome-equality to bring every individual up to a reference point. This concept focuses on individuals, unlike the first two concepts, whose specific causes of difficulties can be hard to identify.

Because of this diversity of disadvantages and of the forms of justice underlying them, the concrete conceptions of "basic well-being" become plural. Take for example, individuals who have suffered disadvantages as a result of having been victims of an atomic bomb. This event, as many say, has completely changed their life plans and goals, and they have decided to live as witnesses of this social disaster in order to prevent it from ever happening again at any other place or time. In such cases, air tickets to fly to New York, which holds the "No more Hiroshima/Nagasaki Congress," or a grant for publishing their memoirs may be counted as a necessity for securing their basic capability. This suggests that, under a common concept of "basic well-being," special needs must be addressed relative to the different types of disadvantages.

Lastly, it should be noted that an individual might actually suffer from all three types of disadvantages mentioned above and as a result will be included in all three types of groups. This implies that such an individual's basic well-being consists of three aspects which cannot be compared intra-personally, while each of the three aspects permits inter-personal comparison within each group. In this case, the individual can participate in the process of making an evaluation of each group, and moreover, deserves taking advantage of social policies which are chosen in terms of all three types of disadvantages, though the actual amount of provision might be reduced considering combination effects of the three policies. Of course, the three types of disadvantages do not necessarily completely characterize such an individual's personality. Individuals have the freedom to evaluate their own well-being in terms of their personal conception of the good. Furthermore, individuals have the

[13] For example, article 25 of the Japanese Constitution, stipulates "the right to the minimum standards of wholesome and cultured living."

freedom not to participate in the process of making group evaluations or not to take advantage of social policies which give a certain provision to that group. In our framework a group is nothing more than an informational basis for making social evaluation sensitive to particularity so that an individual is not fully characterized by the so-called group identity of the group they belong to, as Sen carefully points out (Sen, 1999, p.29, 2006, p.18f.).

3 The Basic Model

Consider a society with population $N = \{1, 2, ..., i, ...n\}$, where $2 \leq n < +\infty$. Let us denote a social state by x, and the set of all possible social states by X, where $3 \leq \#X < +\infty$. Each $x \in X$ may be interpreted as representing an admissible social policy. Thus, we will sometimes call each $x \in X$ a social policy x. Note that a social policy x does not necessarily represent a single policy. For instance, it may present a bundle of multiple social policies; or it may present a state of resource allocation realized by a certain bundle of social policies.

For each $i \in N$, let $Z_i \subseteq \mathbb{R}^m_+$ with $m > 1$ be the set of conceivable well-being contents for i. Let $Z \equiv \cup_{i \in N} Z_i$. For each $i \in N$, let i's well-being transformation be a mapping $C_i : X \twoheadrightarrow Z_i$ such that for each $x \in X$, $\varnothing \neq C_i(x) \subseteq Z_i$ holds. C_i represents an individual's ability to transform each social policy to a content of well-being and $C_i(x)$ represents individual i's well-being available under the social policy x. Note that C_i could be *single-valued*, though it is defined by a multi-valued mapping as a general form. Whether C_i is single-valued or multi-valued depends on the types of well-being indicators presumed in the context. For instance, if we consider capabilities as well-being indicators, then Z_i is the space of *functioninig vectors* (Sen, 1980; 1985) for i, and C_i should be multi-valued in that it is i's *capability correspondence* that associates each social policy x to a capability $C_i(x)$ defined on Z_i.[14]

Let $\mathcal{C}_i$ be the set of such transformations and let $\mathbf{C} \equiv (C_i)_{i \in N}$ be a profile of the well-being transformations. Denote the admissible set of profiles of

[14] Individual capability correspondence is formulated and discussed in, for example, Herrero (1996) and Gotoh and Yoshihara (2003). Basu and Lopez-Calva (2011) provide an illuminating survey on the formulation of functionings and capabilities.

well-being transformations by $\mathcal{C}$. Further let

$$\mathcal{Z} \equiv \{C | \exists i \in N, \exists C_i \in \mathcal{C}_i, \exists x \in X, \text{ s.t. } C_i(x) = C\},$$

which is the universal class of well-being contents.[15]

Given $\mathcal{Z}$, for each $i \in N$, let us define a binary relation $\succsim_i$ on $\mathcal{Z}$, which is *reflexive* and *transitive*. We call this $\succsim_i$ a *well-being appraisal* of i. The interpretation of the well-being appraisal $\succsim_i$ is that, for any $C, C' \in \mathcal{Z}$, $C \succsim_i C'$ if and only if C is at least as good as C' for i. Given $\succsim_i$ defined on $\mathcal{Z}$, let $C \succ_i C'$ if and only if $C \succsim_i C'$ holds but $C' \succsim_i C$ does not hold; let $C \sim_i C'$ if and only if $C \succsim_i C'$ and $C' \succsim_i C$ hold. If $\mathcal{Z}$ is a partially ordered set endowed with a partial ordering $\geq$ on $\mathcal{Z}$,[16] it may be assumed that for any $i \in N$ and any $C, C' \in \mathcal{Z}$, if $C \geq C'$, then $C \succsim_i C'$, and if $C > C'$, then $C \succ_i C'$.

The well-being appraisal $\succsim_i$ reflects a bundle of criteria for comparing i's well-being contents. Let us discuss this point in more detail by assuming that well-being contents are represented by capabilities. Then, the rankings of capabilities can be varied depending on the relative weights of different functionings. For example, suppose that the basic capability of persons with limited vision consists of two functionings: *moving* and *using IT*. Moreover, suppose that the maximal achievement of *moving* in C is higher than the maximal achievement of this functioning in C', whereas the maximal achievement of *using IT* in C' is higher than the maximal achievement of this functioning in C. In this context, $\succsim_i$ indicates the weight on these functionings, which individuals attach based on their own conception of well-being.

Next, let us define the concepts of a group and group appraisal. Given society N, there exists a set of characteristics T with generic element t such that (1) $0 < \#T \leqq \#N$; and (2) for each $\mathbf{C} \in \mathcal{C}$ and each $t \in T$, there exists a unique subset $N^t_{\mathbf{C}}$ of N. Note that $N^t_{\mathbf{C}}$ may be empty for some $t \in T$, and $N^t_{\mathbf{C}}$ may be identical to N for some $t \in T$. As a typical interpretation, t represents a type of conceivable disadvantage, and $N^t_{\mathbf{C}}$ represents the set of t-type disadvantaged individuals in society N with $\mathbf{C}$. Thus, the set

[15] If $\mathcal{C}_i$ consists solely of single-valued mappings, then $\mathcal{Z}$ should be defined as $\mathcal{Z} \equiv Z$.

[16] The precise definitions of $\geq$ and its asymmetric part $>$ depend on the mathematical structure of the space $\mathcal{Z}$. For instance, if each C represents a vector on Z, so that $\mathcal{Z} = Z$, then $(\geq, >)$ represents the standard *vector inequality.* If each C represents a subset in Z, so that $\mathcal{Z} = 2^Z \backslash \varnothing$, then $(\geq, >)$ represents the standard *set-inclusion* as $C \geq C'$ if and only if for any $z \in C'$, $z \in C$ holds; and $C > C'$ if and only if $C \geq C'$ and $C' \not\geq C$.

$N \backslash (\cup_{t \in T} N^t_{\mathbf{C}})$, if non-empty, is the set of non-disadvantaged individuals in society N with $\mathbf{C}$.

As mentioned above, here, we assume that a group of individuals that have a certain disadvantage in common can construct a shared criterion for comparing their well-being contents. Hence, let $\succsim_t \equiv \cap_{i \in t} \succsim_i$ be the well-being appraisal of group t, which, based on the above argument, is assumed to be non-empty.[17] In addition, let $\succsim \equiv \left((\succsim_i)_{i \in N}, (\succsim_t)_{t \in T} \right)$ be a profile of well-being appraisals.

With the well-being appraisal of the group, the least advantaged within the group can be defined as follows. Given society N with $\mathbf{C} \in \mathcal{C}$ and a profile of well-being appraisals $\succsim$ on $\mathcal{Z}$, the set of the least advantaged individuals of type t under social policy $x \in X$ is defined by:

$$L^t_{\mathbf{C}}(x; \succsim_t) \equiv \left\{ i \in N^t_{\mathbf{C}} \mid \nexists j \in N^t_{\mathbf{C}} : C_i(x) \succ_t C_j(x) \right\}.$$

That is, the least advantaged under social policy x is defined as an individual whose well-being content never dominates the well-being contents of others. Note that $L^t_{\mathbf{C}}(x; \succsim_t)$ is non-empty for each $x \in X$ and for each $t \in T$ with $N^t_{\mathbf{C}} \neq \varnothing$. Moreover, it is not necessarily a singleton, because as in the above example of the two functionings, if a group attaches equal weight to *moving* and *using IT*, and under social policy x, individual i is better than j in terms of *moving,* but j is better than i in terms of *using IT*, and the others are better than i and j in both *moving* and *using IT*, then both individuals i and j can be identified as the least advantaged under social policy x.

Lastly, given society N with $\mathbf{C} \in \mathcal{C}$ and a profile of well-being appraisals $\succsim$ on $\mathcal{Z}$, for each $t \in T$, the *group evaluation* is defined as a *reflexive* relation $R^t_{\mathbf{C}}$ on X. The interpretation of $R^t_{\mathbf{C}}$ is that it represents an evaluation of alternative social policies, which is defined on the domain respective to this group, and which can be agreed upon by all individuals in this group, $N^t_{\mathbf{C}}$. Given the relation $R^t_{\mathbf{C}}$ on X, let $P^t_{\mathbf{C}}$ and $I^t_{\mathbf{C}}$ be respectively the strict and the indifferent parts of $R^t_{\mathbf{C}}$. Moreover, let $NR^t_{\mathbf{C}}$ denote the *non-comparable part* of $R^t_{\mathbf{C}}$; that is, $xNR^t_{\mathbf{C}}y$ if and only if neither $xR^t_{\mathbf{C}}y$ nor $yR^t_{\mathbf{C}}x$. Given society N with $\mathbf{C} \in \mathcal{C}$ and a profile of well-being appraisals $\succsim$ on $\mathcal{Z}$, let us denote the admissible class of type t's such relations on X by $D^t_{\mathbf{C}}(\succsim_t)$. Moreover, let $D_{\mathbf{C}}(\succsim) \equiv \times_{t \in T} D^t_{\mathbf{C}}(\succsim_t)$.

[17] The idea behind this formulation is that each individual of each group appraises the well-being contents of the members of the group, including her own, not in terms of a personal conception of the good but in terms of a shared conception of the good based on some commonality among members.

With this basic framework, we are ready to formally define the scheme of social relation functions and social evaluations as follows:

Definition 1: *Given the profile of well-being transformations* $\mathbf{C} \in \mathcal{C}$, *the* social relation function (**SRF**) *is the mapping* F *which associates each well-being appraisal* $\succsim$ *on* $\mathcal{Z}$ *and each profile of group evaluaitons* $(R^t_{\mathbf{C}})_{t \in T} \in D_{\mathbf{C}}(\succsim)$ *to the reflexive and transitive relation* $R_{\mathbf{C}}$ *defined on* X.

$R_{\mathbf{C}}$ is called a *social evaluation over* X in society with $\mathbf{C} \in \mathcal{C}$.

Definition 1 shows that a social evaluation is formulated based not only on the profile of well-being appraisals of all individuals, but also on the profiles of groups' well-being appraisals as well as of group evaluations of social policies.

Taking for granted that disadvantaged groups are given prior treatment in incorporating their information into a social policy, the two types of informational bases for disadvantaged groups may play different functional roles. Group appraisals are necessary to identify the least advantaged members in each group based on each group's own conception of the good. In contrast, each group evaluation is formed on the basis of its own group appraisal by focusing on the least advantaged members of this group. Then, each group is given the chance to take the initiative, by revealing its own group evaluation, in realizing a social policy relevant to its members. Finally, the policy maker can choose appropriate social policies based on the social evaluation derived from the **SRF**, into which she can incorporate each individual's appraisal symmetrically as well as each disadvantaged group's evaluation asymmetrically.

Note that the introduction of group evaluations in addition to group appraisals implies that the **SRF** scheme allows each group a certain degree of "group autonomy," in that the deliberative process of forming a policy evaluation relevant to a certain group's least advantaged members is in the hands of the group. In fact, the information of group appraisals seems sufficient for identifying the ranking of alternative policies in terms of the well-being contents assigned to a certain group's least advantaged members, and making such a ranking can be implemented by the policy maker with the information of group appraisals. Thus, the introduction of group evaluations also makes explicit the recognition of "group autonomy" as an essential element of political liberalism.

4 Axioms for Group Evaluations and Social Relation Functions

In this section, let us define several conditions which we assume to be publicly imposed on the social relation function. The first set of conditions are conditions that restrict the domain of the social relation function through the restriction of the available class of group evaluations. One of the key ideas to define conditions is the concept of *basic well-being* $BC^t \in \mathcal{Z}$, which is unique to each $t \in T$.

Based on this concept, for each $t \in T$, social policies are classified into three categories: (1) social policies under which the well-being contents of the least advantaged members of this type are at least as good as basic well-being; (2) social policies under which the well-being contents of the least advantaged members of this type are worse than basic well-being; (3) social policies under which the well-being contents of the least advantaged members of this type cannot be compared with basic well-being.

Note that, by the transitivity of $\succsim_t$ and the definition of $L^t_{\mathbf{C}}(x; \succsim_t)$, $C_i(x) \succsim_t BC^t$ for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$ implies that individuals in $L^t_{\mathbf{C}}(x; \succsim_t)$ are *either* all better than basic well-being *or* all indifferent to well-being. The same argument also applies to the case that $BC^t \succsim_t C_i(x)$ for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$. Considering this point and the above three categories of social policies, the domain of group evaluations is classified as follows. For each $t \in T$ and each $x, y \in X$,

α) 1) $C_i(x) \succsim_t BC^t$ for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$, and $BC^t_t \succsim_t C_j(y)$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$; and
2) under α)-1), $C_i(x) \succ_t BC^t$ for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$ or $BC^t \succ_t C_j(y)$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$;

β) $C_i(x) \succsim_t BC^t$ for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$, and *not* $[BC^t \succsim_t C_j(y)]$ & *not* $[C_j(y) \succsim_t BC^t]$ for some $j \in L^t_{\mathbf{C}}(y; \succsim_t)$;

γ) *not* $[BC^t \succsim_t C_i(x)]$ & *not* $[C_i(x) \succsim_t BC^t]$ for some $i \in L^t_{\mathbf{C}}(x; \succsim_t)$, and $BC^t \succ_t C_j(y)$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$;

δ) $BC^t \succ_t C_i(x)$ for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$, $BC^t \succ_t C_j(y)$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$;

ϵ) $C_i(x) \succ_t BC^t$ for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$ and $C_j(y) \succ_t BC^t$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$; and

ε) otherwise.

That is, α)-1) refers to the domain where the least advantaged individuals' well-beings in policy x are all at least as good as basic well-being, while the least advantaged individuals' well-beings in policy y are all at least as bad as basic well-being. On the other hand, α)-2) refers to the domain where α)-1) applies, and the least advantaged individuals' well-beings in policy x are all better than basic well-being or the least advantaged individuals' well-beings in policy y are all worse than basic well-being. β) refers to the domain where the least advantaged individuals' well-beings in policy x are *either* all better than basic well-being *or* all indifferent to basic well-being, while there exists at least one of the least advantaged individuals' well-beings in policy y which is non-comparable with basic well-being. γ) refers to the domain where there exists at least one of the least advantaged individuals' well-beings in policy x which is non-comparable with basic well-being, while the least advantaged individuals' well-beings in policy y are all worse than basic well-being. δ) refers to the domain where the least advantaged individuals' well-beings are all worse than basic well-being in both policies x and y. ϵ) refers to the domain where the least advantaged individuals' well-beings in both policy x policy y are all better than basic well-being.

Based on this classification, let us introduce three conditions imposed on group evaluations, which result in restricting the domain of the social relation function F.

Basic Well-being Condition (BWC): For each $\mathbf{C} \in \mathcal{C}$ and each $t \in T$, and for each $x, y \in X$, $xR^t_{\mathbf{C}}y$ holds if at least one of α), β), and γ) holds, and $xP^t_{\mathbf{C}}y$ holds if at least one of α)-2), β), and γ) holds.

Restricted Monotonicity (RM): For each $\mathbf{C} \in \mathcal{C}$ and each $t \in T$, and for each $x, y \in X$,
1) $xR^t_{\mathbf{C}}y$ holds if δ) holds and (i) $C_i(x) \succsim_t C_j(y)$ holds for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$ and all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$;
2) $xP^t_{\mathbf{C}}y$ holds if δ) and (ii) $C_i(x) \succ_t C_j(y)$ holds for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$ and all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$.

Refrain Condition (RC): For each $\mathbf{C} \in \mathcal{C}$ and each $t \in T$, and for each $x, y \in X$ with $x \neq y$, $xNR^t_{\mathbf{C}}y$ holds if ϵ) or ε) holds.

BWC requires each group to evaluate a social policy x, under which the well-being contents of the least advantaged are at least as good as basic well-being, as being *more just* than another social policy y, under which the well-being contents of the least advantaged *either* fall beneath basic well-being *or* cannot be compared with it. Furthermore, it requires each group to evaluate a social policy y, under which the well-being contents of the least advantaged fall beneath basic well-being, as being *less just* than another social policy x in which the well-being contents of the least advantaged cannot be compared with basic well-being.

RM requires each group to evaluate a social policy x as being *more just* than another social policy y whenever the corresponding profile of the least advantaged members' well-beings is better in x than y, given that all of their well-being contents derived from both policies fall beneath basic well-being. **RM** represents a kind of monotonicity criterion,[18] though its applicability is constrained to a proper domain of alternatives.

Lastly, **RC** requires a group evaluation *not* to make pair-wise rankings of the social policies if the well-being contents of the least advantaged corresponding to these social policies are better than basic well-being or they cannot be compared with basic well-being.[19]

Thus, the three conditions together define the available class of group evaluations by identifying the types of non-comparable binary pairs of social policies, and by restricting the types of ranking over some specified pairs of social policies. In this way, the three conditions together make the available group evaluations consistent with the common goal of securing basic well-being for all members of each group, and prohibit any further ranking over policies beyond this goal.

Given these three conditions, however, it is not immediately obvious whether these conditions are mutually consistent. Let us examine this problem.

Lemma 1: *Let the reflexive* $R^t_{\mathbf{C}}$ *satisfy* **BWC**, **RM**, *and* **RC**. *Then, it is*

[18] The concept of dominance proposed by Pattanaik and Xu (2007, p.361-362), which is closely related to Sen's idea of "dominance partial ordering" (Sen, 1987, pp.29-30) is a good example.

[19] This condition is similar to the "focus axiom" proposed by Sen (1981; p. 186) in the sense that the difference between two social states, both of which bring about capabilities at least as good as Basic Capability, is not reflected in the social evaluation. We are grateful to James Foster and Prasanta Pattanaik for pointing this out. See Foster (1984; p. 217) and Sen (1997; p. 172).

transitive.

Due to this lemma, each group can form its own evaluation based on the three conditions which are rational in terms of logical consistency.

The next task for us is to introduce two basic axioms regarding how to aggregate plural group evaluations as well as diverse individual well-being appraisals in order to form a consistent social evaluation. To explore this problem, let us introduce the following conditions.

Non-negative Response (NR): For each $\mathbf{C} \in \mathcal{C}$, each $(R^t_{\mathbf{C}})_{t \in T} \in D_{\mathbf{C}}(\succsim)$, and each $x, y \in X$, if there exists $t' \in T$ such that $xR^{t'}_{\mathbf{C}}y$ (*resp.* $xP^{t'}_{\mathbf{C}}y$) and there exists no $t'' \in T$ such that $yP^{t''}_{\mathbf{C}}x$, then $xR_{\mathbf{C}}y$ (*resp.* $xP_{\mathbf{C}}y$) holds, where $R_{\mathbf{C}} = F(\mathbf{C}, \succsim, (R^t_{\mathbf{C}})_{t \in T})$.

Weak Pareto (WP): For each $\mathbf{C} \in \mathcal{C}$, each $(R^t_{\mathbf{C}})_{t \in T} \in D_{\mathbf{C}}(\succsim)$, and each $x, y \in X$, if $C_i(x) \succ_i C_i(y)$ holds for all $i \in N$, then $xP_{\mathbf{C}}y$ holds, where $R_{\mathbf{C}} = F(\mathbf{C}, \succsim, (R^t_{\mathbf{C}})_{t \in T})$.

Recall that each $t \in T$ represents a particular type of disadvantage, so $N \setminus (\cup_{t \in T} N^t_{\mathbf{C}})$ is the set of non-disadvantaged individuals in society N with $\mathbf{C}$. Hence, **NR** requires giving priority to the evaluations of disadvantaged groups over the evaluations of non-disadvantaged individuals in the aggregation procedure, while treating the evaluation of each group as fully symmetrical to one another. That is, even if the well-being contents of all non-disadvantaged individuals become worse in y than in x, the social evaluation must be that y is at least as good as x whenever a group t evaluates y as being at least as good as x and no other group evaluates x as *more just* than y. Such a requirement seems quite reasonable whenever persons with a particular disadvantage can be considered as "experts" on that disadvantage and these persons are expected to provide a reasonable group evaluation. In this respect, **NR** together with the available class of group evaluations constrained by **BWC**, **RM**, and **RC** guarantee the reasonableness of asymmetric treatments of specific types of groups in the aggregation procedure.

In contrast, **WP** requires treating symmetrically the well-being appraisals of all individuals. Indeed, if the well-being contents of all individuals are better in x than in y, **WP** states that the social evaluation must be that x is more just than y. In terms of respecting the plurality of the conceptions of the good, **WP** also seems quite reasonable.

5 Main Results

As argued above, the combination of **NR** and **WP** may produce a quite reasonable social evaluation if **NR** and **WP** do not compete with each other. In fact, with regard to ranking x and y, it seems reasonable to give priority to the evaluations of disadvantaged groups, as suggested by **NR**, whenever at least one of the two policies assigns to a member of a disadvantaged group a lower level of well-being content than that of basic well-being. It also seems reasonable to treat symmetrically the well-being appraisals of all individuals, as suggested by **WP**, whenever all disadvantaged groups refrain from ranking x and y due to **RC**.

The aim of this section therefore is to verify the compatibility of **NR** and **WP**. As discussed in Section 1, this issue resembles the structure of the Pareto-liberal paradox, which can be summarized as the conflict between the principle of the symmetric treatment of individuals' decisive powers (due to the Pareto principle) and the prior treatment of specific individuals in their decisive powers over some (local) sphere of the domain of alternatives (due to Minimal liberty). In our framework, given the several conditions imposed on group evaluations, **NR** specifies the types of individuals who are assigned the prior treatments as well as to what extent they should have decisive powers in the social choice procedure. In contrast, **WP** represents the principle of symmetric treatment of individuals' appraisals and, moreover, there is no constraint for the application of this principle. Thus, it is not obvious whether the compatibility of **WP** and **NR**, provided that the group evaluations satisfy **BWC**, **RM**, and **RC**, is verified or not, even if the latter axiom seems rather weak as the claim for the local decisiveness of specific types of individuals.

We address this issue in the following four steps. In the first step, we examine whether or not there exists a **SRF** which satisfies **NR**. To do this, we introduce another axiom, the *Positive Response* **(PR)**, which is even weaker than **NR**. Theorem 1 discussed below will show that there is no **SRF** which satisfies **PR**.

Therefore in the second step, to avoid this negative result, we introduce an additional condition, *Full Comparability of Destitution* (**FCD**), which guarantees the full comparability of policies when the well-beings of all of the least advantaged members under those policies become worse than basic well-being. Theorem 2 proves that under the presumption of **FCD**, there exists a **SRF** which satisfies **NR**.

However, in the third step, we will show in Theorem 3 that it is impossible to guarantee the compatibility of **NR** and **WP** even under the presumption of **FCD.** Given these results, Theorem 4 clarifies what kind of further condition is required for the compatibility of these two axioms.

Assume, for the sake of simplicity, that the profile of the disadvantaged groups $(N^t)_{t\in T}$ is fixed independently of the types of **SRF**s. As our first step, let us introduce the following axiom for **SRF**s:

Positive Response (PR): For each $\mathbf{C} \in \mathcal{C}$, each $(R^t_{\mathbf{C}})_{t\in T} \in D_{\mathbf{C}}(\succsim)$, and each $x, y \in X$, if there exists $t' \in T$ such that $xP^{t'}_{\mathbf{C}}y$ and there is no $t'' \in T$ such that $yP^{t''}_{\mathbf{C}}x$, then $xP_{\mathbf{C}}y$ holds, where $R_{\mathbf{C}} = F(\mathbf{C}, \succsim, (R^t_{\mathbf{C}})_{t\in T})$.

PR is a weaker version of **NR**. This condition, as well as **NR**, seems quite reasonable, given that persons with a particular disadvantage can be considered as "experts" on that disadvantage.

Then:

Theorem 1: *There exists a profile of well-being appraisals $\succsim$ under which there is no SRF $F^{\succsim}$ satisfying* **PR**.

Given this impossibility, let us introduce an additional condition on well-being appraisals:

Full Comparability of Destitution (FCD): For each $t \in T$ and each $x, y \in X$, if δ) holds, then for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$ and all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$, $C_i(x) \succsim_t C_j(y)$ or $C_j(y) \succsim_t C_i(x)$.

δ) is the case where in each policy the well-being contents of the least advantaged are all worse than basic well-being. In this situation of "destitution," **FCD** requires that the well-beings of the least advantaged are all comparable. Since the plurality of evaluations of social policies tends to be reduced in a situation of "destitution," **FCD** seems to be natural. Moreover, it would be desirable to guarantee that relatively "less unjust" policies can be selected under the situation of "destitution," and **FCD** guarantees the feasibility of such a social choice.

The next theorem proves that if we introduce **FCD** into the group appraisal, we can guarantee the existence of a **SRF** $F^{\succsim}$ which satisfies **NR**, a strong version of **PR**.

Theorem 2: *Let* **FCD** *hold. Then, there exists a SRF* $F^{\succsim}$ *which satisfies* **NR**.

To show Theorem 2, let us define $F_{NR}^{\succsim}$ as follows: for each $\mathbf{C} \in \mathcal{C}$, each $(R_{\mathbf{C}}^t)_{t \in T} \in D_{\mathbf{C}}(\succsim)$, and each $x, y \in X$, $xR_{\mathbf{C}}^{NR}y$ holds if and only if there exists $t' \in T$ such that $xR_{\mathbf{C}}^{t'}y$ and there is no $t'' \in T$ such that $yP_{\mathbf{C}}^{t''}x$, where $R_{\mathbf{C}}^{NR} = F_{NR}(\mathbf{C}, \succsim, (R_{\mathbf{C}}^t)_{t \in T})$. In the proof of Theorem 2 developed in Appendix below, we will show that this $R_{\mathbf{C}}^{NR}$ is transitive. In other words, we will show that *for each* $\mathbf{C} \in \mathcal{C}$ *and each* $(R_{\mathbf{C}}^t)_{t \in T} \in D_{\mathbf{C}}(\succsim)$*, if* $(x, y), (y, z) \in R_{\mathbf{C}}^{NR}$*, then* $(x, z) \in R_{\mathbf{C}}^{NR}$.

Given Theorem 2, our next step is to examine whether **FCD** is sufficient for the existence of a **SRF** $F^{\succsim}$ satisfying **NR** and **WP**. Unfortunately, the next theorem proves that **FCD** is not sufficient for this purpose.

Theorem 3: *Suppose* **FCD**. *Then there exists a profile of well-being appraisals* $\succsim$ *under which there is no* ***SRF*** *satisfying* **NR** *and* **WP**.

The main reason for this theorem is that there might be at least two least advantaged members within a group whose well-being positions relative to basic well-being are different. To see this, let us suppose that i and j are the least advantaged members within a group t, where i's well-being is worse than BC^t and j's well-being is non-comparable with BC^t in policy x, according to the group appraisal $\succsim_t$. Moreover, let us suppose that all individuals' well-being contents improve as a result of the change from policy x to policy y, and j's well-being becomes better than BC^t in y, though i's is still worse than BC^t and i is the unique least advantaged member in y. This situation corresponds to case γ), so **BWC** applies and this group evaluates that x is more just than y. Thus, if no other groups make any objection, **NR** requires that x is better than y. Yet, **WP** requires that y is better than x.

For the purpose of our four steps, let us introduce an additional condition which requires even greater comparability of each group's well-being appraisal:

Dominance (D): For each $t \in T$, each $x \in X$, and each $i, j \in t$ with $i \in L_{\mathbf{C}}^t(x; \succsim_t)$, if $BC^t \succ_t C_i(x)$ and *not* $BC^t \succsim_t C_j(x)$, then $C_j(x) \succ_t C_i(x)$.

D requires that if a well-being content C is worse than BC^t and another well-being content C' is not, C' should be appraised to be better than C. Recalling that *not* $BC^t \succsim_t C_j(x)$ means $C_j(x) \succsim_t BC^t$ or *not* $[BC^t \succsim_t C_j(x)$

or $C_j(x) \succsim_t BC^t$], the underlying idea of this condition is clear. That is, if one well-being content is appraised to be worse than BC^t, while another is not, a comparative judgment should be made between the two, in that the latter is better than the former. Note that, unless **D** is required, it is possible that j is deemed least advantaged even if j's well-being content is non-comparable with basic well-being and there is another least advantaged member i whose well-being content is worse than basic well-being.

With condition **D** in addition to **FCD**, we can guarantee the existence of a **SRF** $F^{\succsim}$ which satisfies **NR** and **WP** as follows:

Theorem 4: *Let* **FCD** *and* **D** *hold. Then there exists a* ***SRF*** $F^{\succsim}$ *which satisfies* **NR** *and* **WP**.

To show Theorem 4, let us define F_{WP} as follows: for each $\mathbf{C} \in \mathcal{C}$, each $(R^t_{\mathbf{C}})_{t\in T} \in D_{\mathbf{C}}(\succsim)$, and each $x, y \in X$, $xP^{WP}_{\mathbf{C}}y$ holds if and only if $C_i(x) \succ_i C_i(y)$ holds for all $i \in N$, and $xI^{WP}_{\mathbf{C}}y$ holds if and only if $x = y$, where $R^{WP}_{\mathbf{C}} = F_{WP}(\mathbf{C}, \succsim, (R^t_{\mathbf{C}})_{t\in T})$. Moreover, let us define $F^{\succsim}_*$ as follows: for each $\mathbf{C} \in \mathcal{C}$ and each $(R^t_{\mathbf{C}})_{t\in T} \in D_{\mathbf{C}}(\succsim)$, $F_*(\mathbf{C}, \succsim, (R^t_{\mathbf{C}})_{t\in T}) = R^*_{\mathbf{C}}$, where $R^*_{\mathbf{C}} \equiv R^{NR}_{\mathbf{C}} \cup R^{WP}_{\mathbf{C}}$.

In the following discussion, we will show that this $R^*_{\mathbf{C}}$ is transitive. Let $(x, y), (y, z) \in R^*_{\mathbf{C}}$. Then there are the following four possible cases:

1)$(x, y), (y, z) \in R^{NR}_{\mathbf{C}}$

2)$(x, y), (y, z) \in R^{WP}_{\mathbf{C}}$;

3) $(x, y) \in R^{NR}_{\mathbf{C}}$ and $(y, z) \in R^{WP}_{\mathbf{C}}$; and

4) $(x, y) \in R^{WP}_{\mathbf{C}}$ and $(y, z) \in R^{NR}_{\mathbf{C}}$.

Theorem 2 shows that if case 1) applies, $(x, z) \in R^{NR}_{\mathbf{C}}$ holds. Moreover, it is easy to see that if case 2) applies, then $(x, z) \in R^{WP}_{\mathbf{C}}$ holds. Next, let us consider cases 3) and 4):

Lemma 2: For each $\mathbf{C} \in \mathcal{C}$ and each $(R^t_{\mathbf{C}})_{t\in T} \in D_{\mathbf{C}}(\succsim)$, if $(x, y) \in R^{NR}_{\mathbf{C}}$ and $(y, z) \in P^{WP}_{\mathbf{C}}$, *then* $(x, z) \in P^*_{\mathbf{C}}$.

Lemma 3: For each $\mathbf{C} \in \mathcal{C}$ and each $(R^t_{\mathbf{C}})_{t\in T} \in D_{\mathbf{C}}(\succsim)$, if $(x, y) \in P^{WP}_{\mathbf{C}}$ and $(y, z) \in R^{NR}_{\mathbf{C}}$, then $(x, z) \in P^*_{\mathbf{C}}$.

Proof of Theorem 4: By **Lemmas 2** and **3**, it holds. ■

Theorems 3 and 4 indicate that, given the incompleteness of the informational basis, the moderate asymmetric treatment of disadvantaged groups is unable to guarantee consistent and Paretian social decision-making for social policies. This impossibility, however, does not necessarily imply that there is an *intrinsic* conflict between the claim of the prior treatment of specific individuals and the symmetric treatment of all. Rather, it may originate from a lack of sufficient information on the part of a disadvantaged group to make a deliberate appraisal of their own states. In fact, as condition **D** and Theorem 4 show, the main reason for the impossibility is the existence of the least disadvantaged member whose well-being content is deemed non-comparable to basic well-being despite the existence of another least advantaged whose well-being content is deemed worse than basic well-being. Hence, if such "tentative" non-comparability can be resolved via further scrutiny of this member's condition, consistent and Paretian social decision-making for desired polices can be compatible with the prior treatment of specific people. In other words, the difficulty of constructing the desired social choice due to "tentative" non-comparability within the same disadvantaged group could be resolved by technical progress at least in the future, which should be discriminated from the more intrinsic types of impossibility problems.

6 Conclusion

In concluding the paper, we firstly comment on another prominent feature of our framework of **SRF**s. In our **SRF**s, a social choice is made depending on the information of the characteristics of social alternatives and the characteristics of individuals. In contrast, in Arrovian social welfare functions, a social choice is made simply based on the structure of preference profiles revealed by individuals and thus independently of information on those characteristics. Note that this property of Arrovian social welfare functions derives from the three conditions imposed by Arrow, namely the universal domain, the Pareto principle, and the independence of irrelevant alternatives. It is known that these three conditions lead to *neutrality*,[20] that is, these conditions together require that individuals' ordinal rankings of alternatives are the sole

[20] The basic idea of neutrality can be summarized as follows: if the individual preferences over (x, y) in one case are "identical" to the individual preferences over (a, b) in another case, then the social preference in the latter would place a and b respectively where x and y figured in the former (Sen, 2002; p. 333).

See Fleurbaey and Mongin (2005; p. 386) for an excellent survey of studies on neutrality. As pointed out,.Sen has examined the essentail nature of neutrality in term of "welfarism" (for example, Sen, 1970, chs. 5 and 5*).

relevant information to make a social choice. This structure of Arrovian models well represents the spirit of traditional liberalism, which puts first priority on individual autonomy and prohibits arbitrarily unequal treatment before the law.[21] In contrast, our formulation allows us to explore another possibility of liberalism, *i.e.,* substantive equality of political freedom that allows the asymmetrical treatment of ordinal rankings, while the reasonableness of such asymmetrical treatment is guaranteed by the introduction of an explicit device for public scrutiny, which is represented in this paper by the observability of well-being indicators and the three component-structure of the informational basis.

Secondly, let us clarify the basic ideas underlying this paper. The first idea is relevant to two kinds of "incomparability." In this paper, a "(disadvantaged) group" is defined as the maximum unit within which inter-personal comparison of well-being contents is possible to the extent that the least advantaged —individuals whose well-being contents never dominate the well-beings of others— can be identified in each social policy. Yet, due to the multiplicity of attributes for defining the notion of well-being, there could remain incomparability among the least advantaged even within a group. However, the meaning of incomparability within a group should be kept distinct from incomparability (also called "incommensurability") between groups. The reason is that the former is a technical or political problem and certain conditions of compromise can be introduced to deal with it, as we have done by introducing **FDC** and **D** in this paper. On the other hand, the latter is a kind of incomparability for which no compromise can be found as long as the plurality of disadvantages is taken seriously. This distinction between these two forms of incomparability corresponds to the distinction introduced by Sen between "tentative incompleteness" derived from "some pairs of alternatives which are not yet ranked but may all get ranked with more deliberation or information," and "assertive incompleteness" derived from "some pairs of alternatives which are asserted to be 'non-rankable"' (Sen, 2002, p.182) .

The second idea concerns two different kinds of conflicts between groups. One kind of conflict is the one that arises from each group's need to achieve "basic well-being," the other kind of conflict is the one that derives from each group's desire to enjoy well-being beyond "basic well-being." The former

[21] According to Arrow, "[T]he decision as to which preferences are relevant and which are not is itself a value judgement and cannot be settled on an a priori basis" (Arrow, 1963, 18).

type of conflict is avoidable if there are sufficient resources to secure basic well-being for all groups, while the latter is not, if the desire towards better well-being is without limit. The former deserves consideration in terms of justice that this paper is concerened with, while the latter does not. This is the reason why in this paper the application of the monotonicity condition is restricted to the domain below "basic well-being" through the application of Restricted Monotonicity, while in the domain above "basic well-being," conflicts are avoided through the application of the Refrain Condition, which prohibits groups from making rankings.

The third point concerns two kinds of comparative adjectives, "better or worse" and "less or more unjust." Recently, Sen proposed a "comparative approach" to justice in place of a "transcendental approach" (Sen 2009a, pp.15-18, Sen, 2009b, p. 46f.). According to Sen, the latter is traditional approach in ethics, which focuses on the description of an ideal just state, while the former is a new approach in ethics, but very familiar to economics, which ranks alternative social states in terms of justice. This paper is an attempt to formulate a "comparative approach" to justice, which compares "unjust" policies with one another and ranks "less unjust" policies over "more unjust" policies. More precisely, "better or worse" is used in the comparison of individual well-being contents, while "less or more unjust" is used in the comparison of social policies. This usage is derived from the distinction of conceptions of "the good" and of "justice" according to Rawls (Rawls, 1971, pp. 396-397). In the context of the political liberalism of Rawls, conceptions of "justice" is expected to be realized as an overlapping consensus of plural conceptions of the good. We suppose that "well-being" — and even "basic well-being" — essentially corresponds to conceptions of the good, whose plurality should be respected. In contrast, the goal of "securing basic well-being for all" corresponds to the conception of justice, which is expected to be realized as an overlapping consensus of plural concrete conceptions of "basic well-being". It is a kind of consequential justice that this paper is concerned with, which focuses on the current well-being of individuals; yet, as mentioned in Section 2, the current well-being of individuals is broad enough to cope with historical injustices so as to secure the well-being of victims now and in the future, and thereby ensure that such injustices are not repeated in the future.

7 Appendix

Proof of Lemma 1. Let $xR^t_{\mathbf{C}}yR^t_{\mathbf{C}}z$. We will show $xR^t_{\mathbf{C}}z$ holds. Note that $xR^t_{\mathbf{C}}y$ is derived from applying either **BWC** or **RM** to the pair of (x, y).

First, suppose that $xR^t_{\mathbf{C}}y$ is derived from applying **BWC** to the pair of (x, y). Then, $C_i(x) \succsim_t BC^t$ holds for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$, or *not* $[BC^t \succsim_t C_i(x)]$ & *not* $[C_i(x) \succsim_t BC^t]$ holds for some $i \in L^t_{\mathbf{C}}(x; \succsim_t)$. Moreover, regarding y, one of the following three cases holds:

1) $BC^t \sim_t C_j(y)$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$;

2) *not* $[C_j(y) \succsim_t BC^t]$ and *not* $[BC^t \succsim_t C_j(y)]$ for some $j \in L^t_{\mathbf{C}}(y; \succsim_t)$;

3) $BC^t \succ_t C_j(y)$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$.

Let 3) hold for y. Then, the case α)-2) or γ) is applied to the pair of (x, y), so that $xP^t_{\mathbf{C}}y$ holds. Moreover, $yR^t_{\mathbf{C}}z$ is then derived only by applying **RM** to the pair of (y, z), which implies that $C_i(y) \succsim_t C_j(z)$ holds for all $i \in L^t_{\mathbf{C}}(y; \succsim_t)$ and all $j \in L^t_{\mathbf{C}}(z; \succsim_t)$, so that $BC^t \succ_t C_j(z)$ holds for all $j \in L^t_{\mathbf{C}}(z; \succsim_t)$ by the transitivity of $\succsim_t$. Thus, the pair of (x, z) corresponds to the case α)-2) or γ), so that $xP^t_{\mathbf{C}}z$ holds by **BWC**.

Let 1) hold for y. Then, $xR^t_{\mathbf{C}}y$ is derived by applying **BWC** under the case α)-1) or α-2). Suppose that α)-1) is applied but α)-2) cannot be applied. Hence, $C_i(x) \sim_t BC^t$ holds for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$, which implies that $yR^t_{\mathbf{C}}x$ is also derived from applying **BWC** under the case α)-1). Thus, $xI^t_{\mathbf{C}}y$. Moreover, $yR^t_{\mathbf{C}}z$ is only derived from applying **BWC** to the pair of (y, z). Therefore, the pair of (x, z) corresponds to the case α), so that $xR^t_{\mathbf{C}}z$ holds by **BWC**. In particular, if $yP^t_{\mathbf{C}}z$, then $xP^t_{\mathbf{C}}z$ also holds, while if $yI^t_{\mathbf{C}}z$, then $xI^t_{\mathbf{C}}z$ also holds. Suppose that α)-2) is applied. Then, $C_i(x) \succ_t BC^t$ holds for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$ and $xP^t_{\mathbf{C}}y$. Again, $yR^t_{\mathbf{C}}z$ is only derived from applying **BWC** to the pair of (y, z). Therefore, the pair of (x, z) corresponds to the case α)-2), so that $xP^t_{\mathbf{C}}z$ holds by **BWC**.

Let 2) hold for y. Then, the case β) is applied to the pair of (x, y), so that $xP^t_{\mathbf{C}}y$ holds. Moreover, $yR^t_{\mathbf{C}}z$ is only derived by applying **BWC** under the case γ) to the pair of (y, z), where $BC^t \succ_t C_j(z)$ must hold for all $j \in L^t_{\mathbf{C}}(z; \succsim_t)$, so that $yP^t_{\mathbf{C}}z$ holds. Therefore, the pair of (x, z) corresponds to the case α)-2), so that $xP^t_{\mathbf{C}}z$ holds by **BWC**.

Second, suppose that $xR^t_{\mathbf{C}}y$ is derived by applying **RM** to the pair of (x, y). Then, $yR^t_{\mathbf{C}}z$ should also be derived by applying **RM** to the pair of (y, z). Hence, the pair of (x, z) corresponds to the case δ). Moreover, $C_i(x) \succsim_t C_j(y) \succsim_t C_h(z)$ holds for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$, all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$, and all $h \in L^t_{\mathbf{C}}(z; \succsim_t)$. Then, by the transitivity of $\succsim_t$, $C_i(x) \succsim_t C_h(z)$ holds for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$ and all $h \in L^t_{\mathbf{C}}(z; \succsim_t)$. Thus, $xR^t_{\mathbf{C}}z$ holds by **RM**. Moreover, if $xP^t_{\mathbf{C}}y$ or $yP^t_{\mathbf{C}}z$, then $xP^t_{\mathbf{C}}z$ holds, while if $xI^t_{\mathbf{C}}y$ and $yI^t_{\mathbf{C}}z$, then $xI^t_{\mathbf{C}}z$ holds by the transitivity of $\succsim_t$ and **RM**. ■

Proof of Theorem 1. Let us define $F^{\succsim}_{PR}$ as follows: for each $\mathbf{C} \in \mathcal{C}$, each $(R^t_{\mathbf{C}})_{t \in T} \in D_{\mathbf{C}}(\succsim)$, and each $x, y \in X$, let $P^{PR}_{\mathbf{C}}$ be defined as: $xP^{PR}_{\mathbf{C}}y$ holds if and only if there exists $t' \in T$ such that $xP^{t'}_{\mathbf{C}}y$ and there is no $t'' \in T$ such that $yP^{t''}_{\mathbf{C}}x$. Any **SRF** F satisfying **PR** associates with a profile $(\mathbf{C}, \succsim, (R^t_{\mathbf{C}})_{t \in T})$ a quasi-ordering $R_{\mathbf{C}} = F(\mathbf{C}, \succsim, (R^t_{\mathbf{C}})_{t \in T})$ which contains $P^{PR}_{\mathbf{C}}$ as a subrelation. Therefore, if $P^{PR}_{\mathbf{C}}$ is not transitive, this theorem holds.

Let $\{t^1, t^2, t^3\} = T$, and let us consider $(\mathbf{C}, \succsim, (R^t_{\mathbf{C}})_{t \in T})$ satisfying the following properties:

(1) Let $BC^{t^1} \succ_t C_i(x)$ for all $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$; $BC^{t^1} \succ_{t^1} C_j(y)$ for all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$; and $BC^{t^1} \succ_{t^1} C_h(z)$ for all $h \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. Moreover, let $C_i(x) \succ_{t^1} C_j(y)$ for all $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$ and all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$; *not* $[C_i(x) \succsim_{t^1} C_h(z)]$ & *not* $[C_h(z) \succsim_{t^1} C_i(x)]$ for some $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$ and some $h \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$; and *not* $[C_j(y) \succsim_{t^1} C_h(z)]$ & *not* $[C_h(z) \succsim_{t^1} C_j(y)]$ for some $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$ and some $h \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$.

(2) Let $BC^{t^2} \succ_{t^2} C_i(x)$ for all $i \in L^{t^2}_{\mathbf{C}}(x; \succsim_{t^2})$; $BC^{t^2} \succ_{t^2} C_j(y)$ for all $j \in L^{t^2}_{\mathbf{C}}(y; \succsim_{t^2})$; and $BC^{t^2} \succ_{t^2} C_h(z)$ for all $h \in L^{t^2}_{\mathbf{C}}(z; \succsim_{t^2})$. Moreover, let $C_i(y) \succ_{t^2} C_h(z)$ for all $i \in L^{t^2}_{\mathbf{C}}(y; \succsim_{t^2})$ and all $h \in L^{t^2}_{\mathbf{C}}(z; \succsim_{t^2})$; *not* $[C_i(x) \succsim_{t^2} C_j(y)]$ & *not* $[C_j(y) \succsim_{t^2} C_i(x)]$ for some $i \in L^{t^2}_{\mathbf{C}}(x; \succsim_{t^2})$ and some $j \in L^{t^2}_{\mathbf{C}}(y; \succsim_{t^2})$; and *not* $[C_i(x) \succsim_{t^2} C_h(z)]$ & *not* $[C_h(z) \succsim_{t^2} C_i(x)]$ for some $i \in L^{t^2}_{\mathbf{C}}(y; \succsim_{t^2})$ and some $h \in L^{t^2}_{\mathbf{C}}(z; \succsim_{t^2})$.

(3) Let $BC^{t^3} \succ_{t^3} C_i(x)$ for all $i \in L^{t^3}_{\mathbf{C}}(x; \succsim_{t^3})$; $BC^{t^3} \succ_{t^3} C_j(y)$ for all $j \in L^{t^3}_{\mathbf{C}}(y; \succsim_{t^3})$; and $BC^{t^3} \succ_{t^3} C_h(z)$ for all $h \in L^{t^3}_{\mathbf{C}}(z; \succsim_{t^3})$. Moreover, let $C_h(z) \succ_{t^3} C_i(x)$ for all $h \in L^{t^3}_{\mathbf{C}}(z; \succsim_{t^3})$ and all $i \in L^{t^3}_{\mathbf{C}}(x; \succsim_{t^3})$; *not* $[C_j(y) \succsim_{t^3} C_h(z)]$ & *not* $[C_h(z) \succsim_{t^3} C_j(y)]$ for some $h \in L^{t^3}_{\mathbf{C}}(z; \succsim_{t^3})$ and some $j \in L^{t3}_{\mathbf{C}}(y; \succsim_{t^3})$; and *not* $[C_i(x) \succsim_{t^3} C_j(y)]$ & *not* $[C_j(y) \succsim_{t^3} C_i(x)]$ for some $i \in L^{t^3}_{\mathbf{C}}(x; \succsim_{t^3})$ and some $j \in L^{t^3}_{\mathbf{C}}(y; \succsim_{t^3})$.

Under (1), $(x, y) \in P^{t^1}_{\mathbf{C}}$ by **RM**, and $(z, x) \in NR^{t^1}_{\mathbf{C}}$ and $(y, z) \in NR^{t^1}_{\mathbf{C}}$ by **RC**. Under (2), $(y, z) \in P^{t^2}_{\mathbf{C}}$ by **RM**, and $(z, x) \in NR^{t^2}_{\mathbf{C}}$ and $(x, y) \in NR^{t^2}_{\mathbf{C}}$ by **RC**. Under (3), $(z, x) \in P^{t^3}_{\mathbf{C}}$ by **RM**, and $(y, z) \in NR^{t^3}_{\mathbf{C}}$ and $(x, y) \in NR^{t^3}_{\mathbf{C}}$ by **RC**. Therefore, by the definition of **PR**, $(x, y), (y, z), (z, x) \in P^{PR}_{\mathbf{C}}$ holds, which implies that $P^{PR}_{\mathbf{C}}$ is not transitive. ■

Proof of Theorem 2. Let $(x, y), (y, z) \in R_{\mathbf{C}}^{NR}$. This implies that there exists $t^1 \in T$ such that $(x, y) \in R_{\mathbf{C}}^{t^1}$ and $(y, x) \notin P_{\mathbf{C}}^{t}$ for any other $t \in T$, and there exists $t^2 \in T$ such that $(y, z) \in R_{\mathbf{C}}^{t^2}$ and $(z, y) \notin P_{\mathbf{C}}^{t}$ for any other $t \in T$. Moreover, $(x, y) \in R_{\mathbf{C}}^{t^1}$ (resp. $(y, z) \in R_{\mathbf{C}}^{t^2}$) is derived from **BWC** by applying either of α), β), or γ); or it is derived from **RM** by applying δ).

1. First of all, let us show that if $(x, y) \in R_{\mathbf{C}}^{t^1}$ and $(z, y) \notin P_{\mathbf{C}}^{t^1}$, then $(x, z) \in R_{\mathbf{C}}^{t^1}$.

Case 1: Let $(x, y) \in R_{\mathbf{C}}^{t^1}$ be derived from **BWC** by applying α)-1). Then, $BC^{t^1} \succsim_{t^1} C_j(y)$ for all $j \in L_{\mathbf{C}}^{t^1}(y; \succsim_{t^1})$, and $C_i(x) \succsim_{t^1} BC^{t^1}$ for all $i \in L_{\mathbf{C}}^{t^1}(x; \succsim_{t^1})$. Suppose $(z, x) \in P_{\mathbf{C}}^{t^1}$. Note that neither of α), β), γ), δ) can derive $(z, x) \in P_{\mathbf{C}}^{t^1}$, thus (z, x) corresponds to ϵ) or ε), which leads to $(x, z) \in NR_{\mathbf{C}}^{t^1}$ by **RC**, a contradiction. Thus, $(x, z) \in NR_{\mathbf{C}}^{t^1}$ or $(x, z) \in R_{\mathbf{C}}^{t^1}$ holds. Suppose $(x, z) \in NR_{\mathbf{C}}^{t^1}$. This implies that (x, z) corresponds to ϵ), so that $C_i(z) \succ_{t^1} BC^{t^1}$ for all $i \in L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1})$. Then, $(z, y) \in R_{\mathbf{C}}^{t^1}$ from **BWC** by applying α). Since $(z, y) \notin P_{\mathbf{C}}^{t^1}$, $(z, y) \in I_{\mathbf{C}}^{t^1}$ holds, so that $(x, z) \in R_{\mathbf{C}}^{t^1}$ holds, since $R_{\mathbf{C}}^{t^1}$ is transitive by Lemma 1. Moreover, by transitivity, if $(x, y) \in P_{\mathbf{C}}^{t^1}$, then $(x, z) \in P_{\mathbf{C}}^{t^1}$ holds.

Case 2: Let $(x, y) \in P_{\mathbf{C}}^{t^1}$ be derived from **BWC** by applying β). Then, *not* $\left[C_j(y) \succsim_{t^1} BC^{t^1}\right]$ for some $j \in L_{\mathbf{C}}^{t^1}(y; \succsim_{t^1})$, and $C_i(x) \succsim_{t^1} BC^{t^1}$ for all $i \in L_{\mathbf{C}}^{t^1}(x; \succsim_{t^1})$. Suppose $(z, x) \in P_{\mathbf{C}}^{t^1}$. Note that neither of α), β), γ), δ) can derive $(z, x) \in P_{\mathbf{C}}^{t^1}$, thus (z, x) corresponds to ϵ) or ε), which leads to $(x, z) \in NR_{\mathbf{C}}^{t^1}$ by **RC**, a contradiction. Thus, $(x, z) \in NR_{\mathbf{C}}^{t^1}$ or $(x, z) \in R_{\mathbf{C}}^{t^1}$ holds. Suppose $(x, z) \in NR_{\mathbf{C}}^{t^1}$. This implies that (x, z) corresponds to ϵ), so that $C_i(z) \succ_{t^1} BC^{t^1}$ for all $i \in L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1})$. Then, $(z, y) \in P_{\mathbf{C}}^{t^1}$ from **BWC** by applying β), which is a contradiction from $(z, y) \notin P_{\mathbf{C}}^{t^1}$. Thus, $(x, z) \in R_{\mathbf{C}}^{t^1}$ holds. Finally, suppose that $(z, x) \in R_{\mathbf{C}}^{t^1}$. This is only available by applying α)-1), and $C_i(x) \sim_{t^1} BC^{t^1}$ for all $i \in L_{\mathbf{C}}^{t^1}(x; \succsim_{t^1})$ and $C_i(z) \sim_{t^1} BC^{t^1}$ for all $i \in L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1})$. Then, $(z, y) \in P_{\mathbf{C}}^{t^1}$ from **BWC** by applying β), which is a contradiction from $(z, y) \notin P_{\mathbf{C}}^{t^1}$. Thus, $(z, x) \in R_{\mathbf{C}}^{t^1}$ is impossible, so that $(x, z) \in P_{\mathbf{C}}^{t^1}$ holds.

Case 3: Let $(x, y) \in P_{\mathbf{C}}^{t^1}$ be derived from **BWC** by applying α)-2) or γ). First, suppose that $BC^{t^1} \sim_{t^1} C_i(y)$ for all $i \in L_{\mathbf{C}}^{t^1}(y; \succsim_{t^1})$. Then, $C_i(x) \succ_{t^1} BC^{t^1}$ for all $i \in L_{\mathbf{C}}^{t^1}(x; \succsim_{t^1})$ so that $(z, x) \in R_{\mathbf{C}}^{t^1}$ is never possible. Moreover, if $(z, x) \in NR_{\mathbf{C}}^{t^1}$, then $C_i(z) \succ_{t^1} BC^{t^1}$ for all $i \in L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1})$ due to **RC** with ϵ). Then, $(z, y) \in P_{\mathbf{C}}^{t^1}$ by **BWC** with α)-2), which is a contradiction from $(z, y) \notin P_{\mathbf{C}}^{t^1}$. Thus, $(x, y) \in P_{\mathbf{C}}^{t^1}$ is only possible.

Next, suppose that $BC^{t^1} \succ_{t^1} C_i(y)$ for all $i \in L_{\mathbf{C}}^{t^1}(y; \succsim_{t^1})$.

Suppose $(z, x) \in R_{\mathbf{C}}^{t^1}$. If *not* $\left[C_j(x) \succsim_{t^1} BC^{t^1}\right]$ for some $j \in L_{\mathbf{C}}^{t^1}(x; \succsim_{t^1})$, then, in order to $(z, x) \in R_{\mathbf{C}}^{t^1}$, $C_i(z) \succsim_{t^1} BC^{t^1}$ holds for all $i \in L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1})$. Thus, $(z, y) \in P_{\mathbf{C}}^{t^1}$ from **BWC** by applying α)-2), a contradiction from $(z, y) \notin P_{\mathbf{C}}^{t^1}$. If $C_j(x) \succsim_{t^1} BC^{t^1}$ for all $j \in L_{\mathbf{C}}^{t^1}(x; \succsim_{t^1})$, then $(z, x) \in R_{\mathbf{C}}^{t^1}$ is possible only from **BWC** by applying α)-2). Then, it implies that $C_j(x) \succsim_{t^1} BC^{t^1}$ for all $j \in L_{\mathbf{C}}^{t^1}(x; \succsim_{t^1})$, which again implies $(z, y) \in P_{\mathbf{C}}^{t^1}$ by **BWC** with applying α)-2), a contradiction. In summary, $(z, x) \in R_{\mathbf{C}}^{t^1}$ is impossible.

Suppose $(z, x) \in NR_{\mathbf{C}}^{t^1}$. This implies that $(z, x) \in NR_{\mathbf{C}}^{t^1}$ is derived from **RC** by applying ϵ) or ε). If ϵ) is applied, then $C_i(z) \succ_{t^1} BC^{t^1}$ holds for all $i \in L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1})$, so that $(z, y) \in P_{\mathbf{C}}^{t^1}$ from **BWC** by applying α)-2), a contradiction. If ε) is applied, then

$$\begin{aligned} \textit{not}\left[C_j(z) \succsim_{t^1} BC^{t^1}\right] \text{ for some } j &\in L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1})\text{; or} \\ \textit{not}\left[BC^{t^1} \succsim_{t^1} C_i(z)\right] \text{ for some } i &\in L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1}). \end{aligned}$$

Let both the former and the latter hold. Then, there is a common $i \in L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1})$ such that *not* $\left[C_i(z) \succsim_{t^1} BC^{t^1}\right]$ and *not* $\left[BC^{t^1} \succsim_{t^1} C_i(z)\right]$. Then, since the application of ε) implies that γ) is applied for having $(x, y) \in P_{\mathbf{C}}^{t^1}$, it follows that $(z, y) \in P_{\mathbf{C}}^{t^1}$ holds from **BWC** by applying γ), a contradiction.

Hence, either the former does not hold or the latter does not hold. Let the former do not hold, so that $C_i(z) \succsim_{t^1} BC^{t^1}$ holds for all $i \in L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1})$. Then, $(z, y) \in P_{\mathbf{C}}^{t^1}$ from **BWC** by applying α)-2), a contradiction. Let the latter do not hold, so that $BC^{t^1} \succsim_{t^1} C_i(z)$ for all $i \in L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1})$. Thus, $(z, y) \notin P_{\mathbf{C}}^{t^1}$ is derived from **RM** by applying δ). Then, by **FCD**, $(y, z) \in R_{\mathbf{C}}^{t^1}$. Thus, by transitivity, $(x, z) \in P_{\mathbf{C}}^{t^1}$, which is a contradiction from $(z, x) \in NR_{\mathbf{C}}^{t^1}$. In summary, $(z, x) \in NR_{\mathbf{C}}^{t^1}$ is impossible. Thus, $(x, z) \in P_{\mathbf{C}}^{t^1}$ holds.

Case 4: Let $(x, y) \in R_{\mathbf{C}}^{t^1}$ be derived from **RM** by applying δ). Then, $BC^{t^1} \succ_{t^1} C_i(x)$ for all $i \in L_{\mathbf{C}}^{t^1}(x; \succsim_{t^1})$; and also, $BC^{t^1} \succ_{t^1} C_i(y)$ for all $i \in L_{\mathbf{C}}^{t^1}(y; \succsim_{t^1})$. Since $(z, y) \notin P_{\mathbf{C}}^{t^1}$, either $(y, z) \in R_{\mathbf{C}}^{t^1}$ or $(y, z) \in NR_{\mathbf{C}}^{t^1}$. Suppose $(y, z) \in NR_{\mathbf{C}}^{t^1}$. Since $BC^{t^1} \succ_{t^1} C_i(y)$ for all $i \in L_{\mathbf{C}}^{t^1}(y; \succsim_{t^1})$, $(y, z) \in NR_{\mathbf{C}}^{t^1}$ implies that $BC^{t^1} \succ_{t^1} C_i(z)$ for all $i \in L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1})$. However, by **FCD**, **RM** can be applied to evaluate (y, z), which implies $(y, z) \notin NR_{\mathbf{C}}^{t^1}$, a contradiction. Thus, $(y, z) \in R_{\mathbf{C}}^{t^1}$. Then, by transitivity of $R_{\mathbf{C}}^{t^1}$, $(x, z) \in R_{\mathbf{C}}^{t^1}$ holds.

In summary, if $(x, y) \in R_{\mathbf{C}}^{t^1}$ and $(z, y) \notin P_{\mathbf{C}}^{t^1}$, then $(x, z) \in R_{\mathbf{C}}^{t^1}$.

2. Second, let us show that if $(y, z) \in R_{\mathbf{C}}^{t^2}$ and $(y, x) \notin P_{\mathbf{C}}^{t^1}$, then $(z, x) \notin$

$P_{\mathbf{C}}^{t^2}$. Suppose $(z,x) \in P_{\mathbf{C}}^{t^2}$. Since $(y,z) \in R_{\mathbf{C}}^{t^2}$, it follows from transitivity of $R_{\mathbf{C}}^{t^2}$ that $(y,x) \in P_{\mathbf{C}}^{t^2}$, which is a contradiction. Thus $(z,x) \notin P_{\mathbf{C}}^{t^2}$.

3. Third, let us show that for any $t \in T \backslash \{t^1\}$, if $(y,x) \notin P_{\mathbf{C}}^{t}$ and $(z,y) \notin P_{\mathbf{C}}^{t}$, then $(z,x) \notin P_{\mathbf{C}}^{t}$. Suppose, in the contrary, $(z,x) \in P_{\mathbf{C}}^{t}$. Then, it is derived from **BWC** by applying either of α)-2), β), or γ); or it is derived from **RM**-2) by applying δ).

Let $(z,x) \in P_{\mathbf{C}}^{t}$ be derived from **BWC**. Suppose that α)-2) or γ) is applied with $BC^t \succ_t C_i(x)$ for all $i \in L_{\mathbf{C}}^{t}(x; \succsim_t)$. Thus, by **FCD** and **RM**, it is impossible that $(y,x) \in NR_{\mathbf{C}}^{t}$. Thus, $(x,y) \in R_{\mathbf{C}}^{t}$. Then, by transitivity, $(z,y) \in P_{\mathbf{C}}^{t}$, which is a contradiction.

Suppose that $(z,x) \in P_{\mathbf{C}}^{t}$ is derived from **BWC** by applying β). Then, *not* $[C_i(x) \succsim_{t^1} BC^t]$ and *not* $[BC^t \succsim_{t^1} C_i(x)]$ hold for some $i \in L_{\mathbf{C}}^{t}(x; \succsim_{t^1})$. In this case, $(x,y) \in R_{\mathbf{C}}^{t}$ or $(x,y) \in NR_{\mathbf{C}}^{t}$. Let $(x,y) \in R_{\mathbf{C}}^{t}$. This case is derived from **BWC** by applying γ), which implies that $BC^t \succ_t C_i(y)$ for all $i \in L_{\mathbf{C}}^{t}(y; \succsim_t)$. Then, $(z,y) \in P_{\mathbf{C}}^{t}$ holds by **BWC** with applying α-2). Thus, a contradiction. Next, let $(x,y) \in NR_{\mathbf{C}}^{t}$. This is derived from **RC** by applying ε). To apply ε) for (x,y), *not* $[C_j(y) \succsim_t BC^t]$ and *not* $[BC^t \succsim_{t^1} C_j(y)]$ for some $j \in L_{\mathbf{C}}^{t}(y; \succsim_t)$ is necessary. Then, $(z,y) \in P_{\mathbf{C}}^{t}$ is derived from **BWC** by applying β), which is a contradiction.

Suppose that $(z,x) \in P_{\mathbf{C}}^{t}$ is derived from **BWC** by applying α)-2) with $BC^t \sim_t C_i(x)$ for all $i \in L_{\mathbf{C}}^{t}(x; \succsim_t)$. Then, $(y,x) \in NR_{\mathbf{C}}^{t}$ is impossible. Thus, $(x,y) \in R_{\mathbf{C}}^{t}$. Then, by transitivity, $(z,y) \in P_{\mathbf{C}}^{t}$, which is a contradiction. In summary, $(z,x) \in P_{\mathbf{C}}^{t}$ cannot be derived from **BWC**.

Let $(z,x) \in P_{\mathbf{C}}^{t}$ be derived from **RM**-2) by applying δ). Then, $BC^t \succ_t C_i(x)$ for all $i \in L_{\mathbf{C}}^{t}(x; \succsim_t)$. Thus, by **FCD** and **RM**, it is impossible that $(y,x) \in NR_{\mathbf{C}}^{t}$. Thus, $(x,y) \in R_{\mathbf{C}}^{t}$. Then, by transitivity, $(z,y) \in P_{\mathbf{C}}^{t}$, which is a contradiction. Thus, $(z,x) \in P_{\mathbf{C}}^{t}$ cannot be derived from **RM**-2).

In summary, for any $t \in T \backslash \{t^1\}$, if $(y,x) \notin P_{\mathbf{C}}^{t}$ and $(z,y) \notin P_{\mathbf{C}}^{t}$, then $(z,x) \notin P_{\mathbf{C}}^{t}$.

4. By the above arguments of **1.** and **3.**, we have $(x,z) \in R_{\mathbf{C}}^{t^1}$ and $(z,x) \notin P_{\mathbf{C}}^{t}$ for any $t \in T \backslash \{t^1\}$. Thus, $(x,z) \in R_{\mathbf{C}}^{NR}$ holds. ■

Proof of Theorem 3. Let $T = \{t^1\}$, $\{i,j,h,h'\} \subset t^1 = N$, $L_{\mathbf{C}}^{t1}(x; \succsim_{t^1}) = \{i,j,h\}$, $L_{\mathbf{C}}^{t1}(y; \succsim_{t^1}) = \{i,j,h\}$, and $L_{\mathbf{C}}^{t1}(z; \succsim_{t^1}) = \{i,j,h,h'\}$. Suppose that:

$$\begin{aligned} BC^{t^1} &\succ {}_{t^1}C_i(x) \sim_{t^1} C_j(x) \sim_{t^1} C_h(x); \\ BC^{t^1} &\succ {}_{t^1}C_i(y) \sim_{t^1} C_j(y) \sim_{t^1} C_h(y); \end{aligned}$$

$$\begin{aligned}
BC^{t^1} &\succ {}_{t^1}C_i\left(z\right) \sim_{t^1} C_j(z) \sim_{t^1} C_h\left(z\right); \\
\textit{not}\ [BC^{t^1} &\succsim {}_{t^1}C_{h'}\left(z\right)\ \textit{or}\ C_{h'}\left(z\right) \succsim_{t^1} BC^{t^1}]; \\
C_i\left(x\right) &\succ {}_iC_i(y) \succ_i C_i\left(z\right); \\
C_j\left(x\right) &\succ {}_jC_j(y) \succ_j C_j\left(z\right); \\
C_h\left(x\right) &\succ {}_hC_h(y) \succ_h C_h\left(z\right); \text{and} \\
C_{h'}\left(x\right) &\succ {}_{h'}C_{h'}(y) \succ_{h'} C_{h'}\left(z\right).
\end{aligned}$$

Moreover, for any $k \in t^1 \backslash \{i, j, h, h'\}$, let $C_k(z) \succ {}_{t^1}C_{h'}\left(z\right)$, $C_k(x) \succ_{t^1} BC^{t^1}$, $C_k(y) \succ_{t^1} BC^{t^1}$, $C_k(z) \succ_{t^1} BC^{t^1}$, and $C_k\left(x\right) \succ_k C_k(y) \succ_k C_k\left(z\right)$.

Then, since $BC^{t^1} \succ_{t^1} C_i\left(x\right)$ for all $i \in L_{\mathbf{C}}^{t^1}(x; \succsim_{t^1})$, and *not* $BC^{t^1} \succsim_{t^1} C_{h'}\left(z\right)$ & *not* $C_{h'}\left(z\right) \succsim_{t^1} BC^{t^1}$ for some $h' \in L_{\mathbf{C}}^{t^1}(x; \succsim_{t^1})$, it follows that $(z, x) \in P_{\mathbf{C}}^{t^1}$ from **BWC** with γ). Thus, since $T = \{t^1\}$ by **NR**, $(z, x) \in P_{\mathbf{C}}$, while by **WP**, $(x, z) \in P_{\mathbf{C}}$. Thus, a contradiction, which implies **NR** and **WP** are incompatible. ■

Proof of Lemma 2. 1. Let $(x, y) \in R_{\mathbf{C}}^{NR}$ and $(y, z) \in P_{\mathbf{C}}^{WP}$. This implies that there exists $t^1 \in T$ such that $(x, y) \in R_{\mathbf{C}}^{t^1}$ and $(y, x) \notin P_{\mathbf{C}}^t$ for any other $t \in T$, and $C_i(y) \succ_i C_i(z)$ holds for all $i \in N$. Moreover, $(x, y) \in R_{\mathbf{C}}^{t^1}$ is derived from **BCC** by applying either of α), β), or γ); or it is derived from **RM** by applying δ).

2. Show that for any $t \in T$, (i) $(y, x) \notin P_{\mathbf{C}}^t$, and (ii) $C_i(y) \succ_i C_i(z)$ holds for all $i \in N$ together imply that $(x, z) \in R_{\mathbf{C}}^t \cup NR_{\mathbf{C}}^t$. First of all, $(y, x) \notin P_{\mathbf{C}}^t$ if and only if $(x, y) \in R_{\mathbf{C}}^t \cup NR_{\mathbf{C}}^t$. If $(x, y) \in R_{\mathbf{C}}^t$, then it is derived from **BCC** by applying α), β), or γ), or from **RM** by applying δ). If $(x, y) \in NR_{\mathbf{C}}^t$, then it is derived from **RC** by applying ϵ) or ε).

2-i). Suppose $(x, y) \in R_{\mathbf{C}}^t$ is derived from **BWC** by applying α), β), or γ). Then, *not* $[BC^t \succsim_t C_i(x)]$ for some $i \in L_{\mathbf{C}}^t(x; \succsim_t)$. Note, since $C_i(y) \succ_i C_i(z)$ holds for all $i \in N^t$, we have *not* $[C_h\left(z\right) \succ_t C_j\left(y\right)]$ holds for any $j \in L_{\mathbf{C}}^t(y; \succsim_t)$ and any $h \in L_{\mathbf{C}}^t(z; \succsim_t)$. (In fact, if $[C_h\left(z\right) \succ_t C_j\left(y\right)]$ holds for some $j \in L_{\mathbf{C}}^t(y; \succsim_t)$ and some $h \in L_{\mathbf{C}}^t(z; \succsim_t)$, then $C_h\left(z\right) \succ_t C_j\left(z\right)$ holds, which is a contradiction from $h \in L_{\mathbf{C}}^t(z; \succsim_t)$.) Then, since $BC^t \succsim_t C_j\left(y\right) \succ_t C_j(z)$ for all $j \in L_{\mathbf{C}}^t(y; \succsim_t)$, then *not* $[C_h\left(z\right) \succsim_t BC^t]$ holds for any $h \in L_{\mathbf{C}}^t(z; \succsim_t)$. (In fact, if $[C_h\left(z\right) \succsim_t BC^t]$ holds for some $h \in L_{\mathbf{C}}^t(z; \succsim_t)$, it implies $C_h\left(z\right) \succ_t C_j(z)$ for all $j \in L_{\mathbf{C}}^t(y; \succsim_t)$, which is a contradiction from $h \in L_{\mathbf{C}}^t(z; \succsim_t)$.) In summary, the above arguments imply that only either of α), β), γ), or ε) is applied to (x, z), thus $(x, z) \in R_{\mathbf{C}}^t \cup NR_{\mathbf{C}}^t$.

2-ii). Suppose $(x, y) \in NR_{\mathbf{C}}^t$ is derived from **RC** by applying ϵ) or ε). If

ϵ) is applied, then $[C_i(x) \succ_t BC^t]$ holds for any $i \in L^t_{\mathbf{C}}(x; \succsim_t)$. Then, $(x,z) \in R^t_{\mathbf{C}} \cup NR^t_{\mathbf{C}}$ is derived by applying either of α), β), or ϵ). If ε) is applied, then [*not* $[BC^t \succsim_t C_i(x)]$ for some $i \in L^t_{\mathbf{C}}(x; \succsim_t)$ and *not* $[C_j(x) \succsim_t BC^t]$ for some $j \in L^t_{\mathbf{C}}(x; \succsim_t)$] and [*not* $[BC^t \succsim_t C_i(y)]$ for some $i \in L^t_{\mathbf{C}}(y; \succsim_t)$ and *not* $[C_j(y) \succsim_t BC^t]$ for some $j \in L^t_{\mathbf{C}}(y; \succsim_t)$]. Then, by either **RC** with ε) or **BWC** with γ), $(x,z) \in R^t_{\mathbf{C}} \cup NR^t_{\mathbf{C}}$ is derived.

2-iii). Suppose $(x,y) \in R^t_{\mathbf{C}}$ is derived from **RM** by applying δ). Then, since $C_i(y) \succ_i C_i(z)$ holds for all $i \in N^t$, we have, by combining with **RM** with δ), $C_i(x) \succsim_t C_j(y) \succ_t C_j(z)$ for any $i \in L^t_{\mathbf{C}}(x; \succsim_t)$ and any $j \in L^t_{\mathbf{C}}(y; \succsim_t)$. Note that, for each $j \in L^t_{\mathbf{C}}(y; \succsim_t)$, there exists $h \in L^t_{\mathbf{C}}(z; \succsim_t)$ such that $C_j(z) \succ_t C_h(z)$. Thus, there exists $h \in L^t_{\mathbf{C}}(z; \succsim_t)$ such that $C_i(x) \succ_t C_h(z)$ for any $i \in L^t_{\mathbf{C}}(x; \succsim_t)$. Thus, if $BC^t_t \succ_t C_h(z)$ for any $h \in L^t_{\mathbf{C}}(z; \succsim_t)$, then by **RM** with δ), $(x,z) \in P^t_{\mathbf{C}}$. Moreover, since $BC^t \succsim_t C_j(y) \succ_t C_j(z)$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$, *not* $[C_i(z) \succsim_t BC^t]$ holds for any $i \in L^t_{\mathbf{C}}(z; \succsim_t)$. Suppose that there exists $h' \in L^t_{\mathbf{C}}(z; \succsim_t)$ such that *not* $[BC^t \succsim_t C_{h'}(z)]$. However, since $h \in L^t_{\mathbf{C}}(z; \succsim_t)$ has $BC^t \succsim_t C_j(z) \succ_t C_h(z)$, **D** implies $C_{h'}(z) \succ_t C_h(z)$, which is a contradiction from $h' \in L^t_{\mathbf{C}}(z; \succsim_t)$. Thus, there is no $h' \in L^t_{\mathbf{C}}(z; \succsim_t)$ such that *not* $[BC^t \succsim_t C_{h'}(z)]$. Thus, from **RC** by applying ε), $(x,z) \in NR^t_{\mathbf{C}}$.

3. Let $(x,y) \in R^{t^1}_{\mathbf{C}}$ be derived from **BWC** by applying either of α), β), or γ).

3-i): Let $(x,y) \in R^{t^1}_{\mathbf{C}}$ be derived from **BWC** by applying α). Then, $BC^{t^1} \succsim_{t^1} C_j(y)$ for all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$, and $C_i(x) \succsim_{t^1} BC^{t^1}$ for all $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$. Then, we can show that $C_i(x) \succsim_t BC^{t^1}$ holds for every $i \in N^{t^1}$. First, if $i \in N^{t^1}$ is $C_i(x) \succ_t C_j(x)$ for some $j \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$, it is obvious. If $i \in N^{t^1}$ is *not* $[C_i(x) \succ_{t^1} C_j(x)]$ for any $j \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$, then $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$, so that $C_i(x) \succsim_{t^1} BC^{t^1}$ holds.

Let us consider $L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. Since $C_i(y) \succ_{t^1} C_i(z)$ holds for all $i \in N$, it follows that, for any $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$ and any $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$, *not* $[C_i(z) \succ_{t^1} C_j(y)]$ holds. In fact, if $[C_i(z) \succ_{t^1} C_j(y)]$ holds for some $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$ and some $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$, then $C_i(z) \succ_{t^1} C_j(z)$ holds, which is a contradiction from $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. Then, since $BC^{t^1} \succsim_{t^1} C_j(y) \succ_{t^1} C_j(z)$ for all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$, *not* $\left[C_i(z) \succsim_{t^1} BC^{t^1}\right]$ holds for any $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. In fact, if $\left[C_i(z) \succsim_{t^1} BC^{t^1}\right]$ holds for some $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$, it implies $C_i(z) \succ_{t^1} C_j(z)$, which is a contradiction from $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. In summary, $(x,z) \in P^{t^1}_{\mathbf{C}}$ is derived from **BWC** by applying α)-2) or β).

3-ii): Let $(x,y) \in P^{t^1}_{\mathbf{C}}$ be derived from **BWC** by applying β). Then, *not* $\left[C_j(y) \succsim_{t^1} BC^{t^1}\right]$ for some $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$, and $C_i(x) \succsim_{t^1} BC^{t^1}$ for all $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$. Suppose $(z, x) \in R^{t^1}_{\mathbf{C}}$. Note that neither of α), β), γ), δ) can derive $(z, x) \in R^{t^1}_{\mathbf{C}}$, thus (z, x) corresponds to ϵ) or ε), which leads to $(x, z) \in NR^{t^1}_{\mathbf{C}}$ by **RC**, a contradiction. Thus, $(x, z) \in NR^{t^1}_{\mathbf{C}}$ or $(x, z) \in P^{t^1}_{\mathbf{C}}$ holds. Suppose $(x, z) \in NR^{t^1}_{\mathbf{C}}$. This implies that (x, z) corresponds to ϵ) , so that $C_i(z) \succ_{t^1} BC^{t^1}$ for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. However, *not* $\left[C_j(y) \succsim_{t^1} BC^{t^1}\right]$ for some $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$ and $C_j(y) \succ_{t^1} C_j(z)$, which implies that *not* $\left[C_j(z) \succsim_{t^1} BC^{t^1}\right]$. Then, since $C_i(z) \succ_{t^1} BC^{t^1}$ for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$, $j \notin L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$ holds, which further implies that there exists $h \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$ such that $C_j(z) \succ_{t^1} C_h(z) \succ_{t^1} BC^{t^1}$, a contradiction. Thus, $(x, z) \in NR^{t^1}_{\mathbf{C}}$ is impossible, so that $(x, z) \in P^{t^1}_{\mathbf{C}}$ holds.

3-iii): Let $(x, y) \in P^{t^1}_{\mathbf{C}}$ be derived from **BWC** by applying α)-2) or γ). This implies *not* $\left[BC^{t^1} \succsim_{t^1} C_i(x)\right]$ for some $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$, and $BC^{t^1} \succ_{t^1} C_j(y)$ for all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$.

Suppose $(z, x) \in R^{t^1}_{\mathbf{C}}$. In order to $(z, x) \in R^{t^1}_{\mathbf{C}}$, $C_i(z) \succsim_{t^1} BC^{t^1}$ holds for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. However, since $BC^{t^1} \succ_{t^1} C_j(y) \succ_{t^1} C_j(z)$ for all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$, $C_i(z) \succ_{t^1} C_j(z)$ for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$, which is a contradiction. Thus, in summary, $(z, x) \notin R^{t^1}_{\mathbf{C}}$.

Suppose $(z, x) \in NR^{t^1}_{\mathbf{C}}$. This implies that $(z, x) \in NR^{t^1}_{\mathbf{C}}$ is derived from **RC** by applying ϵ) or ε). If ϵ) is applied, then $C_i(z) \succ_{t^1} BC^{t^1}$ holds for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. Then, since $BC^{t^1} \succ_{t^1} C_j(y) \succ_{t^1} C_j(z)$ for all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$, $C_i(z) \succ_{t^1} C_j(z)$ for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$, which is a contradiction. If ε) is applied, then

$$\begin{aligned} \textit{not} \left[C_j(z) \succsim_{t^1} BC^{t^1}\right] \text{ for some } j &\in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1}); \\ \text{or } \textit{not} \left[BC^{t^1} \succsim_{t^1} C_i(z)\right] \text{ for some } i &\in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1}). \end{aligned}$$

Let the former do not hold, so that $C_i(z) \succsim_{t^1} BC^{t^1}$ holds for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. Then, again, since $BC^{t^1} \succ_{t^1} C_j(y) \succ_{t^1} C_j(z)$ for all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$, $C_i(z) \succ_{t^1} C_j(z)$ for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$, which is a contradiction. Let the latter do not hold, so that $BC^{t^1} \succsim_{t^1} C_i(z)$ for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. Thus, $(z, y) \notin P^{t^1}_{\mathbf{C}}$ is derived from **RM** by applying δ). Then, by **FCD**, $(y, z) \in R^{t^1}_{\mathbf{C}}$. Thus, by transitivity, $(x, z) \in P^{t^1}_{\mathbf{C}}$, which is a contradiction from $(z, x) \in NR^{t^1}_{\mathbf{C}}$. In summary, $(z, x) \in NR^{t^1}_{\mathbf{C}}$ does not hold. Thus, $(x, z) \in P^{t^1}_{\mathbf{C}}$ holds.

4. Let $(x,y) \in R_{\mathbf{C}}^{t^1}$ be derived from **RM** by applying δ). Then, $BC^{t^1} \succ_{t^1} C_i(x)$ for all $i \in L_{\mathbf{C}}^{t^1}(x;\succsim_{t^1})$; and also, $BC^{t^1} \succ_{t^1} C_i(y)$ for all $i \in L_{\mathbf{C}}^{t^1}(y;\succsim_{t^1})$. Since $BC^{t^1} \succ_{t^1} C_j(y) \succ_{t^1} C_j(z)$ for all $j \in L_{\mathbf{C}}^{t^1}(y;\succsim_{t^1})$, **D** implies that $BC^{t^1} \succ_{t^1} C_h(z)$ for all $h \in L_{\mathbf{C}}^{t^1}(z;\succsim_{t^1})$. Thus, by **FCD** and **RM** with δ), $(y,z) \in R_{\mathbf{C}}^{t^1}$ is derived. Hence, the pair of (x,z) corresponds to the case δ). Moreover, $C_i(x) \succsim_{t^1} C_j(y) \succ_{t^1} C_h(z)$ holds for all $i \in L_{\mathbf{C}}^{t^1}(x;\succsim_{t^1})$, all $j \in L_{\mathbf{C}}^{t^1}(y;\succsim_{t^1})$, and all $h \in L_{\mathbf{C}}^{t^1}(z;\succsim_{t^1})$. Then, by the transitivity of $\succsim_{t^1}$, $C_i(x) \succ_{t^1} C_h(z)$ holds for all $i \in L_{\mathbf{C}}^{t^1}(x;\succsim_{t^1})$ and all $h \in L_{\mathbf{C}}^{t^1}(z;\succsim_{t^1})$. Thus, $(x,z) \in P_{\mathbf{C}}^{t^1}$ holds by **RM**.

5. In combining the above arguments, if $(x,y) \in R_{\mathbf{C}}^{NR}$ and $(y,z) \in P_{\mathbf{C}}^{WP}$, then $(x,z) \in P_{\mathbf{C}}^{t^1}$ holds and $(x,z) \in R_{\mathbf{C}}^{t} \cup NR_{\mathbf{C}}^{t}$ for any $t \in T \backslash \{t^1\}$, thus $(x,z) \in P_{\mathbf{C}}^{NR}$. ■

Proof of Lemma 3. Let $(x,y) \in P_{\mathbf{C}}^{WP}$ and $(y,z) \in R_{\mathbf{C}}^{NR}$. This implies that there exists $t^1 \in T$ such that $(y,z) \in R_{\mathbf{C}}^{t^1}$ and $(z,y) \notin P_{\mathbf{C}}^{t}$ for any other $t \in T$, and $C_i(x) \succ_{t^1} C_i(y)$ holds for all $i \in N$.

1. We will show that, in this case, $(x,z) \in P_{\mathbf{C}}^{t^1}$ holds.

First, if $(y,z) \in P_{\mathbf{C}}^{t^1}$ is derived from **BWC** by applying either of α)-2), β), or γ), then $(x,z) \in P_{\mathbf{C}}^{t^1}$ is derived from **BWC** by applying either of α)-2), β), or γ).

Second, if $(y,z) \in I_{\mathbf{C}}^{t^1}$ is derived from **BWC** by applying α)-1), then $(x,z) \in R_{\mathbf{C}}^{t^1}$ is derived from **BWC** by applying α). Moreover, since $C_i(x) \succ_{t^1} BC^{t^1}$ for all $i \in L_{\mathbf{C}}^{t^1}(x;\succsim_{t^1})$, $(x,z) \in P_{\mathbf{C}}^{t^1}$ holds by **BWC** with α)-2).

Third, let $(y,z) \in R_{\mathbf{C}}^{t^1}$ be derived from **RM** by applying δ). If $C_i(x) \prec_t BC^t$ for all $i \in L_{\mathbf{C}}^{t}(x;\succsim_t)$, then $(x,z) \in P_{\mathbf{C}}^{t^1}$ is derived from **RM**-2) and **FCD**; otherwise, then $(x,z) \in P_{\mathbf{C}}^{t^1}$ is derived from **BWC** and **FCD** by applying α)-2) or γ).

In summary, $(x,z) \in P_{\mathbf{C}}^{t^1}$ holds for $t^1 \in T$.

2. Next, we will show that $(z,x) \notin P_{\mathbf{C}}^{t}$ for any $t \in T \backslash \{t^1\}$. Note that for any $t \in T \backslash \{t^1\}$, (i) $(z,y) \notin P_{\mathbf{C}}^{t}$, and (ii) $C_i(x) \succ_i C_i(y)$ holds for all $i \in N$ together imply that $(x,z) \in R_{\mathbf{C}}^{t} \cup NR_{\mathbf{C}}^{t}$. First of all, $(z,y) \notin P_{\mathbf{C}}^{t}$ if and only if $(y,z) \in R_{\mathbf{C}}^{t} \cup NR_{\mathbf{C}}^{t}$.

If $(y,z) \in R_{\mathbf{C}}^{t}$, then it is derived from **BWC** by applying α), β), or γ), or from **RM** by applying δ). Then, as shown in the case of **1.**, we can see that $(x,z) \in P_{\mathbf{C}}^{t}$ holds for any $t \in T \backslash \{t^1\}$.

If $(y,z) \in NR_{\mathbf{C}}^{t}$, then it is derived from **RC** by applying ϵ) or ε). If ϵ) is applied for $(y,z) \in NR_{\mathbf{C}}^{t}$, then $(x,z) \in NR_{\mathbf{C}}^{t}$ also holds by **RC**. If ε) is applied for $(y,z) \in NR_{\mathbf{C}}^{t}$, then [*not* $[BC^t \succsim_t C_i(y)]$ for some $i \in L_{\mathbf{C}}^{t}(y;\succsim_t)$

and *not* $[C_j(y) \succsim_t BC^t]$ for some $j \in L^t_{\mathbf{C}}(y; \succsim_t)]$ and $[$*not* $[BC^t \succsim_t C_i(z)]$ for some $i \in L^t_{\mathbf{C}}(z; \succsim_t)$ and *not* $[C_j(z) \succsim_t BC^t]$ for some $j \in L^t_{\mathbf{C}}(z; \succsim_t)]$. Then, since $C_i(x) \succ_t C_i(y)$ holds for all $i \in N$, **RC** with ε) is again applied for $(x, z) \in NR^t_{\mathbf{C}}$ or **BWC** with β) is applied for $(x, z) \in R^t_{\mathbf{C}}$.

Thus, in summary, for any $t \in T \setminus \{t^1\}$, $(x, z) \in R^t_{\mathbf{C}} \cup NR^t_{\mathbf{C}}$ holds.

3. From **1.** and **2.**, $(x, z) \in P^{NR}_{\mathbf{C}}$ holds, so that $(x, z) \in P^*_{\mathbf{C}}$. ■

8 References

Arrow, K. J. (1951/1963) *Social Choice and Individual Values*, 2nd ed., Wiley, New York.

Basu, K. and Lopez-Calva, L. F. (2011) "Functionings and capabilities," in K.J. Arrow, A. Sen, and K. Suzumura (eds.), *Handbook of Social Choice and Welfare*, vol. 2, North-Holland, Amsterdam, 154-187.

d'Aspremont, C. and Gevers, L. (2002) "Social welfare functionals and interpersonal comparability," in K.J. Arrow, A. Sen, and K. Suzumura (eds.), *Handbook of Social Choice and Welfare*, vol. 1, North-Holland, Amsterdam, 459- 541.

Fleurbaey, M. (2007) "Social choice and the indexing dilemma," *Social Choice and Welfare*, 29, 633-648.

Fleurbaey, M. and Hammond, P. J. (2004) "Interpersonally comparable utility," in S. Barbera, P.J. Hammond, and C. Seidl (eds.), *Handbook of Utility Theory*, vol. 2, Kluwer, Boston.

Fleurbaey, M. and Mongin, P. (2005) "The news of the death of welfare economics is greatly exaggerated," *Social Choice and Welfare*, 25, 381-418.

Foster, J. (1984) "On economic poverty: a survey of aggregate measures," *Advances in Econometrics*, 3, 215-251.

Gotoh, R. and Yoshihara, N. (2003) "A class of fair distribution rules a la Rawls and Sen," *Economic Theory*, 22, 63-88.

Herrero, C. (1996) "Capabilities and utilities," *Economic Design*, 2, 69-88.

Pattanaik, P.K. and Xu, Y. (1990) "On ranking opportunity sets in terms of freedom of choice," *Recherches Economiques de Louvain*, 56, 383-390.

Pattanaik, P.K. and Xu, Y. (2007): "Minimal Relativism, Dominance, and Standard of Living Comparisons Based on Functionings," *Oxford Economic Papers*, 59, 354-374.

Pattanaik P. K. and Xu, Y. (2012) "On dominance and context-dependence in decisions involving multiple attributes," *Economics and Philosophy*, 28, 117-132.

Rawls, J. (1971) *A Theory of Justice*, Harvard University Press, Cambridge, Mass.

Sen, A.K. (1970) *Collective Choice and Social Welfare*, Holden-Day, San Francisco.

Sen, A.K. (1980) "Equality of what?" *The Tanner Lectures on Human Values*, vol. 1, University of Utah Press, Salt Lake City (reprinted in 1982, *Choice, Welfare, and Measurement*, Blackwell, Oxford, 353-369).

Sen, A.K. (1981) *Poverty and Famines: An Essay on Entitlement and Deprivation*, Oxford University Press, Oxford.

Sen, A.K. (1985) *Commodities and Capabilities*, North-Holland, Amsterdam.

Sen, A.K. (1984) "The Living Standard," *Oxford Economic Papers*, 36, 198, 74-90.

Sen, A.K. (1997) *On Economic Inequality, expanded edition with a sub- stantial annex* by James E. Foster and Amartya K. Sen, Clarendon Press, Oxford.

Sen, A.K. (1999) *Reason Before Identity, The Romanes Lecture for 1998*, Oxford University Press, Oxford.

Sen, A.K. (2002) *Rationality and Freedom*, Harvard University Press, Cambridge, Mass.

Sen, A.K. (2006) *Identity and Violence*: *The Illusion of Destiny*, W.W. Norton & Company, New York.

Sen, A.K. (2009a) *The Idea of Justice*, Allen Lane, London.

Sen, A.K. (2009b) *Economics, ethics and law*, in R. Gotoh and P. Dumouchel (eds.), *Against Injustice: The New Economics of Amartya Sen*, Cambridge University Press, Cambridge, 39-54.