Discussion Paper Series A No.591

Securing Basic Well-being for All

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October 2014
(First version: July 2013)

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Current Version: October 2014

Abstract

The purpose of this paper is to examine the possibility of a social choice rule to implement a social policy for "securing basic well-being for all." The paper introduces a new scheme of social choice, called a *social relation function* (**SRF**), which associates a reflexive and transitive binary relation over a set of social policies to each profile of *individual well-being appraisals* and each profile of *group evaluations*. As part of the domains of **SRF**s, the available class of group evaluations is constrained by three conditions. Furthermore, the *non-negative response* (**NR**) and the *weak Pareto condition* (**WP**) are introduced. **NR** demands giving priority to group evaluation, while treating the groups as formally equal relative to each other. **WP** requires treating impartially the well-being appraisals of all individuals. In conclusion, this paper shows that under some reasonable assumptions, there exists an **SRF** that satisfies **NR** and **WP**.

JEL: D63.

Keywords: basic well-being; individual well-being appraisals; social relation functions.

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*We are extremely grateful to Marc Fleurbaey, James Foster, Peter Hammond, Carmen Herrero, Laurence Kranich, Prasanta Pattanaik, Yoshi Saijo, and Yongsheng Xu for their fruitful discussions relevant to this paper. Special thanks also go to the thorough referee of this journal, whose comments and suggestions have led to substantial improvements in the paper.

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1 Introduction

Despite the United Nations' declaration of universal human rights in 1948, persons with disabilities have long been restricted in their effective exercise of these rights. The Convention on the Rights of Persons with Disabilities, which was adopted by the United Nations in 2006, has brought about new insights on human rights as well as democracy. The convention is innovative in that it requires the effective exercise of human rights for persons with disabilities by, for example, removing discriminatory practices that have built up over time and implementing "reasonable accommodations" in public places.[1] Further, a remarkable aspect of drafting this convention is that, persons with disabilities have taken the initiative and offered their expertise in assessing alternative articles, going by the slogan "Nothing about us, without us."

The above example urges us to reconsider the appropriateness of the standard framework of social choice theory, as there is little discussion about the relationship between asymmetrical prior treatments of individual preferences and the different types of social choice problems they are admissible in. In addition, it indicates that the asymmetrical prior treatment of individual preferences could be appropriate when the given social choice problem is on the effective exercise of universal human rights with respect to the particularity of those individuals. The main purpose of this paper is to formulate a social choice procedure that permits asymmetrical prior treatments for disadvantaged groups not as exceptions but as a general rule under some reasonable and socially imposed conditions. More specifically, we focus on a specific type of social choice problem: selecting a public policy in terms of securing basic well-being for all and defining the concept of a "group" as a representation of particularity that requires an asymmetrical prior treatment in order to achieve universal aims such as this.

The framework of this paper is as follows. First, the key concept of this paper, an individual's "well-being,"[2] is defined as a function of individuals' abilities and social policies (called *well-being transformations*). While no particular type of a well-being indicator is presumed, it is generically multi-dimensional in the space of plural attributes,[3] each of which is observable in public. For the sake of simplicity and without loss of generality, individuals'

[1] See Convention on the Rights of Persons with Disabilities (Article 2).

[2] The concept of "functionings vector" or "capability" a la Amartya Sen are typical examples of the well-being indicator (Sen, 1980, 1985).

[3] For a detailed discussion of evaluative attributes, see Pattanaik and Xu (2007, 2012).

well-being transformations are assumed to be fixed and the profile of each individual well-being is identified corresponding to each alternative social policy. The paper also refers to "basic well-being," which represents a critical reference point of multi-dimensional well-beings that one can legitimately claim to have met by social policies, and each group can refer to it to identify the "injustice" of social policies.[4]

Second, a social choice rule to select a social policy for securing basic well-being for all is introduced and examined. This social choice rule, which we call a *social relation function* (**SRF**), is defined as having three elements as its informational basis: the individual appraisal of well-being contents, the group appraisal of well-being contents, and the group evaluation of social policies. The individual appraisal is formulated as a binary relation defined over the universal class of well-being contents; the group appraisal is formulated as the intersection of its members' appraisals; and the group evaluation is formulated as a binary relation defined over social policies, focusing on its least advantaged members who are identified on the basis of the group appraisal. By using the three elements of information, an **SRF** forms a social evaluation, which is a binary relation defined over social alternatives.

Some remarks on the **SRF** framework are necessary. First, the individual appraisal of well-being contents is based on her own conception of the good, while the group appraisal of well-being contents is based on the conception shared by the members. In contrast, the group evaluation of social policies is supposed to correspond to conceptions of justice shared in the society, which are embodied explicitly by the axioms and conditions and implicitly by public reasoning. Thus, throughout this paper, "better or worse" is used in the comparative evaluation of well-being contents, while "more or less just" or "less or more unjust" is used in the comparative evaluation of social policies.[5]

Second, because of the multiplicity of attributes for well-being, the types of disadvantages may be diversified, which could generate different types of "the least advantaged." Although they share the common feature that

Also see Fleurbaey (2007) and Fleurbaey and Hammond (2004) for a discussion of well-being indicators.

[4]For instance, if the well-being indicator is specified by "capability," basic well-being implies "basic capability" (Sen, 1980, p. 367).

[5]This usage is derived from the distinction of concepts of "the good" and of "justice" according to Rawls (Rawls, 1971, pp. 396-397), while "less or more unjust" is derived from Sen, as mentioned in section 4.

they lack access to basic well-being, as illustrated in section 2, the concrete contents (the lists, scales, and sizes) of basic well-being might differ from each other, depending upon the types of disadvantages and their corresponding conceptions of justice.[6] Therefore, in this paper, the concept of "group" is operationally defined as a maximum unit that can commonly share a concrete content of basic well-being and can identify "the least advantaged" within the group.

Third, due to the multi-dimensionality of well-being contents, the individual and group appraisals could be *incomplete*,[7] which implies that intra-personal full comparability of these appraisals cannot be generally presumed, while inter-personal comparability of these appraisals can be legitimately presumed to some extent at least in a group. To what extent should we assume inter-personal and intra-personal comparability is one of our research questions.

Due to the three-component structure of the informational basis, **SRF**s allow the appropriate asymmetric and prior treatment of specific groups of individuals relevant to the underlying social choice problem in question as well as the symmetric treatment of individual appraisals. To incorporate this idea formally, we introduce two basic axioms, the *non-negative response* (**NR**) and the *weak Pareto* (**WP**) axioms for **SRF**s within this framework. **NR** requires that **SRF**s should give priority to a disadvantaged group's evaluation whenever any other groups' evaluations are not completely opposite to this group's, while **WP** requires that **SRF**s should treat every individual's appraisal symmetrically.

Given the possibility of a prior treatment of disadvantaged groups, we introduce domain conditions of group evaluations to restrict groups' "decisive powers" so as not to depart from the general societal goal. We name the domain conditions the *basic well-being condition,* the *restricted monotonicity,* and the *refrain condition.* These conditions together stipulate that any specific disadvantaged group should evaluate social state x as "more just"

[6]Our framework follows John Rawls' difference principle in that securing basic well-being of the least advantaged respective to each policy is necessary and sufficient for achieving the social goal of securing basic well-being for all. Yet, although Rawls' model assumes inter-personal level-comparability for society as a whole, our model starts from the possibility of different types of "the least advantaged" derived from different types of disadvantages.

[7]Note that the line of research on ranking opportunity sets initiated by Pattanaik and Xu (1990) also does not presume completeness of binary relations over opportunity sets.

than social state y whenever (i) its least advantaged members' well-being contents under x (*resp.* y) are better or at least not worse (*resp.* not better or even worse) than under basic well-being; or (ii) its least advantaged members' well-being contents are better under x than under y, given that their respective well-being contents are worse under x and y respectively than under basic well-being. Moreover, this group should refrain from comparing x and y whenever its least advantaged members' well-being contents are better under x and y respectively than under basic well-being.

An interesting question is to examine the general existence of an **SRF** satisfying the three domain conditions as well as the two basic axioms. On the one hand, such an existence problem may have some similarity to the dominance and context-dependence paradox observed by Pattanaik and Xu (2007; 2012). As a typical example of this kind of paradox, recall the Pareto-liberal paradox initiated by Sen (1970), which points out the incompatibility of minimal liberty and the Pareto principle, where the former is formulated as the local decisiveness of some individuals, while the latter is formulated as the global decisiveness of all individuals. Incidentally, in our framework, the three conditions of group evaluations and **NR** together imply that a disadvantaged group is given locally decisive power, in a weak sense,[8] over the specific pairs of alternatives.

On the other hand, the existence issue of **SRF**s should not be argued analogical to the original Pareto-liberal paradox. For, firstly, the locally decisive power of a disadvantaged group is much weaker than the standard notion of local decisive power discussed in Arrow (1951/1963) and Sen (1970); secondly, the least advantaged members of each group may vary owing to the change of social policies, which makes it more complicated to identify each group evaluation; and finally, the domain of **SRF**s is not universal but restricted by the three conditions. Moreover, among other things, the key factor of this existence issue is the incompleteness of binary relations as the informational basis of **SRF**s.[9] In fact, our paper shows the extent to which the incompleteness of group appraisals is acceptable so as not to rule out the existence of **SRF**s that are compatible with **NR** and **WP**.

In the following discussion, Section 2 provides remarks on the concept of

[8]The intention of "in a weak sense" here is that this group's 'local decisiveness' over such pairs is conditional on there being no resistance of any other groups.

[9]Incompleteness of binary relations as the informational basis is not assumed in the context of the Pareto-liberal paradox as well as other types of the dominance and context-dependence paradox discussed by Pattanaik and Xu (2012).

"group" with regard to the aim of securing basic well-being for all. Section 3 provides the basic **SRF** framework and section 4 the three conditions for group evaluations and the relevant two axioms. Section 5 discusses the existence problem of **SRF**s satisfying these properties. Section 6 provides some philosophical implications of this paper, and section 7 concludes the paper.

2 Group Characteristics: Differences in "Basic Well-being" and the Corresponding Conceptions of Justice

As mentioned in section 1, basic well-being represents a critical reference point that one can legitimately claim to have met by social policies. Assuming three types of disadvantages, this section illustrates the contents of basic well-being and their corresponding conceptions of justice.

The first type of disadvantage is closely related to what Aristotle called "justice as redress." It is based on recognizing the cause of the suffered disadvantage as an injustice that needs to be redressed. Examples are disadvantages that derive from historical injustices such as colonial exploitation and the ill-treatment of indigenous populations and victims of social diseases.

The second type of disadvantage is related to the conception of "justice as compensation." This concept implies that individuals should be recognized as disadvantaged if their vulnerability is due to the failure of social institutions to protect them from social discrimination, such as persons with disabilities or certain diseases, or those discriminated on the basis of age, nationality, gender, or being a single parent.

The third type of disadvantage relates to the concept of "justice as protection." This concept considers it unjust that some individuals have less than what is necessary for a minimum standard of wholesome and cultured living.[10] Redressing this requires a form of outcome equality to bring every individual up to a reference point. This concept focuses on individuals, unlike the first two concepts, whose specific causes of difficulties can be hard to identify.

Because of this diversity of disadvantages and of the forms of justice underlying them, the concrete conception of basic well-being becomes plural.

[10]For example, article 25 of the Japanese Constitution stipulates "the right to the minimum standards of wholesome and cultured living."

Moreover, under a common concept of basic well-being, special needs must be addressed relative to the different types of disadvantages. To demonstrate this, consider some examples of each disadvantage type. Individuals who have suffered historical injustice due to an atomic bomb, an event that completely changed their life-goals, are affected by the first type of disadvantage. Many have decided to live as witnesses of this social disaster in order to prevent it from ever happening again at any other place or time. In such cases, air tickets to fly to New York, which holds the "No more Hiroshima/Nagasaki Congress," or a grant for publishing their memoirs may be counted as a necessity for securing their basic well-being. Similarly, with respect to the second type of disadvantage, the full and equal enjoyment of all human rights and fundamental freedoms should be promoted, protected, and ensured for individuals with disabilities to function as active members of society. For example, for individuals using the wheelchair, one of the essential claims to secure basic well-being would be to remove environmental barriers such as inaccessible buses or staircases. Finally, for individuals affected by the third disadvantage type, such as the homeless, it is important to claim basic needs including food, closing, shelter, and health care to protect their right to the minimum standards of wholesome and cultured living as a means to secure basic well-being.

Lastly, it should be noted that an individual might actually suffer from all three types of disadvantages mentioned above and as a result will be included in each of the three groups. This implies that such an individual's basic well-being consists of three aspects that cannot be compared intra-personally, while each of the three aspects permits inter-personal comparison within each group. In this case, the individual can participate in the process of making an evaluation of each group, also deserving to take advantage of social policies addressing all three types of disadvantages, although the actual amount of provision might be reduced considering combination effects of the three policies.

3 The Basic Model

Consider a society with population $N = \{1, 2, ..., i, ...n\}$, where $2 \leq n < +\infty$. Let us denote a *social state* by x, and the set of all possible social states by X, where $3 \leq \#X < +\infty$. Each $x \in X$ may be interpreted as representing an admissible social policy. Thus, we sometimes call each $x \in X$ a *social* *policy* x. Note that a social policy x does not necessarily represent a single policy. For instance, it may present a bundle of multiple social policies or a state of resource allocation realized by a certain bundle of social policies.

For each $i \in N$, let Z_i be a product of some multiple topological spaces, which represents the *set of conceivable well-being contents for* i. Let $Z \equiv \cup_{i \in N} Z_i$. For each $i \in N$, let i's *well-being transformation* be a mapping $C_i : X \to Z_i$ such that for each $x \in X$, $C_i(x)$ is a vector in Z_i. C_i represents an individual's ability to transform each social policy to a content of well-being and $C_i(x)$ represents individual i's well-being available under the social policy x.[11] Let $\mathbf{C} \equiv (C_i)_{i \in N}$ be a profile of the well-being transformations. Denote the admissible set of profiles of well-being transformations by $\mathcal{C}$.

Given Z, for each $i \in N$, let us define a binary relation $\succsim_i$ on Z, which is *reflexive* and *transitive*. We call this $\succsim_i$ a *well-being appraisal of* i. The interpretation of the well-being appraisal $\succsim_i$ is that, for any $C, C' \in Z$, $C \succsim_i C'$ if and only if C is at least as good as C' for i. Given $\succsim_i$ defined on Z, let $C \succ_i C'$ if and only if $C \succsim_i C'$ holds but $C' \succsim_i C$ does not hold; let $C \sim_i C'$ if and only if $C \succsim_i C'$ and $C' \succsim_i C$ hold. If Z is a partially ordered set endowed with a partial ordering $\geq$ on Z,[12] it may be assumed that for any $i \in N$ and any $C, C' \in Z$, if $C \geq C'$, then $C \succsim_i C'$, and if $C > C'$, then $C \succ_i C'$. The well-being appraisal $\succsim_i$ reflects a bundle of criteria for comparing i's well-being contents.

Next, let us define the concepts of group and group appraisal. Given society N, there exists a set of characteristics T with generic element t such that (1) $0 < \#T \leqq \#N$; and (2) for each $\mathbf{C} \in \mathcal{C}$ and each $t \in T$, there exists a unique subset $N^t_{\mathbf{C}}$ of N. Note that $N^t_{\mathbf{C}}$ may be empty for some $t \in T$, and $N^t_{\mathbf{C}}$ may be identical to N for some $t \in T$. As a typical interpretation, t represents a type of conceivable disadvantage, and $N^t_{\mathbf{C}}$ represents the *set of*

[11] Although $C_i(x)$ is formulated as a vector in Z_i, this formulation may allow an interpretation that Z_i is a Hausdorff topological space whose elements are all non-empty, compact, and comprehensive subsets of $\mathbb{R}^m_+$, and each $C_i(x) \in Z_i$ represents i's *capability* (Sen, 1980; 1985) associated with the social policy x, as Herrero (1996) and Gotoh and Yoshihara (2003) did. Basu and Lopez-Calva (2011) provide an illuminating survey on the formulation of functionings and capabilities.

[12] The precise definitions of $\geq$ and its asymmetric part $>$ depend on the mathematical structure of the space Z. For instance, if each C represents a vector on $Z \subseteq \mathbb{R}^m_+$, then $(\geq, >)$ represents the standard *vector inequality*. If each C represents a compact and comprehensive subset in $\mathbb{R}^m_+$, so that $Z = 2^{\mathbb{R}^m_+} \backslash \varnothing$, then $(\geq, >)$ represents the standard *set-inclusion* as $C \geq C'$ if and only if for any $z \in C'$, $z \in C$ holds; and $C > C'$ if and only if $C \geq C'$ and $C' \ngeq C$.

t-type disadvantaged individuals in society N with $\mathbf{C}$. Thus, $N \backslash (\cup_{t \in T} N_{\mathbf{C}}^t)$ is the set of non-disadvantaged individuals in society N with $\mathbf{C}$.

As mentioned above, here, we assume that any group of individuals that have a certain disadvantage in common can construct a shared criterion for comparing their well-being contents. Hence, let $\succsim_t \equiv \cap_{i \in t} \succsim_i$ be the *well-being appraisal of group* t, which, based on the above argument, is assumed to be non-empty.[13] Since each $\succsim_i$ is reflective and transitive, so is the well-being appraisal of each group. Finally, let $\succsim \equiv \left((\succsim_i)_{i \in N}, (\succsim_t)_{t \in T} \right)$ be a *profile of well-being appraisals*. Denote the admissible set of profiles of well-being appraisals by $\mathcal{A}$.

With the well-being appraisal of the group, the least advantaged within the group can be defined as follows. Given society N with $\mathbf{C} \in \mathcal{C}$ and a profile of well-being appraisals $\succsim \in \mathcal{A}$, the *set of the least advantaged individuals of type* t *under social policy* $x \in X$ is defined by

$$L_{\mathbf{C}}^t(x; \succsim_t) \equiv \left\{ i \in N_{\mathbf{C}}^t \mid \nexists j \in N_{\mathbf{C}}^t : C_i(x) \succ_t C_j(x) \right\}.$$

That is, the least advantaged under social policy x is defined as an individual whose well-being content never dominates the well-being contents of others. Note that $L_{\mathbf{C}}^t(x; \succsim_t)$ is non-empty for each $x \in X$ and for each $t \in T$ with $N_{\mathbf{C}}^t \neq \varnothing$. Moreover, it is not necessarily a singleton.

Lastly, given society N with $\mathbf{C} \in \mathcal{C}$ and a profile of well-being appraisals $\succsim \in \mathcal{A}$, for each $t \in T$, the *group evaluation* is defined as a *reflexive* relation $R_{\mathbf{C}}^t$ on X. The interpretation of $R_{\mathbf{C}}^t$ is that it represents an evaluation of alternative social policies, which is defined on the domain respective to this group, and which can be agreed upon by all individuals in this group, $N_{\mathbf{C}}^t$. Given the relation $R_{\mathbf{C}}^t$ on X, let $P_{\mathbf{C}}^t$ and $I_{\mathbf{C}}^t$ be the strict and the indifferent parts of $R_{\mathbf{C}}^t$, respectively. Moreover, let $NR_{\mathbf{C}}^t$ denote the *non-comparable part* of $R_{\mathbf{C}}^t$; that is, $xNR_{\mathbf{C}}^t y$ if and only if neither $xR_{\mathbf{C}}^t y$ nor $yR_{\mathbf{C}}^t x$. Given society N with $\mathbf{C} \in \mathcal{C}$ and a profile of well-being appraisals $\succsim \in \mathcal{A}$, let us denote such relations of the admissible class of type t on X by $D_{\mathbf{C}}^t(\succsim_t)$. Moreover, let $D_{\mathbf{C}}(\succsim) \equiv \times_{t \in T} D_{\mathbf{C}}^t(\succsim_t)$ and $R_{\mathbf{C}}^T \equiv (R_{\mathbf{C}}^t)_{t \in T}$.

With this basic framework, we are ready to formally define our scheme of social choice rules as follows:

[13] The idea behind this formulation is that each individual of each group appraises the well-being contents of the members of the group, including her own, not in terms of a personal conception of the good but in terms of a shared conception of the good, on the basis of some commonality among members.

Definition 1: *Given a society* N *with a profile of well-being transformations* $\mathbf{C} \in \mathcal{C}$, *the* social relation function (**SRF**) *is the mapping* F,*which associates each well-being appraisal* $\succsim \in \mathcal{A}$ *and each profile of group evaluations* $R_{\mathbf{C}}^{T} \in D_{\mathbf{C}}(\succsim)$ *to the reflexive and transitive relation* $R_{\mathbf{C}}$ *on* X.

$R_{\mathbf{C}}$ is called a *social evaluation over* X in a society with $\mathbf{C} \in \mathcal{C}$.

By assuming that disadvantaged groups are given prior treatment in incorporating their information into a social policy, the two types of informational bases for disadvantaged groups may play different functional roles. Group appraisals are necessary to identify the least advantaged members in each group on the basis of its own conception of the good. In contrast, each group evaluation is formed on the basis of its own group appraisal by focusing on the least advantaged members of this group. Then, to realize a social policy relevant to its members, each group is given the chance to take the initiative by revealing its own group evaluation. Finally, the policy maker can choose appropriate social policies based on the social evaluation derived from the **SRF**, into which she can incorporate each individual's appraisal symmetrically as well as each disadvantaged group's evaluation asymmetrically.

4 Axioms for Group Evaluations and Social Relation Functions

In this section, we define several conditions assumed to be publicly imposed on **SRF**s. One of the key ideas to define conditions is the concept of *basic well-being* $BC^t \in Z$, which is unique to each $t \in T$.

Based on this concept, for each $t \in T$, social policies are classified into three categories: (1) social policies under which the well-being contents of the least advantaged members of a type are at least as good as its basic well-being; (2) social policies under which the well-being contents of the least advantaged members of a type are worse than its basic well-being; (3) social policies under which the well-being contents of the least advantaged members of a type cannot be compared with its basic well-being.

By the transitivity of $\succsim_t$ and the definition of $L_{\mathbf{C}}^t(x; \succsim_t)$, $C_i(x) \succsim_t BC^t$ for all $i \in L_{\mathbf{C}}^t(x; \succsim_t)$ implies that the well-being contents of individuals in $L_{\mathbf{C}}^t(x; \succsim_t)$ are *either* all better than their basic well-being *or* all indifferent to their basic well-being. The same argument also applies to the case that

$BC^t \succsim_t C_i(x)$ for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$. Considering this point and the above three categories of social policies, the domain of group evaluations is classified for each $\succsim \in \mathcal{A}$ as follows. For each $t \in T$ and each $x, y \in X$,

α)-1) $C_i(x) \succsim_t BC^t$ for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$, and $BC^t_t \succsim_t C_j(y)$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$; and

α)-2) under α)-1), $C_i(x) \succ_t BC^t$ for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$ or $BC^t \succ_t C_j(y)$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$;

β) $C_i(x) \succsim_t BC^t$ for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$, and *not* $[BC^t \succsim_t C_j(y)]$ & *not* $[C_j(y) \succsim_t BC^t]$ for some $j \in L^t_{\mathbf{C}}(y; \succsim_t)$;

γ) *not* $[BC^t \succsim_t C_i(x)]$ & *not* $[C_i(x) \succsim_t BC^t]$ for some $i \in L^t_{\mathbf{C}}(x; \succsim_t)$, and $BC^t \succ_t C_j(y)$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$;

δ) $BC^t \succ_t C_i(x)$ for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$, $BC^t \succ_t C_j(y)$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$;

ϵ) $C_i(x) \succ_t BC^t$ for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$ and $C_j(y) \succ_t BC^t$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$; and

ε) otherwise.

That is, α)-1) refers to the domain where the least advantaged individuals' well-beings in policy x are all at least as good as their basic well-being. The least advantaged individuals' well-beings in policy y are all at least as bad as their basic well-being. On the other hand, α)-2) refers to the domain where α)-1) applies, and the least advantaged individuals' well-beings in policy x are all better than their basic well-being, or the least advantaged individuals' well-beings in policy y are all worse than their basic well-being. β) refers to the domain where the least advantaged individuals' well-beings in policy x are *either* all better than their basic well-being *or* all indifferent to their basic well-being, while there exists at least one of the least advantaged individuals' well-beings in policy y that is non-comparable with their basic well-being. γ) refers to the domain where there exists at least one of the least advantaged individuals' well-beings in policy x that is non-comparable with their basic well-being, while the least advantaged individuals' well-beings in policy y are all worse than their basic well-being. δ) refers to the domain where the least advantaged individuals' well-beings are all worse than their basic well-being in both policies x and y. ϵ) refers to the domain where the least advantaged individuals' well-beings in both policy x policy y are all better than their basic well-being.

Based on this classification, let us introduce three conditions imposed on group evaluations, which result in restricting the domain of the **SRF** F.

Basic Well-being Condition (BWC): For each $\mathbf{C} \in \mathcal{C}$, each $\succsim \in \mathcal{A}$, and each $t \in T$, and for each $x, y \in X$, $xR^t_{\mathbf{C}}y$ (*resp.* $xP^t_{\mathbf{C}}y$) holds if at least one of α) (*resp.* α)-2)), β), and γ) holds.

Restricted Monotonicity (RM): For each $\mathbf{C} \in \mathcal{C}$, each $\succsim \in \mathcal{A}$, and each $t \in T$, and for each $x, y \in X$, $xR^t_{\mathbf{C}}y$ (*resp.* $xP^t_{\mathbf{C}}y$) holds if δ) holds and $C_i(x) \succsim_t C_j(y)$ (*resp.* $C_i(x) \succ_t C_j(y)$) holds for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$ and all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$.

Refrain Condition (RC): For each $\mathbf{C} \in \mathcal{C}$, each $\succsim \in \mathcal{A}$, and each $t \in T$, and for each $x, y \in X$ with $x \neq y$, $xNR^t_{\mathbf{C}}y$ holds if ϵ) or ε) holds.

BWC requires each group to evaluate a social policy x, under which the well-being contents of the least advantaged are at least as good as their basic well-being, as being *more just* than another social policy y, under which the well-being contents of the least advantaged *either* fall beneath their basic well-being *or* cannot be compared with it. Furthermore, it requires each group to evaluate a social policy y, under which the well-being contents of the least advantaged fall beneath their basic well-being, as being *less just* than another social policy x in which the well-being contents of the least advantaged cannot be compared with their basic well-being.

RM requires each group to evaluate a social policy x as being *more just* than another social policy y whenever the corresponding profile of the least advantaged members' well-beings is better in x than y, given that all of their well-being contents derived from both policies fall beneath their basic well-being. **RM** represents a kind of monotonicity criterion,[14] although its applicability is constrained to a proper domain of alternatives.

Lastly, **RC** requires a group evaluation *not* to make pair-wise rankings of the social policies if the well-being contents of the least advantaged cor-

[14] The concept of dominance proposed by Pattanaik and Xu (2007, p. 361-362), which is closely related to Sen's idea of "dominance partial ordering" (Sen, 1987, pp. 29-30) is a good example.

responding to these social policies are better than their basic well-being or they cannot be compared with their basic well-being.[15]

Thus, the three conditions together define the available class of group evaluations, which make their rankings over pairs of social policies deserve being called *more* or *less unjust*. The last comparative adjective is motivated from the "comparative approach to justice" proposed by Sen in place of a "transcendental approach to justice" (Sen 2009a, pp. 15-18, Sen, 2009b, p. 46f.). According to Sen, the latter is a traditional approach in ethics, which focuses on the description of an ideal just state, while the former is a new approach in ethics, which ranks alternative social states in terms of justice but does not necessarily identify an ideal just state. The three conditions constitute an attempt to formulate a "comparative approach to justice," in that they together make the available group evaluations consistent with the common goal of securing basic well-being for all members of each group, on the one hand by identifying "unjust" policies as those fall short of basic well-being and comparing them with one another to make evaluations of them as "less unjust" or "more unjust,"on the other hand by refraining from identifying any of the policies as ideal just when they warrant everyone's well-being beyond basic well-being.

We examine the mutual consistency of these three conditions.

Lemma 1: *Let the reflexive* $R^t_{\mathbf{C}}$ *satisfy* **BWC**, **RM**, *and* **RC**. *Then, it is transitive.*

Due to this lemma, each group can form its own evaluation based on the three conditions that are rational in terms of logical consistency.

The next task for us is to introduce two basic axioms regarding how to aggregate plural group evaluations as well as diverse individual well-being appraisals in order to form a consistent social evaluation. To explore this problem, let us introduce the following conditions.

Non-negative Response (NR): For each $\mathbf{C} \in \mathcal{C}$, each $\succsim \in \mathcal{A}$, each $R^T_{\mathbf{C}} \in D_{\mathbf{C}}(\succsim)$, and each $x, y \in X$, if there exists $t' \in T$ such that $xR^{t'}_{\mathbf{C}}y$ (*resp.* $xP^{t'}_{\mathbf{C}}y$)

[15] This condition is similar to the "focus axiom" proposed by Sen (1981; p. 186), which requests that the difference between two social states, both of which bring about capabilities at least as good as basic capability, is not reflected in the social evaluation. We are grateful to James Foster and Prasanta Pattanaik for pointing this out. See Foster (1984; p. 217) and Sen (1997; p. 172).

and there exists no $t'' \in T$ such that $yP_{\mathbf{C}}^{t''}x$, then $xR_{\mathbf{C}}y$ (*resp.* $xP_{\mathbf{C}}y$) holds, where $R_{\mathbf{C}} = F(\succsim, R_{\mathbf{C}}^T)$.

Weak Pareto (WP): For each $\mathbf{C} \in \mathcal{C}$, each $\succsim \in \mathcal{A}$, each $R_{\mathbf{C}}^T \in D_{\mathbf{C}}(\succsim)$, and each $x, y \in X$, if $C_i(x) \succ_i C_i(y)$ holds for all $i \in N$, then $xP_{\mathbf{C}}y$ holds, where $R_{\mathbf{C}} = F(\succsim, R_{\mathbf{C}}^T)$.

Recall that each $t \in T$ represents a particular type of disadvantage, so $N \setminus (\cup_{t \in T} N_{\mathbf{C}}^t)$ is the set of non-disadvantaged individuals in society N with $\mathbf{C}$. Hence, **NR** requires giving priority to the evaluations of disadvantaged groups over the evaluations of non-disadvantaged individuals in the aggregation procedure, while treating the evaluation of each group as fully symmetrical to one another. That is, even if the well-being contents of all non-disadvantaged individuals become worse in y than in x, the social evaluation must be that y is at least as just as x whenever a group t evaluates y as being at least as just as x and no other group evaluates x as *more just* than y. Such a requirement seems quite reasonable whenever persons with a particular disadvantage can be considered as "experts" on that disadvantage and these persons are expected to provide a reasonable group evaluation. In this respect, **NR** together with the available class of group evaluations constrained by **BWC**, **RM**, and **RC** warrant the reasonableness of asymmetric treatments of specific types of groups in the aggregation procedure.

In contrast, **WP** requires treating symmetrically the well-being appraisals of all individuals. Indeed, if the well-being contents of all individuals are better in x than in y, **WP** states that the social evaluation must be that x is more just than y. In terms of respecting the plurality of the conceptions of the good, **WP** also seems quite reasonable.

It is also worth mentioning that, although **WP** is a weaker condition of welfarism, the requirement of **NR** with the scheme of group evaluations makes **SRF**s *non-welfaristic*. To see this point, remember that the neutrality property of social choice rules is necessary for welfarism in the standard Arrovian framework.[16] In our framework, *neutrality* of **SRF**s is defined as

[16] The basic idea of neutrality can be summarized as follows: if the individual preferences over (x, y) in one case are "identical" to the individual preferences over (a, b) in another case, then the social preference in the latter would place a and b respectively where x and y figured in the former (Sen, 2002; p. 333).

See Fleurbaey and Mongin (2005; p. 386) for an excellent survey of studies on neutrality. As pointed out, Sen has examined the essential nature of neutrality in terms of "welfarism" (e.g., Sen, 1970, chs. 5 and 5*).

follows: for any $\mathbf{C} \in \mathcal{C}$, for every $x, y, z, w \in X$, and for any two profiles of well-being appraisals $\succsim, \succsim' \in \mathcal{A}$, if $[C_i(x) \succsim_i C_i(y)$ if and only if $C_i(z) \succsim'_i C_i(w)]$ holds for every $i \in N$, then $[xR_{\mathbf{C}}y$ if and only if $zR'_{\mathbf{C}}w]$ holds, where $R_{\mathbf{C}} = F(\succsim, R^T_{\mathbf{C}})$ and $R'_{\mathbf{C}} = F(\succsim', R'^T_{\mathbf{C}})$.

The following example shows that if **SRF**s satisfies **NR** with the scheme of group evaluations satisfying **BWC**, **RM**, and **RC**, then it does not satisfy neutrality.

Example 1: Let $N = \{1, 2, 3, 4\}$ with $\mathbf{C} \in \mathcal{C}$, $T = \{t^1, t^2, t^3\}$ with $N^{t^1}_{\mathbf{C}} = \{1\}$, $N^{t^2}_{\mathbf{C}} = \{2\}$, and $N^{t^3}_{\mathbf{C}} = \{3\}$, and $X = \{x, y, z, w\}$. Let us define two profiles of well-being appraisals $\succsim, \succsim' \in \mathcal{A}$ as follows:

$$\begin{aligned} C_i(z) &\succ_i C_i(x) \succ_i BC^{t^i} \succ_i C_i(w) \succ_i C_i(y) \text{ for } i \in \{1, 3\}\,; \\ C_2(w) &\succ_2 C_2(z) \succ_2 BC^{t^2} \succ_2 C_2(x) \succ_2 C_2(y); \\ C_4(x) &\succ_4 C_4(y) \succ_4 C_4(w) \succ_4 C_4(z); \text{ and} \end{aligned}$$

$\succsim_{t^i} = \succsim_i$ for $i \in \{1, 2, 3\}$, and

$$\begin{aligned} C_i(x) &\succ'_i C_i(z) \succ'_i BC^{t^i} \succ'_i C_i(w) \succ'_i C_i(y) \text{ for } i \in \{1, 3\}\,; \\ C_2(z) &\succ'_2 C_2(w) \succ'_2 BC^{t^2} \succ'_2 C_2(x) \succ'_2 C_2(y); \\ C_4(z) &\succ'_4 C_4(y) \succ'_4 C_4(w) \succ'_4 C_4(x); \text{ and} \end{aligned}$$

$\succsim'_{t^i} = \succsim'_i$ for $i \in \{1, 2, 3\}$. Given this structure, note that $[C_i(z) \succsim_i C_i(w)$ if and only if $C_i(x) \succsim'_i C_i(z)]$ holds for every $i \in N$. Therefore, if an **SRF** F satisfies neutrality, then $[zR_{\mathbf{C}}w$ if and only if $xR'_{\mathbf{C}}z]$ must hold, where $R_{\mathbf{C}} = F(\succsim, R^T_{\mathbf{C}})$ and $R'_{\mathbf{C}} = F(\succsim', R'^T_{\mathbf{C}})$. In contrast, by **BWC**, $C_1(z) \succ_{t^1} BC^{t^1} \succ_{t^1} C_1(w)$ implies $zP^{t^1}_{\mathbf{C}}w$, and $C_3(z) \succ_{t^3} BC^{t^3} \succ_{t^3} C_3(w)$ implies $zP^{t^3}_{\mathbf{C}}w$; while by **RC**, $C_2(w) \succ_2 C_2(z) \succ_2 BC^{t^2}$ implies $zNP^{t^2}_{\mathbf{C}}w$. Similarly, by **BWC**, $C_2(z) \succ'_2 BC^{t^2} \succ'_2 C_2(x)$ implies $zP'^{t^2}_{\mathbf{C}}x$; while by **RC**, $C_1(x) \succ'_1 C_1(z) \succ'_1 BC^{t^1}$ implies $zNP'^{t^1}_{\mathbf{C}}x$, and $C_3(x) \succ'_3 C_3(z) \succ'_3 BC^{t^3}$ implies $zNP'^{t^3}_{\mathbf{C}}x$. Therefore, $zP_{\mathbf{C}}w$ and $zP'_{\mathbf{C}}x$, where $R_{\mathbf{C}} = F(\succsim, R^T_{\mathbf{C}})$ and $R'_{\mathbf{C}} = F(\succsim', R'^T_{\mathbf{C}})$, hold by **NR**. Thus, this F does not satisfy neutrality. ■

In summary, the group evaluations scheme constrained by **BWC**, **RM**, and **RC** and the axiom **NR** together imply non-welfarism.

5 Main Results

The combination of **NR** and **WP** may produce a quite reasonable social evaluation if **NR** and **WP** do not compete with each other. The aim of this section therefore is to verify the compatibility of **NR** and **WP**. In our framework, **NR** specifies the types of individuals whose group evaluations are assigned the prior treatments as well as to what extent their evaluations should have decisive powers in the social choice procedure. In contrast, **WP** represents the principle of symmetric treatment of individuals' appraisals and, moreover, there is no constraint for the application of this principle. Thus, provided that the group evaluations satisfy **BWC**, **RM**, and **RC**, it is not obvious whether the compatibility of **WP** and **NR** is verified, even if the latter axiom seems rather weak as the claim for the local decisiveness of specific types of individuals.

We examine whether or not there exists an **SRF** that satisfies **NR**. To do this, we introduce another axiom, the *Positive Response* **(PR)**, which is even weaker than **NR**. Theorem 1 discussed below shows that there is no **SRF** that satisfies **PR**.

Therefore, in the second step, to avoid this negative result, we introduce an additional condition, *Full Comparability of Destitution* (**FCD**), which insures the full comparability of policies when the well-beings of all of the least advantaged members under those policies become worse than their basic well-being. Theorem 2 proves that under the presumption of **FCD**, there exists an **SRF** that satisfies **NR**.

However, in the third step, we show in Theorem 3 that it is impossible to guarantee the compatibility of **NR** and **WP** even under the presumption of **FCD**. Given these results, Theorem 4 clarifies what kind of further condition is required for the compatibility of these two axioms.

Assume, for the sake of simplicity, that the profile of the disadvantaged groups $(N^t_{\mathbf{C}})_{t\in T}$ is fixed independent of the types of **SRF**s. As our first step, let us introduce the following axiom for **SRF**s:

Positive Response (PR): For each $\mathbf{C} \in \mathcal{C}$, each $\succsim \in \mathcal{A}$, each $R^T_{\mathbf{C}} \in D_{\mathbf{C}}(\succsim)$, and each $x, y \in X$, if there exists $t' \in T$ such that $xP^{t'}_{\mathbf{C}}y$ and there is no $t'' \in T$ such that $yP^{t''}_{\mathbf{C}}x$, then $xP_{\mathbf{C}}y$ holds, where $R_{\mathbf{C}} = F(\succsim, R^T_{\mathbf{C}})$.

PR is a weaker version of **NR**. This condition, as well as **NR**, seems quite reasonable, given that persons with a particular disadvantage can be considered as "experts" on that disadvantage.

Then,

Theorem 1: *There exists a profile of well-being appraisals $\succsim$ under which no* ***SRF*** *F satisfies* **PR**.

This impossibility theorem holds whenever there are at least three different disadvantaged groups, $t^1, t^2, t^3 \in T$, and also at least three alternatives, $x, y, z \in X$. To focus on the simplest case, let there be no other group, $T = \{t^1, t^2, t^3\}$, and each of the three groups have only one member, $N_{\mathbf{C}}^{t^i} = \{i\}$ for $i = 1, 2, 3$. Then, each individual $i = 1, 2, 3$ is also the least advantaged member of her group, and her appraisal is identical to her group's appraisal.

Given this setting, assume that the well-being contents of all the three individuals under any of the three alternatives are worse than their corresponding basic well-being contents: $BC^{t^i} \succ_{t^i} C_i(w)$ for any $w \in \{x, y, z\}$ and for each $i = 1, 2, 3$. Moreover, assume that, according to group t^1's appraisal, x is better than y, y and z are non-comparable, and z and x are non-comparable; according to group t^2's appraisal, y is better than z, z and x are non-comparable, and x and y are non-comparable; and according to group t^3's appraisal, z is better than x, x and y are non-comparable, and so are y and z. Then, **RM** implies that $xP_{\mathbf{C}}^{t^1}y$, $yP_{\mathbf{C}}^{t^2}z$, and $zP_{\mathbf{C}}^{t^3}x$. Moreover, let $yNR_{\mathbf{C}}^{t^1}z$ and $zNR_{\mathbf{C}}^{t^1}x$; $zNR_{\mathbf{C}}^{t^2}x$ and $xNR_{\mathbf{C}}^{t^2}y$; and $xNR_{\mathbf{C}}^{t^3}y$ and $yNR_{\mathbf{C}}^{t^3}z$. Such group evaluations are available due to the assumptions of the three groups' appraisals. However, by **PR**, the corresponding social evaluation $R_{\mathbf{C}}$ should have $xP_{\mathbf{C}}y$, $yP_{\mathbf{C}}z$, and $zP_{\mathbf{C}}x$, which implies that there is no **SRF** satisfying **PR**.

The above arguments indicate that the incompleteness of group appraisals is the key factor to generate such a cyclical social evaluation.[17] Given this impossibility, let us introduce an additional condition on well-being appraisals:

Full Comparability of Destitution (FCD): For each $t \in T$ and each $x, y \in X$, if δ) holds, then for all $i \in L_{\mathbf{C}}^t(x; \succsim_t)$ and all $j \in L_{\mathbf{C}}^t(y; \succsim_t)$, $C_i(x) \succsim_t C_j(y)$ or $C_j(y) \succsim_t C_i(x)$.

δ) is the case where in each policy, the well-being contents of the least advantaged are all worse than their basic well-being. In such a situation of

[17]Indeed, in the three-group and three-alternative example, for each group, if at least one non-comparable pair is changed to be comparable, then the cyclical social evaluation is no longer generated.

"destitution," **FCD** requires that the well-beings of the least advantaged are all comparable. This condition seems reasonable, since the plurality of evaluations over social policies tends to be reduced under a situation of "destitution." Moreover, it would be desirable that relatively "less unjust" policies can be selected under the situation of "destitution," and **FCD** insures the feasibility of such a social choice.

The next theorem proves that if we introduce **FCD** into the group appraisal, we can warrant the existence of an **SRF** F , which satisfies **NR**, a strong version of **PR**.

Theorem 2: *Let* **FCD** *hold. Then, there exists an* ***SRF*** F *that satisfies* **NR**.

To show Theorem 2, let us define F_{NR} as follows: for each $\mathbf{C} \in \mathcal{C}$, each $\succsim \in \mathcal{A}$, each $R_{\mathbf{C}}^{T} \in D_{\mathbf{C}}(\succsim)$, and each $x, y \in X$, $xR_{\mathbf{C}}^{NR}y$ holds if and only if there exists $t' \in T$ such that $xR_{\mathbf{C}}^{t'}y$ and there is no $t'' \in T$ such that $yP_{\mathbf{C}}^{t''}x$, where $R_{\mathbf{C}}^{NR} = F_{NR}(\succsim, R_{\mathbf{C}}^{T})$. This $R_{\mathbf{C}}^{NR}$ is shown to be transitive.

Given Theorem 2, our next step is to examine whether **FCD** is sufficient for the existence of an **SRF** F satisfying **NR** and **WP**. Unfortunately, the next theorem proves that **FCD** is not sufficient for this purpose.

Theorem 3: *Suppose* **FCD**. *Then, there exists a profile of well-being appraisals* $\succsim$ *under which no* ***SRF*** *satisfies* **PR** *and* **WP**.

The essential factor to generate this theorem is that at least two least advantaged members may exist within a group under a social policy, whose well-being positions relative to basic well-being are different, and at least one of such members is no longer the least advantaged under another social policy. To see this, let us suppose that i and j are the least advantaged members within a group t, where i's well-being is worse than BC^{t} and j's well-being is non-comparable with BC^{t} in policy x, according to the group appraisal $\succsim_{t}$. Moreover, let us suppose that all individuals' well-being contents are improved as a result of the change from policy x to policy y, and j's well-being becomes better than BC^{t} in y, although i's is still worse than BC^{t}. Moreover, let i be the unique least advantaged member of the group t under y, so that j is no longer the least advantaged of this group. This situation corresponds to case γ), so **BWC** applies and this group evaluates that x is more just than y. Then, if no other groups make any objection,

PR requires that x is better than y. Yet, **WP** requires that y is better than x.

This argument suggests that, unlike the Pareto-liberal paradox, **PR** and **WP** can be incompatible even if the social evaluation $R_{\mathbf{C}}$ is not requested to be acyclic. Note also that the existence of only one disadvantaged group is sufficient to generate this incompatibility; in contrast, in the Pareto-liberal paradox, at least two agents must exercise their local decisive powers to generate a conflict with the Pareto principle.

For the purpose of our four steps, let us introduce an additional condition that requires even greater comparability of each group's well-being appraisal:

Dominance (D): For each $\mathbf{C} \in \mathcal{C}$, each $t \in T$, each $x \in X$, and each $i, j \in N_{\mathbf{C}}^{t}$ with $i \in L_{\mathbf{C}}^{t}(x; \succsim_t)$, if $BC^t \succ_t C_i(x)$ and *not* $BC^t \succsim_t C_j(x)$, then $C_j(x) \succ_t C_i(x)$.

D requires that if a well-being content C is worse than BC^t and another well-being content C' is not, C' should be appraised to be better than C. Recalling that *not* $BC^t \succsim_t C_j(x)$ means $C_j(x) \succsim_t BC^t$ or *not* $[BC^t \succsim_t C_j(x)$ or $C_j(x) \succsim_t BC^t]$, the underlying idea of this condition is clear. That is, if one well-being content is appraised to be worse than BC^t, while another is not, a comparative judgment should be made between the two, in that the latter is better than the former. Note that, unless **D** is required, it is possible that j is deemed least advantaged even if j's well-being content is non-comparable with basic well-being and there is another least advantaged member i whose well-being content is worse than basic well-being.

With condition **D** in addition to **FCD**, we can insure the existence of an **SRF** F that satisfies **NR** and **WP** as follows:

Theorem 4: *Let* **FCD** *and* **D** *hold. Then, there exists an* ***SRF*** F *that satisfies* **NR** *and* **WP**.

To show Theorem 4, let us define F_{WP} as follows: for each $\mathbf{C} \in \mathcal{C}$, each $\succsim \in \mathcal{A}$, each $R_{\mathbf{C}}^{T} \in D_{\mathbf{C}}(\succsim)$, and each $x, y \in X$, $xP_{\mathbf{C}}^{WP}y$ holds if and only if $C_i(x) \succ_i C_i(y)$ holds for all $i \in N$, and $xI_{\mathbf{C}}^{WP}y$ holds if and only if $x = y$, where $R_{\mathbf{C}}^{WP} = F_{WP}(\succsim, R_{\mathbf{C}}^{T})$. Moreover, let us define F_* as follows: for each $\mathbf{C} \in \mathcal{C}$ and each $R_{\mathbf{C}}^{T} \in D_{\mathbf{C}}(\succsim)$, $F_*(\succsim, R_{\mathbf{C}}^{T}) = R_{\mathbf{C}}^{*}$, where $R_{\mathbf{C}}^{*} \equiv R_{\mathbf{C}}^{NR} \cup R_{\mathbf{C}}^{WP}$.

In the following discussion, we show that this $R_{\mathbf{C}}^{*}$ is transitive. Let $(x, y), (y, z) \in R_{\mathbf{C}}^{*}$. Then, there are the following four possible cases:

1)$(x, y), (y, z) \in R_{\mathbf{C}}^{NR}$;

2)$(x, y), (y, z) \in R_{\mathbf{C}}^{WP}$;
3) $(x, y) \in R_{\mathbf{C}}^{NR}$ and $(y, z) \in R_{\mathbf{C}}^{WP}$; and
4) $(x, y) \in R_{\mathbf{C}}^{WP}$ and $(y, z) \in R_{\mathbf{C}}^{NR}$.

Theorem 2 shows that if case (1) applies, $(x, z) \in R_{\mathbf{C}}^{NR}$ holds. Moreover, it is easy to see that if case (2) applies, then $(x, z) \in R_{\mathbf{C}}^{WP}$ holds. Next, let us consider cases (3) and (4):

Lemma 2: For each $\mathbf{C} \in \mathcal{C}$, each $\succsim \in \mathcal{A}$, and each $R_{\mathbf{C}}^{T} \in D_{\mathbf{C}}(\succsim)$, if $(x, y) \in R_{\mathbf{C}}^{NR}$ and $(y, z) \in P_{\mathbf{C}}^{WP}$, *then* $(x, z) \in P_{\mathbf{C}}^{*}$.

Lemma 3: For each $\mathbf{C} \in \mathcal{C}$, each $\succsim \in \mathcal{A}$, and each $R_{\mathbf{C}}^{T} \in D_{\mathbf{C}}(\succsim)$, if $(x, y) \in P_{\mathbf{C}}^{WP}$ and $(y, z) \in R_{\mathbf{C}}^{NR}$, then $(x, z) \in P_{\mathbf{C}}^{*}$.

Proof of Theorem 4: By **Lemmas 2** and **3**, it holds. ■

Theorems 3 and 4 indicate that, given the incompleteness of the informational basis, the moderate asymmetric treatment of disadvantaged groups is unable to guarantee consistent and Paretian social decision-making for social policies. This impossibility, however, does not necessarily imply that there is an *intrinsic* conflict between the claim of the prior treatment of specific individuals and the symmetric treatment of all. Rather, it may originate from a lack of sufficient information on the part of a disadvantaged group to make a deliberate appraisal of their own states. As condition **D** and Theorem 4 show, the main reason for the impossibility is the existence of the least disadvantaged member whose well-being content is deemed non-comparable to basic well-being despite the existence of another least advantaged whose well-being content is deemed worse than basic well-being. If such "tentative" non-comparability can be resolved via further scrutiny of this member's condition, consistent and Paretian social decision-making for desired polices can be compatible with the prior treatment of specific people.[18] In other words, the difficulty of constructing the desired social choice due to "tentative" non-comparability within the same disadvantaged group could be resolved by technical progress at least in the future, which should be discriminated from the more intrinsic types of impossibility problems.

[18] The notions of "tentative incompleteness" and "assertive incompleteness" are introduced by Sen, where the former consists of "some pairs of alternatives that are not yet ranked (although all may get ranked with more deliberation or information)," while the latter consists of "some pairs of alternatives that are asserted to be 'non-rankable'" (Sen, 2002, p. 182).

It is also worth noting that, while **FCD** and **D** make any pairs of social policies within the domains α), β), γ), δ) of group evaluations comparable, the remaining non-comparable parts in group evaluations is "assertive" rather than "tentative,"as warranted by **RC** for group evaluations. Interestingly, however, such "assertive" non-comparability is also indispensable for the existence of **SRF**s satisfying **NR** and **WP**. Indeed, regardless of whether **FCD** and **D** are imposed, allowing comparability for group evaluations even within the domains ϵ) and ε) may restore the incompatibility of **NR** and **WP**, as the following example suggests.

Example 2: Let $N = \{h, i^1, j^1, i^2, j^2\}$ with $\mathbf{C} \in \mathcal{C}$, $T = \{t^1, t^2\}$ with $N_{\mathbf{C}}^{t^1} = \{i^1, j^1\}$ and $N_{\mathbf{C}}^{t^2} = \{i^2, j^2\}$, and $X = \{x, y, z\}$. Let $Z \equiv \mathbb{R}^2$ and

$$\begin{aligned}
C_{i^1}(x) &= (4,4); C_{i^1}(y) = (1,3.3); C_{i^1}(z) = (5,3);\\
C_{j^1}(x) &= (3.9,2); C_{j^1}(y) = (2,3.6); C_{j^1}(z) = (6,3.5);\\
C_{i^2}(x) &= (4,0.5); C_{i^2}(y) = (3,3); C_{i^2}(z) = (1,1);\\
C_{j^2}(x) &= (5,2); C_{j^2}(y) = (1.5,1.5); C_{j^2}(z) = (1.5,4);\\
C_h(x) &= (6,6); C_h(y) = (5,5); C_h(z) = (4,4).
\end{aligned}$$

For each $k \in \{h, i^1, j^2\}$, let $\succsim_k$ be given by $C_k(v) \succsim_k C_k(u)$ if and only if $C_k(v) \geq C_k(u)$; and $C_k(v) \succ_k C_k(u)$ if and only if $C_k(v) > C_k(u)$, where $\geq$ (*resp.* $>$) is the vector inequality (*resp.* the strict vector inequality). For each $l \in \{j^1, i^2\}$, let $\succsim_l$ be given by $C_l(v) \succsim_l C_l(u)$ if and only if $(C_l(v))_1 \geq (C_l(u))_1$; and $C_l(v) \succ_l C_l(u)$ if and only if $(C_l(v))_1 > (C_l(u))_1$, where $(C_l(v))_1$ represents the first component of the vector $C_l(v)$. Finally, let $\succsim_{t^1} \equiv \succsim_{i^1}$ and $\succsim_{t^2} \equiv \succsim_{j^2}$. Moreover, let $BC^{t^1} \equiv (0, 0.5)$ and $BC^{t^2} \equiv (0.5, 0)$.

Given these settings, it follows that

$$\begin{aligned}
C_\ell(x) &\succ {}_\ell C_\ell(y) \text{ for all } \ell \in N;\\
L_{\mathbf{C}}^{t^1}(x; \succsim_{t^1}) &= \{j^1\}; L_{\mathbf{C}}^{t^1}(y; \succsim_{t^1}) = \{i^1\}; L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1}) = \{i^1\}; \text{ and}\\
L_{\mathbf{C}}^{t^2}(x; \succsim_{t^2}) &= \{i^2\}; L_{\mathbf{C}}^{t^2}(y; \succsim_{t^2}) = \{j^2\}; L_{\mathbf{C}}^{t^2}(z; \succsim_{t^2}) = \{i^2\}.
\end{aligned}$$

Then,

$$not\,[C_{j^1}(x) \succsim_{t^1} C_{i^1}(y)] \;\&\; not\,[C_{i^1}(y) \succsim_{t^1} C_{j^1}(x)]$$

by $C_{j^1}(x) \ngeq C_{i^1}(y)$ & $C_{j^1}(x) \nleq C_{i^1}(y)$;

$$not\,[C_{i^1}(y) \succsim_{t^1} C_{i^1}(z)] \;\&\; not\,[C_{i^1}(z) \succsim_{t^1} C_{i^1}(y)]$$

by $C_{i^1}(y) \not\geq C_{i^1}(z)$ & $C_{i^1}(y) \not\leq C_{i^1}(z)$; and

$$C_{i^1}(z) \succ_{t^1} C_{j^1}(x) \text{ by } C_{i^1}(z) > C_{j^1}(x).$$

Similarly,

$$not\,[C_{i^2}(x) \succsim_{t^2} C_{j^2}(y)] \;\&\; not\,[C_{j^2}(y) \succsim_{t^2} C_{i^2}(x)]$$

by $C_{i^2}(x) \not\geq C_{j^2}(y)$ & $C_{i^2}(x) \not\leq C_{j^2}(y)$;

$$not\,[C_{i^2}(x) \succsim_{t^2} C_{i^2}(z)] \;\&\; not\,[C_{i^2}(z) \succsim_{t^2} C_{i^2}(x)]$$

by $C_{i^2}(z) \not\geq C_{i^2}(x)$ & $C_{i^2}(z) \not\leq C_{i^2}(x)$; and

$$C_{j^2}(y) \succ_{t^2} C_{i^2}(z) \text{ by } C_{j^2}(y) > C_{i^2}(z).$$

Now, let $R^{t^1}_{\mathbf{C}}$ and $R^{t^2}_{\mathbf{C}}$ do not satisfy **RC**, and so $zP^{t^1}_{\mathbf{C}}x$ and $yP^{t^2}_{\mathbf{C}}z$, while $xNR^{t^1}_{\mathbf{C}}y$, $yNR^{t^1}_{\mathbf{C}}z$, $xNR^{t^2}_{\mathbf{C}}y$, and $zNR^{t^2}_{\mathbf{C}}x$. Then, **NR** implies that $zP_{\mathbf{C}}x$ and $yP_{\mathbf{C}}z$, while $xP_{\mathbf{C}}y$ follows from **WP**, where $R_{\mathbf{C}} = F(\succsim, R^T_{\mathbf{C}})$. Thus, $R_{\mathbf{C}}$ violates transitivity. ■

The above example implies that, under the imposition of **FCD** and **D**, a further increase of comparable parts in group evaluations via the elimination of **RC** results in the impossibility of **SRF**s satisfying **NR** and **WP**.

6 Discussion

Before concluding the paper, we first comment on another prominent feature of our **SRF** framework. In Arrovian social welfare functions, a social choice is made simply on the basis of the structure of preference profiles revealed by individuals and thus is independent of information on the characteristics of individuals and alternatives. Note that this property of Arrovian social welfare functions derives from the three conditions imposed by Arrow, namely the universal domain, the Pareto principle, and the independence of irrelevant alternatives. It is known that these three conditions lead to *neutrality* or *welfarism,* that is, these conditions together require that individuals' ordinal rankings of alternatives are the sole relevant information to make a social choice. This structure of Arrovian models well represents the spirit of traditional liberalism, which gives priority to individual autonomy and prohibits arbitrarily unequal legal treatment.[19] In contrast, in our **SRF**s, a social choice is made on the basis of not only the preference profiles revealed by individuals but also the information of the characteristics of individuals and of alternatives. We have shown that this change is a clue to avoid *welfaristic* nature of social choice. Besides, this formulation allows us to explore another possibility of liberalism, that is, substantive equality of political freedom that allows the asymmetrical treatment of ordinal rankings. The reasonableness of such asymmetrical treatment is guaranteed by the introduction of an explicit device for public scrutiny, which is represented in this paper by the observability of well-being indicators and the three-component structure of the informational basis.

Second, let us clarify the basic ideas underlying this paper. The first idea is relevant to two kinds of "incomparability." In this paper, the least advantaged are identified as individuals whose well-being contents never dominate the well-beings of others in each social policy. Due to the multiplicity of attributes that define the notion of well-being, there could remain incomparability among the least advantaged even within a group. However, the meaning of incomparability within a group should be kept distinct from incomparability (also called "incommensurability") between groups. The reason is that the former is a technical or political problem and certain conditions of compromise can be introduced to deal with it, as we have done by introducing **FDC** and **D** in this paper. On the other hand, the latter is a kind of incomparability for which no compromise can be found as long as the plurality of disadvantages is taken seriously. This distinction between these two forms of incomparability corresponds to the distinction introduced by Sen (2002) between "tentative incompleteness" and "assertive incompleteness."

The second idea concerns two types of conflicts between groups—one arises from each group's need to achieve basic well-being, while the other derives from each group's desire to enjoy well-being beyond basic well-being. While the former type of conflict is avoidable if there are sufficient resources to secure basic well-being for all groups, the latter is not, if the desire towards better well-being is without limit. The former deserves consideration in terms of justice that this paper is concerned with, while the latter does not. This is the reason why in this paper, the application of the monotonic-

[19] According to Arrow, "The decision as to which preferences are relevant and which are not is itself a value judgement and cannot be settled on an a priori basis" (Arrow, 1963, p. 18).

ity condition is restricted to the domain below basic well-being through the application of **RM**, while in the domain above basic well-being, conflicts are avoided through the application of **RC**, which prohibits groups from making rankings.

7 Conclusion

This paper addressed and formulated a social choice problem for "securing basic well-being for all," where individual well-being was defined in the space of multi-dimensional attributes; the notion of basic well-being was introduced as a critical reference point of multi-dimensional well-beings in order to identify the "injustice" of social policies; and the mutually incommensurable types of disadvantages were allowed to exist. Given such an environment, a new scheme of social choice rule, a *social relation function* (**SRF**), was introduced with three elements of informational basis: individuals' appraisals of their respective well-being contents; each disadvantaged group's appraisal of its members' well-being contents to identify the least advantaged members on the basis of the members' shared conception of the good; and each group's evaluation of alternative social policies, formed on the basis of its own appraisal by focusing on the least advantaged members. In the scheme of **SRF**, while individuals' appraisal is symmetrically treated, group evaluations are allowed an asymmetrical prior treatment as experts on their own disadvantages. The former property is formulated as *weak Pareto condition* (**WP**) and the latter as *non-negative response* (**NR**) along with three constraints, *basic well-being condition* (**BWC**), *restricted monotonocity* (**RM**), and *refrain condition* (**RC**), on the admissible group evaluations. It was shown that **NR** together with **BWC**, **RM**, and **RC** makes an **SRF** *non-welfaristic.*

Then, the paper showed that the three constraints are compatible in that a group evaluation is transitive if it satisfies these constraints, while there exists an **SRF** that satisfies **NR** if full comparability among the disadvantaged individuals' well-being contents within a group is allowed under the situation of "destitution." It also showed that any **SRF** cannot simultaneously satisfy **WP** and **NR** together with **BWC**, **RM**, and **RC** if "tentative" non-comparability of the least advantaged's well-being contents within the same group remains, while there exists an **SRF** satisfying these axioms if the remaining non-comparability is "assertive," where the "assertive" property of the remaining non-comparability is insured by **RC**. Finally, it was shown that the existence of the "assertive" non-comparability is indispensable for the existence of such an **SRF**.

Essentially, this study presents the profiles of individual and group appraisals and group evaluations along with the concrete conceptions of basic well-being contents for the respective groups as parameters in the framework of **SRF**s. It would be interesting to develop an analytical framework to address the structure of interaction in the formation process of these profiles and lists, as the adoption of the Convention on the Rights of Persons with Disabilities and its ratification by each party suggest. This is thus a topic worth discussing in future research.

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Addendum: Securing Basic Well-being for All

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1 Appendix: Proofs

Proof of Lemma 1. Let $xR^t_{\mathbf{C}}yR^t_{\mathbf{C}}z$. We will show $xR^t_{\mathbf{C}}z$ holds. Note that $xR^t_{\mathbf{C}}y$ is derived from applying either **BWC** or **RM** to the pair of (x, y).

First, suppose that $xR^t_{\mathbf{C}}y$ is derived from applying **BWC** to the pair of (x, y). Then, $C_i(x) \succsim_t BC^t$ holds for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$, or *not* $[BC^t \succsim_t C_i(x)]$ & *not* $[C_i(x) \succsim_t BC^t]$ holds for some $i \in L^t_{\mathbf{C}}(x; \succsim_t)$. Moreover, regarding y, one of the following three cases holds:

1) $BC^t \sim_t C_j(y)$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$;

2) *not* $[C_j(y) \succsim_t BC^t]$ and *not* $[BC^t \succsim_t C_j(y)]$ for some $j \in L^t_{\mathbf{C}}(y; \succsim_t)$;

3) $BC^t \succ_t C_j(y)$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$.

Let 3) hold for y. Then, the case α)-2) or γ) is applied to the pair of (x, y), so that $xP^t_{\mathbf{C}}y$ holds. Moreover, $yR^t_{\mathbf{C}}z$ is then derived only by applying **RM** to the pair of (y, z), which implies that $C_i(y) \succsim_t C_j(z)$ holds for all $i \in L^t_{\mathbf{C}}(y; \succsim_t)$ and all $j \in L^t_{\mathbf{C}}(z; \succsim_t)$, so that $BC^t \succ_t C_j(z)$ holds for all $j \in L^t_{\mathbf{C}}(z; \succsim_t)$ by the transitivity of $\succsim_t$. Thus, the pair of (x, z) corresponds to the case α)-2) or γ), so that $xP^t_{\mathbf{C}}z$ holds by **BWC**.

Let 1) hold for y. Then, $xR^t_{\mathbf{C}}y$ is derived by applying **BWC** under the case α)-1) or α-2). Suppose that α)-1) is applied but α)-2) cannot be applied. Hence, $C_i(x) \sim_t BC^t$ holds for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$, which implies that $yR^t_{\mathbf{C}}x$ is also derived from applying **BWC** under the case α)-1). Thus, $xI^t_{\mathbf{C}}y$. Moreover, $yR^t_{\mathbf{C}}z$ is only derived from applying **BWC** to the pair of (y, z). Therefore, the pair of (x, z) corresponds to the case α), so that $xR^t_{\mathbf{C}}z$ holds by **BWC**. In particular, if $yP^t_{\mathbf{C}}z$, then $xP^t_{\mathbf{C}}z$ also holds, while if $yI^t_{\mathbf{C}}z$, then $xI^t_{\mathbf{C}}z$ also holds. Suppose that α)-2) is applied. Then, $C_i(x) \succ_t BC^t$ holds for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$ and $xP^t_{\mathbf{C}}y$. Again, $yR^t_{\mathbf{C}}z$ is only derived from applying **BWC** to the pair of (y, z). Therefore, the pair of (x, z) corresponds to the case α)-2), so that $xP^t_{\mathbf{C}}z$ holds by **BWC**.

Let 2) hold for y. Then, the case β) is applied to the pair of (x, y), so that $xP^t_{\mathbf{C}}y$ holds. Moreover, $yR^t_{\mathbf{C}}z$ is only derived by applying **BWC** under the case γ) to the pair of (y, z), where $BC^t \succ_t C_j(z)$ must hold for all $j \in L^t_{\mathbf{C}}(z; \succsim_t)$, so that $yP^t_{\mathbf{C}}z$ holds. Therefore, the pair of (x, z) corresponds to the case α)-2), so that $xP^t_{\mathbf{C}}z$ holds by **BWC**.

Second, suppose that $xR^t_{\mathbf{C}}y$ is derived by applying **RM** to the pair of (x, y). Then, $yR^t_{\mathbf{C}}z$ should also be derived by applying **RM** to the pair of (y, z). Hence, the pair of (x, z) corresponds to the case δ). Moreover, $C_i(x) \succsim_t C_j(y) \succsim_t C_h(z)$ holds for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$, all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$, and all $h \in L^t_{\mathbf{C}}(z; \succsim_t)$. Then, by the transitivity of $\succsim_t$, $C_i(x) \succsim_t C_h(z)$ holds for all $i \in L^t_{\mathbf{C}}(x; \succsim_t)$ and all $h \in L^t_{\mathbf{C}}(z; \succsim_t)$. Thus, $xR^t_{\mathbf{C}}z$ holds by **RM**. Moreover, if $xP^t_{\mathbf{C}}y$ or $yP^t_{\mathbf{C}}z$, then $xP^t_{\mathbf{C}}z$ holds, while if $xI^t_{\mathbf{C}}y$ and $yI^t_{\mathbf{C}}z$, then $xI^t_{\mathbf{C}}z$ holds by the transitivity of $\succsim_t$ and **RM**. ■

Proof of Theorem 1. Let us define F_{PR} as follows: for each $\mathbf{C} \in \mathcal{C}$, each $(R^t_{\mathbf{C}})_{t \in T} \in D_{\mathbf{C}}(\succsim)$, and each $x, y \in X$, let $P^{PR}_{\mathbf{C}}$ be defined as: $xP^{PR}_{\mathbf{C}}y$ holds if and only if there exists $t' \in T$ such that $xP^{t'}_{\mathbf{C}}y$ and there is no $t'' \in T$ such that $yP^{t''}_{\mathbf{C}}x$. Any **SRF** F satisfying **PR** associates with a profile $(\mathbf{C}, \succsim, (R^t_{\mathbf{C}})_{t \in T})$ a quasi-ordering $R_{\mathbf{C}} = F(\mathbf{C}, \succsim, (R^t_{\mathbf{C}})_{t \in T})$ which contains $P^{PR}_{\mathbf{C}}$ as a subrelation. Therefore, if $P^{PR}_{\mathbf{C}}$ is not transitive, this theorem holds.

Let $\{t^1, t^2, t^3\} = T$, and let us consider $(\mathbf{C}, \succsim, (R^t_{\mathbf{C}})_{t \in T})$ satisfying the following properties:

(1) Let $BC^{t^1} \succ_{t^1} C_i(x)$ for all $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$; $BC^{t^1} \succ_{t^1} C_j(y)$ for all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$; and $BC^{t^1} \succ_{t^1} C_h(z)$ for all $h \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. Moreover, let $C_i(x) \succ_{t^1} C_j(y)$ for all $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$ and all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$; *not* $[C_i(x) \succsim_{t^1} C_h(z)]$ & *not* $[C_h(z) \succsim_{t^1} C_i(x)]$ for some $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$ and some $h \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$; and *not* $[C_j(y) \succsim_{t^1} C_h(z)]$ & *not* $[C_h(z) \succsim_{t^1} C_j(y)]$ for some $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$ and some $h \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$.

(2) Let $BC^{t^2} \succ_{t^2} C_i(x)$ for all $i \in L^{t^2}_{\mathbf{C}}(x; \succsim_{t^2})$; $BC^{t^2} \succ_{t^2} C_j(y)$ for all $j \in L^{t^2}_{\mathbf{C}}(y; \succsim_{t^2})$; and $BC^{t^2} \succ_{t^2} C_h(z)$ for all $h \in L^{t^2}_{\mathbf{C}}(z; \succsim_{t^2})$. Moreover, let $C_i(y) \succ_{t^2} C_h(z)$ for all $i \in L^{t^2}_{\mathbf{C}}(y; \succsim_{t^2})$ and all $h \in L^{t^2}_{\mathbf{C}}(z; \succsim_{t^2})$; *not* $[C_i(x) \succsim_{t^2} C_j(y)]$ & *not* $[C_j(y) \succsim_{t^2} C_i(x)]$ for some $i \in L^{t^2}_{\mathbf{C}}(x; \succsim_{t^2})$ and some $j \in L^{t^2}_{\mathbf{C}}(y; \succsim_{t^2})$; and *not* $[C_i(x) \succsim_{t^2} C_h(z)]$ & *not* $[C_h(z) \succsim_{t^2} C_i(x)]$ for some $i \in L^{t^2}_{\mathbf{C}}(y; \succsim_{t^2})$ and some $h \in L^{t^2}_{\mathbf{C}}(z; \succsim_{t^2})$.

(3) Let $BC^{t^3} \succ_{t^3} C_i(x)$ for all $i \in L^{t^3}_{\mathbf{C}}(x; \succsim_{t^3})$; $BC^{t^3} \succ_{t^3} C_j(y)$ for all $j \in L^{t^3}_{\mathbf{C}}(y; \succsim_{t^3})$; and $BC^{t^3} \succ_{t^3} C_h(z)$ for all $h \in L^{t^3}_{\mathbf{C}}(z; \succsim_{t^3})$. Moreover, let $C_h(z) \succ_{t^3} C_i(x)$ for all $h \in L^{t^3}_{\mathbf{C}}(z; \succsim_{t^3})$ and all $i \in L^{t^3}_{\mathbf{C}}(x; \succsim_{t^3})$; *not* $[C_j(y) \succsim_{t^3} C_h(z)]$ & *not* $[C_h(z) \succsim_{t^3} C_j(y)]$ for some $h \in L^{t^3}_{\mathbf{C}}(z; \succsim_{t^3})$ and some $j \in L^{t3}_{\mathbf{C}}(y; \succsim_{t^3})$; and *not* $[C_i(x) \succsim_{t^3} C_j(y)]$ & *not* $[C_j(y) \succsim_{t^3} C_i(x)]$ for some $i \in L^{t^3}_{\mathbf{C}}(x; \succsim_{t^3})$ and some $j \in L^{t^3}_{\mathbf{C}}(y; \succsim_{t^3})$.

Under (1), $(x, y) \in P^{t^1}_{\mathbf{C}}$ by **RM**, and $(z, x) \in NR^{t^1}_{\mathbf{C}}$ and $(y, z) \in NR^{t^1}_{\mathbf{C}}$. by **RC**. Under (2), $(y, z) \in P^{t^2}_{\mathbf{C}}$ by **RM**, and $(z, x) \in NR^{t^2}_{\mathbf{C}}$ and $(x, y) \in NR^{t^2}_{\mathbf{C}}$. by **RC**. Under (3), $(z, x) \in P^{t^3}_{\mathbf{C}}$ by **RM**, and $(y, z) \in NR^{t^3}_{\mathbf{C}}$ and $(x, y) \in NR^{t^3}_{\mathbf{C}}$. by **RC**. Therefore, by the definition of **PR**, $(x, y), (y, z), (z, x) \in P^{PR}_{\mathbf{C}}$ holds, which implies that $P^{PR}_{\mathbf{C}}$ is not transitive. ■

Proof of Theorem 2. Let $(x, y), (y, z) \in R_{\mathbf{C}}^{NR}$. This implies that there exists $t^1 \in T$ such that $(x, y) \in R_{\mathbf{C}}^{t^1}$ and $(y, x) \notin P_{\mathbf{C}}^{t}$ for any other $t \in T$, and there exists $t^2 \in T$ such that $(y, z) \in R_{\mathbf{C}}^{t^2}$ and $(z, y) \notin P_{\mathbf{C}}^{t}$ for any other $t \in T$. Moreover, $(x, y) \in R_{\mathbf{C}}^{t^1}$ (resp. $(y, z) \in R_{\mathbf{C}}^{t^2}$) is derived from **BWC** by applying either of α), β), or γ); or it is derived from **RM** by applying δ).

1. First of all, let us show that if $(x, y) \in R_{\mathbf{C}}^{t^1}$ and $(z, y) \notin P_{\mathbf{C}}^{t^1}$, then $(x, z) \in R_{\mathbf{C}}^{t^1}$.

Case 1: Let $(x, y) \in R_{\mathbf{C}}^{t^1}$ be derived from **BWC** by applying α)-1). Then, $BC^{t^1} \succsim_{t^1} C_j(y)$ for all $j \in L_{\mathbf{C}}^{t^1}(y; \succsim_{t^1})$, and $C_i(x) \succsim_{t^1} BC^{t^1}$ for all $i \in L_{\mathbf{C}}^{t^1}(x; \succsim_{t^1}$). Suppose $(z, x) \in P_{\mathbf{C}}^{t^1}$. Note that neither of α), β), γ), δ) can derive $(z, x) \in P_{\mathbf{C}}^{t^1}$, thus (z, x) corresponds to ϵ) or ε), which leads to $(x, z) \in NR_{\mathbf{C}}^{t^1}$ by **RC**, a contradiction. Thus, $(x, z) \in NR_{\mathbf{C}}^{t^1}$ or $(x, z) \in R_{\mathbf{C}}^{t^1}$ holds. Suppose $(x, z) \in NR_{\mathbf{C}}^{t^1}$. This implies that (x, z) corresponds to ϵ) , so that $C_i(z) \succ_{t^1} BC^{t^1}$ for all $i \in L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1})$. Then, $(z, y) \in R_{\mathbf{C}}^{t^1}$ from **BWC** by applying α). Since $(z, y) \notin P_{\mathbf{C}}^{t^1}$, $(z, y) \in I_{\mathbf{C}}^{t^1}$ holds, so that $(x, z) \in R_{\mathbf{C}}^{t^1}$ holds, since $R_{\mathbf{C}}^{t^1}$ is transitive by Lemma 1. Moreover, by transitivity, if $(x, y) \in P_{\mathbf{C}}^{t^1}$, then $(x, z) \in P_{\mathbf{C}}^{t^1}$ holds.

Case 2: Let $(x, y) \in P_{\mathbf{C}}^{t^1}$ be derived from **BWC** by applying β). Then, *not* $\left[C_j(y) \succsim_{t^1} BC^{t^1}\right]$ for some $j \in L_{\mathbf{C}}^{t^1}(y; \succsim_{t^1})$, and $C_i(x) \succsim_{t^1} BC^{t^1}$ for all $i \in L_{\mathbf{C}}^{t^1}(x; \succsim_{t^1})$. Suppose $(z, x) \in P_{\mathbf{C}}^{t^1}$. Note that neither of α), β), γ), δ) can derive $(z, x) \in P_{\mathbf{C}}^{t^1}$, thus (z, x) corresponds to ϵ) or ε), which leads to $(x, z) \in NR_{\mathbf{C}}^{t^1}$ by **RC**, a contradiction. Thus, $(x, z) \in NR_{\mathbf{C}}^{t^1}$ or $(x, z) \in R_{\mathbf{C}}^{t^1}$ holds. Suppose $(x, z) \in NR_{\mathbf{C}}^{t^1}$. This implies that (x, z) corresponds to ϵ) , so that $C_i(z) \succ_{t^1} BC^{t^1}$ for all $i \in L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1})$. Then, $(z, y) \in P_{\mathbf{C}}^{t^1}$ from **BWC** by applying β), which is a contradiction from $(z, y) \notin P_{\mathbf{C}}^{t^1}$. Thus, $(x, z) \in R_{\mathbf{C}}^{t^1}$ holds. Finally, suppose that $(z, x) \in R_{\mathbf{C}}^{t^1}$. This is only available by applying α)-1), and $C_i(x) \sim_{t^1} BC^{t^1}$ for all $i \in L_{\mathbf{C}}^{t^1}(x; \succsim_{t^1})$ and $C_i(z) \sim_{t^1} BC^{t^1}$ for all $i \in L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1})$. Then, $(z, y) \in P_{\mathbf{C}}^{t^1}$ from **BWC** by applying β), which is a contradiction from $(z, y) \notin P_{\mathbf{C}}^{t^1}$. Thus, $(z, x) \in R_{\mathbf{C}}^{t^1}$ is impossible, so that $(x, z) \in P_{\mathbf{C}}^{t^1}$ holds.

Case 3: Let $(x, y) \in P_{\mathbf{C}}^{t^1}$ be derived from **BWC** by applying α)-2) or γ). First, suppose that $BC^{t^1} \sim_{t^1} C_i(y)$ for all $i \in L_{\mathbf{C}}^{t^1}(y; \succsim_{t^1})$. Then, $C_i(x) \succ_{t^1} BC^{t^1}$ for all $i \in L_{\mathbf{C}}^{t^1}(x; \succsim_{t^1})$ so that $(z, x) \in R_{\mathbf{C}}^{t^1}$ is never possible. Moreover, if $(z, x) \in NR_{\mathbf{C}}^{t^1}$, then $C_i(z) \succ_{t^1} BC^{t^1}$ for all $i \in L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1})$ due to **RC** with ϵ). Then, $(z, y) \in P_{\mathbf{C}}^{t^1}$ by **BWC** with α)-2), which is a contradiction from $(z, y) \notin P_{\mathbf{C}}^{t^1}$. Thus, $(x, y) \in P_{\mathbf{C}}^{t^1}$ is only possible.

Next, suppose that $BC^{t^1} \succ_{t^1} C_i(y)$ for all $i \in L_{\mathbf{C}}^{t^1}(y; \succsim_{t^1})$.

Suppose $(z, x) \in R_{\mathbf{C}}^{t^1}$. If *not* $\left[C_j(x) \succsim_{t^1} BC^{t^1}\right]$ for some $j \in L_{\mathbf{C}}^{t^1}(x; \succsim_{t^1})$, then, in order to $(z, x) \in R_{\mathbf{C}}^{t^1}$, $C_i(z) \succsim_{t^1} BC^{t^1}$ holds for all $i \in L_{\mathbf{C}}^{t^1}(z; \succsim_{t^1})$. Thus, $(z, y) \in P_{\mathbf{C}}^{t^1}$ from **BWC** by applying α)-2), a contradiction from $(z, y) \notin$

$P_{\mathbf{C}}^{t^1}$. If $C_j(x) \succsim_{t^1} BC^{t^1}$ for all $j \in L_{\mathbf{C}}^{t^1}(x;\succsim_{t^1})$, then $(z,x) \in R_{\mathbf{C}}^{t^1}$ is possible only from **BWC** by applying α)-2). Then, it implies that $C_j(x) \succsim_{t^1} BC^{t^1}$ for all $j \in L_{\mathbf{C}}^{t^1}(x;\succsim_{t^1})$, which again implies $(z,y) \in P_{\mathbf{C}}^{t^1}$ by **BWC** with applying α)-2), a contradiction. In summary, $(z,x) \in R_{\mathbf{C}}^{t^1}$ is impossible.

Suppose $(z,x) \in NR_{\mathbf{C}}^{t^1}$. This implies that $(z,x) \in NR_{\mathbf{C}}^{t^1}$ is derived from **RC** by applying ϵ) or ε). If ϵ) is applied, then $C_i(z) \succ_{t^1} BC^{t^1}$ holds for all $i \in L_{\mathbf{C}}^{t^1}(z;\succsim_{t^1})$, so that $(z,y) \in P_{\mathbf{C}}^{t^1}$ from **BWC** by applying α)-2), a contradiction. If ε) is applied, then

$$\begin{aligned} \textit{not}\left[C_j(z) \succsim_{t^1} BC^{t^1}\right] \text{ for some } j &\in L_{\mathbf{C}}^{t^1}(z;\succsim_{t^1})\text{; or} \\ \textit{not}\left[BC^{t^1} \succsim_{t^1} C_i(z)\right] \text{ for some } i &\in L_{\mathbf{C}}^{t^1}(z;\succsim_{t^1}). \end{aligned}$$

Let both the former and the latter hold. Then, there is a common $i \in L_{\mathbf{C}}^{t^1}(z;\succsim_{t^1})$ such that *not* $\left[C_i(z) \succsim_{t^1} BC^{t^1}\right]$ and *not* $\left[BC^{t^1} \succsim_{t^1} C_i(z)\right]$. Then, since the application of ε) implies that γ) is applied for having $(x,y) \in P_{\mathbf{C}}^{t^1}$, it follows that $(z,y) \in P_{\mathbf{C}}^{t^1}$ holds from **BWC** by applying γ), a contradiction.

Hence, either the former does not hold or the latter does not hold. Let the former do not hold, so that $C_i(z) \succsim_{t^1} BC^{t^1}$ holds for all $i \in L_{\mathbf{C}}^{t^1}(z;\succsim_{t^1})$. Then, $(z,y) \in P_{\mathbf{C}}^{t^1}$ from **BWC** by applying α)-2), a contradiction. Let the latter do not hold, so that $BC^{t^1} \succsim_{t^1} C_i(z)$ for all $i \in L_{\mathbf{C}}^{t^1}(z;\succsim_{t^1})$. Thus, $(z,y) \notin P_{\mathbf{C}}^{t^1}$ is derived from **RM** by applying δ). Then, by **FCD**, $(y,z) \in R_{\mathbf{C}}^{t^1}$. Thus, by transitivity, $(x,z) \in P_{\mathbf{C}}^{t^1}$, which is a contradiction from $(z,x) \in NR_{\mathbf{C}}^{t^1}$. In summary, $(z,x) \in NR_{\mathbf{C}}^{t^1}$ is impossible. Thus, $(x,z) \in P_{\mathbf{C}}^{t^1}$ holds.

Case 4: Let $(x,y) \in R_{\mathbf{C}}^{t^1}$ be derived from **RM** by applying δ). Then, $BC^{t^1} \succ_{t^1} C_i(x)$ for all $i \in L_{\mathbf{C}}^{t^1}(x;\succsim_{t^1})$; and also, $BC^{t^1} \succ_{t^1} C_i(y)$ for all $i \in L_{\mathbf{C}}^{t^1}(y;\succsim_{t^1})$. Since $(z,y) \notin P_{\mathbf{C}}^{t^1}$, either $(y,z) \in R_{\mathbf{C}}^{t^1}$ or $(y,z) \in NR_{\mathbf{C}}^{t^1}$. Suppose $(y,z) \in NR_{\mathbf{C}}^{t^1}$. Since $BC^{t^1} \succ_{t^1} C_i(y)$ for all $i \in L_{\mathbf{C}}^{t^1}(y;\succsim_{t^1})$, $(y,z) \in NR_{\mathbf{C}}^{t^1}$ implies that $BC^{t^1} \succ_{t^1} C_i(z)$ for all $i \in L_{\mathbf{C}}^{t^1}(z;\succsim_{t^1})$. However, by **FCD**, **RM** can be applied to evaluate (y,z), which implies $(y,z) \notin NR_{\mathbf{C}}^{t^1}$, a contradiction. Thus, $(y,z) \in R_{\mathbf{C}}^{t^1}$. Then, by transitivity of $R_{\mathbf{C}}^{t^1}$, $(x,z) \in R_{\mathbf{C}}^{t^1}$ holds.

In summary, if $(x,y) \in R_{\mathbf{C}}^{t^1}$ and $(z,y) \notin P_{\mathbf{C}}^{t^1}$, then $(x,z) \in R_{\mathbf{C}}^{t^1}$.

2. Second, let us show that if $(y,z) \in R_{\mathbf{C}}^{t^2}$ and $(y,x) \notin P_{\mathbf{C}}^{t^1}$, then $(z,x) \notin P_{\mathbf{C}}^{t^2}$. Suppose $(z,x) \in P_{\mathbf{C}}^{t^2}$. Since $(y,z) \in R_{\mathbf{C}}^{t^2}$, it follows from transitivity of $R_{\mathbf{C}}^{t^2}$ that $(y,x) \in P_{\mathbf{C}}^{t^2}$, which is a contradiction. Thus $(z,x) \notin P_{\mathbf{C}}^{t^2}$.

3. Third, let us show that for any $t \in T\backslash\left\{t^1\right\}$, if $(y,x) \notin P_{\mathbf{C}}^{t}$ and $(z,y) \notin P_{\mathbf{C}}^{t}$, then $(z,x) \notin P_{\mathbf{C}}^{t}$. Suppose, in the contrary, $(z,x) \in P_{\mathbf{C}}^{t}$. Then, it is derived from **BWC** by applying either of α)-2), β), or γ); or it is derived from **RM**-2) by applying δ).

Let $(z,x) \in P_{\mathbf{C}}^{t}$ be derived from **BWC**. Suppose that α)-2) or γ) is applied with $BC^t \succ_t C_i(x)$ for all $i \in L_{\mathbf{C}}^{t}(x;\succsim_t)$. Thus, by **FCD** and **RM**, it is impossible that $(y,x) \in NR^t_{\mathbf{C}}$. Thus, $(x,y) \in R^t_{\mathbf{C}}$. Then, by transitivity, $(z,y) \in P^t_{\mathbf{C}}$, which is a contradiction.

Suppose that $(z,x) \in P^t_{\mathbf{C}}$ is derived from **BWC** by applying β). Then, *not* $[C_i(x) \succsim_{t^1} BC^t]$ and *not* $[BC^t \succsim_{t^1} C_i(x)]$ hold for some $i \in L^t_{\mathbf{C}}(x;\succsim_{t^1})$. In this case, $(x,y) \in R^t_{\mathbf{C}}$ or $(x,y) \in NR^t_{\mathbf{C}}$. Let $(x,y) \in R^t_{\mathbf{C}}$. This case is derived from **BWC** by applying γ), which implies that $BC^t \succ_t C_i(y)$ for all $i \in L^t_{\mathbf{C}}(y;\succsim_t)$. Then, $(z,y) \in P^t_{\mathbf{C}}$ holds by **BWC** with applying α-2). Thus, a contradiction. Next, let $(x,y) \in NR^t_{\mathbf{C}}$. This is derived from **RC** by applying ε). To apply ε) for (x,y), *not* $[C_j(y) \succsim_t BC^t]$ and *not* $[BC^t \succsim_{t^1} C_j(y)]$ for some $j \in L^t_{\mathbf{C}}(y;\succsim_t)$ is necessary. Then, $(z,y) \in P^t_{\mathbf{C}}$ is derived from **BWC** by applying β), which is a contradiction.

Suppose that $(z,x) \in P^t_{\mathbf{C}}$ is derived from **BWC** by applying α)-2) with $BC^t \sim_t C_i(x)$ for all $i \in L^t_{\mathbf{C}}(x;\succsim_t)$. Then, $(y,x) \in NR^t_{\mathbf{C}}$ is impossible. Thus, $(x,y) \in R^t_{\mathbf{C}}$. Then, by transitivity, $(z,y) \in P^t_{\mathbf{C}}$, which is a contradiction. In summary, $(z,x) \in P^t_{\mathbf{C}}$ cannot be derived from **BWC**.

Let $(z,x) \in P^t_{\mathbf{C}}$ be derived from **RM**-2) by applying δ). Then, $BC^t \succ_t C_i(x)$ for all $i \in L^t_{\mathbf{C}}(x;\succsim_t)$. Thus, by **FCD** and **RM**, it is impossible that $(y,x) \in NR^t_{\mathbf{C}}$. Thus, $(x,y) \in R^t_{\mathbf{C}}$. Then, by transitivity, $(z,y) \in P^t_{\mathbf{C}}$, which is a contradiction. Thus, $(z,x) \in P^t_{\mathbf{C}}$ cannot be derived from **RM**-2)**.**

In summary, for any $t \in T \setminus \{t^1\}$, if $(y,x) \notin P^t_{\mathbf{C}}$ and $(z,y) \notin P^t_{\mathbf{C}}$, then $(z,x) \notin P^t_{\mathbf{C}}$.

4. By the above arguments of **1.** and **3.**, we have $(x,z) \in R^{t^1}_{\mathbf{C}}$ and $(z,x) \notin P^t_{\mathbf{C}}$ for any $t \in T \setminus \{t^1\}$. Thus, $(x,z) \in R^{NR}_{\mathbf{C}}$ holds. ■

Proof of Theorem 3. Let $T = \{t^1\}$, $\{i,j,h,h'\} \subset t^1 = N$, $L^{t^1}_{\mathbf{C}}(x;\succsim_{t^1}) = \{i,j,h\}$, $L^{t^1}_{\mathbf{C}}(y;\succsim_{t^1}) = \{i,j,h\}$, and $L^{t1}_{\mathbf{C}}(z;\succsim_{t^1}) = \{i,j,h,h'\}$. Suppose that:

$$
\begin{aligned}
BC^{t^1} &\succ {}_{t^1}C_i(x) \sim_{t^1} C_j(x) \sim_{t^1} C_h(x)\,; \\
BC^{t^1} &\succ {}_{t^1}C_i(y) \sim_{t^1} C_j(y) \sim_{t^1} C_h(y)\,; \\
BC^{t^1} &\succ {}_{t^1}C_i(z) \sim_{t^1} C_j(z) \sim_{t^1} C_h(z)\,; \\
not\ [BC^{t^1} &\succsim {}_{t^1}C_{h'}(z)\ \ or\ C_{h'}(z) \succsim_{t^1} BC^{t^1}]; \\
C_i(x) &\succ {}_iC_i(y) \succ_i C_i(z)\,; \\
C_j(x) &\succ {}_jC_j(y) \succ_j C_j(z)\,; \\
C_h(x) &\succ {}_hC_h(y) \succ_h C_h(z)\,; \text{and} \\
C_{h'}(x) &\succ {}_{h'}C_{h'}(y) \succ_{h'} C_{h'}(z)\,.
\end{aligned}
$$

Moreover, for any $k \in t^1 \setminus \{i,j,h,h'\}$, let $C_k(z) \succ {}_{t^1}C_{h'}(z)$, $C_k(x) \succ_{t^1} BC^{t^1}$, $C_k(y) \succ_{t^1} BC^{t^1}$, $C_k(z) \succ_{t^1} BC^{t^1}$, and $C_k(x) \succ_k C_k(y) \succ_k C_k(z)$.

Then, since $BC^{t^1} \succ_{t^1} C_i(x)$ for all $i \in L^{t^1}_{\mathbf{C}}(x;\succsim_{t^1})$, and *not* $BC^{t^1} \succsim_{t^1} C_{h'}(z)$ & *not* $C_{h'}(z) \succsim_{t^1} BC^{t^1}$ for some $h' \in L^{t^1}_{\mathbf{C}}(x;\succsim_{t^1})$, it follows that $(z,x) \in P^{t^1}_{\mathbf{C}}$ from **BWC** with γ). Thus, since $T = \{t^1\}$ by **NR**, $(z,x) \in P_{\mathbf{C}}$, while by **WP**, $(x,z) \in P_{\mathbf{C}}$. Thus, a contradiction, which implies **NR** and **WP** are incompatible. ■

Proof of Lemma 2. 1. Let $(x, y) \in R^{NR}_{\mathbf{C}}$ and $(y, z) \in P^{WP}_{\mathbf{C}}$. This implies that there exists $t^1 \in T$ such that $(x, y) \in R^{t^1}_{\mathbf{C}}$ and $(y, x) \notin P^t_{\mathbf{C}}$ for any other $t \in T$, and $C_i(y) \succ_i C_i(z)$ holds for all $i \in N$. Moreover, $(x, y) \in R^{t^1}_{\mathbf{C}}$ is derived from **BCC** by applying either of α), β), or γ); or it is derived from **RM** by applying δ).

2. Show that for any $t \in T$, (i) $(y, x) \notin P^t_{\mathbf{C}}$, and (ii) $C_i(y) \succ_i C_i(z)$ holds for all $i \in N$ together imply that $(x, z) \in R^t_{\mathbf{C}} \cup NR^t_{\mathbf{C}}$. First of all, $(y, x) \notin P^t_{\mathbf{C}}$ if and only if $(x, y) \in R^t_{\mathbf{C}} \cup NR^t_{\mathbf{C}}$. If $(x, y) \in R^t_{\mathbf{C}}$, then it is derived from **BCC** by applying α), β), or γ), or from **RM** by applying δ). If $(x, y) \in NR^t_{\mathbf{C}}$, then it is derived from **RC** by applying ϵ) or ε).

2-i). Suppose $(x, y) \in R^t_{\mathbf{C}}$ is derived from **BWC** by applying α), β), or γ). Then, *not* $[BC^t \succsim_t C_i(x)]$ for some $i \in L^t_{\mathbf{C}}(x; \succsim_t)$. Note, since $C_i(y) \succ_i C_i(z)$ holds for all $i \in N^t$, we have *not* $[C_h(z) \succ_t C_j(y)]$ holds for any $j \in L^t_{\mathbf{C}}(y; \succsim_t)$ and any $h \in L^t_{\mathbf{C}}(z; \succsim_t)$. (In fact, if $[C_h(z) \succ_t C_j(y)]$ holds for some $j \in L^t_{\mathbf{C}}(y; \succsim_t)$ and some $h \in L^t_{\mathbf{C}}(z; \succsim_t)$, then $C_h(z) \succ_t C_j(z)$ holds, which is a contradiction from $h \in L^t_{\mathbf{C}}(z; \succsim_t)$.) Then, since $BC^t \succsim_t C_j(y) \succ_t C_j(z)$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$, then *not* $[C_h(z) \succsim_t BC^t]$ holds for any $h \in L^t_{\mathbf{C}}(z; \succsim_t)$. (In fact, if $[C_h(z) \succsim_t BC^t]$ holds for some $h \in L^t_{\mathbf{C}}(z; \succsim_t)$, it implies $C_h(z) \succ_t C_j(z)$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$, which is a contradiction from $h \in L^t_{\mathbf{C}}(z; \succsim_t)$.) In summary, the above arguments imply that only either of α), β), γ), or ε) is applied to (x, z), thus $(x, z) \in R^t_{\mathbf{C}} \cup NR^t_{\mathbf{C}}$.

2-ii). Suppose $(x, y) \in NR^t_{\mathbf{C}}$ is derived from **RC** by applying ϵ) or ε). If ϵ) is applied, then $[C_i(x) \succ_t BC^t]$ holds for any $i \in L^t_{\mathbf{C}}(x; \succsim_t)$. Then, $(x, z) \in R^t_{\mathbf{C}} \cup NR^t_{\mathbf{C}}$ is derived by applying either of α), β), or ϵ). If ε) is applied, then $[$*not* $[BC^t \succsim_t C_i(x)]$ for some $i \in L^t_{\mathbf{C}}(x; \succsim_t)$ and *not* $[C_j(x) \succsim_t BC^t]$ for some $j \in L^t_{\mathbf{C}}(x; \succsim_t)]$ and $[$*not* $[BC^t \succsim_t C_i(y)]$ for some $i \in L^t_{\mathbf{C}}(y; \succsim_t)$ and *not* $[C_j(y) \succsim_t BC^t]$ for some $j \in L^t_{\mathbf{C}}(y; \succsim_t)]$. Then, by either **RC** with ε) or **BWC** with γ), $(x, z) \in R^t_{\mathbf{C}} \cup NR^t_{\mathbf{C}}$ is derived.

2-iii). Suppose $(x, y) \in R^t_{\mathbf{C}}$ is derived from **RM** by applying δ). Then, since $C_i(y) \succ_i C_i(z)$ holds for all $i \in N^t$, we have, by combining with **RM** with δ), $C_i(x) \succsim_t C_j(y) \succ_t C_j(z)$ for any $i \in L^t_{\mathbf{C}}(x; \succsim_t)$ and any $j \in L^t_{\mathbf{C}}(y; \succsim_t)$. Note that, for each $j \in L^t_{\mathbf{C}}(y; \succsim_t)$, there exists $h \in L^t_{\mathbf{C}}(z; \succsim_t)$ such that $C_j(z) \succ_t C_h(z)$. Thus, there exists $h \in L^t_{\mathbf{C}}(z; \succsim_t)$ such that $C_i(x) \succ_t C_h(z)$ for any $i \in L^t_{\mathbf{C}}(x; \succsim_t)$. Thus, if $BC^t_t \succ_t C_h(z)$ for any $h \in L^t_{\mathbf{C}}(z; \succsim_t)$, then by **RM** with δ), $(x, z) \in P^t_{\mathbf{C}}$. Moreover, since $BC^t \succsim_t C_j(y) \succ_t C_j(z)$ for all $j \in L^t_{\mathbf{C}}(y; \succsim_t)$, *not* $[C_i(z) \succsim_t BC^t]$ holds for any $i \in L^t_{\mathbf{C}}(z; \succsim_t)$. Suppose that there exists $h' \in L^t_{\mathbf{C}}(z; \succsim_t)$ such that *not* $[BC^t \succsim_t C_{h'}(z)]$. However, since $h \in L^t_{\mathbf{C}}(z; \succsim_t)$ has $BC^t \succsim_t C_j(z) \succ_t C_h(z)$, **D** implies $C_{h'}(z) \succ_t C_h(z)$, which is a contradiction from $h' \in L^t_{\mathbf{C}}(z; \succsim_t)$. Thus, there is no $h' \in L^t_{\mathbf{C}}(z; \succsim_t)$ such that *not* $[BC^t \succsim_t C_{h'}(z)]$. Thus, from **RC** by applying ε), $(x, z) \in NR^t_{\mathbf{C}}$.

3. Let $(x, y) \in R^{t^1}_{\mathbf{C}}$ be derived from **BWC** by applying either of α), β), or γ).

3-i): Let $(x, y) \in R^{t^1}_{\mathbf{C}}$ be derived from **BWC** by applying α). Then, $BC^{t^1} \succsim_{t^1} C_j(y)$ for all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$, and $C_i(x) \succsim_{t^1} BC^{t^1}$ for all $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1}$). Then, we can show that $C_i(x) \succsim_t BC^{t^1}$ holds for every $i \in N^{t^1}$. First, if

$i \in N^{t^1}$ is $C_i(x) \succ_t C_j(x)$ for some $j \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$, it is obvious. If $i \in N^{t^1}$ is *not* $[C_i(x) \succ_{t^1} C_j(x)]$ for any $j \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$, then $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$, so that $C_i(x) \succsim_{t^1} BC^{t^1}$ holds.

Let us consider $L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. Since $C_i(y) \succ_{t^1} C_i(z)$ holds for all $i \in N$, it follows that, for any $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$ and any $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$, *not* $[C_i(z) \succ_{t^1} C_j(y)]$ holds. In fact, if $[C_i(z) \succ_{t^1} C_j(y)]$ holds for some $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$ and some $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$, then $C_i(z) \succ_{t^1} C_j(z)$ holds, which is a contradiction from $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. Then, since $BC^{t^1} \succsim_{t^1} C_j(y) \succ_{t^1} C_j(z)$ for all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$, *not* $\left[C_i(z) \succsim_{t^1} BC^{t^1}\right]$ holds for any $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. In fact, if $\left[C_i(z) \succsim_{t^1} BC^{t^1}\right]$ holds for some $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$, it implies $C_i(z) \succ_{t^1} C_j(z)$, which is a contradiction from $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. In summary, $(x,z) \in P^{t^1}_{\mathbf{C}}$ is derived from **BWC** by applying α)-2) or β).

3-ii): Let $(x,y) \in P^{t^1}_{\mathbf{C}}$ be derived from **BWC** by applying β). Then, *not* $\left[C_j(y) \succsim_{t^1} BC^{t^1}\right]$ for some $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$, and $C_i(x) \succsim_{t^1} BC^{t^1}$ for all $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$. Suppose $(z,x) \in R^{t^1}_{\mathbf{C}}$. Note that neither of α), β), γ), δ) can derive $(z,x) \in R^{t^1}_{\mathbf{C}}$, thus (z,x) corresponds to ϵ) or ε), which leads to $(x,z) \in NR^{t^1}_{\mathbf{C}}$ by **RC**, a contradiction. Thus, $(x,z) \in NR^{t^1}_{\mathbf{C}}$ or $(x,z) \in P^{t^1}_{\mathbf{C}}$ holds. Suppose $(x,z) \in NR^{t^1}_{\mathbf{C}}$. This implies that (x,z) corresponds to ϵ) , so that $C_i(z) \succ_{t^1} BC^{t^1}$ for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. However, *not* $\left[C_j(y) \succsim_{t^1} BC^{t^1}\right]$ for some $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$ and $C_j(y) \succ_{t^1} C_j(z)$, which implies that *not* $\left[C_j(z) \succsim_{t^1} BC^{t^1}\right]$. Then, since $C_i(z) \succ_{t^1} BC^{t^1}$ for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$, $j \notin L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$ holds, which further implies that there exists $h \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$ such that $C_j(z) \succ_{t^1} C_h(z) \succ_{t^1} BC^{t^1}$, a contradiction. Thus, $(x,z) \in NR^{t^1}_{\mathbf{C}}$ is impossible, so that $(x,z) \in P^{t^1}_{\mathbf{C}}$ holds.

3-iii): Let $(x,y) \in P^{t^1}_{\mathbf{C}}$ be derived from **BWC** by applying α)-2) or γ). This implies *not* $\left[BC^{t^1} \succsim_{t^1} C_i(x)\right]$ for some $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$, and $BC^{t^1} \succ_{t^1} C_j(y)$ for all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$.

Suppose $(z,x) \in R^{t^1}_{\mathbf{C}}$. In order to $(z,x) \in R^{t^1}_{\mathbf{C}}$, $C_i(z) \succsim_{t^1} BC^{t^1}$ holds for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. However, since $BC^{t^1} \succ_{t^1} C_j(y) \succ_{t^1} C_j(z)$ for all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$, $C_i(z) \succ_{t^1} C_j(z)$ for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$, which is a contradiction. Thus, in summary, $(z,x) \notin R^{t^1}_{\mathbf{C}}$.

Suppose $(z,x) \in NR^{t^1}_{\mathbf{C}}$. This implies that $(z,x) \in NR^{t^1}_{\mathbf{C}}$ is derived from **RC** by applying ϵ) or ε). If ϵ) is applied, then $C_i(z) \succ_{t^1} BC^{t^1}$ holds for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. Then, since $BC^{t^1} \succ_{t^1} C_j(y) \succ_{t^1} C_j(z)$ for all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$, $C_i(z) \succ_{t^1} C_j(z)$ for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$, which is a contradiction. If ε) is applied, then

$$\begin{aligned} \textit{not}\left[C_j(z) \succsim_{t^1} BC^{t^1}\right] \text{ for some } j &\in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1}); \\ \text{or } \textit{not}\left[BC^{t^1} \succsim_{t^1} C_i(z)\right] \text{ for some } i &\in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1}). \end{aligned}$$

Let the former do not hold, so that $C_i(z) \succsim_{t^1} BC^{t^1}$ holds for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. Then, again, since $BC^{t^1} \succ_{t^1} C_j(y) \succ_{t^1} C_j(z)$ for all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$, $C_i(z) \succ_{t^1} C_j(z)$ for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$, which is a contradiction. Let the latter do not hold, so that $BC^{t^1} \succsim_{t^1} C_i(z)$ for all $i \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. Thus, $(z, y) \notin P^{t^1}_{\mathbf{C}}$ is derived from **RM** by applying δ). Then, by **FCD**, $(y, z) \in R^{t^1}_{\mathbf{C}}$. Thus, by transitivity, $(x, z) \in P^{t^1}_{\mathbf{C}}$, which is a contradiction from $(z, x) \in NR^{t^1}_{\mathbf{C}}$. In summary, $(z, x) \in NR^{t^1}_{\mathbf{C}}$ does not hold. Thus, $(x, z) \in P^{t^1}_{\mathbf{C}}$ holds.

4. Let $(x, y) \in R^{t^1}_{\mathbf{C}}$ be derived from **RM** by applying δ). Then, $BC^{t^1} \succ_{t^1} C_i(x)$ for all $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$; and also, $BC^{t^1} \succ_{t^1} C_i(y)$ for all $i \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$. Since $BC^{t^1} \succ_{t^1} C_j(y) \succ_{t^1} C_j(z)$ for all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$, **D** implies that $BC^{t^1} \succ_{t^1} C_h(z)$ for all $h \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. Thus, by **FCD** and **RM** with δ), $(y, z) \in R^{t^1}_{\mathbf{C}}$ is derived. Hence, the pair of (x, z) corresponds to the case δ). Moreover, $C_i(x) \succsim_{t^1} C_j(y) \succ_{t^1} C_h(z)$ holds for all $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$, all $j \in L^{t^1}_{\mathbf{C}}(y; \succsim_{t^1})$, and all $h \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. Then, by the transitivity of $\succsim_{t^1}$, $C_i(x) \succ_{t^1} C_h(z)$ holds for all $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$ and all $h \in L^{t^1}_{\mathbf{C}}(z; \succsim_{t^1})$. Thus, $(x, z) \in P^{t^1}_{\mathbf{C}}$ holds by **RM**.

5. In combining the above arguments, if $(x, y) \in R^{NR}_{\mathbf{C}}$ and $(y, z) \in P^{WP}_{\mathbf{C}}$, then $(x, z) \in P^{t^1}_{\mathbf{C}}$ holds and $(x, z) \in R^{t}_{\mathbf{C}} \cup NR^{t}_{\mathbf{C}}$ for any $t \in T \backslash \{t^1\}$, thus $(x, z) \in P^{NR}_{\mathbf{C}}$. ■

Proof of Lemma 3. Let $(x, y) \in P^{WP}_{\mathbf{C}}$ and $(y, z) \in R^{NR}_{\mathbf{C}}$. This implies that there exists $t^1 \in T$ such that $(y, z) \in R^{t^1}_{\mathbf{C}}$ and $(z, y) \notin P^{t}_{\mathbf{C}}$ for any other $t \in T$, and $C_i(x) \succ_{t^1} C_i(y)$ holds for all $i \in N$.

1. We will show that, in this case, $(x, z) \in P^{t^1}_{\mathbf{C}}$ holds.

First, if $(y, z) \in P^{t^1}_{\mathbf{C}}$ is derived from **BWC** by applying either of α)-2), β), or γ), then $(x, z) \in P^{t^1}_{\mathbf{C}}$ is derived from **BWC** by applying either of α)-2), β), or γ).

Second, if $(y, z) \in I^{t^1}_{\mathbf{C}}$ is derived from **BWC** by applying α)-1), then $(x, z) \in R^{t^1}_{\mathbf{C}}$ is derived from **BWC** by applying α). Moreover, since $C_i(x) \succ_{t^1} BC^{t^1}$ for all $i \in L^{t^1}_{\mathbf{C}}(x; \succsim_{t^1})$, $(x, z) \in P^{t^1}_{\mathbf{C}}$ holds by **BWC** with α)-2).

Third, let $(y, z) \in R^{t^1}_{\mathbf{C}}$ be derived from **RM** by applying δ). If $C_i(x) \prec_t BC^t$ for all $i \in L^{t}_{\mathbf{C}}(x; \succsim_t)$, then $(x, z) \in P^{t^1}_{\mathbf{C}}$ is derived from **RM**-2) and **FCD**; otherwise, then $(x, z) \in P^{t^1}_{\mathbf{C}}$ is derived from **BWC** and **FCD** by applying α)-2) or γ).

In summary, $(x, z) \in P^{t^1}_{\mathbf{C}}$ holds for $t^1 \in T$.

2. Next, we will show that $(z, x) \notin P^{t}_{\mathbf{C}}$ for any $t \in T \backslash \{t^1\}$. Note that for any $t \in T \backslash \{t^1\}$, (i) $(z, y) \notin P^{t}_{\mathbf{C}}$, and (ii) $C_i(x) \succ_i C_i(y)$ holds for all $i \in N$ together imply that $(x, z) \in R^{t}_{\mathbf{C}} \cup NR^{t}_{\mathbf{C}}$. First of all, $(z, y) \notin P^{t}_{\mathbf{C}}$ if and only if $(y, z) \in R^{t}_{\mathbf{C}} \cup NR^{t}_{\mathbf{C}}$.

If $(y, z) \in R^{t}_{\mathbf{C}}$, then it is derived from **BWC** by applying α), β), or γ), or from **RM** by applying δ). Then, as shown in the case of **1.**, we can see that $(x, z) \in P^{t}_{\mathbf{C}}$ holds for any $t \in T \backslash \{t^1\}$.

If $(y,z) \in NR^t_{\mathbf{C}}$, then it is derived from **RC** by applying ϵ) or ε). If ϵ) is applied for $(y,z) \in NR^t_{\mathbf{C}}$, then $(x,z) \in NR^t_{\mathbf{C}}$ also holds by **RC**. If ε) is applied for $(y,z) \in NR^t_{\mathbf{C}}$, then $[\mathit{not}\,[BC^t \succsim_t C_i\,(y)]$ for some $i \in L^t_{\mathbf{C}}(y;\succsim_t)$ and $\mathit{not}\,[C_j\,(y) \succsim_t BC^t]$ for some $j \in L^t_{\mathbf{C}}(y;\succsim_t)]$ and $[\mathit{not}\,[BC^t \succsim_t C_i\,(z)]$ for some $i \in L^t_{\mathbf{C}}(z;\succsim_t)$ and $\mathit{not}\,[C_j\,(z) \succsim_t BC^t]$ for some $j \in L^t_{\mathbf{C}}(z;\succsim_t)]$. Then, since $C_i(x) \succ_t C_i(y)$ holds for all $i \in N$, **RC** with ε) is again applied for $(x,z) \in NR^t_{\mathbf{C}}$ or **BWC** with β) is applied for $(x,z) \in R^t_{\mathbf{C}}$.

Thus, in summary, for any $t \in T \backslash \left\{t^1\right\}$, $(x,z) \in R^t_{\mathbf{C}} \cup NR^t_{\mathbf{C}}$ holds.

3. From **1.** and **2.**, $(x,z) \in P^{NR}_{\mathbf{C}}$ holds, so that $(x,z) \in P^*_{\mathbf{C}}$. ■