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Abstract

In this paper, we consider a natural procedure of decision-making, called a “Grouping Choice Method”, which leads to a kind of bounded rational choices. In this procedure a decision-maker (DM) first divides the set of available alternatives into some groups and in each group she chooses the best element (winner) for her preference relation. Then, among the winners in the first round, she selects the best one as her final choice. We characterize Grouping Choice Methods in three different ways. First, we show that a choice function is a Grouping Choice Method if and only if it is a Rational Shortlist Method (Manzini and Mariotti, 2007) in which the first rationale is transitive. Second, Grouping Choice Methods are axiomatically characterized by means of a new axiom called Elimination, in addition to two well-known axioms, Expansion and Weak WARP (Manzini and Mariotti, 2007). Third, Grouping Choice Methods are also characterized by a weak version of Path Independence.

JEL Classification: D01.

Keywords: grouping of alternatives, preference, bounded rationality.

1 Introduction

To construct models to explain (seemingly) irrational choices of individuals or societies is one of the central themes in economic theory recently. In this paper, we consider a natural procedure of decision-making, called “Grouping Choice Methods”, which leads to a kind of bounded rational choices. In this procedure, a decision-maker (DM) first divides the set of available alternatives into some groups and in each group she chooses the best element (winner) for her preference relation. Then, among the winners in the first round, she selects the best one as her final choice.

Such choice behaviors are often observed in real life. For example, suppose that a family would like to buy a house. Three houses $\{x, y, z\}$ are available, of which x and y are located in town A, and z in town B. They first choose the best house in each town, and then make a final choice from the “winners” in the first round. Suppose that they prefer x to y , y to z , and z to x .¹ Now, when y and z are available, each of them is the only house in each town. Hence, y is chosen from $\{y, z\}$ because they prefer y to z . On the other hand, when all three houses are available, they first choose x as the best house in town A since they prefer x to y . Because z is the only house in town B, hence the best, the set of winners in the first round is $\{x, z\}$. Then they choose z because they prefer z to x . Thus, z is selected from $\{x, y, z\}$. Notice that the family’s preference relation is cyclic in this example and yet a final choice can be determined by this procedure of choice with grouping. However, these choices are inconsistent with Samuelson’s (1938) Weak Axiom of Revealed Preferences (WARP), which requires that if y is chosen when z is available, then z should not be chosen whenever y is available.

In this paper, we formalize and analyze decision-making procedures as described above. First, we define a *grouping rule* as a correspondence that specifies for each set S of available alternatives a grouping in S (a set of subsets of S). Several natural requirements are imposed on admissible grouping rules.

¹Note that a family’s preference relation may become cyclic because it is a collective preference relation (if they decide by majority voting, for instance).

We assume that each DM is endowed with a *single* preference relation. Given a grouping rule and a preference relation, and for each set of available alternatives, a *Grouping Choice Method* first takes a maximal element in each group in the set for the preference relation, and then selects a maximal element among these first-round maximums.

We characterize Grouping Choice Methods in three different ways. First, we show that a choice function is a Grouping Choice Method if and only if it is a Rational Shortlist Method (Manzini and Mariotti, 2007) in which the first rationale is transitive. In Rational Shortlist Methods, a DM is endowed with *two* preference relations, called “rationales”, and for each set of available alternatives, she sequentially applies the two rationales to make the selection.

Second, we axiomatically characterize Grouping Choice Methods. Manzini and Mariotti (2007) showed that Rational Shortlist Methods are characterized by a weak version of WARP and a standard choice-consistency property under the expansion of the set of alternatives, simply called “Expansion”. Their Weak WARP requires that if an alternative x is chosen in binary comparison with y , as well as in a set S containing both x and y , then y should not be chosen in any “intermediate” set T between $\{x, y\}$ and S (that is, $\{x, y\} \subseteq T \subseteq S$). Because the class of Grouping Choice Methods is a restricted class of Rational Shortlist Methods, Grouping Choice Methods also satisfy Weak WARP and Expansion. In addition to the above two axioms, we introduce a new axiom called “Elimination”. This property means that if an alternative y is never chosen in the presence of another alternative x , then (i) whenever y is chosen in a menu (without x) and then x becomes newly available, x should be chosen in the new menu, or (ii) whenever y and x are present, eliminating y from the menu does not affect the choice. We show that these three axioms fully characterize Grouping Choice Methods. In the literature, the papers close to this part of our results are Au and Kawai (2011) and Horan (2013). Both of them characterize Rational Shortlist Methods in which both of the two rationales are transitive by distinct sets of axioms respectively. Horan (2013) also provides a characterization of Rational Shortlist Methods in which the

first rationale is transitive (and the second is unrestricted) by a list of axioms that is different from ours.²

Third, we consider a weak version of Path Independence, which we call Grouping Path Independence. The original version of Path Independence was introduced by Plottt (1973). It means that final outcomes should be independent of the “paths” to lead to them. To describe our version, assume that a grouping rule G and a set S of available alternatives are given. As an example, let the groups specified by G for S be $G(S) = \{S_1, S_2, S_3\}$. Consider the following path to a final choice: first, apply a choice rule C to each group S_i ($i \in \{1, 2, 3\}$) to select an alternative $C(S_i)$; second, apply the rule to the set of the alternatives selected from the groups in the first round, namely $\{C(S_1), C(S_2), C(S_3)\}$, to make a final choice $C(\{C(S_1), C(S_2), C(S_3)\})$. Now, let us change the path either by merging or by splitting some groups in the original grouping $G(S) = \{S_1, S_2, S_3\}$. For instance, by merging S_1 and S_2 , we obtain $\{S_1 \cup S_2, S_3\}$ as the new grouping. Then, apply the choice rule C the same way as above, but under the new grouping in S . Grouping Path Independence requires that this type of change in grouping should not affect the final choice, that is, $C(\{C(S_1), C(S_2), C(S_3)\}) = C(\{C(S_1 \cup S_2), C(S_3)\})$. Both our Grouping Path Independence and the original version of Path Independence require the final choice to be unchanged with changes in the grouping of the set of available alternatives. The difference between the two conditions is that the former considers only departures by merging or by splitting from the initial groups specified by the given grouping rule, whereas the latter allows any changes in grouping. Hence, our version is weaker than the original one. We show that, given a grouping rule G , a choice function satisfies Grouping Path Independence for G if and only if it is a Grouping Choice Method with G and some preference relation.

²In the first version of Horan (2013), he characterizes Rational Shortlist Methods in which the first rationale is transitive by Expansion, Weak WARP, and the axiom called “Choice Symmetry”. In the latest version which we have just known, he strengthens our Elimination axiom and defines the axiom called “Exclusivity”. Then, he provides a characterization by means of Expansion and Exclusivity.

In the literature on individual or social decision-making, many authors have proposed and studied models to explain choice behaviors that are inconsistent with single preference maximization over the sets of feasible alternatives. Among them, sequential applications of multiple criteria are often considered in both individual and social choices.³ It is interesting that choices by sequential maximization of two rationales with the first one being transitive may be alternatively described as decision-making by single preference maximization with a grouping procedure. The two distinct procedures may explain the same set of choice outcomes.

Manzini and Mariotti (2012b) consider yet another decision-making procedure that looks similar to ours. In their procedure, a DM first “categorizes” alternatives. Here “categorization” is the same as “grouping” in our procedure. In their model, however, a DM is endowed with *two* distinct preference relations, one over the “categories” (subsets of the set of alternatives) and the other over alternatives. Then, she first eliminates all alternatives in categories dominated by another category, and chooses an alternative that is maximal among the remaining ones. In contrast to their model, a DM is endowed with a *single* preference relation over the alternatives in our model, just like the standard choice theory. Our point of departure from the classical theory is to introduce a grouping process before maximization.

In social choice contexts, agenda setting is crucial for determining final outcomes, especially when a social preference relation contains cycles as in the case of majority voting. Here agenda setting is the same as grouping in our model. Hence, our results may shed some light on bounded rationality of social choice under various agenda settings.

The rest of this paper is organized as follows. Section 2 introduces basic notation and definitions, and defines Grouping Choice Methods. Sections 3,

³Such contributions include Suzumura (1983), Aizerman (1985), Aizerman and Aleskerov (1995), Roelofsma and Read (2000), Kalai, Rubinstein, and Spiegler (2002), Tadenuma (2002, 2005), Manzini and Mariotti (2007, 2012a), Houy (2007), Houy and Tadenuma (2009), Au and Kawai (2011), Apestequia and Ballester (2013), Dutta and Horan (2013), and Horan (2013).

4, and 5 present three characterizations of Grouping Choice Methods, respectively. Section 6 discusses relationship between grouping rules and rationality of choice. The final section contains concluding remarks. All proofs are relegated in the Appendix.

2 Grouping Choice Methods

First, we introduce basic notation and definitions throughout the paper. Let X be a finite set of *alternatives*, and \mathcal{X} the set of all nonempty subsets of X . A *choice function* is a function $C : \mathcal{X} \rightarrow X$ such that for every $S \in \mathcal{X}$, $C(S) \in S$. A *binary relation* (or *rationale*) on X is a set $P \subseteq X \times X$. For simplicity, $(x, y) \in P$ is written as $x P y$. A binary relation P is *asymmetric* if $x P y$ implies not $[y P x]$. Let \mathcal{P} be the set of all asymmetric binary relations on X .

We say that $x \in X$ and $y \in X$ with $x \neq y$ are *comparable* in $P \in \mathcal{P}$ if $x P y$ or $y P x$ holds. An asymmetric binary relation $P \in \mathcal{P}$ is *complete* if for all $x, y \in X$ with $x \neq y$, x and y are comparable in P . It is *transitive* if for all $x, y, z \in X$, $x P y$ and $y P z$ implies $x P z$. It contains a *cycle* if there exist an integer n with $n \geq 3$ and n alternatives $x_1, \dots, x_n \in X$ such that $x_i P x_{i+1}$ for all $i \in \{1, \dots, n-1\}$ and $x_n P x_1$. It is *acyclic* if it contains no cycle.

For each $P \in \mathcal{P}$ and each $S \in \mathcal{X}$, let $M(S; P) \subseteq S$ denote the set of maximal elements in S for P :

$$M(S; P) = \{x \in S \mid \nexists y \in S \text{ such that } y P x\}.$$

For each $S \in \mathcal{X}$, let $|S|$ denote the number of elements in S .

Next, we introduce a new procedure of decision making, which we call a “Grouping Choice Method”. In this procedure, a DM first divides the set of feasible alternatives into some groups, and from each group, she selects an element (winner). Then, she chooses an element among the winners in the first round. In order to formally define the new procedure, we first introduce “grouping rules”.

Definition 1. A **grouping rule** is a correspondence G that associates with every $S \in \mathcal{X}$ a family $G(S)$ of subsets of S , and that satisfies the following three conditions.

(G1) For every $S \in \mathcal{X}$, $\cup_{S_k \in G(S)} S_k = S$,

(G2) For every $S \in \mathcal{X}$, there exist no $S_i, S_j \in G(S)$ with $S_i \neq S_j$ and $S_i \subseteq S_j$.

(G3) For all $S, T \in \mathcal{X}$, if there exists $S_i \in G(S)$ such that $\{x, y\} \subseteq S_i$ and $\{x, y\} \subseteq T$, then there exists $T_j \in G(T)$ such that $\{x, y\} \subseteq T_j$

For each $S \in \mathcal{X}$, $G(S)$ is called the grouping in S , and each member of $G(S)$ a **group** in S .

Condition (G1) means that every element in S belongs to some group in $G(S)$. Condition (G2) says that no group is a strict subset of another group. Condition (G3) is consistency in grouping. To motivate the condition, consider again the situation for a family to buy a house. Let $X = \{x, y, z, w\}$ be the set of all houses where x, y are located in town A while z, w are in town B. At first, all houses are available, and they divide X into $\{x, y\}$ and $\{z, w\}$ by location. But then, w is sold so that $S = \{x, y, z\}$ becomes the new set of available houses. Then, if they still divide S according to location, $\{x, y\}$ should be a group in S . That is, x and y are always in a group as long as both are available. Condition (G3) requires this kind of consistency in grouping procedures.

Now we are ready to define our new decision procedure.

Definition 2. A choice function C is a **Grouping Choice Method** if and only if there exist a grouping rule G and an asymmetric binary relation P such that for every $S \in \mathcal{X}$, $C(S) = M(\cup_{S_k \in G(S)} M(S_k; P); P)$.

3 Grouping Choice Methods and Sequential Applications of Multiple Criteria

Sequential applications of multiple criteria in individual or social choices have been studied by many authors as cited in the Introduction. In this section, we clarify the relationship of Grouping Choice Methods with the models of sequential applications of multiple criteria.

Manzini and Mariotti (2007) defined and analyzed Rational Shortlist Methods. In the Methods, a DM is endowed with a pair of preference relations, called “rationale”, and for each set of available alternatives, she first takes all maximal elements for the first rationale in the set, then among these elements, she selects the maximum for the second rationale. An obvious difference of Grouping Choice Methods from Rational Shortlist Methods is that a DM is endowed with *single* preference relation in the former, whereas with *two* preference relations in the latter. Despite this difference, there is a strong connection between the two methods of choice. Our first theorem shows that a choice function is a Grouping Choice Method if and only if it is a Rational Shortlist Method in which the first rationale is transitive. It is interesting that the same set of choice outcomes may be explained in either of the two models of decision-making: the model of sequential maximization with a pair of rationales and that of maximization of a single preference relation with a grouping procedure.

Theorem 1. *A choice function is a Grouping Choice Method if and only if it is a Rational Shortlist Method in which the first rationale is transitive.*

It is worth noting the relationship of two rationales in a Rational Shortlist Method with the single preference relation in the corresponding Grouping Choice Method. Let a pair of rationales (P_1, P_2) be given. Construct the single preference P_{12} as follows: For all $x, y \in X$,

$$x P_{12} y \Leftrightarrow x P_1 y \text{ or } [\text{not}(y P_1 x) \text{ and } x P_2 y].$$

The preference relation P_{12} was defined and studied in Tadenuma (2002) and Houy and Tadenuma (2009), and called the *lexicographic composition* of (P_1, P_2) . In this composition, an alternative x is superior to another alternative y if and only if (1) x is superior to y by the first criterion P_1 or (2) x is superior to y by the second criterion P_2 when x and y are not comparable by P_1 . Houy and Tadenuma (2009) scrutinize differences between the two ways of decision-making with a given pair of preference relations: one is a Rational Shortlist Method and the other is to construct the (single) lexicographic composition and then maximize it. Despite the differences, the lexicographic composition plays a key role to connect Rational Shortlist Methods with Grouping Choice Methods, as seen below.

Let a pair of preference relations (P_1, P_2) be given. Suppose that there exist a grouping rule G and a preference relation P such that $M(M(S; P_1); P_2) = M(\cup_{S_k \in G(S)} M(S_k; P); P)$ for every $S \in \mathcal{X}$. Then, we have $M(M(\{x, y\}; P_1); P_2) = M(\{x, y\}; P)$ for all $x, y \in X$. This means that $x P y$ if and only if $[x P_1 y]$ or $[\text{not}(y P_1 x) \text{ and } x P_2 y]$. Hence, $P = P_{12}$. Thus, as a corollary of Theorem 1, we have the following:

Corollary 1. *If a choice function is a Rational Shortlist Method with a pair of rationales (P_1, P_2) in which the first rationale is transitive, then it is a Grouping Choice Method with the lexicographic composition of (P_1, P_2) and some grouping rule.*

4 Axiomatic Characterization of Grouping Choice Methods

In this section, we define three natural properties of choice functions. Then, we show that Grouping Choice Methods satisfy all these properties, and conversely, every choice function satisfying the three properties is a Grouping Choice Method.

The first property is standard choice-consistency under the expansion of

available alternatives. It says that if an alternative is chosen in each of the two sets of available alternatives, then it should be chosen in the union of the two sets.

Expansion: For all $S, T \in \mathcal{X}$, if $x = C(S) = C(T)$, then $x = C(S \cup T)$.

The next property is a weaker version of Samuelson’s WARP, which was introduced by Manzini and Mariotti (2007). This axiom requires that if an alternative x is chosen in $\{x, y\}$ and a set S containing both x and y , then y should not be chosen in any “intermediate” set between $\{x, y\}$ and S .

Weak WARP: For all $x, y \in X$ and all $S, T \in \mathcal{X}$, if $\{x, y\} \subseteq T \subseteq S$ and $x = C(\{x, y\}) = C(S)$, then $y \notin C(T)$.

The third property says that if an alternative y is never chosen in the presence of another alternative x , then (1) whenever y is chosen in a menu (in the absence of x) and then x becomes newly available, x should be chosen in the new menu (that is, x “eliminates” the initial winner y), or (2) whenever both x and y are present, eliminating y from the menu does not affect the choice.

Elimination: For all $x, y \in X$, if $y \notin C(S)$ for every $S \in \mathcal{X}$ with $x \in S$, then (1) for every $S \in \mathcal{X}$ with $x \notin S$ and $y = C(S)$, $x = C(S \cup \{x\})$, or (2) for every $S \in \mathcal{X}$ with $x \in S$ and $y \in S$, $C(S) = C(S \setminus \{y\})$.

Our next theorem shows that the above three properties characterize Grouping Choice Methods.

Theorem 2. *A choice function satisfies Expansion, Weak WARP, and Elimination if and only if it is a Grouping Choice Method.*

In the Appendix, we show that the three axioms in Theorem 2 are independent.

5 Grouping Path Independence

To present our final characterization of Grouping Choice Methods, we need to introduce some additional definitions. The idea behind the following definitions is simple. Given a grouping in some set S , we consider two ways to change it. In one way, we merge two groups into one and iterate this operation to obtain a new grouping. In the other way, we split a group into two groups and iterate it.

Let $S \in \mathcal{X}$. Let \mathcal{S}_1 and \mathcal{S}_2 be two families of subsets of S . We say that (1) \mathcal{S}_2 is *obtained by merging from* \mathcal{S}_1 if (i) $T_i = S_j \cup S_k$ for some $T_i \in \mathcal{S}_2$ and some $S_j, S_k \in \mathcal{S}_1$ and (ii) $\mathcal{S}_1 \setminus \{S_j, S_k\} = \mathcal{S}_2 \setminus \{T_i\}$; and that

(2) \mathcal{S}_2 is *obtained by splitting from* \mathcal{S}_1 if (i) $S_i = T_j \cup T_k$ for some $S_i \in \mathcal{S}_1$ and some $T_j, T_k \in \mathcal{S}_2$, and (ii) $\mathcal{S}_1 \setminus \{S_i\} = \mathcal{S}_2 \setminus \{T_j, T_k\}$.

Then, we say that a family \mathcal{T} of subsets of S is *obtained by iteratively merging (resp., iteratively splitting) from* a family \mathcal{S} if there exists a sequence of families of subsets of S , $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_\ell$ such that $\mathcal{S}_1 = \mathcal{S}$, $\mathcal{S}_\ell = \mathcal{T}$, and for all $h \in \{1, \dots, \ell - 1\}$, \mathcal{S}_{h+1} is obtained by merging (resp., splitting) from \mathcal{S}_h . Let $\mathcal{G}(\mathcal{S})$ be the set of all families of subsets of S that are obtained either by iteratively merging or by iteratively splitting from \mathcal{S} .

The following property requires that the above types of changes in grouping should not affect the final outcomes.

Grouping Path Independence: Let a grouping rule G be given. We say that a choice function C satisfies *Grouping Path Independence for* G if the following condition holds: For every $S \in \mathcal{X}$, if $\mathcal{T} \in \mathcal{G}(G(S))$, then $C(\{C(T_j) \mid T_j \in \mathcal{T}\}) = C(\{C(S_i) \mid S_i \in G(S)\})$.

Notice that the grouping consisting only of the whole set S is obtained by iteratively merging from any grouping in S . Hence, the above condition is equivalent to the following: For every $S \in \mathcal{X}$, and every $\mathcal{T} \in \{G(S)\} \cup \mathcal{G}(G(S))$, $C(S) = C(\{C(T_j) \mid T_j \in \mathcal{T}\})$.

We are now ready to state our third characterization of Grouping Choice Methods.

Theorem 3. *Let a grouping rule G be given. A choice function C satisfies Grouping Path Independence for G if and only if it is a Grouping Choice Method with G and some asymmetric binary relation P .*

We note that the necessity part of the above theorem does not rely on the property (G3) of grouping rules. Hence, this part holds in the class of grouping rules that satisfy (G1) and (G2) but not necessarily (G3). This means that a choice function satisfying Grouping Path Independence can be rationalized by a preference relation in more general cases of grouping.

6 Properties of Grouping Rules and Rationality of Choice

A key to determine properties of a Grouping Choice Method is the grouping rule. If we take the finest grouping $\{\{x\} \mid x \in S\}$ or the coarsest grouping $\{S\}$, then the grouping choice method is simply the classical rational choice function. Between the two extremes, there are a variety of cases. Depending on which grouping rules are admissible, the degree of rationality of grouping choice methods varies.

We consider grouping rules that satisfy three conditions (G1) to (G3). However, there may be some situations to which it is not appropriate to require (G3). Consider again the example in which a family buys a house. They divide available houses into groups by their locations. Suppose that the set of all houses is $X = \{x, y, z, w\}$ where x and y are located in district a of town A , z is located in another district b of town A , and w is located in another town B . If the set of available houses is $\{x, z, w\}$, then she divides it into $\{x, z\}$ and $\{w\}$ because x and z are located in the same town and w is located in the other town. But when she faces $\{x, y, z\}$, she might divide $\{x, y, z\}$ into $\{x, y\}$ and $\{z\}$ because x and y are located in the same district and z is located in another district. This is a violation of (G3). The condition (G3) requires that any two alternatives that are in a group in some situation of choice should be

in a group in any other situations. However, there may be cases in which two alternatives in a group get separated with a change of the whole menu.

Without (G3), however, much of rationality of choice will be lost. In fact, we can construct an example of a Grouping Choice Method that violates both Expansion and Weak WARP. Consider again the above example of choosing a house. Let G be such that $G(X) = \{\{x, y\}, \{z\}, \{w\}\}$, $G(\{x, y, z\}) = \{\{x, y\}, \{z\}\}$, $G(\{x, y, w\}) = \{\{x, y\}, \{w\}\}$, $G(\{x, z, w\}) = \{\{x, z\}, \{w\}\}$, and $G(\{y, z, w\}) = \{\{y, z\}, \{w\}\}$. Define $P = \{(y, x), (x, z), (w, x), (z, y), (w, y), (z, w)\}$. Let C be the Grouping Choice Method with G and P . Then, we have $z = C(X) = C(\{z, w\})$ and $w = C(\{x, z, w\}) = C(\{x, y, w\})$. Thus, C violates both Weak WARP and Expansion.

7 Concluding Remarks

In this paper, we introduce a new, natural procedure of decision making, called a Grouping Choice Method. We clarify the relationships between the two distinct procedures, Rational Shortlist Methods and Grouping Choice Methods. We also axiomatically characterize Grouping Choice Methods by using new properties.

An advantages of our grouping procedure lies in its simplicity. Grouping before maximization is quite common in every day decision-makings. Moreover, our method assumes only one preference relation for a DM, as in the classical theory. We do not need to imagine more complex DMs with multiple criteria. Yet the outcomes of choice are the same as those in the case where the DMs would sequentially maximize their multiple preference relations.

The properties of Grouping Choice Methods depends crucially on grouping rules. We impose three conditions (G1) to (G3) on grouping rules. If we weaken the requirements on grouping rules, we may explain a broader range of choice behaviors, including more “irrational” ones. On the other hand, we impose additional conditions, we may characterize a more restricted class of

choice functions. It may be an interesting future topic to study what kind of bounded rationality Grouping Choice Methods keep with weaker or stronger requirements on grouping rules.

8 Appendix

8.1 Proof of Theorems 1 and 2

We first introduce additional definitions and notation. Let a grouping rule G be given. For all $x, y \in X$, define $x \leftrightarrow y$ as

$$x \leftrightarrow y \Leftrightarrow \exists S \in \mathcal{X} : \exists S_i \in G(S) \text{ such that } \{x, y\} \subseteq S_i$$

That is, $x \leftrightarrow y$ means that x and y belong to the same group in some subset $S \in \mathcal{X}$. Notice that the following relation holds since the grouping rule satisfies (G3).

$$x \leftrightarrow y \Leftrightarrow \forall S \in \mathcal{X} : \exists S_i \in G(S) \text{ such that } \{x, y\} \subseteq S_i$$

That is, the relation $x \leftrightarrow y$ also means x and y belong to the same group in every subset of X . We write $\text{not}[x \leftrightarrow y]$ as $x \not\leftrightarrow y$:

$$x \not\leftrightarrow y \Leftrightarrow \forall S \in \mathcal{X} : \nexists S_i \in G(S) \text{ such that } \{x, y\} \subseteq S_i$$

Given a pair of asymmetric binary relations (P_1, P_2) , define the binary relation P_2^* as follows (Houy and Tadenuma, 2009, p.1776): for all $x, y \in X$,

$$x P_2^* y \Leftrightarrow x \text{ and } y \text{ are not comparable in } P_1 \text{ and } x P_2 y.$$

Given $P \in \mathcal{P}$, define the *transitive closure* $T(P)$ of P as follows: for all $x, y \in X$,

$$x T(P) y \Leftrightarrow \exists z_1, \dots, z_k \in X \text{ with } k \geq 2 \text{ such that } x = z_1, y = z_k \text{ and} \\ \forall i \in \{1, \dots, k-1\}, z_i P z_{i+1}.$$

To prove theorems, we need some lemmas.

Lemma 1. *Assume that a pair of asymmetric binary relations (P_1, P_2) sequentially rationalizes a choice function C . Then the following claims hold.*

- (a) P_1 is acyclic.
- (b) For all $S \in \mathcal{X}$, $M(M(S; P_1); P_2^*) = M(M(S; P_1); P_2)$ holds. That is, (P_1, P_2^*) also sequential rationalizes C .
- (c) For all $x, y \in X$ with $x \neq y$, x and y are comparable in one and only one of P_1 and P_2^* .

Proof. Assume that a pair of asymmetric binary relations (P_1, P_2) sequentially rationalized a choice function C .

- (a) If P_1 contains a cycle $x_1, \dots, x_n \in X$, then $M(S; P_1) = \emptyset$ where $S = \{x_1, \dots, x_n\}$, and hence $C(S) = M(M(S; P_1); P_2) = \emptyset$. This contradicts nonemptiness of C .
- (b) Let $S \in \mathcal{X}$. For all $x, y \in M(S; P_1)$, x and y are not comparable in P_1 , and hence $x P_2 y$ holds if and only if $x P_2^* y$ holds. Therefore, $M(M(S; P_1); P_2) = M(M(S; P_1); P_2^*)$.
- (c) Assume that x and y are neither comparable in P_1 nor in P_2^* . By the above claim, (P_1, P_2^*) sequentially rationalizes C . Then, we have $\{x, y\} = M(M(\{x, y\}; P_1); P_2^*) = C(S)$. This contradicts $C(S)$ is a singleton for every $S \in \mathcal{X}$. Hence, x and y must be comparable in P_1 or P_2^* . By the definition of P_2^* , x and y are comparable in only one of P_1 and P_2^* .

□

Lemma 2. *Assume that a pair of asymmetric binary relations (P_1, P_2) sequentially rationalizes a choice function C . Let $S \in \mathcal{X}$ and $x = C(S)$. Then, the following claims hold.*

- (d) There exists no $y \in S$ such that $y P_1 x$.
- (e) For every $y \in S$, if $y P_2^* x$, then there exist k alternatives $z_1, \dots, z_k \in S \setminus \{x, y\}$ with $k \geq 1$ such that
 - (i) $z_1 P_1 y$,

- (ii) $z_{i+1} P_1 z_i$ and $z_i P_2^* x$ for all $i = 1, \dots, k-1$ if $k \geq 2$, and
- (iii) $x P_1 z_k$ or $x P_2^* z_k$.

Proof. Since $x = C(S) = M(M(S; P_1); P_2)$, we have $x \in M(S; P_1)$. Hence, Claim (d) follows.

Assume that $y \in S$ and $y P_2^* x$. By claim (b) in Lemma 1, $x = M(M(S; P_1); P_2^*)$. If there exists no $z_1 \in S$ with $z_1 P_1 y$, then $y \in M(S; P_1)$ holds, and $x \notin M(M(S; P_1); P_2^*)$, which is a contradiction. Hence, there exists $z_1 \in S$ with $z_1 P_1 y$. If $z_1 = x$, then $x P_1 y$, which contradicts $y P_2^* x$. Thus, $z_1 \neq x$. If $x P_1 z_1$ or $x P_2^* z_1$, we are done.

Assume that neither $x P_1 z_1$ nor $x P_2^* z_1$ holds. By claim (d), $z_1 P_1 x$ does not hold. Then, by claim (c) in Lemma 1, we have $z_1 P_2^* x$. If there exists no $z_2 \in S$ with $z_2 P_1 z_1$, then $z_1 \in M(S; P_1)$ and $x \notin M(M(S; P_1); P_2^*)$, which is a contradiction. Thus, there exists $z_2 \in S$ with $z_2 P_1 z_1$. It follows from $z_2 P_1 z_1$ and $z_1 P_2^* x$ that $x \neq z_2$. If $x P_1 z_2$ or $x P_2^* z_2$, we are done. If not, then by the same argument as above, there exists z_3 with $z_3 P_1 z_2$. Iterating this procedure, we have a sequence z_1, z_2, \dots such that $z_{i+1} P_1 z_i$ and $z_i P_2^* x$ for all $i = 1, 2, \dots$. Because P_1 is acyclic by claim (a) in Lemma 1, it must be the case that $z_i \neq y$ for all $i = 1, 2, \dots$ and $z_i \neq z_j$ for all i, j with $i \neq j$. Moreover, since S is finite, this procedure must terminate. Let the k th iteration terminate the procedure. Then, we have $x P_1 z_k$ or $x P_2^* z_k$. \square

Lemma 3. *Assume that a pair of asymmetric binary relations (P_1, P_2) sequentially rationalizes a choice function C and satisfies the following property:*

$$\text{Property T: } \forall x, y \in X; x T(P_1) y \Rightarrow x P_1 y \text{ or } x P_2^* y.$$

Then, $(T(P_1), P_2^)$ also sequentially rationalizes C .*

Proof. Assume that (P_1, P_2) sequentially rationalizes C and satisfies Property T. Let $S \in \mathcal{X}$ and $x = C(S)$. By claim (b) in Lemma 1, $x = M(M(S; P_1); P_2^*)$. Now we prove $x = M(M(S; T(P_1)); P_2^*)$.

First we show $x \in M(S; T(P_1))$. Suppose, on the contrary, that there exists $y \in S$ with $y T(P_1) x$. By claim (d) in Lemma 2, $y P_1 x$ does not hold. Then,

by Property T, we have $y P_2^* x$. It follows from claim (e) in Lemma 2 that there exists $z \in S$ such that $z T(P_1) y$ and $[x P_1 z \text{ or } x P_2^* z]$. By $z T(P_1) y$ and $y T(P_1) x$, we have $z T(P_1) x$. Then, Property T implies that $[z P_1 x \text{ or } z P_2^* x]$. However, $[x P_1 z \text{ or } x P_2^* z]$ and $[z P_1 x \text{ or } z P_2^* x]$ are incompatible because (i) both P_1 and P_2^* are asymmetric, and (ii) by the definition of P_2^* , $x P_1 z$ and $z P_2^* x$ are incompatible, and so as $x P_2^* z$ and $z P_1 x$. Hence, we have $x \in M(S; T(P_1))$.

Since $P_1 \subseteq T(P_1)$, we have $M(S; T(P_1)) \subseteq M(S; P_1)$. It follows from $x = M(M(S; P_1); P_2^*)$ that $x \in M(M(S; T(P_1)); P_2^*)$. Notice that P_2^* is complete in $M(S; P_1)$ by claim (c) in Lemma 1. Hence, it is also complete in $M(S; T(P_1))$. Thus, we have $x = M(M(S; T(P_1)); P_2^*)$. \square

Lemma 4. *Assume that a choice function C is a Grouping Choice Method with an asymmetric binary relation P and a grouping rule G . Then, the following claims hold:*

- (f) P is complete.
- (g) For every $S \in \mathcal{X}$, if $x \leftrightarrow y$ for all $x, y \in S$, then $M(S; P) \neq \emptyset$.
- (h) For every $S \in \mathcal{X}$ and all $x, y \in S$, if $x \leftrightarrow y$ and $y P x$, then $x \neq C(S)$.
- (i): For every $S \in \mathcal{X}$ and all $x, y \in S$, if $y P x$ and $x = C(S)$, then there exists $z \in S \setminus \{x, y\}$ such that $x P z$, $z P y$, and $y \leftrightarrow z$.

Proof.

Claim (f). For all $x, y \in X$, $C(\{x, y\})$ is a single element in $\{x, y\}$. Hence, we have $x P y$ or $y P x$.

Claim (g). Suppose, on the contrary, that $S \in \mathcal{X}$ and for all $x, y \in S$, $x \leftrightarrow y$ but $M(S; P) = \emptyset$. Since S is finite and P is asymmetric and complete, P contains a cycle in A , that is, there exist $x_1, x_2, \dots, x_n \in S$ with $n \geq 3$ such that $x_i P x_{i+1}$ for all $i \in \{1, \dots, n-1\}$ and $x_n P x_1$. Take minimal $j \in \{3, \dots, n\}$ such that $x_j P x_1$. Because P is complete, we have $x_1 P x_{j-1}$, $x_{j-1} P x_j$, and $x_j P x_1$. Let $T = \{x_1, x_{j-1}, x_j\}$. By the initial supposition, we have $v \leftrightarrow w$ for all $v, w \in T$. Then it must

be the case that $T \in G(T)$ or $\{\{x_1, x_{j-1}\}, \{x_{j-1}, x_j\}, \{x_1, x_j\}\} \subseteq G(T)$. If $T \in G(T)$, it follows from (G2) in the definition of Grouping Rules, $\{T\} = G(T)$. Thus, we have $C(T) = M(T; P) = \emptyset$, which contradicts non-emptiness of C . If $\{\{x_1, x_{j-1}\}, \{x_{j-1}, x_j\}, \{x_1, x_j\}\} \subseteq G(T)$, then we also have $C(T) = M(\cup_{T_k \in G(T)} M(T_k; P); P) = M(T; P) = \emptyset$, which is a contradiction. Therefore, it must be the case that $M(S; P) \neq \emptyset$.

Claim (h). Assume that $S \in \mathcal{X}$, $x, y \in S$, $x \leftrightarrow y$, and $y P x$. Let $S_i \in G(S)$ be a group in S such that $x, y \in S_i$. Then, $x \notin M(S_i; P)$. Because G satisfies (G3), we have $v \leftrightarrow w$ for all $v, w \in S_i$. By Claim (g), $M(S_i; P) \neq \emptyset$. Let $z \in M(S_i; P)$. Since P is complete, we have $z P x$. Thus, $x \notin M(\cup_{S_k \in G(S)} M(S_k; P); P) = C(S)$.

Claim (i). Assume that $S \in \mathcal{X}$, $x, y \in S$, $y P x$, and $x = C(S)$. If $y \in \cup_{S_k \in G(S)} M(S_k; P)$, then $y P x$ and $x = C(S)$ are incompatible. Hence, $y \notin \cup_{S_k \in G(S)} M(S_k; P)$. Let $S_i \in G(S)$ be a group with $y \in S_i$. By Claim (g), $M(S_i; P) \neq \emptyset$. Then, there exists $z \in M(S_i; P)$. By completeness of P , we have $z P y$. Then, $z \neq x$ since $y P x$. Because G satisfies (G3), we have $y \leftrightarrow z$. Moreover, from $x = C(S) = M(\cup_{S_k \in G(S)} M(S_k; P); P)$ and $z \in \cup_{S_k \in G(S)} M(S_k; P)$, we have $x P z$.

□

We now prove Theorems 1 and 2 together in the following three parts:

Part 1: to show that every Grouping Choice Method satisfies Expansion, Weak WARP, and Elimination.

Part 2: to show that if a choice function satisfies Expansion, Weak WARP, and Elimination, then it is a Rational Shortlist Method in which the first rationale is transitive.

Part 3: to show that if a choice function is a Rational Shortlist Method in which the first rationale is transitive, then it is a Grouping Choice Method.

Part 1: We show that every Grouping Choice Method satisfies Weak WARP, Expansion, and Elimination.

Let C be a Grouping Choice Function with an asymmetric binary relation P and a grouping rule G .

Weak WARP:

Assume $T \in \mathcal{X}$ and $x = C(\{x, y\}) = C(T)$. Let $S \in \mathcal{X}$ be a set such that $\{x, y\} \subseteq S \subseteq T$. By $x = C(\{x, y\})$, we have $x P y$. It follows from $x = C(T)$ and Claim (h) in Lemma 4 that there exists no $z \in T$ such that $x \leftrightarrow z$ and $z P x$. Because $S \subseteq T$, there exists no $z \in S$ with $x \leftrightarrow z$ and $z P x$. Then, Claim (i) in Lemma 4 and $x P y$ together imply $y \neq C(S)$.

Expansion:

Suppose, on the contrary, that $x = C(S) = C(T)$ but $y = C(S \cup T) \neq x$. Since C satisfies Weak WARP, $y \neq C(\{x, y\})$. Hence, $x = C(\{x, y\})$ and $x P y$. Then, by Claim (i) in Lemma 4, there exists $z \in S \cup T$ with $z \leftrightarrow x$ and $z P x$. Without loss of generality, assume $z \in S$. It follows from Claim (h) in Lemma 4 that $x \neq C(S)$, which is a contradiction. Thus, $x = C(S \cup T)$ must hold.

Elimination:

Let $x, y \in X$. Assume that $y \neq C(A)$ for all $A \in \mathcal{X}$ with $x \in A$. Suppose, on the contrary, that there exist $S, T \in \mathcal{X}$ such that $y = C(S)$, $x \neq C(S \cup \{x\})$, and $x \in T$, $C(T) \neq C(T \cup \{y\})$. Let $v = C(S \cup \{x\})$ and $w = C(T)$.

In the following, we have seven steps to derive a contradiction.

Step 1: We show that $v \neq x, y$ and $w \neq x, y$.

By the assumption and the supposition, $v \neq x, y$ and $w \neq y$. Moreover, if $x = w = C(T)$, then we have $x = C(T) = C(T \cup \{y\})$ because $x = C(\{x, y\})$ by the assumption and C satisfies Expansion. Therefore, we have $w \neq x$.

Step 2: We show $x P y$, $x \leftrightarrow y$, $y P v$, $v \not\leftrightarrow y$, and $v P x$.

Since $v \neq x$, we have $v \in S$. Because $y = C(S)$ and C satisfies Weak WARP, it must be the case that $y = C(\{v, y\})$. Hence, $y P v$.

It follows from $v = C(S \cup \{x\})$ and Claim (i) in Lemma 4 that there exists

$a \in S \cup \{x\}$ such that $v P a$, $a P y$ and $a \leftrightarrow y$. If $a \in S$, then it contradicts $y = C(S)$ by Claim (h) in Lemma 4. Therefore, we have $a = x$. Thus, $v P x$, $x P y$ and $x \leftrightarrow y$.

If $v \leftrightarrow y$, then by Claim (h) in Lemma 4, we have $v \neq C(S \cup \{x\})$, which is a contradiction. Thus, we have $v \not\leftrightarrow y$.

Step 3: We show $y P w$ and $w \leftrightarrow y$.

Let $z = C(T \cup \{y\})$. By the initial supposition, $z \neq w = C(T)$. From the initial assumption and $x \in T$, we have $z \neq y$, and hence $z \in T$. If $z P w$, then $C(\{z, w\}) = z$, which contradicts the fact that C satisfies Weak WARP. Thus, $w P z$. It follows from $z = C(T \cup \{y\})$ and Claim (i) in Lemma 4 that there exists $b \in T \cup \{y\}$ such that $z P b$, $b P w$ and $b \leftrightarrow w$. If $b \in T$, then it contradicts $w = C(T)$ by Claim (h) in Lemma 4. Therefore, we have $b = y$. Thus, $y P w$, and $w \leftrightarrow y$.

Step 4: We show $w \not\leftrightarrow x$.

Assume not: let $w \leftrightarrow x$. Then, by this assumption and the above steps, we have $w \leftrightarrow x$, $w \leftrightarrow y$, and $x \leftrightarrow y$. It follows from Claim (g) in Lemma 4 that $M(\{w, x, y\}; P) \neq \emptyset$. From Step 2 and Step 3, we have $x P y$ and $y P w$ which imply $x = M(\{w, x, y\}; P)$. Then, by asymmetry of P , we have $x P w$. From the combination of $x P w$ and $w \leftrightarrow x$, Claim (h) in Lemma 4 states that $w \neq C(T)$. It is a contradiction. Hence, we have $w \not\leftrightarrow x$.

Step 5: We show $v \neq w$, $x P w$, $w P v$, $v \leftrightarrow w$, and $x = C(\{v, w, x, y\})$.

By Steps 2 and 3, we have $v \not\leftrightarrow y$ and $w \leftrightarrow y$. Hence, $v \neq w$.

Consider $C(\{v, w, x, y\})$. By Step 2, $x P y$ and $x \leftrightarrow y$. It follows from Claim (h) in Lemma 4 that $y \neq C(\{v, w, x, y\})$. Similarly, since $y P w$ and $w \leftrightarrow y$ by Step 3, we have $w \neq C(\{v, w, x, y\})$.

Since $w \leftrightarrow y$ by Step 3, there exists $A_i \in G(\{v, w, x, y\})$ with $\{w, y\} \subseteq A_i$. Because $v \not\leftrightarrow y$ and $w \not\leftrightarrow x$ by Steps 2 and 4, we have $\{w, y\} = A_i$. From $y P w$ in Step 3, we have $y = M(A_i; P)$, and hence $y \in \cup_{A_k \in G(\{v, w, x, y\})} M(A_k; P)$. Then, since $y P v$ by Step 2, $v \neq M(\cup_{A_k \in G(\{v, w, x, y\})} M(A_k; P); P)$.

Thus, it must be the case that $x = C(\{v, w, x, y\})$. It follows from $v P x$ in Step 2 and Claim (i) in Lemma 4 that there exists $d \in \{w, y\}$ such that $x P d$, $d P v$ and $d \leftrightarrow v$. However, since $v \not\leftrightarrow y$ by Step 2, we have $d \neq y$. Hence, $x P w$, $w P v$ and $v \leftrightarrow w$.

Step 6: We show that there exists $t \in T \setminus \{x, w\}$ such that $t \neq v, y$, $w P t$, $t P x$, and $t \leftrightarrow x$.

Since $w = C(T)$ and $x P w$ by Step 5, It follows from Claim (h) in Lemma 4 that there exists $t \in T \setminus \{x, w\}$ such that $w P t$, $t P x$, and $t \leftrightarrow x$. Because $x = C(\{v, w, x, y\})$ by Step 5, it must be the case that $t \neq v, y$.

Step 7: We show that none of t, v, w, x , and y is equal to $C(\{t, v, w, x, y\})$, which is a contradiction.

By Step 2, $x P y$ and $x \leftrightarrow y$. It follows from Claim (h) in Lemma 4 that $y \neq C(\{t, v, w, x, y\})$. By a similar argument, we can show that none of v, w , and x is equal to $C(\{t, v, w, x, y\})$.

Since $v \leftrightarrow w$ by Step 5, there exists $B_j \in G(\{t, v, w, x, y\})$ with $\{v, w\} \subseteq B_j$. Suppose $t \in B_j$. It follows from $w \leftrightarrow t$, $w P t$ by Step 6, and Claim (h) in Lemma 4 that $t \neq C(\{t, v, w, x, y\})$.

Next, suppose $t \notin B_j$. Because $v \not\leftrightarrow y$ and $w \not\leftrightarrow x$ by Steps 2 and 4, we have $x, y \notin B_j$. Then, $B_j = \{v, w\}$. Since $w P v$ by Step 5, we have $w = M(B_j; P)$, and hence $w \in \cup_{B_k \in G(\{u, v, w, x, y\})} M(B_k; P)$. Because $w P t$ in Step 6, we have $t \neq M(\cup_{B_k \in G(\{u, v, w, x, y\})} M(B_k; P); P) = C(\{t, v, w, x, y\})$.

Part 2: We show that if a choice function satisfies Expansion, Weak WARP, and Elimination, then it is a Rational Shortlist Method in which the first rational is transitive.

Assume that a choice function C satisfies Expansion, Weak WARP, and Elimination. By Manzini and Mariotti (2007, Theorem 1), C is a Rational Shortlist Method.

Define P_1 and P_2 as follows: For all $a, b \in X$ with $a \neq b$,

$$a P_1 b \Leftrightarrow \exists A \in \mathcal{X} \text{ such that } C(A) = b \text{ and } C(A \cup \{a\}) \notin \{a, b\}.$$

$$a P_2 b \Leftrightarrow C(\{a, b\}) = a.$$

Then, by Dutta and Horan (2013, Proposition 1 and Lemma 4), (P_1, P_2^*) sequentially rationalizes C .⁴ By Lemma 3, if (P_1, P_2^*) satisfies Property T:

$$\forall a, b \in X : a T(P_1) b \Rightarrow a P_1 b \text{ or } a P_2^* b,^5$$

then $(T(P_1), P_2^*)$ also sequentially rationalizes C , which means that C is a Rational Shortlist Method in which the first rationale is transitive. Therefore, it remains to show that (P_1, P_2^*) satisfies Property T.

Suppose, on the contrary, that there exist $z_1, \dots, z_n \in X$ such that $z_i P_1 z_{i+1}$ for all $i \in \{1, \dots, n-1\}$ but neither $z_1 P_1 z_n$ nor $z_1 P_2^* z_n$ holds. Then, $n \geq 3$. Because (P_1, P_2^*) sequentially rationalizes C , we have $z_n = C(\{z_1, z_n\})$ and $z_1 = C(\{z_1, z_2, \dots, z_n\})$. Then, there exists $j \in \{1, \dots, n-2\}$ such that (i) $C(\{z_1, \dots, z_j, z_n\}) \neq C(\{z_1, \dots, z_j, z_n\} \cup \{z_{j+1}\})$. Moreover, by the definition of P_1 , there exists $U \in \mathcal{X}$ such that (ii) $z_{j+1} = C(U)$ but $z_j \neq C(U \cup \{z_j\})$.

However, since $z_j P_1 z_{j+1}$, it follows that $z_{j+1} \neq C(S)$ for all $S \in \mathcal{X}$ with $z_j \in S$. Then, by Elimination, one of the claims (i) and (ii) cannot hold, which is a contradiction. Thus, (P_1, P_2^*) satisfies Property T.

Part 3: We show that if a choice function is a Rational Shortlist Method in which the first rationale is transitive, then it is a Grouping Choice Method.

Assume that a choice function C is sequentially rationalized by a pair of asymmetric binary relations (P_1, P_2) and P_1 is transitive. Define a binary relation P as follows: For all $x, y \in X$ with $x \neq y$,

$$x P y \Leftrightarrow x = C(\{x, y\}).$$

Then, P is asymmetric.

For every $S \in \mathcal{X}$, let $g(S)$ be the class of all subsets S_i of S such that for all $x, y \in S_i$ with $x \neq y$, either $x P_1 y$ or $y P_1 x$ holds. Then, we define a

⁴We are grateful to Sean Horan for suggesting that this part of the proof could be shortened by using the results in Dutta and Horan (2013).

⁵Note that $(P_2^*)^* = P_2^*$ by definition.

correspondence G as follows:

$$G(S) = \{S_i \in g(S) \mid \nexists S_j \in g(S) \text{ such that } S_j \neq S_i \text{ and } S_i \subseteq S_j\}.$$

That is, $G(S)$ is the class of “maximal” subsets (in inclusion relations) of S in which every element is comparable with every other element in P_1 .

Now we check that G is a grouping rule, that is, for every $S \in \mathcal{X}$, $G(S)$ satisfies the conditions (G1), (G2), and (G3) in Definition 2.

(G1) By the definition of G , $\cup_{S_i \in G(S)} S_i \subseteq S$. For every $x \in S$, because $\{x\} \in g(S)$ by the definition of $g(S)$, there exists $S_i \in G(S)$ with $x \in S_i$. Therefore, we have $S \subseteq \cup_{S_i \in G(S)} S_i$.

(G2) By the definition of G , it satisfies (G2).

(G3) Assume that there exists $S_i \in G(S) \subseteq g(S)$ such that $\{x, y\} \subseteq S_i$, and $\{x, y\} \subseteq T$. By definition, either $x P_1 y$ or $y P_1 x$ holds. This implies $\{x, y\} \in g(T)$. Then, there exists $T_j \in G(T)$ such that $\{x, y\} \subseteq T_j$.

Next, we show that C is a grouping choice method with P and G . Let $S \in \mathcal{X}$ and $x = C(S)$. First, we prove that there exists $S_k \in G(S)$ such that $x = M(S_k; P)$. Since $G(S)$ satisfies (G1), there exists $S_i \in G(S)$ with $x \in S_i$. If $S_i = \{x\}$, then obviously, $x = M(S_k; P)$. Assume $|S_i| \geq 2$. Let $y \in S_i \setminus \{x\}$. It follows from $x = C(S)$ and Lemma 2 that $y P_1 x$ does not hold. Because y is comparable with x in P_1 , it must be the case that $x P_1 y$. Then, we have $x = C(\{x, y\})$. By the definition of P , $x P y$. This holds for every $y \in S_i \setminus \{x\}$. Thus, we have $x = M(S_i; P)$.

Second, we show $x P y$ for every $y \in S \setminus \{x\}$ such that $y = M(S_j; P)$ for some $S_j \in G(S)$. Suppose, on the contrary, that there exists $y \in S$ such that $y = M(S_j; P)$ for some $S_j \in G(S)$ but $x P y$ does not hold. By the definition of P , $x \neq C(\{x, y\})$, and hence $y = C(\{x, y\})$. Then, either $y P_1 x$ or $y P_2^* x$. Since $x = C(S)$, it cannot be the case that $y P_1 x$. It follows from $y P_2^* x$ and Claim (e) in Lemma 2 that there exists $w \in S$ with $w P_1 y$. Then, $w = C(\{y, w\})$. Hence, we have $w P y$. Because $y = M(S_j; P)$, it follows that $w \notin S_j$.

If $S_j = \{y\}$, it contradicts $\{w, y\} \in g(S)$ and the construction of G . Hence, we have $|S_j| \geq 2$. Let $z \in S_j \setminus \{y\}$. Then, either $z P_1 y$ or $y P_1 z$ holds. If $z P_1 y$,

then $z = C(\{y, z\})$, which implies $z P y$, which contradicts $y = M(S_j; P)$. Thus, $y P_1 z$ must be the case. It follows from $w P_1 y$ and transitivity of P_1 that $w P_1 z$. This holds for every $z \in S_j \setminus \{y\}$. Hence, we have $S_j \cup \{w\} \in g(S)$, which contradicts $S_j \in G(S)$.

8.2 Independence of the Axioms in Theorem 2

The three axioms, Expansion, Weak WARP, and Elimination in Theorem 2 are independent in the sense that no two axioms imply the other. The following examples illustrate choice functions that satisfy two of the axioms while violating the third.⁶ All the three choice functions are defined on $X = \{x, y, z, w\}$.

Expansion:

Define $x = C(x, y) = C(x, z) = C(x, w) = C(x, y, z) = C(x, y, w) = C(x, z, w)$, $y = C(y, z) = C(y, w) = C(y, z, w) = C(X)$, $z = C(z, w)$. Then, we can check that C satisfies Weak WARP and Elimination. However, C violates Expansion because $x = C(x, y, z) = C(x, z, w)$ but $y = C(X)$.

Weak WARP:

Define $x = C(x, y) = C(x, w) = C(x, y, w)$, $y = C(y, z) = C(y, w) = C(y, z, w)$, $z = C(x, z) = C(z, w) = C(x, y, z) = C(x, z, w) = C(X)$. Then, we can check that C satisfies Expansion and Elimination. However, C violates Weak WARP because $z = C(x, z) = C(X)$ but $y = C(y, z, w)$.

Elimination:

Define $x = C(x, y) = C(x, z) = C(x, y, z)$, $y = C(y, z) = C(y, z, w) = C(X)$, $z = C(z, w) = C(x, z, w)$, $w = C(x, w) = C(y, w) = C(x, y, w)$. Then, we can check that C satisfies Expansion and Weak WARP. However, C violates Elimination because $w \neq C(S)$ for every S with $z \in S$ but (1) $w = C(x, y, w)$ and $z \neq C(X)$, and (2) $C(x, z, w) \neq C(x, z)$.

⁶For simplicity of notation, $C(\{x, y\})$ is written as $C(x, y)$ in this subsection.

8.3 Proof of Theorem 3

To prove the theorem, we use a standard property of choice consistency. It says that if an alternative in a set S “wins” over every other alternative in S in binary choices, then it should be chosen in S .

Condorcet Consistency: For every $S \in \mathcal{X}$, if there exists $x \in S$ such that $x = C(\{x, y\})$ for every $y \in S \setminus \{x\}$, then $x = C(S)$.

The following lemma may be interesting of itself.

Lemma 5. *If a choice function C satisfies Grouping Path Independence for a grouping rule G , then it satisfies Condorcet Consistency.*

Proof. The proof is by induction. Assume that a choice function C satisfies Grouping Path Independence for a grouping rule G . Let $S \in \mathcal{X}$. Assume that there exists $x \in S$ such that $x = C(\{x, y\})$ for every $y \in S \setminus \{x\}$. If $|S|=2$, then by the above assumption, $x = C(S)$. Assume $x = C(S)$ holds if $|S| \leq k - 1$ where $k \geq 3$. We show that it also holds if $|S| = k$.

We divide two cases: (i) $G(S) = \{S\}$ and (ii) $G(S) \neq \{S\}$. First, assume $G(S) = \{S\}$. Consider the family of subsets of S , $\{\{x\}, S \setminus \{x\}\}$, which is obtained by iteratively splitting from $G(S)$. Since C satisfies Grouping Path Independence for G , we have $C(S) = C(\{C(\{x\}), C(S \setminus \{x\})\}) = C(\{x, v\})$ where $v = C(S \setminus \{x\})$. By the initial assumption, we have $x = C(\{x, v\})$. Hence, we have $x = C(S)$.

Second, assume $G(S) \neq \{S\}$. Let $G(S) = \{S_1, \dots, S_n\}$ where $n \geq 2$. Without loss of generality, assume $x \in S_1$. Consider the family of subsets of S , $\{S_1, \cup_{S_k \in G(S), S_k \neq S_1} S_k\}$, which is obtained by iteratively merging from $G(S)$. From condition (G2) in the definition of grouping rules, it cannot be the case that $S_1 = S$. Hence, we have $|S_1| \leq k - 1$. By the assumption of induction, we have $x = C(S_1)$. Now, since C satisfies Grouping Path Independence for G , we have $C(S) = C(\{C(S_1), C(\cup_{S_k \in G(S), S_k \neq S_1} S_k)\}) = C(\{x, y\})$ where $y = C(\cup_{S_k \in G(S), S_k \neq S_1} S_k)$. By the initial assumption, we have $x = C(\{x, y\})$. Therefore we have $x = C(S)$. \square

We now prove Theorem 3.

Let a grouping rule G be given.

[Sufficiency]

Assume that a choice function C is a Grouping Choice Method with G and some asymmetric binary relation P . We show that C satisfies Grouping Path Independence for G .

Let $S \in \mathcal{X}$. Let $\Sigma = \{T_1, \dots, T_m\}$ be obtained either by iteratively merging or by iteratively splitting from $G(S) = \{S_1, \dots, S_n\}$. We need to show that $C(\cup_{T_i \in \Sigma} C(T_i)) = C(S)$. Let $x = C(S) = M(\cup_{S_j \in G(S)} M(S_j; P); P)$.

Without loss of generality, assume $x \in T_1 \in \Sigma$. We show $x = C(T_1)$.

Assume that Σ is obtained by iteratively merging from $G(S)$. Suppose, on the contrary, that $x \neq C(T_1) = y$. By Claim (f) in Lemma 4, P is complete.

Case 1: $y P x$.

There exists $S_k \in G(S)$ with $y \in S_k \subseteq T_1$. From Claims (f) and (g) in Lemma 4, $M(S_k; P) = \{z\}$ for some $z \in S_k \subseteq T_1$. Then, $z \neq y$ because $x = M(\cup_{S_j \in G(S)} M(S_j; P); P)$ and $y P x$. Hence, we have $z \in T_1$, $y \leftrightarrow z$, and $z P y$. It follows from Claim (h) in Lemma 4 that $y \neq C(T_1)$, which is a contradiction.

Case 2: $x P y$.

We have $x = C(\{x, y\}) = C(S)$ and $\{x, y\} \subseteq T_1 \subseteq S$. It follows from Weak WARP that $y \neq C(T_1)$, which is a contradiction.

Hence, it must be the case that $x = C(T_1)$.

Next, assume that Σ is obtained by iteratively splitting from $G(S)$. Then, there exists $S_j \in G(S)$ such that $T_1 \subseteq S_j$. It follows from $x = C(S)$ and Claims (f) and (h) in Lemma 4 that $x P y$ for every $y \in S_j \setminus \{x\}$ and hence for every $y \in T_1 \setminus \{x\}$. Thus, it must be case that $x = C(T_1)$.

We have shown $x \in \cup_{T_i \in \Sigma} C(T_i)$. To show $x = C(\cup_{T_i \in \Sigma} C(T_i))$, suppose, on the contrary, $C(\cup_{T_i \in \Sigma} C(T_i)) = y \neq x$. As in Case 2 above, supposing $x P y$ leads to a contradiction. Suppose $y P x$. As in Case 1 above, if Σ is obtained by iteratively merging from $G(S)$, then we have a contradiction. Suppose that Σ is obtained by iteratively splitting from $G(S)$. There exists

$S_k \in G(S)$ with $y \in S_k$. From Claims (f) and (g) in Lemma 4, $M(S_k; P) = \{z\}$ for some $z \in S_k$. Because $y P x$ and $x = M(\cup_{S_j \in G(S)} M(S_j; P); P)$, it must be the case that $z \neq y$. Hence, $z P y$ and $y \leftrightarrow z$. There exists $T_h \in \Sigma$ such that $z \in T_h \subseteq S_k$. Then, $z P w$ for every $w \in T_h \setminus \{z\}$. Therefore, $z = C(T_h) \subseteq \cup_{T_i \in \Sigma} C(T_i)$. It follows from $z P y$, $y \leftrightarrow z$, $z \in \cup_{T_i \in \Sigma} C(T_i)$, and Claim (h) in Lemma 4 that $y \neq C(\cup_{T_i \in \Sigma} C(T_i))$, which is a contradiction. Thus, we have $x = C(\cup_{T_i \in \Sigma} C(T_i))$.

[Necessity]

Assume that a choice function C satisfies Grouping Path Independence for G . Define a binary relation P as follows: For all $x, y \in X$, $x P y \Leftrightarrow x = C(\{x, y\})$. Because either $C(\{x, y\}) = x$ or $C(\{x, y\}) = y$ holds, P is complete and asymmetric. Hence, for every $A \in \mathcal{X}$, $|M(A; P)| \leq 1$.

Let $S \in \mathcal{X}$ and $x = C(S)$. We show that $\{x\} = M(\cup_{S_j \in G(S)} M(S_j; P); P)$. First, we show that $\{x\} = M(S_k; P)$ for some $S_k \in G(S)$. Suppose, on the contrary, $x \notin M(S_j; P)$ for all $S_j \in G(S)$ with $x \in S_j$. Then, for every $S_j \in G(S)$ with $x \in S_j$, there exists $y_j \in S_j$ such that $y_j P x$. By definition, $y_j = C(\{x, y_j\})$. Construct a family Σ of subsets of S as follows:

$$\Sigma = \{\{x, y_j\}, S_j \setminus \{x, y_j\} \mid S_j \in G(S) \text{ and } x \in S_j\} \cup \{S_j \mid S_j \in G(S) \text{ and } x \notin S_j\}.$$

Then, Σ is obtained by iteratively splitting from $G(S)$. Now we have $x \neq C(T_i)$ for all $T_i \in \Sigma$, and hence $x \neq C(\cup_{T_i \in \Sigma} C(T_i))$. However, by Grouping Path Independence, we have $x = C(S) = C(\cup_{T_i \in \Sigma} C(T_i)) \neq x$, which is a contradiction. Thus, we have $\{x\} = M(S_k; P)$ for some $S_k \in G(S)$.

Second, we show $x P y$ for every $y \in S$ such that $\{y\} = M(S_j; P)$ for some $S_j \in G(S)$. Suppose, on the contrary, that there exists $y \in S$ such that $\{y\} = M(S_j; P)$ for some $S_j \in G(S)$ and $y P x$. Because P is complete, it follows that for every $z \in S_j$, $y P z$, and hence $y = C(\{y, z\})$. Since C satisfies Condorcet Consistency by Lemma 5, we have $y = C(S_j)$. Consider the family of subsets of S , $\left\{S_j, \cup_{S_h \in G(S), S_h \neq S_j} S_h\right\}$. This family is obtained by iteratively merging from $G(S)$. Because C satisfies Grouping Path Independence for G , we have $C(S) = C(\{C(S_j), C(\cup_{S_h \in G(S), S_h \neq S_j} S_h)\}) =$

$C(\{y, C(\cup_{S_h \in G(S), S_h \neq S_j} S_h)\})$. However, since $y P x$, we have $y = C(\{y, x\})$. It follows that $C(\{y, C(\cup_{S_h \in G(S), S_h \neq S_j} S_h)\}) \neq x$, which contradicts $x = C(S) = C(\{y, C(\cup_{S_h \in G(S), S_h \neq S_j} S_h)\})$. Thus, $x P y$ for every $y \in S$ such that $\{y\} = M(S_j; P)$ for some $S_j \in G(S)$. Therefore, $\{x\} = M(\cup_{S_j \in G(S)} M(S_j; P); P)$.

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