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COMPETITIVE EQUILIBRIUM WITH AN ATOMLESS MEASURE SPACE OF AGENTS AND INFINITE DIMENSIONAL COMMODITY SPACES WITHOUT CONVEX AND COMPLETE PREFERENCES*

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Abstract

We prove the existence of a competitive equilibrium for an economy with an atomless measure space of agents and an infinite dimensional commodity space. The commodity space is a separable Banach space with a non-empty interior in its positive cone. We dispense with convexity and completeness assumptions on preferences. We employ a saturated probability space for the space of agents which enables us to utilize the convexifying effect on aggregation. By applying the Gale-Nikaido-Debreu lemma, we provide a direct proof of the existence of a competitive equilibrium.

Keywords: convexifying effect, saturated probability space, the Gale-Nikaido-Debreu lemma
JEL Classification Codes: C62, D51

I. *Introduction*

In this paper, we prove the existence of a competitive equilibrium in an economy with an infinite dimensional commodity space and an atomless measure space of agents whose preferences are not necessarily convex and complete.

A convexifying effect on aggregation or the convexity of the integral of a correspondence over an atomless measure space enables one to prove the existence of a competitive equilibrium without convex preferences. For a finite dimensional space valued correspondence, the convexity of the integral is a result of the Lyapunov theorem on the range of an atomless vector measure. By appealing to this convexifying effect Aumann (1966) showed the existence of a competitive equilibrium for an economy with a finite dimensional commodity space and an

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atomless measure space of agents whose preferences are non-convex. Schmeidler (1969) extended the existence result to an economy without complete preferences.

Since the Lyapunov theorem holds only in finite dimensional spaces,[1] the extension of the convexifying effect to infinite dimensional spaces had been open. To address this issue, Rustichini and Yannelis (1991) suggested the 'many more agents than commodities' requirement on the set of agents. Podczeck (2008) remarked that this requirement is "stronger than necessary." Podczeck (1997) introduced the 'many agents of every type' requirement on the set of agents to achieve the convexifying effect. He added agents' types and sharpened the atomless measure on the set of agents by decomposing it into a family of atomless measures on the types of agents.

On the other hand, Sun (1997) proved a number of desirable properties such as convexity, compactness and preservation of upper semicontinuity of the integral of Banach space valued correspondences. He worked with these correspondences over Loeb measure spaces. Recently, Podczeck (2008) showed convexity and compactness of the integral of these correspondences on a super-atomless measure space. Sun and Yannelis (2008) showed that the results in Sun (1997) are valid on an arbitrary saturated probability space. As Sun and Yannelis (2008) wrote, a probability space is saturated if and only if it is super-atomless.[2, 3]

Our commodity space is a separable Banach space with a non-empty interior in its positive cone. Khan and Yannelis (1991), Podczeck (1997), Martins-da-Rocha (2003) used this space and provided the existence of a competitive equilibrium in an economy with an atomless measure space of agents. Due to the lack of the convexifying effect, Khan and Yannelis (1991) assumed convex as well as complete preferences. With his convexifying effect, Podczeck (1997) dropped the convex preferences assumption from the modelof Khan and Yannelis (1991). Both papers applied the infinite dimensional version of the Gale-Nikaido-Debreu lemma for their proofs. Martins-da-Rocha (2003) extended Khan and Yannelis (1991) and Podczeck (1997) to a large production economy. He proved the existence of competitive equilibria for economies with non-ordered but convex preferences as well as possibly incomplete but non-convex preferences. In his proof, the author appealed to the approximation of finite economies with a finite number of agents. He took advantage of the Edgeworth equilibria existence result reported by Florenzano (1990) and obtained a sequence of competitive equilibria for such finite economies. It was proved that the limit of the sequence is the equilibrium for the large economy.

We employ a saturated probability space for the space of agents which enables us to utilize the results in Sun and Yannelis (2008). Therefore, we do not need any additional conditions such as the 'many more agents than commodities' in Rustichini and Yannelis (1991) and the 'many agents of every type' in Podczeck (1997) to revive the convexifying effect. Moreover, we do not rely on the approximation method as in Martins-da-Rocha (2003). Hence, in this paper we show that one can have the convexifying effect in a separable Banach space with a proper formulation of economic negligibility, and provide a proof of competitive

[1] The Lyapunov theorem fails in every infinite dimensional Banach space. See Diestel and Uhl (1977) p.261.

[2] See Theorem 3B.7 in Fajardo and Keisler (2002).

[3] In the literature there are several expressions which are equivalent to the saturation property: "$\aleph_1$-atomless" in Hoover and Keisler (1984), "nowhere countably generated" in Loeb and Sun (2009), "rich" in Noguchi (2009), and "super-atomless" in Podczeck (2008). See Sun and Yannelis (2008) and Keisler and Sun (2009).

equilibrium existence without completeness of preferences. In this respect, this paper can be seen as an extension of Schmeidler (1969) to infinite dimensional commodity spaces. For exchange economies with possibly incomplete and non-convex preferences, this paper directly obtained the same result as in Theorem 3.1 of Martins-da-Rocha (2003) by applying the Gale-Nikaido-Debreu lemma.

This paper proceeds as follows. In section II, we provide notation and definitions. Section III contains our model. An existence theorem is stated in section IV.

II. *Notation and Definitions*

We denote by 2^A the set of all non-empty subsets of the set A and / denotes the set theoretic subtraction. Let X be a Banach space ordered by its positive cone X_+. If $A \subset X$, cl$A$ denotes the norm closure of A. The dual space of X is denoted by X' which is the set of all continuous linear functionals on X. If $p \in X'$ and $x \in X$, the value of p at x is denoted by $p \cdot x$. We denote by X'_+ the set $\{p \in X' : p \cdot x \geq 0 \ \ \forall x \in X_+\}$.

Let X and Y be topological spaces. A correspondence $F : X \to 2^Y$ is said to be *weakly upper semicontinuous* if x_n converges to x, $y_n \in F(x_n)$ for each n and y_n weakly converges to y, then $y \in F(x)$. The graph of F is denoted by $G_F = \{(x, y) \in X \times Y : y \in F(x)\}$.

Let $(T, \mathcal{T}, \mu)$ be a finite measure space and Y be a Banach space. A function $f : (T, \mathcal{T}, \mu) \to Y$ is said to be *μ-measurable* if there exists a sequence of simple functions $f_n : T \to Y$ such that $\lim_{n\to\infty} \|f_n(t) - f(t)\| d\mu = 0$ for almost all $t \in T$. A μ-measurable function f is called *Bochner integrable* if there exists a sequence of simple functions $\{f_n\}_{n=1,2,\ldots}$ such that

$$\lim_{n\to\infty} \int_T \| f_n(t) - f(t) \| d\mu = 0.$$

Then we define for each $A \in \mathcal{T}$ the integral to be $\int_A f(t) d\mu = \lim_{n\to\infty} \int_A f_n(t) d\mu$. It can be shown that if $f : T \to Y$ is a μ-integrable then f is Bochner integrable if and only if $\int_T \| f(t) \| d\mu < \infty$.[4] We denote by $\mathcal{L}_1(\mu, Y)$ the space of equivalence class of Y-valued Bochner integrable functions $f : T \to Y$ normed by $\| f \| = \int_T \| f(t) \| d\mu$. A correspondence $F : T \to 2^Y$ is said to be *integrably bounded* if there is a real-valued integrable function h on $(T, \mathcal{T}, \mu)$ such that for μ-almost all $t \in T$, $\sup\{\| y \| : y \in F(t)\} \leq h(t)\}$. The correspondence F is said to have a *measurable graph* if G_F belongs to product σ-algebra $\mathcal{T} \otimes \mathcal{B}(Y)$, where $\mathcal{B}(Y)$ denotes the Borel σ-algebra on Y. The correspondence F is said to be *lower measurable* if for every open subset U of Y, the set $\{t \in T : F(t) \cap U \neq \emptyset\} \in \mathcal{T}$. A measurable function f from $(T, \mathcal{T}, \mu)$ to Y is called a *measurable selection* of the correspondence F if $f(t) \in F(t)$ for almost all $t \in T$. We denote by S_F the set of all Y-valued Bochner integrable selections for the correspondence $F : T \to 2^Y$, i.e.,

$$S_F = \{f \in \mathcal{L}_1(\mu, Y) : f(t) \in F(t) \ \mu - \text{a.e.}\}.$$

The integral of the correspondence $F : T \to 2^Y$ is defined by

$$\int_T F(t) d\mu = \{\int_T f(t) d\mu : f \in S_F\}.$$

[4] See Diestel and Uhl (1977), Theorem 2, p.45.

Let X and Y be complete separable metric spaces and $\mathcal{M}(X)$ the space of all Borel probability measures on X with the Prohorov metric ρ. For each $\lambda\in\mathcal{M}(X\times Y)$, we denote by λ_X the marginal of λ in $\mathcal{M}(X)$. Let $(T, \mathcal{T}, \mu)$ be a countably additive complete probability space and $L^0(T, X)$ the space of all measurable functions $f: T\rightarrow X$ with the metric of convergence in probability. For $f\in L^0(T, X)$, the law (or distribution) of f is defined by $\mathrm{law}(f)(B)=\mu(f^{-1}(B))$ for each Borel set B in X. A probability space $(T, \mathcal{T}, \mu)$ is said to be *saturated* if $(T, \mathcal{T}, \mu)$ is atomless, and for any complete separable metric spaces X and Y, any $\lambda\in\mathcal{M}(X\times Y)$, any $f\in L^0(T, X)$ with $\mathrm{law}(f)=\lambda_X$, there exists $g\in L^0(T, Y)$ such that $\mathrm{law}(f, g)=\lambda$.[5]

III. *The Model*

The commodity space E is an ordered separable Banach Space with an interior point v in its positive cone E_+.[6] We employ a saturated probability space (T, T, μ) for the space of agents. Let X be a correspondence from T to E. The consumption set of agent t is given by $X(t)\subset E_+$. The initial endowment of each agent is given by a Bochner integrable function $e: T\rightarrow E$ and $e(t)\in X(t)$ for all $t\in T$. The aggregate initial endowment is $\int_T e(t)d\mu$. An economy $\mathcal{E}$ is a pair $[(T, \mathcal{T}, \mu), (X(t), \succ_t, e(t))_{t\in T}]$ where $\succ_t\subset X(t)\times X(t)$ is the preference relation of agent t.

An allocation for the economy $\mathcal{E}$ is a Bochner integrable function $f: T\rightarrow E$ such that $f(t)\in X(t)$ for all $t\in T$ and $\int_T f(t)d\mu\leq\int_T e(t)d\mu$. A price is $p\in E'_+/\{0\}$. The budget set of agent t at a price p is $B(p, t)=\{x\in X(t): p\cdot x\leq p\cdot e(t)\}$ and the demand set is defined by $D(p, t)=\{x\in B(p, t): y_t\not\succ_t x\ \ \forall y\in B(p, t)\}$.

A competitive equilibrium for the economy $\mathcal{E}$ is a price-allocation pair (p, f) if $f(t)\in D(p, t)$ for almost all $t\in T$.

We assume that the economy E satisfies the following assumptions:

A.1 $X(t)$ is non-empty, convex, integrably bounded and weakly compact for all $t\in T$.
A.2 There is an element $w(t)\in X(t)$ such that $e(t)-w(t)$ is in the norm interior of E_+, $\forall t\in T$.
A.3 $\succ_t$ is irreflexive and transitive $\forall t\in T$.
A.4 $\{y\in X(t): y\succ_t x\}$ and $\{y\in X(t): x\succ_t y\}$ are weakly open in $X(t)$ for each $x\in X(t)$ and $t\in T$.
A.5 $\succ_t$ is measurable, i.e., $\{(t, x, x')\in T\times E\times E: x\succ_t x'\}\in\mathcal{T}\otimes\mathcal{B}(E)\otimes\mathcal{B}(E)$.
A.6 The correspondence $X: T\rightarrow 2^E$ is measurable, i.e., $\{(t, x)\in T\times E: x\in X(t)\}\in\mathcal{T}\otimes\mathcal{B}(E)$.
A.7 If $x\in X(t)$ is a satiation point for $\succ_t$, then $x\geq e(t)\ \ \forall t\in T$.
A.8 If $x\in X(t)$ is not a satiation point for $\succ_t$, then x is an element of the weak closure of $\{x'\in X(t): x'\succ_t x\}$, $\forall t\in T$.

Notice that in A.1 we make the consumption set bounded. This assumption is used in Khan and Yannelis (1991), Rustichini and Yannelis (1991), Podczeck (1997) and Martins-da-Rocha (2003). As Martins-da-Rocha (2003) pointed out, this assumption is a "natural

[5] This paragraph is based on Sun and Yannelis (2008).
[6] The examples of this space include $C(K)$, the set of bounded continuous functions on a Hausdorff compact metric space K equipped with sup norm and a weakly compact subset of $L_\infty(\mu)$ where μ is a finite measure.

framework" to deal with Banach spaces. With this assumption, we can appeal to the results on the integral of correspondences. By A.2, budget sets are non-empty. Notice that in A.3 we do not assume convexity nor completeness of preferences. A.4 is about continuity of preferences. A.5 and A.6 are the measurability conditions. Following Podczeck (1997) we employ A.7 and A.8 whose meanings are clear.

IV. *Results*

The following theorem is our main result 1.6

Theorem 1. *Suppose that the economy $\mathscr{E}$ satisfies A.1-A.8. Then there exists a competitive equilibrium in $\mathscr{E}$.*

For the proof of Theorem 1, we first define a price space. Let $\Delta=\{p\in E'_+/\{0\} : p\cdot v=1\}$ be a price space. Clearly, Δ is convex and it is easy to see that Δ is bounded and weak* closed. Then by Alaoglu's theorem[7] we have the following result.

Lemma 1. *Δ is weak* compact.*

As is well known the evaluation map is not jointly continuous if E is equipped with the weak topology and and Δ with the weak* topology.[8] To avoid this problem, we follow Podczeck (1997) to construct the better-than-set $C(p, t)$ of agent $t\in T$ for a given $p\in\Delta$:

$$C(p, t)=X(t)/\{x\in X(t) : \exists y\in D(p, t) \text{ such that } y \succ_t x\}.$$

Note that $C(p, t)=X(t)\cap(E/\{x\in X(t) : \exists y\in D(p, t)$ such that $y \succ_t x\}$ and thus, from A.4 and the fact that $X(t)$ is weakly closed, $C(p, t)$ is weakly closed. Since $X(t)$ is weakly compact and integrably bounded, $C(p, t)$ is also weakly compact and integrably bounded. It is clear that $D(p, t)\subset C(p, t)$ for all $p\in\Delta$ and $t\in T$. The following lemma shows the existence of maximal elements in a weakly compact set. The proof of the following lemma is similar to that of Lemma 2 in Schmeidler (1969).

Lemma 2. *If K is a non-empty weakly compact subset of $X(t)$, then the set $M=\{x\in K : y\not\succ x \ \forall y\in K\}$ is non-empty.*

Observe that $B(p, t)$ is a weakly closed subset of $X(t)$ and hence, it is weakly compact. Then by Lemma 2, $D(p, t)$ is not empty. It follows that $C(p, t)$ is not empty either for all $p\in\Delta$ and for all $t\in T$.

Lemma 3. *For each $t\in T$, $C(\cdot, t)$ is weakly upper semicontinuous in p.*

Proof. This proof is analogous to the proof of Lemma 6 in Schmeidler (1969). Consider $p_n\to p$ in the weak* topology and $x_n\to x$ in the weak topology with $x_n\in C(p_n, t)$ for all n. We want to prove $x\in C(p, t)$. Suppose $x\notin C(p, t)$. Then there is $y\in X(t)$ such that $y\succ_t x$ and $p\cdot y\leq p\cdot e(t)$. From A.2 and A.4, there exists z sufficiently close to y such that $z\succ_t x$ and $p\cdot z<p\cdot e(t)$. For sufficiently large n, we have $p_n\cdot z\leq p_n\cdot e(t)$ and again from A.4, $z\succ_t x_n$ which contradicts

[7] See Theorem 5.105 in Aliprantis and Border (2006).

[8] See Aliprantis and Border (2006) pp.241-242.

$x_n \in C(p_n, t)$. Thus $C(\cdot, t)$ is weakly upper semicontinuous. ■

Lemma 4. *For given p, C(p, ·) has a measurable graph.*

Proof. See Appendix. ■

Lemma 5. *For any* $p\in\Delta$, $\int_T C(p, \cdot)d\mu$ *is non-empty, convex, weakly compact and weakly upper semicontinuous in p.*

Proof. From Lemma 4, $C(p, \cdot)$ has a measurable graph for all $p\in\Delta$ and recall that it is non-empty. By Aumann's (1969) measurable selection theorem, there exists a measurable function $g^p : T\rightarrow E$ such that $g^p(t)\in C(p, t)$ for almost all $t\in T$. Since $C(p, t)$ is integrably bounded, g^p is Bochner integrable. Hence, $\int_T g^p(t)d\mu\in\int_T C(p, \cdot)d\mu$. It follows from Proposition 1 in Sun and Yannelis (2008) that $\int_T C(p, \cdot)d\mu$ is convex for any p in Δ. As we already mentioned, $C(p, \cdot)$ is weakly compact for each $p\in\Delta$. By Lemma 3, $C(\cdot, t)$ is weakly upper semicontinuous in p for each $t\in T$. Recall that for fixed $p\in\Delta$, $C(p, t)\subset X(t)$ for all $t\in T$ and that, by A.1, $X(t)$ is integrably bounded and weakly compact for all $t\in T$. Then it follows from Proposition 1 in Sun and Yannelis (2008) that $\int_T C(p, \cdot)d\mu$ is weakly compact and weakly upper semicontinuous in p. ■

We now consider a correspondence $\zeta : \Delta\rightarrow 2^E$ defined by

$$\zeta(p)=\int_T C(p, t)d\mu-\int_T e(t)d\mu.$$

Proposition 1. *ζ is non-empty, convex-valued, weakly compact-valued and weakly upper semicontinuous in p.*

Proof. This is a direct consequence of Lemma 5. ■

The following is the Gale-Nikaido-Debreu Lemma for infinite dimensional spaces proved by Yannelis (1985).

Lemma 6. *Let E be a Hausdorff locally convex linear topological space whose positive cone* E_+ *has an interior point v. Let* $P=\{p\in E'_+/\{0\} : p\cdot v=1\}$. *Suppose that the correspondence* $\phi : P\rightarrow 2^E$ *satisfies the following conditions:*

- *(i) For all* $p\in P$, *there exists* $z\in\varphi(p)$ *such that* $p\cdot z\leq 0$;
- *(ii)* $\varphi : P \rightarrow 2^E$ *is upper semi-continuous in the weak* topology (i.e.,* $\varphi : (P, w^*) \rightarrow 2^E$ *is upper semi-continuous);*
- *(iii) for all* $p\in P$, $\varphi(p)$ *is non-empty, convex and compact.*

Then there exists $\bar{p}\in P$, *such that* $\varphi(\bar{p})\cap(-E_+)\neq\emptyset$.

We now turn to the proof of Theorem 1.

Proof of Theorem 1. We first show that there exists $z\in\zeta(p)$ such that $p\cdot z\leq 0$ for all $p\in\Delta$. Since $D(p, t)$ is non-empty and has a measurable graph (see the proof of Lemma 4 in Appendix), we appeal to Aumann's (1969) measurable selection theorem to have a measurable function $g^p : T \rightarrow E$ such that $g^p(t)\in D(p, t)$ for each $p\in\Delta$ and for all $t\in T$. Then by definition of $D(p, t)$, we obtain $p \cdot g^p(t) \leq p \cdot e(t)$ which implies $\int_T p \cdot g^p d\mu \leq \int_T p \cdot e(t) d\mu$. Since $E \subset E''$(dual of E') and g^p is Bochner integrable,[9] we have $p \cdot (\int_T g^p(t) d\mu - \int_T e(t) d\mu) \leq 0$. Since $D(p, t) \subset C(p, t)$, it is clear that $(\int_T g^p(t) d\mu - \int_T e(t) d\mu) \in \zeta(p)$.

Proposition 1 assures that ζ is non-empty, convex-valued, weakly compact-valued and weakly upper semicontinuous. Consequently ζ satisfies all the assumptions of Lemma 6 and therefore there exists $p^* \in \Delta$ and a Bochner integrable function f^* such that $\bar{z} \in \zeta(p^*)$ with

$$\bar{z} = \int_T f^*(t) d\mu - \int_T e(t) d\mu \leq 0 \tag{1}$$

where $f^*(t) \in C(p^*, t)$ for almost all $t \in T$.

Notice that for any $x \in C(p, t)$,

$$p \cdot x \geq p \cdot e(t). \tag{2}$$

Consider the case where x is a satiation point. Then by A.7, $x \geq e(t)$. With $p \in \Delta$ being positive, we can assert $p \cdot x \geq p \cdot e(t)$. Now consider the case where x is not a satiation point. Suppose $p \cdot x < p \cdot e(t)$. By A.8,we know that x belongs to a weak closure of $\{x' \in X(t) : x' \succ_t x\}$. Note that $x \in B(p, t)$ and thus there exists $z \in B(p, t)$ such that $z \succ_t x$ which implies $x \notin C(p, t)$, a contradiction. Observe that $D(p, t) = C(p, t) \cap \{x \in E : p \cdot x = p \cdot e(t)\}$. By Combining (1) and (2) and with the fact that $p^* \in E'_+$, we have

$$p^* \cdot f^*(t) = p^* \cdot e(t) \quad \text{and} \quad f^*(t) \in D(p^*, t)$$

for almost all . Hence (p^*, f^*) is a competitive equilibrium. ■

[9] If E is embedded in E'' and $f : T \to E$ is Bochnerintegrable, then f is Gelfand integrable and two integrals coincide. Hence $\int_T p \cdot f d\mu = p \cdot \int_T f d\mu$. See Aliprantis and Border (2006) p.430.

Appendix

Proof of Lemma 4

Proof. The following proof is based on the proof in Khan and Yannelis (1991). We will show that $C(p, \cdot)$ is has a measurable graph. We first show that the budget set $B(p, \cdot)$ has a measurable graph, i.e., $\{(t, x) \in T \times X(t) : p \cdot x \leq p \cdot e(t)\} \in \mathcal{T} \otimes \mathcal{B}(E)$. Let $g_p : T \times E \to [-\infty, \infty]$ defined by $g_p(t, x) = p \cdot x - p \cdot e(t)$. Then g_p is measurable in t because $e(\cdot)$ is measurable in t. In addition, g_p is continuous in x since the map $x \to p \cdot x$ is continuous in x. Hence by Lemma III.14 in Castaing and Valadier (1977), g_p is jointly measurable, i.e., $g_p^{-1}((-\infty, 0]) \in \mathcal{T} \otimes \mathcal{B}(E)$. It can be shown that

$$G_{B(p, \cdot)} = \{(t, x) \in T \times X(t) : p \cdot x \leq p \cdot e(\cdot)\} = g_p^{-1}((-\infty, 0]) \cap G_X$$

where G_X is the graph of $X(t)$. By A.6, $G_X \in \mathcal{T} \otimes \mathcal{B}(E)$. It now follows that for each $p \in \Delta$, $G_{B(p, \cdot)} \in \mathcal{T} \otimes \mathcal{B}(E)$.

We now show that $D(p, \cdot)$ has a measurable graph. Let $H(p, t) = \{x \in B(p, t) : \exists y \in B(p, t)$ suchthat $y \succ_t x\}$. Observe that $G_{D(p, \cdot)}$, the graph of $D(p, \cdot)$, can be written as

$$G_{D(p, \cdot)} = G_{B(p, \cdot)} / G_{H(p, \cdot)}$$

where $G_{H(p, \cdot)}$ is the graph of $H(p, \cdot)$. Since $(T, \mathcal{T}, \mu)$ is a measure space and $B(p, \cdot)$ has a measurable graph and is weakly closed valued and, thus, norm closed, we can appeal to Lemma 3.1 in Yannelis (1991) to assert that for each $p \in \Delta$, $B(p, \cdot)$ is lower measurable. Therefore, by Castaing's Representation Theorem,[10] there exists a family $\{f_n : n = 1, 2, ...\}$ of measurable functions $f_n : T \to E$ such that for all $t \in T$, $\mathrm{cl}\{f_n(t) : n = 1, 2, ...\} = B(p, t)$. For $n (= 1, 2, ...)$, let $H_n(p, t) = \{x \in B(p, t) : f_n(t) \succ_t x\}$. Note that $B(p, \cdot)$ and, by A.5, $\succ_t$ have measurable graphs. Therefore $H_n(p, \cdot)$ also has a measurable graph. We shall prove $H(p, t) = \cup_{n=1}^{\infty} H_n(p, t)$ for any $t \in T$. It is clear that $H_n(p, t) \subset H(p, t)$ for each n so that $\cup_{n=1}^{\infty} H_n(p, t) \subset H(p, t)$. We now show $H(p, t) \subset \cup_{n=1}^{\infty} H_n(p, t)$. Suppose otherwise. Then there exists $z \in E$ such that $z \in H(p, t)$ and $z \notin \cup_{n=1}^{\infty} H_n(p, t)$. Then there exists $y \in B(p, t)$ such that $y \succ_t z$ but there does not exist n such that $f_n(t) \succ_t z$. Since the family $\{f_n(t) : n = 1, 2, ...\}$ is norm dense in $B(p, t)$, we can find an n_0 such that $f_{n_0}(t) \in B(p, t)$ and $f_{n_0}(t)$ is sufficiently close to y in the norm topology and hence in the weak topology. Then with the continuity of preference, we have $f_{n_0}(t) \succ_t z$, a contradiction. Thus $H(p, t) = \cup_{n=1}^{\infty} H_n(p, t)$. Since for each $p \in \Delta$, $H_n(p, \cdot) (n = 1, 2, ...)$ has a measurable graph, so does $H(p, \cdot)$. Hence we conclude $G_{D(p, \cdot)} = G_{B(p, \cdot)} / G_{H(p, \cdot)} \in \mathcal{T} \otimes \mathcal{B}(E)$.

We now turn to $C(p, \cdot)$. Let $J(p, t) = \{x \in X(t) : \exists y \in D(p, t)$ suchthat $y \succ_t x\}$. Then $G_{C(p, \cdot)} = G_X / G_{J(p, \cdot)}$. Recall that $B(p, t)$ is weakly closed. By A.4, $\cup_{n=1}^{\infty} H_n(p, t)$ is weakly open and thus, so is $H(p, t)$. It is clear that $D(p, t) = B(p, t) / H(p, t)$. It follows that $D(p, t)$ is weakly closed and hence norm closed. The similar argument in the above applies to $D(p, \cdot)$ to assert that $D(p, \cdot)$ is lower measurable. Then again by Castaing's Representation Theorem, there exists a family $\{g_n : n = 1, 2, ...\}$ of measurable function $g_n : T \to E$ such that for all $t \in T$, $\mathrm{cl}\{g_n(t) : n = 1, 2, ...\} = D(p, t)$. For $n (= 1, 2, ...)$, let $J_n(p, t) = \{y \in X(t) : g_n(t) \succ_t y\}$. By the similar argument above, we have $J(p, t) = \cup_{n=1}^{\infty} J_n(p, t)$. It follows that $G_{J(p, \cdot)} \in \mathcal{T} \otimes \mathcal{B}(E)$. Therefore, $G_{C(p, \cdot)} \in \mathcal{T} \otimes \mathcal{B}(E)$. ■

[10] See Theorem III. 9 in Castaing and Valadier (1977).

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