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VANISHING THEOREMS IN ASYMPTOTIC ANALYSIS AND THEIR APPLICATIONS TO DIFFERENTIAL EQUATIONS*

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About one century ago, H. Poincaré[22] obtained the concept of asymptotic expansions of holomorphic functions in one-dimensional sectors. He proved the existence of a solution to an ordinary differential equation of 'Poincaré rank 1' which has an asymptotic expansion. Since then, the asymptotic expansions of one-variable functions have been used to express local behavior of solutions to functional equations (cf. Wasow[30], Olver[21]). Among the works, on the ordinary differential equations with singular points, the theorem of existence of asymptotic solutions is significant, which is due to Pioncaré[22], Birkhoff[1], Trjitzinsky[27], Malmquist]19], Hukuhara[8], Turrittin[28], Iwano[10], and Sibuya[24]. In the former 1970's, the index theorems of linear ordinary differential equations were proved by several authors: Deligne[3], Malgrange[17], Komatsu[13]. Moreover, the notion of regular singularity of linear ordinary differential equations was characterized by validity of the comparison theorem or cohomology vanishing theorem[3, 17, 13]. These well suggested that there was an absence of a tool in the theory of asymptotic expansions of one-variable functions. In the middle 1970's, via Sibuya[25], Malgrange[17] introduced the sheaf of germs of functions having asymptotic expansions and proved a kind of vanishing theorem for it. Thereafter, the theory of asymptotic expansions of one-variable functions became more powerful. As is written in the encyclopedical dictionary of mathematics by the Mathematical Society of Japan, there are many ways to define the notion of asymptotic expansions of several-variable functions. Hukuhara[9] gave a definition for the study of non-linear ordinary differential equations. In the latter 1970's, some works on the Pfaffian systems with certain kind of irregular singularities appeared: Gérard-Sibuya[7], Takano[26]. In these articles, the asymptotic expansions of several-variable functions were adopted as in [9]. In 1981, there appeared another theory of asymptotic expansions of several-variable functions[14]. It is considered as powerful as the asymptotic theory of one-variable functions.

I. *Definition of Asymptotic Developability of Functions*

Let $S[\theta,\bar{\theta},r]$ and $S(\underline{\theta},\bar{\theta},r)$ be a closed and an open sector with summit at the origin in $\boldsymbol{C}$.

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We say that a holomorphic function $f(x)$ in $S[\underline{\theta},\bar{\theta},r]$ is asymptotically developable as x tends to 0 if there exists a formal power series $\hat{f}(x)=\sum_{q=0}^{\infty}f_q x^q$ such that for any non-negative integer N the following inequality is satisfied

$$\sup\ \{|x|^{-N}|f(x)-\sum_{q=0}^{N-1}f_q x^q|;\ x\in S[\underline{\theta},\bar{\theta},r]\}<+\infty. \tag{1.1}$$

By using the Cauchy's integral formula, this condition is translated to the following condition that for any N there exists a limit

$$\lim\ (d/dx)^N f(x)(x\in S[\underline{\theta},\bar{\theta},r],\ x\to 0)=N!f_N \tag{1.2}$$

Then, $\hat{f}(x)$ is called the asymptotic series for $f(x)$, and $f(x)$ is asymptotically developable to $\hat{f}(x)$ as x tends to 0. A function in $S(\underline{\theta},\bar{\theta},r)$ is said to be asymptotically developable as x tends to 0 if $f(x)$ is holomorphic and asymptotically developable in any closed subsector S' in the open sector. Now, let $\underline{\theta}_i,\bar{\theta}_i,(i=1,\ldots,k)$ and $r_i(i=1,\ldots,n)$ be real numbers and positive real numbers, respectively. The subset of $\boldsymbol{C}^n$

$$S'=\prod_{i=1}^{k}S[\underline{\theta}_i,\bar{\theta}_i,r_i]\times\prod_{j=k+1}^{n}D[r_j] \tag{1.3}$$

is called an n-dimensional closed polysector with the edge in $V=\{(x_1,\ldots,x_n)\in\boldsymbol{C}^n;\ x_1\cdots x_k=0\}$, where $D[r_j]$ denotes the closed disc at the origin with the radius r_j in x_j-plane. Let $[1,n]$ denote the subset $\{i=1,\ldots,n\}$ in $\boldsymbol{N}$. For a subset J of $[1,k]$, $\boldsymbol{N}^J$ denotes the set of all $q_J=(q_j)_{j\in J}$. Put $x_J=(x_j)$ and put $x_J{}^{q_J}=\prod_{j\in J}x_j{}^{q_j}$ for $q_J\in\boldsymbol{N}^J$. For the complement I of J in $[1,n]$, put $S'_{[I]}=\prod_{i\in I\cap[1,k]}S[\underline{\theta}_i,\bar{\theta}_i,r_i]\times\prod_{j=k+1}{}^{n}D[r_j]$, $x_I=(x_i)_{i\in J}$. Write x for $x_{[1,n]}$.

DEFINITION 1.1. A holomorphic function in S' is said to be *asymptotically developable* as x tends to V if there exists a family of functions

$$F=\{f(x_I;q_J);\ \phi\neq J\subset[1,k],\ q_J\in\boldsymbol{N}^J,\ I=[1,n]-J\} \tag{1.4}$$

such that $f(x_I;q_J)$ is holomorphic in $S'_{[I]}$, and for any $N=(N_1,\ldots N_k,0,\ldots,0)\in\boldsymbol{N}^n$,

$$\sup\ \{|x^{-N}(f(x)-App_N(x;F))|;\ x\in S'\}<+\infty \tag{1.5}$$

is satisfied, where

$$App_N(x;F)=\sum_{\phi\neq J\subset[1,k]}(-1)^{\sharp J+1}\sum_{j\in J}\sum_{q_j=0}^{N_j-1}f(x_I;q_J)x_J{}^{q_J}. \tag{1.6}$$

This condition is equivalent to the existence of the limit

$$\lim\ (\partial/\partial x_J)^{q_J}f(x)(x\in S',\ x_J\to 0)=q_J!f(x_J,q_J) \tag{1.7}$$

for any non-empty subset J of $[1,k]$ and any $q_J\in\boldsymbol{N}^J$. Then, F is called the family of the total coefficients of asymptotic expansion for f and $f(x_I;q_J)$ is called the coefficient of the degree q_J with respect to J. These are denoted by $TA(f),TA(f)_{q_J}$, respectively. The series

$$FA_J(f)=\sum_{q_J\in\boldsymbol{N}^J}f(x_I;q_J)x_J{}^{q_J} \tag{1.8}$$

is called the asymptotic series of f with respect to J. $App_N(x;TA(f))$ is called the approximate function of the degree N. If the above condition is satisfied, then the function $f(x_I;q_J)$ in $S'_{[I]}$ satisfies the same type of condition as (1.5) for

$$F(q_J)=\{f(x_K;q_{J\cup L});\phi\neq L\subset[1,k]-J, q_L\in N^L, K=[1,n]-J\cup L\} \tag{1.9}$$

as x tends to $V_I=\{x_I;\prod_{i\in I\cup[1,k]}x_i=0\}$.

DEFINITION 1.2. A holomorphic function $f(x)$ in an open polysector with the edge in V

$$S=\prod_{i=1}^{k}S(\theta_i,\bar{\theta}_i,r_i)\times\prod_{j=k+1}^{n}D(r_j) \tag{1.10}$$

is said to be *asymptotically developable as x tends to V*, if $f(x)$ is asymptotically developable in any closed subpolysector S' in the sence of the definition 1.1, where $D(r_j)$ denotes the open disc at the origin in the x_j-plane.

If the function $f(x)$ is asymptotically developable in S, for any non-empty subset J in $[1,k]$ and for any q_J, there exists a holomorphic $f(x_I,q_J)$ in the open polysector

$$S_{(I)}=\prod_{i\in I\cap(1,k)}S(\theta_i,\bar{\theta}_i,r_i)\times\prod_{j=k+1}^{n}D(r_j)$$

with respect to $I=[1,n]-J$ such that the family of the restrictions to $S'_{(I)}$ satisfies the condition (1.5). And, as for the closed polysector, $TA(f)_{q_J}=f(x_I;q_J)$, $TA(f)$, $FA_J(f)$, $App_N(x;TA(f))$ are defined.

DEFINITION 1.3. Let be given a family of functions

$$F=\{f(x_I;q_J);\phi\neq=J\subset[1,k], q_J\in N^J, I=[1,n]-J\} \tag{1.11}$$

in an open or closed polysector S with the edge in V, where $f(x_I;q_J)$ is holomorphic in $S_{(I)}$ or $S_{[I]}$. The family F is said to be a *consistent* family if, for any non-empty subset J in $[1,k]$ and for any $q_J\in N^J$, $f(x_I;q_J)$ is asymptotically developable and the family of the total coefficients $TA(f(x_I;q_J))$ is equal to $\{f(x_K;q_{J\cup L});\phi\neq L\subset[1,k]-J, q_L\in N^L, K=[1,n]-J\cup L\}$.

THEOREM 1.1. *Let S be an n-dimensional open or closed polysector with the edge in V and let F be a consistent family in S. Then, there exists a holomorphic function $f(x)$ such that it is asymptotically developable in S and $TA(f)=F$.*

Let $f(x)$ be a asymptotically developable function with the family of the total coefficients $TA(f)$ in an n-dimensional polysector S with the edge in V. If the function $f(x_I;q_J)$ is holomorphic in a disc including $S_{[I]}$ or $S_{(I)}$ for any non-empty subset J in $[1,k]$ and for any $q_J\in N^J$,

$$f(x_I;q_J)=\sum_{q_L\in N^L}f(x_K;q_{J\cup L})x_L{}^{q_L} \tag{1.12}$$

for any non-empty subset L in $[1,k]-J$, the approximate function $App_N(x;F)$ of the degree N is holomorphic in a disc including S and

$$App_N(x;F)=\sum_{q_1=0}^{\infty}\cdots\sum_{q_k=0}^{\infty}f(x_{k+1},\ldots,x_n;q_1,\ldots,q_k)x_1{}^{q_1}\cdots x_k{}^{q_k}$$

$$-\sum_{q_1=N_1}^{\infty}\cdots\sum_{q_k=N_k}^{\infty}f(x_{k+1},\ldots,x_n;q_1,\ldots,q_k)x_1{}^{q_1}\cdots x_k{}^{q_k}. \tag{1.13}$$

Put $D_j=D[r_j]$ or $D(r_j)$. Denote by $\mathcal{O}(\prod_{j\neq i}D_j)[[x_i]]$ the ring of formal power series with respect to x_i with coefficients in $\mathcal{O}(\prod_{j\neq i}D_j)$. Then, the formal power series

$$FA_{[1,k]}(f)(x)=\sum_{q_1=0}^{\infty}\cdots\sum_{q_k=0}^{\infty}f(x_{k+1},\ldots,x_n;q_1,\ldots,q_k)x_1^{q_1}\cdots x_k^{q_k} \tag{1.14}$$

belongs to the intersection of $\mathcal{O}(\prod_{j\neq i}D_j)[[x_i]](i=1,\ldots,k)$. Namely, let $\mathcal{O}$ be the sheaf of germs of holomorphic functions and let $\hat{\mathcal{O}}_V=\text{proj lim}_{N\to\infty}\mathcal{O}/(x_1\cdots x_k)^N\mathcal{O}$ be the formal completion of $\mathcal{O}$ along V. Then, $FA_{[1,k]}(f)$ is a section of $\hat{\mathcal{O}}_V$ on $D=\prod_{i=1}^n D_i$ and $f(x)$ is said to be asymptotically developable to $\hat{f}(x)=FA_{[1,k]}(f)(x)$ in $\hat{\mathcal{O}}_V(D)$ as x tends to V.

THEOREM 1.2. *Let S be an n-dimensional open or closed polysector with the edge in V and let $\hat{f}(x)\in\hat{\mathcal{O}}_V(D)$. Then, there exists a holomorphic function in S which is asymptotically developable to $\hat{f}(x)$ as x tends to V.*

Denote by $A(S)$ the set of all functions asymptotically developable in S. Then, $A(S)$ is closed by fundamental operations: addition, substitution, multiplication and integration. If S is open then $A(S)$ is also closed by differentiation. Moreover, the operations commute with FA_J. Denote by $A'(S)$ ad $A_0(S)$ be the set of all functions asymptotically developable to some formal power series in $\hat{\mathcal{O}}_V$ and to the formal power series 0, respectively. Then, $A_0(S)\subset A'(S)\subset A(S)$ and $A_0(S)$, $A'(S)$ are closed by the fundamental operations.

II. *Sheaves of Asymptotically Developable Functions and Vanishing Theorems*

Let M be a complex analytic manifold and let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on M. Let H be an union of a finite number of non-singular hypersurfaces in M and let $\mathcal{O}_{\widehat{M|H}}$ be the formal completion of $\mathcal{O}$ along H. Namely, let $\mathcal{P}_H$ be the defining ideal of H, then $\mathcal{O}_{\widehat{M|H}}=\text{proj lim}_{N\to\infty}\mathcal{O}/\mathcal{P}_H{}^N$. Suppose that singular points of H are normal crossings. Namely, for any point h on H, there exist a neighborhood U of h and local coordinates $x_1,\ldots,x_n$ such that $H\cap U=\{(x_1,\ldots,x_n)\in U;\ x_1\cdots x_k=0\}$ $(k\leqq n)$. Define the real blowing-up (U^-,pr_U) of U along $H\cap U$ by

$$U^-=\{(x,z)\in U\times(S^1)^k;\ \text{Im}(x_i\bar{z}_i)=0,\ \text{Re}(x_i\bar{z}_i)\geqq 0,\ i=1,\ldots,k\}, \tag{2.1}$$

where pr_U is the natural projection from U^- to U and $S^1=\{z\in\boldsymbol{C};\ |z|=1\}$. If $U=\prod_{i=1}^n U_i$, then

$$U^-=\prod_{i=1}^{k}(U_i-\{0\}\cup S^1)\times\prod_{j=k+1}^{n}U_j, \tag{2.2}$$

where $U_i-\{0\}\cup S^1$ is the space constructed by pulling the origin from U_i and inserting S^1. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open covering of M, and let (U^-_λ,pr_λ) be the real blowing-up of U_λ along $H\cap U_\lambda$. Then, by patching U^-_λ together, the real blowing-up of M along H is constructed. For an open set U^- in M^-, put $\mathscr{A}^-(U^-)=\mathscr{A}'^-(U^-)=\mathscr{A}_0{}^-(U^-)=\mathcal{O}(pr(U^-))$, if the closure of $pr(U^-)$ has no intersection with H. If the closure of $pr(U^-)$ has intersection with H, then denote by $\mathscr{A}^-(U^-)$ the ring of functions asymptotically developable in any n-dimensional polysector in $pr(U^-)-H$ with the edge in H, and denote by $\mathscr{A}'^-(U^-)$ and $\mathscr{A}_0{}^-(U^-)$ the

rings of functions asymptotically developable to some formal power series in $\mathcal{O}_{\widehat{M|H}}$ $(pr(U^-))$ and to the formal power series 0 in any n-dimentional polysector in $pr(U^-)-H$ with the edge in H, respectively. Then, in a natural way we can define the sheaves $\mathscr{A}^-$, $\mathscr{A}'^-$ and $\mathscr{A}_0^-$ whose sets of sections on U^- are equal to $\mathscr{A}^-(U^-)$, $\mathscr{A}'^-(U^-)$ and $\mathscr{A}_0^-(U^-)$, respectively. The vanishing theorems in asymptotic analysis are stated as follows:

THEOREM 2.1. *For any point h on H, put $h^-=pr^{-1}(h)$, then*

$$H^q(h^-, \mathscr{A}_0^-|_{h^-})=\begin{cases}0, & q\neq 1\\ (\mathcal{O}_{\widehat{M|H}})_h/\mathcal{O}_h, & q=1.\end{cases} \tag{2.3}$$

THEOREM 2.2. *If $H^q(M, \mathcal{O})=H^q(M, \mathcal{O}_{\widehat{M|H}})=0$ for $q\geqq 1$, then,*

$$H^q(M, \mathscr{A}_0)=\begin{cases}0, & q\neq 1\\ H^0(M, \mathcal{O}_{\widehat{M|H}})/H^0(M, \mathcal{O}), & q=1.\end{cases} \tag{2.4}$$

These theorems can be stated in anotehr way. Note that Theorem 1.2 means the following:

THEOREM 2.3. *The following sequence of sheaves is exact:*

$$0\to\mathscr{A}_0^-\to\mathscr{A}'^-\to pr^{-1}\mathcal{O}_{\widehat{M|H}}\to 0, \tag{2.5}$$

where i is the canonical inclusion mapping and, on $pr^{-1}(H)$, j assigns to f the asymptotic series and outside j is the zero mapping. From the short exact sequence, we have the long exact sequences:

$$\begin{aligned}&0\to H^0(h^-, \mathscr{A}_0^-|_{h^-})\to H^0(h^-, \mathscr{A}'|_{h^-})\to H^0(h^-, pr^{-1}\mathcal{O}_{\widehat{M|H}}|_{h^-})\\ &\to H^1(h^-, \mathscr{A}_0^-|_{h^-})\to H^1(h^-, \mathscr{A}'^-|_{h^-})\to H^1(h^-, pr^{-1}\mathcal{O}_{\widehat{M|H}}|_{h^-})\\ &\to H^2(h^-, \mathscr{A}_0^-|_{h^-})\to \cdots,\end{aligned} \tag{2.6}$$

$$\begin{aligned}&0\to H^0(M^-, \mathscr{A}_0^-)\to H^0(M^-, \mathscr{A}'^-)\to H^0(M^-, pr^{-1}\mathcal{O}_{\widehat{M|H}})\\ &\to H^1(M^-, \mathscr{A}_0^-)\to H^1(M^-, \mathscr{A}'^-)\to H^1(M^-, pr^{-1}\mathcal{O}_{\widehat{M|H}})\\ &\to H^2(M^-, \mathscr{A}_0^-)\to \cdots,\end{aligned} \tag{2.7}$$

The first three terms are easily calculated:

$$H^0(h^-, \mathscr{A}_0|_{h^-})=0,\ H^0(h^-, \mathscr{A}'^-|_{h^-})=\mathcal{O}_h,\ H^0(h^-, pr^{-1}\mathcal{O}_{\widehat{M|H}}|_{h^-})=(\mathcal{O}_{\widehat{M|H}})_h \tag{2.8}$$

$$H^0(M^-, \mathscr{A}_0^-)=0,\ H^0(M^-, \mathscr{A}'^-)=H^0(M, \mathcal{O}),\ H^0(M^-, pr^{-1}\mathcal{O}_{\widehat{M|H}})=H^0(M, \mathcal{O}_{\widehat{M|H}}) \tag{2.9}$$

Therefore, we can translate Theorems 2.1–2 as follows:

THEOREM 2.4. *For any point h on H, (i) $i_{1,h}$ is a zero mapping and (ii) $j_{q,h}(q\geqq 1)$ are isomorphisms.*

THEOREM 2.5. *If $H^q(M, \mathcal{O})=H^q(M, \mathcal{O}_{\widehat{M|H}})=0(q\geqq 1)$, then i_1 is a zero mapping and $j_q(q\geqq 1)$ are isomorphisms.*

More precisely, we can assert the following:

THEOREM 2.6. *(i) If $H^1(M, \mathcal{O})=0$ then i_1 is a zero mapping, i.e. $H^1(M^-, \mathscr{A}_0^-)\cong H^0(M, \mathcal{O}_{\widehat{M|H}})/H^0(M, \mathcal{O})$, (ii) If $H^q(M, \mathcal{O})=0$ $(1\leqq q\leqq j)$, $H^q(M, \mathcal{O}_{\widehat{M|H}})=0$ $(1\leqq q\leqq j-1)$ then $j_q(1\leqq q\leqq j)$ are isomorphisms, i.e. $H^q(M^-, \mathscr{A}_0^-)=0$ $(2\leqq q\leqq j)$.*

Denote by $\mathcal{O}(*H)$ the sheaf of germs of meromorphic functions which are holomorphic in $M-H$ and have poles on H, and put $\mathcal{O}_{\widehat{M|H}}(*H)=\mathcal{O}_{\widehat{M|H}}\otimes_{\mathcal{O}}\mathcal{O}(*H)$. By the second isomorphism theorem, we have

$$(\mathcal{O}_{\widehat{M|H}})_h/\mathcal{O}_h\cong\mathcal{O}_{\widehat{M|H}}(*H)_h/\mathcal{O}(*H)_h, \tag{2.10}$$

$$H^0(M, \mathcal{O}_{\hat{M}|H})/H^0(M, \mathcal{O}) \cong H^0(M, \mathcal{O}_{\hat{M}|H}(*H))/H^0(M, \mathcal{O}(*H)). \tag{2.11}$$

Put $\mathcal{O}'=pr^{-1}\mathcal{O}$ and $\mathscr{A}'^-(*H)=\mathscr{A}'^-\otimes_{\mathcal{O}} pr^{-1}\mathcal{O}(*H)$, we have the short exact sequence analogous to (2.5):

$$0\to\mathscr{A}_0^-\to\mathscr{A}'^-(*H)\to pr^{-1}(\mathcal{O}_{\hat{M}|H}(*H))\to 0. \tag{2.12}$$

Therefore, we have another verison of the vanishing theorem:

COROLLARY 2.1. *For any point h on H, put $h^-=pr^{-1}(h)$, then*

$$H^1(h^-, \mathscr{A}_0^-|_{h^-}) \cong \mathcal{O}_{\hat{M}|H}(*H)_h/\mathcal{O}(*H)_h, \tag{2.13}$$

$$H^q(h^-, \mathscr{A}'^-(*H)|_{h^-}) \cong H^q(h^-, pr^{-1}\mathcal{O}_{\hat{M}|H}(*H)|_{h^-}), \quad q\geqq 1. \tag{2.14}$$

COROLLARY 2.2. *(i) If $H^1(M, \mathcal{O})=0$ then,*

$$H^1(M^-, \mathscr{A}_0^-) \cong H^0(M, \mathcal{O}_{\hat{M}|H}(*H))/H^0(M, \mathcal{O}(*H)), \tag{2.15}$$

(ii) If $H^q(M, \mathcal{O})=0$ $(1\leqq q\leqq j)$, $H^q(M, \mathcal{O}_{\hat{M}|H})=0$ $(1\leqq q\leqq j-1)$ then,

$$H^q(M^-, \mathscr{A}'^-(*H)) \cong H^q(M^-, pr^{-1}\mathcal{O}_{\hat{M}|H}(*H)), \quad 1\leqq q\leqq j. \tag{2.16}$$

For a positive integer m and $E=\mathscr{A}^-(U^-)$, $\mathscr{A}'^-(U^-)$ and $\mathcal{O}_{\hat{M}|H}(pr(U^-))$, denote by $GL(m,E)$ the ring of invertible m-by-m matrices whose elements belong to E. As above, we obtain the sheaves $GL(m, \mathscr{A}^-)$, $GL(m, \mathscr{A}'^-)$ and $GL(m, \mathcal{O}_{\hat{M}|H})$. Denote by $GL(m, \mathscr{A}^-)_I$ the subsheaf of $GL(m, \mathscr{A}'^-)$ of germs of matricial functions asymptotically developable to the identity matrix I_m. Then, we have the short exact sequence:

$$I_m\to GL(m, \mathscr{A}^-)_I\to GL(m, \mathscr{A}'^-)\to pr^{-1}GL(m, \mathcal{O}_{\hat{M}|H})\to I_m, \tag{2.17}$$

THEOREM 2.7. *For any point h on H, put $h^-=pr^{-1}(h)$, then*

(i) i: $H^1(h^-, GL(m, \mathscr{A}^-)_I|_{h^-})\to H^1(h^-, GL(m, \mathscr{A}'^-)_I|_{h^-})$

is a trivial mapping.

(ii) $H^1(h^-, GL(m, \mathscr{A}^-)_I|_{h^-}) \approx GL(m, \mathcal{O}_{\hat{M}|H})_h/GL(m, \mathcal{O})_h$,

(iii) j: $H^1(h^-, GL(m, \mathscr{A}'^-)_I|_{h^-})\to H^1(h^-, pr^{-1}GL(m, \mathcal{O}_{\hat{M}|H})|_{h^-})$

is injective.

By the long exact sequence deduced from (2.17), (*i*), (*ii*) and (*iii*) are equivalent.

Let $\mathcal{S}$ be a locally free sheaf of $\mathcal{O}$-modules of rank m. Put

$$\mathcal{S}_{\hat{M}|H}=\mathcal{S}\otimes_{\mathcal{O}}\mathcal{O}_{\hat{M}|H},\ \mathcal{S}(*H)=\mathcal{S}\otimes_{\mathcal{O}}\mathcal{O}(*H),\ \mathcal{S}_{\hat{M}|H}(*H)=\mathcal{S}\otimes_{\mathcal{O}}\mathcal{O}_{\hat{M}|H}(*H), \tag{2.18}$$

$$\mathcal{S}_0^-=pr^{-1}\mathcal{S}\otimes_{\mathcal{O}'}\mathscr{A}_0^-,\ \mathcal{S}'^-=pr^{-1}\mathcal{S}\otimes_{\mathcal{O}'}\mathscr{A}'^-,\ \mathcal{S}'^-(*H)=pr^{-1}\mathcal{S}(*H)\otimes_{\mathcal{O}'}\mathscr{A}'^- \tag{2.19}$$

THEOREM 2.8. *The statements obtained by replacing $\mathscr{A}$ and $\mathcal{O}$ with $\mathcal{S}$ from Theorems 2.1–6 and Corollaries 2.1–2 are valid.*

In the theory of asymptotic expansions the origin of this kind of vanishing theorem is 'Preliminary Theorem' in Birkhoff[2] (p. 533). Sibuya[25] obtained (*i*) and (*iii*) of Theorem 2.7 and Malgrange[18] obtained (*ii*) of Theorem 2.7 and Theorem 2.1 in the one-variable case.

III. *Existence of Asymptotically Developable Solutions to Systems of Differential Equations*

Let S be an n-dimensional polysector with the edge in V in $\boldsymbol{C}^n$ with coordinates $x_1, \ldots, x_n$ and let U be a polydisc at the origin in $\boldsymbol{C}^m$ with a coordinate system $u={}^t(u_1, \ldots, u_m)$.

Then, $S\times U$ is an $(n\times m)$-dimensional polysector with the edge in $V\times \boldsymbol{C}^m$. Consider a system of partial differential equations of the first order

$$x^{p_i}e_i(\partial/\partial x_i)u=a_i(x,u),\quad i=1,\ldots,n', \tag{3.1}$$

where

$$a_i(x,u)={}^t(a_{1i}(x,u),\ldots,a_{mi}(x,u)),\quad i=1,\ldots,n'\ (\leqq n),$$
$$p_i=(p_{i1},\ldots,p_{ik},0,\ldots,0),\ e_i=x_i,\ i=1,\ldots,k,\ e_i=1,\ i=k+1,\ldots,n'.$$

Suppose that this system satisfies the integrability condition

$$\begin{aligned}&(\partial/\partial x_i)(x^{-p_j}a_j(x,u))+x^{-p_i-p_j}(\partial/\partial u)a_j(x,u)a_i(x,u)\\&=(\partial/\partial x_j)(x^{-p_i}a_i(x,u))+x^{-p_i-p_j}(\partial/\partial u)a_i(x,u)a_j(x,u),\quad i,j=1,\ldots,n',\end{aligned} \tag{3.2}$$

where

$$(\partial/\partial u)a_j(x,u)=((\partial/\partial u_1)a_j(x,u),\ldots,(\partial/\partial x_m)a_j(x,u)) \tag{3.3}$$

By the holomorphy of $a_i(x,u)$ with respect to u, these functions have the Taylor's expansions

$$a_i(x,u)=a_{i0}(x)+A_i(x)u+\sum_{\substack{q\in N^m\\ |q|>1}}a_{iq}(x)u^q,\quad i=1,\ldots,n', \tag{3.4}$$

where $a_{iq}(x)(q\in N^m,\ |q|\neq 1)$, $A_i(x)$ are asymptotically developable in the polysector S with the edge in V. If $u(x)$ is a holomorphic solution to (3.1) in a subsector of S, then, for a nonempty subset J of $[1,k]$,

$$x_J^{p_{iJ}}x_I^{p_{iI}}e_i(\partial/\partial x_i)FA_J(u)=FA_J(a_{i0})+FA_J(A_i)FA_J(u)+\sum_{\substack{q\in N^m\\ |q|>1}}FA_J(a_{iq})(FA_J(u))^q,\quad i=1,\ldots,n', \tag{3.5}$$

where $I=[1,n]-J$, $p_{iI}=(p_{ik})_{k\in I}$, $p_{iJ}=(p_{ij})_{j\in J}$. Therefore, for $q_J\in N^J$, $TA(u)_{q_J}$ satisfies the equation obtained by formally operating $(\partial/\partial x_J)^{q_J}$ and substituting $x_J=0$, i.e., put

$$LHS_i(q_J)=\begin{cases}q_iw & (p_i=0,\ i\in J)\\ e_i(\partial/\partial x_i)w & (p_i=0,\ i\notin J)\\ 0 & (p_i\neq 0,\ p_{iJ}\neq 0)\\ x_I^{p_{iI}}(x_i\partial/\partial x_i)w & (p_i\neq 0,\ p_{iJ}=0)\end{cases}, \tag{3.6}$$

then, $w=TA(u)_{0_J}$ satisfies the non-linear differential equation

$$LHS_i(0_J)=TA(a_{i0})_{0_J}+TA(A_i)_{0_J}w+\sum_{\substack{q\in N^m\\ |q|>1}}TA(a_{iq})_{0_J}w^J,\quad i=1,\ldots,n', \tag{3.7}$$

and $w=TA(u)_{q_J}(q_J\neq 0_J)$ satisfies the non-homogeneous linear differential equations

$$LHS_i(q_J)=((\partial/\partial u)a_i(x,u)|_{u=u(0_J),x_J=0})w+\textit{terms determined by } TA(u)_{s_J}(|s_J|<|q_J|) \tag{3.8}$$

DEFINITION 3.1. A consistent family

$$F=\{f(x_I;q_J);\ \phi\neq J\subset[1,k],\ q_J\in N^J,\ I=[1,n]-J\}$$

is said to be a family of coefficients of a formal solution to (3.1) if for any non-empty subset J of $[1,k]$, $w=f(x_I;0_J)$ satisfies (3.7) and $w=f(x_I;q_J)$ satisfies (3.8) for $q_J\neq 0_J$.

DEFINITION 3.2. A formal power series $\hat{u}\in(\hat{\mathcal{O}}_V)_0^m$ is a formal power-series solution to (3.1) if $\hat{u}$ satisfies (3.5) instead of $FA_{[1,k]}(u)$.

In the following, suppose that $\lim_{x\to 0}a_{i0}(x)=0$ $(i=1,\ldots,n')$ and put $\lim_{x\to 0}A_i(x)=A_i(i=1,\ldots,n')$.

THEOREM 3.1. *Suppose that $p_i=0$ for an i in $[1,k]\cap[1,n']$. (1) If F is a family of total coefficients of solution to (3.1) in a subsector S' of S, then there exists a unique solution u to (3.1) in S' whose family of total coefficients is equal to F. (2) If $\hat{u}$ is a formal power-series solution to (3.1), then there exists a unique solution to (3.1) in any subsector S' with a sufficiently small radius in S which is asymptotic to $\hat{u}$. (3) If a_i's are holomorphic in a polydisc, then a formal power-series solution to (3.1) is convergent.*

THEOREM 3.2. *Suppose that $p_i=0$ and A_i has no eigenvalues of integers for some i in $[1,k]\cap[1,n']$. Then, there exists a unique family F of total coefficienst of formal solution to (3.1) in any subsector S' of S with $f(0;0_{[1,k]})=0$. If a_i's are asymptotic to formal power series, then there exists a unique formal power series solution $\hat{u}$ to (3.1) with $\hat{u}(0)=0$. For F and $\hat{u}$, Theorem 3.1 is valid.*

If $p_i\neq 0$, then the domain of existence of asymptotically developable solutions to (3.1) could be smaller than the whole sector S. For example, consider a ordinary differential equation of the first order

$$t^{p+1}(d/dt)u=-p\mu u+a(t,u). \tag{3.9}$$

Solutions to this equation satisfy the integral equation

$$u(t)=\exp(\mu t^{-p})\exp(-\mu t_0^{-p})u(t_0)+\exp(\mu t^{-p})\int_{t_0}^{t}\exp(-\mu s^{-p})a(s,u(s))s^{-p-1}ds. \tag{3.10}$$

So, $\exp(\mu t^{-p})$ dominates the domain of existence of asymptotically developable solutions to (3.9).

DEFINITION 3.3. An n-dimensional sector S with the edge in V is said to be a proper domain of the function

$$|\exp(\mu x^{-p})|=\exp \mathrm{Re}(\mu x^{-p})=\exp(|\mu x^{-p}|\cos(\arg\mu-\sum_{i=1}^{k}p_i\arg x_i)) \tag{3.11}$$

if $\{x\in S;\ \mathrm{Re}(\mu x^{-p})>0\}$ has at most one connected component in S, i.e., for some integer k and for all $x\in S$

$$-\frac{3}{2}\pi<\arg\mu-\sum_{i=1}^{k}p_i\arg x_i+2k\pi<\frac{3}{2}\pi. \tag{3.12}$$

Then, $D-V$ is covered by a finite number of proper sectors.

THEOREM OF EXISTENCE OF ASYMPTOTIC SOLUTIONS. *Suppose that a_i's are holomorphic in an $(n+m)$-dimentional polydisc, and that $p_{ii}>0$ and all eigenvalues d_{ij} of A_i $(i=1,\ldots,k\leqq n', j=1,\ldots,m)$ are not zero. Then, for any direction l tends to 0 in D and for an adequate proper domain S_l containing l with respect to $|\exp(-p_{ii}^{-1}d_{ij}x^{-p_i})|$ $(i=1,\ldots k, j=1,\ldots,m)$, there exists a solution u to (3.1) which is holomorphic and asymptotically developable to $\hat{u}$ in S_l.*

This theorem is proved by using the method of Hukuhara[8], cf. Majima[15].

IV. *Structure of Local Solution Matrices to Integrable Pfaffian Systems and Riemann-Hilbert-Birkhoff Problem of Local Version*

Consider a completely integrable system of partial differential equations in D at the origin in $\boldsymbol{C}^n$,

$$x^{p_i}e_i(\partial/\partial x_i)v=C_i(x)v, \quad i=1, \ldots, n, \tag{4.1}$$

where, $p_i=(p_{i1}, \ldots, p_{ik}, 0, \ldots, 0)$, $e_i=x_i$, $i=1, \ldots, k$, $e_i=1$, $i=k+1, \ldots, n'$, and $C_i(x)$ $(i=1, \ldots, n)$ are m-by-m matrices of holomorphic functions in D such that

$$\begin{aligned} &e_i(\partial/\partial x_i)(x^{-p_j}C_j(x))+x^{-p_j-p_i}C_j(x)C_i(x) \\ &=e_j(\partial/\partial x_j)(x^{-p_i}C_i(x))+x^{-p_i-p_j}C_i(x)C_j(x), \quad j, i=1, \ldots, n \end{aligned} \tag{4.2}$$

Put

$$\Omega=\sum_{i=1}^{n} x^{-p_i}e_i^{-1}C_i(x)dx_i, \tag{4.3}$$

then Ω is an m-by-m matrix of meromorphic 1-forms which is holomorphic in $D-V$ and have poles on V, the system (4.1) is re-written as the linear Pfaffian system $(d-\Omega)u=0$ and the condition (4.2) means $d\Omega=\Omega\wedge\Omega$, called the integrability condition of $(d-\Omega)u=0$, where d is the exterior derivative and $\wedge$ is the exterior product. The system (4.1) or $(d-\Omega)u=0$ have no singular points in $D-V$ and so for each point in $D-V$ there exist m linearly independent holomorphic solutions to (4.1). Therefore, by analytic continuation, there exists a fundamental matrix of solutions of the type $\Phi(x)=P(x)x_1^{M_1}\ldots x_k^{M_k}$ to (4.1) in D, where $P(x)$ is an invertible m-by-m matrix of functions holomorphic in $D-V$ and eventually essentially singular on V, and M_i $(i=1, \ldots, k)$ are m-by-m constant matrices such that $M_iM_j=M_jM_i$. In general, it is difficult to calculate M_j's and $P(x)$ and to estimate the singularity of $P(x)$. If $p_i=0$ for some i, then (4.1) is reduced as follows:

PROPOSITION 4.1. *If $p_i=0$ for some $i\in[k+1, n]$, then there exists a holomorphic transformation $v=Q(x)w$ in D such that (4.1) is transformed to*

$$\begin{cases} x^{p_j}e_j(\partial/\partial x_j)w=B_j(x)w, & j\neq i, \\ (\partial/\partial x_i)w=0, \end{cases}$$

where $B_j(x)(j\neq i)$ are independent of x_i.

PROPOSITION 4.2. *Suppose that $p_i=0$ and that the eigenvalues of $C_i(0)$ are $\mu_i+\lambda_{lj}$ $l=1, \ldots, s, j=1, \ldots, s_l$ whose multiplicities are m_{lj}, where λ_{lj}'s are positive integers and $\mu_l-\mu_{l'}$'s are not integer if $l\neq l'$. Then, there exist an invertible matrix $Q(x)$ of holomorphic functions and a diagonal matrix $D(x_i)$ of momonials of x_i such that (4.1) is transformed to*

$$\begin{cases} (x_i\partial/\partial x_i)w=(\oplus_{l=1}^{s}(\mu_lI_{ml}+N_l))w \\ x^{q_j}(e_j\partial/\partial x_j)w=(\oplus_{l=1}^{s}B_{j,l}(x))w, \quad j\neq i, \end{cases} \tag{4.4}$$

by the transformation $v=Q(x)D(x_i)w$, where I_{m_l} is the unit matrix of the order $m_l=m_{l1}+\ldots+m_{ls_l}$, N_l's are strictly upper triangular matrices of the order m_l, $q_j=(q_{j1}, \ldots, q_{j,i-1}, 0, q_{j,i+1}, 0, \ldots, 0)$, $B_{jl}(x)$'s are m_l-by-m_l matrices of holomorphic functions, respectively.

COROLLARY 4.1. *If $p_1=\ldots=p_k=0$, then there exists a fundamental matrix of solutions*

of the type $\Phi(x)=P(x)x_1^{M_1}\ldots x_k^{M_k}$ *to (4.1) in D, where* $P(x)$ *is holomorphic in D and* $P^{-1}(x)$ *has at most poles on V.*

If $p_i\neq 0$ for some $i=1,\ldots,k$, in general, (4.1) is not reduced to a simpler system by a holomorphic transformation in D. However, there is a case that it is transformed to a simpler system by an *asymptotic* transformation. By Proposition 4.2 and the condition (4.2), we can suppose that $C_i(0)C_j(0)=C_j(0)C_i(0)$ $(i,j=1,\ldots,n)$. Suppose that $C_i(0)$ has at least two different eigenvalues $\mu_{i,l}$ with the multiplicity $m_{i,l}$ $(l=1,\ldots,s)$. Then, there exists an invertible constant matrix T,

$$T^{-1}C_j(0)T=\oplus_{l=1}^{s}(\mu_{j,l}I_{ml}+N_{j,l}),\quad j=1,\ldots,n, \tag{4.5}$$

where $\mu_{j,l}$'s are eigenvalues of $C_j(0)$ and $N_{j,l}$ are strictly upper triangular matrix of the order m_l. For simplicity, suppose that $C_i(0)$ is of the form of the right hand side in (4.5). Pose the question whether (4.1) is reduced to a decomposed form as (4.5). If (4.1) is transformed to

$$x^{p_j}(e_j\partial/\partial x_j)w=(\oplus_{l=1}^{s}B_{j,l}(x))w,\ j=1,\ldots,n, \tag{4.6}$$

by a transformation $v=(I_m-(P_{ll'}(x)))w$, then

$$-x^{p_j}(e_j\partial/\partial x_j)(P_{ll'})=C_j(I_m-(P_{ll'}))-(I_m-(P_{ll'}))\oplus_{l=1}^{s}B_{j,l},\quad j=1,\ldots,n. \tag{4.7}$$

Put $P_{ll}=0(l=1,\ldots,s)$, then

$$B_{j,l}=C_{j,ll}-\sum_{h\neq l}C_{j,lh}P_{hl},\quad j=1,\ldots,n,\quad l=1,\ldots,s, \tag{4.8}$$

$$x^{p_j}(e_j\partial/\partial x_j)P_{ll'}=\sum_{h\neq l'}C_{j,lh}P_{hl'}-P_{ll'}B_{j,l'}-C_{j,ll'},\quad j=1,\ldots,n,\quad l,l'=1,\ldots,s,\quad l\neq l'. \tag{4.9}$$

Therefore, $P_{ll'}$'s satisfy the system of partial differential equations

$$x^{p_j}(e_j\partial/\partial x_j)P_{ll'}=\sum_{h\neq l'}C_{j,lh}P_{hl'}-P_{ll'}C_{j,l'l'}+\sum_{h\neq l'}P_{ll'}C_{j,l'h}P_{hl'}-C_{j,ll'},$$
$$j=1,\ldots,s,\quad l,l'=1,\ldots,s,\quad l\neq l'. \tag{4.10}$$

Denote by u the column vector obtained by arranging $P_{ll'}$'s in an adequate order, u satisfies a completely integrable system of partial differential equations of the form

$$x_j^{p_j}(e_j\partial/\partial x_j)u=a_{j0}(x)+A_j(x)u+Q_j(x,u),\quad j=1,\ldots,n, \tag{4.11}$$

where the eigenvalues of $A_j(0)$ are $\mu_{j,l}-\mu_{j,l'}$, $l,l'=1,\ldots,s$, $l\neq l'$ whose multiplicities are $m_lm_{l'}$, respectively. Therefore, in a proper domain S' with respect to

$$|\exp(-p_{ii}^{-1}(\mu_{il}-\mu_{il'})x^{-p_i})|(i=1,\ldots,k,\quad l,l'=1,\ldots,s,\quad l\neq l') \tag{4.12}$$

(4.1) is reduced to a decomposed form by an asymptotically developable transformation.

Suppose that (4.1) satisfies the following condition (4.13): *for all* $i=1,\ldots,k$, $p_{ii}>0$ *and* $C_i(0)$ *has m different eigenvalues or* $p_i=0$, $C_i(0)$ *has m different eigenvalues and the difference of any two eigenvalues is not integer.*

Then, (4.1) has a fundamental matrix of formal solutions of the form

$$Q(x)\exp(x^{-q}\Lambda(x))x_1^{T_1}\ldots x_k^{T_k}. \tag{4.14}$$

Therefore, by Theorem of existence of asymptotic solutions, there exists an open covering $\{S_\tau;\tau=1,\ldots,\sigma\}$ of $D-V$ such that S_τ's are open sectors with edge in V and in it

$$Q_\tau(x)\exp(x^{-q}\Lambda(x))x_1^{T_1}\cdots x_k^{T_k}=\Phi_\tau(x) \tag{4.15}$$

forms a fundamental matrix of solutions to (4.1) for $\tau=1,\ldots,\sigma$. If $S_\tau\cap S_{\tau'}\neq\phi$, there exists an invertible constant matrix $C_{\tau\tau'}$, called a connection matrix, such that $\Phi_{\tau'}(x)=\Phi_\tau(x)C_{\tau\tau'}$. Then,

$$Q_\tau^{-1}(x)Q_{\tau'}(x)=\exp(x^{-q}\Lambda(x))x_1^{T_1}\cdots x_k^{T_k}C_{\tau\tau'}(\exp(x^{-q}\Lambda(x))x_1^{T_1}\cdots x_k^{T_k})^{-1} \tag{4.16}$$

is asymptotically developatble to I_m in $S_\tau\cap S_{\tau'}$, and the 1-cocycle condition

$$C_{\tau\tau'}C_{\tau'\tau''}C_{\tau''\tau}=I_m,\ S_\tau\cap S_{\tau'}\cap S_{\tau''}\neq\phi \tag{4.17}$$

is satisfied. Conversely, let be given $\exp(x^{-q}\Lambda(x))x_1^{T_1}\cdots x_k^{T_k}$, an open covering $\{S_\tau;\tau=1,\ldots,\sigma\}$ of $D-V$ and a system $\{C_{\tau\tau'};\tau,\tau'=1,\ldots,\sigma\}$ of connection matrices such that (4.17) is satisfied and

$$F_{\tau\tau'}(x)=\exp(x^{-q}\Lambda(x))x_1^{T_1}\cdots x_k^{T_k}C_{\tau\tau'}(\exp(x^{-q}\Lambda(x))x_1^{T_1}\cdots x_k^{T_k})^{-1} \tag{4.18}$$

is asymptotically developable to I_m in $S_\tau\cap S_{\tau'}$. Then, does there exist an integrable linear Paffian system $(d-\Omega)v=0$ or (4.1) such that $\exp(x^{-q}\Lambda(x))x_1^{T_1}\cdots x_k^{T_k}$ forms the essentially singular part of the fundamental matrices of solutions to the system and that $\{C_{\tau\tau'};\tau,\tau'=1,\ldots,\sigma,\}$ forms the family of connection matrices for the covering? This is called the Riemann-Hilbert-Birkhoff problem of local version. By using Theorem 2.7, we can solve this problem.

V. *Riemann-Hilbert-Birkhoff Problem of Global Version and Meromorphic Integrable Connections*

Let M and H be as in II. Consider a linear integrable Pfaffian system

$$(d-\Omega)v=0,\quad d\Omega=\Omega\wedge\Omega \tag{5.1}$$

where Ω is an m-by-m matrix of meromorphic 1-forms which are holomorphic in $M-H$ and has poles on H. For any point h on H, there exist an adequate neighborhood U_h and local coordinates such that Ω is represented as (4.3). If Ω satisfies the condition (4.13), then, for each point h on H, there exist a matrice E_h of essentially singular functions represented as $\exp(x^{-q}\Lambda(x))x_1^{T_1}\cdots x_k^{T_k}$ by an adequate local coordinates and a finite open covering $\{S_{h\tau_h};\tau_h=1,\ldots,\sigma_h\}$ of U_h-H such that $Q_{h\tau_h}E_h$ forms a fundamental matrix of solutions to (5.1) in $S_{h\tau_h}$ for some matrice $Q_{h\tau_h}$ of asymptotically developable functions. For a point in $M-H$, there exists a fundamental matrix Q_h of solutions to (5.1) which is holomorphic in a sufficiently small neighborhood U_h. Then, put $E_h=I_m$, $S_{hl}=U_h$ and $\tau=\sigma=1$. If $S_{h\tau_h}\cap S_{h'\tau_{h'}}\neq\phi$ there exists an invertible matrix $C_{h\tau_hh'\tau_{h'}}$ satisfying $Q_{h'\tau_{h'}}E_{h'}=Q_{h\tau_h}E_hC_{h\tau_hh'\tau_{h'}}$.

$$\{C_{h\tau_hh\tau_{h'}};h,h'\in M,\tau_h=1,\ldots,\sigma_h,\tau_{h'}=1,\ldots,\sigma_{h'}\}$$

satisfies the 1-cocycle condition. And, if $h,h'\in H$ and $S_{h\tau_h}\cap S_{h'\tau_{h'}}\neq\phi$, then

$$Q_{h\tau_h}^{-1}Q_{h'\tau_{h'}}=E_hC_{h\tau_hh'\tau_{h'}}E_h^{-1} \tag{5.2}$$

is asymptotically developable to a formal power series, in particular, to I_m for $h=h'$. Conversely, let be given E_h for each h in M, a covering $\{S_{h\tau_h};\tau_h=1,\ldots,\sigma_h\}$ of $M-H$ and

$$\{C_{h\tau_hh'\tau_{h'}};h,h'\in M,\tau_h=1,\ldots,\sigma_h,\tau_{h'}=1,\ldots,\sigma_{h'}\}$$

satisfying the 1-cocycle condition. Suppose that $E_h C_{h\tau_h h'\tau_{h'}} E_{h'}^{-1}$'s satisfy the same asymptotic property as (5.2). Then, does there exist an integrable Pfaffian system (5.1) over M such that, for each point h on H, E_h forms the essential singular part of the fundamental matrix of formal solutions, and that $\{C_{h\tau_h h'\tau_{h'}}\}$ forms the family of connection matrices for the covering? By the result in IV, for each h in M, there exists a local linear integrable Pfaffian system $(d-\Omega_h)v=0$ in a neighborhood U_h of h for the local problem. Moreover, by the construction and the 1-cocycle condition, if $U_h \cap U_{h'} \neq \phi$ for h, h' in M, there exists an invertible matrix of holomorphic functions satisfying

$$dG_{hh'}=\Omega_h G_{hh'}-G_{hh'}\Omega_{h'}, \text{ i.e. } \Omega_{h'}=G_{hh'}^{-1}(\Omega_h G_{hh'}-dG_{hh'}). \tag{5.3}$$

The family $\{G_{hh'}; h, h' \in M\}$ is a 1-cocycle of the covering $\{U_h; h \in M\}$ with coefficients in $GL(m, \mathcal{O})$. If there exists a 0-cochain $\{R_h; h \in M\}$ of the covering $\{U_h; h \in M\}$ with coefficients in $GL(m, \mathcal{O}(*H))$ such that, $G_{hh'}=R_h^{-1}R_{h'}, h, h' \in M$, then the problem can be solved. However, in general, it is impossible. Though it is possible to decompose $G_{hh'}$ as above in M a Stein or projective manifold, R_h's may have poles outside H.

The family $\{(d-\Omega_h)v=0; h \in M\}$ of local linear integrable Pfaffian systems with the family $\{G_{hh'}; h, h' \in M\}$ of transformation matrices is a representation of a meromorphic integrable connection.

Denote by Ω^1 and by $\Omega^1(*H)$ the sheaf of germs of holomorphic 1-forms in M and the sheaf of germs of meromorphic 1-forms which are holomorphic in $M-H$ and have poles on H, respectively. Let $\mathcal{S}$ be a locally free sheaf of $\mathcal{O}$-modules of rank m and put $\mathcal{S}(*H)=\mathcal{S} \otimes_{\mathcal{O}} \mathcal{O}(*H)$. A meromorphic connection ∇ on $\mathcal{S}(*H)$ is by definition a $\boldsymbol{C}$-linear mapping

$$\nabla : \mathcal{S}(*H) \to \mathcal{S} \otimes_{\mathcal{O}} \Omega^1(*H) \tag{5.4}$$

satisfying the Leibniz rule, i.e., take an open set U in M, then, for f in $\mathcal{O}(*H)(U)$ and for e in $\mathcal{O}(*H)(U)$.

$$\nabla(fe)=f\nabla e+e \otimes_{\mathcal{O}} df. \tag{5.5}$$

For a free basis $e_U=\langle e_{U1}, \ldots, e_{Um} \rangle$ of $\mathcal{S}$ on U, there exist $C_{ij} \in H^0(U, \Omega^1(*H))$ such that

$$\nabla e_{Uj}=\sum_{i=1}^{m} C_{ij} e_{Ui}, \quad j=1, \ldots, m. \tag{5.6}$$

The matrix $\Omega_{e_U}=(C_{ij})$ is called a meromorphic connection matrix of ∇ with respect to the free basis e_U. For $u={}^t(u_1, \ldots, u_m) \in (H^0(U, \mathcal{O}(*H)))^m$ and for $e_U u=u_1 e_{U_1}+ \ldots +u_m e_{U_m}$,

$$\nabla(e_U u)=e_U(d+\Omega_{e_U})u, \tag{5.7}$$

and so $\nabla(e_U u)=0$ means $(d+\Omega_{e_U})=0$. Let $f_U=\langle f_{U1}, \ldots, f_{Um} \rangle$ be another free basis of $\mathcal{S}$ on U, and let Ω_{e_U} be the meromorphic connection matrix of ∇ with respect to f_U. Denote by P_{fe} the transformation matrix from e_U to f_U, then

$$\Omega_{e_U}=P_{fe}^{-1}(\Omega_{f_U}P_{fe}-dP_{fe}), \tag{5.8}$$

i.e. $(d+\Omega_{e_U})w=0$ is transformed to $(d+\Omega_{f_U})v=0$ by the transformation $v=P_{fe}w$. The meromorphic connection ∇ is integrable if, for any open set U in M, the corresponding Pfaffian systems are integrable.

Therefore, the family $\{(d-\Omega_h)v=0; h \in M\}$ of local linear integrable Pfaffian systems

constructed as above defines a meromorphic integrable connection. The Riemann-Hilbert-Birkhoff problem can be always solved in the sense of existence of meromorphic integrable connections satisfying the condition.

VI. *Cohomologies of de Rham Complexes deduced from Meromorphic Integrable Connections*

Let $\Omega^q(q=0, \ldots, n)$ be the sheaf of germs of holomorphic q-forms on M. Denote by $\Omega^q(*H)$ the sheaf of germs of meromorphic q-forms which are holomorphic in $M-H$ and have poles on H. We write $\mathcal{O}(*H)$ for $\Omega^0(*H)$. Let $\mathcal{S}$ be a locally free sheaf of $\mathcal{O}$-modules of rank m and put $\mathcal{S}\Omega^q(*H)=\mathcal{S}\otimes_{\mathcal{O}}\Omega^q(*H)$. Consider a meromorphic integrable connection ∇ on $\mathcal{S}\Omega^0(*H)=\mathcal{S}(*H)$, from which we can define naturally

$$\nabla_q\colon \mathcal{S}\Omega^q(*H)\to\mathcal{S}\Omega^{q+1}(*H), \quad q=1, \ldots, n. \tag{6.1}$$

For an open set U of M and a free basis e_U of $\mathcal{S}$ on U, take the meromorphic connection matrix Ω_{e_U} of ∇, then

$$\nabla_q(e_U\omega)=e_U(d\omega+\Omega_{e_U}\wedge\omega), \tag{6.2}$$

where $\omega={}^t(\omega_1, \ldots, \omega_m)\in(H^0(U,\Omega^q(*H)))^m$ and $e_U\omega=\omega_1e_U{}^q+\ldots+\omega_me_{Um}$. By the integrability condition $\nabla_q\circ\nabla_{q-1}=0(q=1, \ldots, n, \nabla_0=\nabla)$, therefore

$$\mathcal{S}(*H)\to\mathcal{S}\Omega^1(*H)\to\ldots\to\mathcal{S}\Omega^n(*H)\to 0 \tag{6.3}$$

becomes a complex of sheaves, denoted by $(\mathcal{S}\Omega^\cdot(*H), \nabla)$ and called the de Rham complex deduced from ∇. Put

$$\mathcal{S}_{\widehat{M|H}}\Omega^q(*H)=\mathcal{S}\Omega^q(*H)\otimes_{\mathcal{O}}\mathcal{O}_{\widehat{M|H}}, \tag{6.4}$$

$$\mathcal{S}^-(*H)=pr^{-1}\mathcal{S}\Omega^q(*H)\otimes_{\mathcal{O}'}\mathcal{A}^-, \tag{6.5}$$

$$\mathcal{S}'^-\Omega^q(*H)=pr^{-1}\mathcal{S}\Omega^q(*H)\otimes_{\mathcal{O}'}\mathcal{A}'^-, \tag{6.6}$$

$$\mathcal{S}_0^-\Omega^q=pr^{-1}\mathcal{S}\Omega^q(*H)\otimes_{\mathcal{O}'}\mathcal{A}_0^-=pr^{-1}(\mathcal{S}\otimes_{\mathcal{O}}\Omega^q)\otimes_{\mathcal{O}'}\mathcal{A}_0^-, \tag{6.7}$$

then we can define naturally the complexes $(\mathcal{S}_{\widehat{M|H}}\Omega^\cdot(*H), \nabla)$, $(\mathcal{S}^-\Omega^\cdot(*H), \nabla)$, $(\mathcal{S}'^-\Omega^\cdot(*H), \nabla)$, and $(\mathcal{S}_0^-\Omega^\cdot, \nabla)$. In the sequel, we suppose that ∇ satisfies the following condition: for any point h on H, an adequate neighborhood U and an adequate free basis e_U of $\mathcal{S}$ on U, the meromorphic connection matrix Ω is represented as

$$\Omega=\sum_{i=1}^{k}x_1{}^{-p_{i1}}\ldots x_k{}^{-p_{ik}}x_i{}^{-1}A_i(x)+\sum_{i=k+1}^{n}x_1{}^{-p_{i1}}\ldots x_k{}^{-p_{ik}}A_i(x) \tag{6.8}$$

with respect to adequate local coordinates $x_1, \ldots, x_n$ and p_{ii} is positive and $A_i(0)$ is invertible, or, $p_{i1}=\ldots=p_{ik}=0$ and $A_i(0)$ has no eigenvalues of integer, for $i=1, \ldots, k$, where $U\cap H=\{x; x_1\ldots x_k=0\}$ and the point h corresponds to $x_1=\ldots=x_n=0$.

Under this condition, we can assert the following:

THEOREM 6.1. *For the complexes* $(\mathcal{S}_{\widehat{M|H}}\Omega^\cdot(*H), \nabla)$, $(\mathcal{S}^-\Omega^\cdot(*H), \nabla)$, $(\mathcal{S}'^-\Omega^\cdot(*H), \nabla)$, $(\mathcal{S}_0^-\Omega^\cdot, \nabla)$, *Poincaré lemma holds. Namely, denote by* $(\mathcal{K}^\cdot, \nabla)$ *one of the complexes, then the q-th cohomology sheaf* $\mathcal{H}^q(\mathcal{K}^\cdot, \nabla)=0$ *for* $q\geqq 1$.

Moreover, by calculating the hypercohomologies of the complexes and by using Theorems 6.1 and 2.8, we have the following isomorphism theorems:

THEOREM 6.2. *For any point h on H,*

$$H^{q-1}((\mathscr{S}_{\widehat{M|H}}\Omega^{\cdot}(*H))_h/\mathscr{S}\Omega^{\cdot}(*H)_h),\ \nabla)\cong H^q(h^-,\ \mathscr{H}^0_0(\mathscr{S}_0{}^-\Omega^{\cdot},\ \nabla)|_{h^-}),\quad q=1,\ \ldots,\ n, \tag{6.9}$$

$$H^q(\mathscr{S}\Omega^{\cdot}(*H)_h),\ \nabla)\cong H^q(h^-,\ \mathscr{H}^0(\mathscr{S}_0\Omega^-,\ \nabla)|_{h^-}),\quad q=0,\ 1,\ \ldots,\ n. \tag{6.10}$$

THEOREM 6.3. *If* $H^q(M,\ \mathscr{S}\otimes_{\mathscr{O}}\Omega^r)=H^q(M,\ \mathscr{S}_{\widehat{M|H}}\otimes_{\mathscr{O}}\Omega^r)=0$ *for* $q\geqq 1$ *and* $r\geqq 0$, *then*

$$H^{q-1}(H^0(M,\ \mathscr{S}_{\widehat{M|H}}\Omega^{\cdot}(*H))/H^0(M,\ \mathscr{S}\Omega^{\cdot}(*H)),\ \nabla)\cong H^q(M^-,\ \mathscr{H}^0(\mathscr{S}_0{}^-\Omega^{\cdot},\ \nabla))\quad (q=1,\ \ldots,\ n), \tag{6.11}$$

$$H^q(H^0(M,\ \mathscr{S}\Omega^{\cdot}(*H)),\ \nabla)\cong H^q(M^-,\ \mathscr{H}^0(\mathscr{S}_0{}^-\Omega^{\cdot},\ \nabla))\quad (q=0,\ \ldots,\ n). \tag{6.12}$$

More precisely, if $H^q(M,\ \mathscr{S}\otimes_{\mathscr{O}}\Omega^r)=0$ $(q+r\leqq j,\ q\geqq 1,\ r\geqq 0)$, $H^q(M,\ \mathscr{S}_{\widehat{M|H}}\otimes_{\mathscr{O}}\Omega^r)=0$ $(q+r\leqq j-1,\ q\geqq 1,\ r\geqq 0)$, then (6.11) and (6.12) are valid for $q\leqq j$.

VII. *Characterization of Regular Singularities for Integrable Connections*

At first, consider the integrable Pfaffian system $(d-\Omega)u=0$, in D with singularities on V as in IV. Let $P(x)x_1^{M_1}\ldots x_k^{M_k}$ be a fundamental matrix of solutions to the system in D. The system is said to be regular singular along V, if $P(x)$ has at most poles on V. Now, consider a meromoprhc integrable connection ∇ on $\mathscr{S}$ with singularities on H over M as in VI. ∇ is said to be regular singular along H, if for any point h on H, there exists a neighborhood U_h of h such that the integrable Pfaffian system of ∇ with respect to a free basis e_{U_h} of $\mathscr{S}$ on U_h is regular singular along $H\cap U_h$ in the above sense. Let H_1 be the set of non-singular points on H.

THEOREM 7.1. *If a meromorphic integrable connection* ∇ *is regular singular along H, then for any point h on H*

$$\mathscr{H}^0(\mathscr{S}_0{}^-\Omega^{\cdot},\ \nabla)|_{h^-}=0, \tag{7,1}$$

$$H^1(h^-,\ \mathscr{H}^0(\mathscr{S}_0{}^-\Omega^{\cdot},\ \nabla)|_{h^-})=0, \tag{7.2}$$

$$H^q(h^-,\ \mathscr{H}^0(\mathscr{S}_0{}^-\Omega^{\cdot},\ \nabla)|_{h^-})=0,\quad q=0,\ 1,\ \ldots,\ n, \tag{7.3}$$

$$H^0(\mathscr{S}_{\widehat{M|H}}\Omega^{\cdot}(*H)_h/\mathscr{S}\Omega^{\cdot}(*H)_h,\ \nabla)=0, \tag{7.4}$$

$$H^q(\mathscr{S}_{\widehat{M|H}}\Omega^{\cdot}(*H)_h/\mathscr{S}\Omega^{\cdot}(*H)_h,\ \nabla)=0,\quad q=0,\ 1,\ \ldots,\ n, \tag{7.5}$$

$$\mathscr{H}^k(\mathscr{S}\Omega^{\cdot}(*H),\ \nabla)_h\cong\mathscr{H}^q(\mathscr{S}_{\widehat{M|H}}\Omega^{\cdot}(*H),\ \nabla)_h,\quad q=0,\ 1,\ \ldots,\ n, \tag{7.6}$$

$$\sum_{q=0}^{n}(-1)^q dim_C\mathscr{H}^q(\mathscr{S}\Omega^{\cdot}(*H),\ \nabla),$$

$$=\sum_{q=0}^{n}(-1)^q dim_C(\mathscr{H}^q(\mathscr{S}_{\widehat{M|H}}\Omega^{\cdot}(*H),\ \nabla)_h,\quad q=0,\ 1,\ \ldots,\ n, \tag{7.7}$$

where $h^-=pr^{-1}(h)$. *Conversely,* ∇ *is regular singular along H, if there exists an open desne set* H' *of* H_1 *such that one of (7.1–7) is satisfied for each point h on* H'.

The equivalence among (7.2–7) is proved by using the following:

THEOREM 7.2. *There exists an open dense set H' of H_1 such that (6.9) is valid at any point h on H'. Moreover, the terms are null for $q \geqq 2$.*

It is deduced from the following theorem that (7.1) or (7.2) means the regular singularities of ∇.

THEOREM 7.3. *There exists an open dense set H' of H_1 such that for any point h on H' we have*

$$\dim_C H^1(h^-, \mathscr{H}^0(\mathscr{S}_0^-\Omega^\cdot, \nabla)|_{h^-}) = \frac{1}{2}\,(\textit{Total Variation of the function } (d_h\colon \theta \in h^- \to \dim_C \mathscr{H}^0(\mathscr{S}_0^-\Omega^\cdot, \nabla))_\theta)). \tag{7.8}$$

REMARK 7.1. Denote by (7.4)′, . . . , (7.7)′ the conditions obtained from (7.4), . . . , (7.7) by replacing $\mathscr{S}\widehat{}_{M|H}\Omega^\cdot(^*H)$ with $i_*i^{-1}(\mathscr{S}\otimes_{\mathscr{O}}\Omega^\cdot)$, where i is the canonical inclusion from $M-H$ to M. Then, (7.4–7)′ also characterize the regular singularities of ∇.

REMARK 7.2. The number given by (7.8) may be called the irregularity at h of ∇.

The notion of regular singularity of ordinary differential equation was introduced essentially by Riemann and definitively by Fuchs. It was characterized by conditions with respect to the order of coefficients by Fuchs, Moser, Lutz, Jurkat, etc. (cf. Kohno-Okubo[12]). By Deligne[3] (Proposition II. 6.20), it was characterized by (7.6–7)′ i.e. the validity of comparison theorem. Malgrange[17] characterized it by (7.4–7), (7.4–7)′. Komatsu[13] obtained the characterization by conditions analogous to (7.4) and (7.6) between hyperfunctions and distribuitions. Gérard-Levelt[6] investigated measures of irregular singularity. In the several-variable case, the definition of regular singularity was due to Gérard[4] for Pfaffian systems, to Deligne[3] for meromorphic connections and to Kawai-Kashiwara[11], Ramis[23], (Mebkhout[20]) and Van den Essen[29] for $\mathscr{D}_M$*-modules.* Nowaday, we know the equivalence (cf.. Ramis[23].) Our conditions (7.1–2), (7.4), (7.2)′ and (7.4)′ are apparently week (Majima[16]).

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