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A REMARK ON THE STANDARD FORM PROBLEM FOR $^2F_4(2^{2n+1})$, $n \geq 1$

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In [10] Miyamoto showed that if a finite group G has a standard subgroup L isomorphic to $^2F_4(2^{2n+1})$, $n \geq 1$, such that $LO(G) \not\trianglelefteq G$ and a Sylow 2-subgroup of $C_G(L)$ is cyclic, then $O^2(G) \neq G$. He obtained much more information. In this paper we proceed to construct a subgroup G_0 isomorphic to $F_4(q)$ or $^2F_4(q) \times {}^2F_4(q)$, $q=2^{2n+1}$. More precisely the following theorem is proved.

THEOREM. *Let G be a finite group with $F^*(G)$ simple and suppose L is a standard subgroup of G isomorphic to $^2F_4(q)$, $q=2^{2n+1} \geq 8$. Assume that a Sylow 2-subgroup of $C_G(L)$ is cyclic and that $L \not\trianglelefteq G$. Then $O^2(G)$ possesses a subgroup G_0 of odd index which is isomorphic to $F_4(q)$ or $^2F_4(q) \times {}^2F_4(q)$.*

By a further fusion argument it might be possible to show that G_0 is normal in G and then $F^*(G) \simeq F_4(q)$. But such an effort is no longer important. The simplicity of $F^*(G)$ in the hypothesis of the theorem is necessary since we use, in the proof of the theorem, the unbalanced group theorem to determine the structure of certain 2-local subgroups of G. The proof of this fundamental theorem has been completed by the work of many authors. The detailed description of the theorem can be found in Harris [8], Solomon [16], Walter [18].

UNBALANCED GROUP THEOREM. *Let X be a finite group with $F^*(X)$ simple. Assume that $O(N_X(T)) \neq 1$ for some 2-subgroup T of X. Then $F^*(X)$ is isomorphic to one of the known finite simple groups.*

Our notation is fairly standard. Possible exceptions are the use of the following: For a subset D of a group, $\mathscr{I}(D)$ denotes the set of involutions in D and for a 2-group P, $\mathscr{E}^*(P)$ denotes the set of maximal elementary abelian subgroups of P.

I. *Preliminary Lemmas*

(1.1) *Let G be a group acting on a group X and let A and B be subgroups of G. If B normalizes $C_X(A)$, then* $[[A, B], C_X(A)]=1$.

PROOF. See [19, (2.4)].

The following lemma is due to Bender [4, 1.1(i'')].

(1.2) *Let P be a p-group acting on a p'-group K. Suppose $X^P=X\lhd K=XY$ with $Y=Y^P$. Then $C_K(P)=C_X(P)_Y(P)$.*

The following lemma can be easily verified.

(1.3) *Suppose $K\simeq SL_2(2^n)$, $n\geq 3$, acts naturally on an elementary abelian group V of order 2^{2n}. Let H be a subgroup of K which corresponds to a diagonal subgroup of $SL_2(2^n)$ and w an involution of $N_K(H)$. Then V has precisely two H-invariant proper subgroups. They are of order 2^n and interchanged by w.*

II. *Properties of* $^2F_4(2^{2n+1})$

For convenience we summarize some properties of $L={}^2F_4(q)$, $q=2^{2n+1}\geq 8$, which can be found in Parrott [12], Ree [13], Shinoda [15], or verified by direct computation.

Let $\alpha_i(t)$, $1\leq i\leq 12$, and θ be as in [13]. Then

$$\alpha_1(t)\alpha_1(u)=\alpha_1(t+u)\alpha_2(tu^{2\theta}),\quad \alpha_4(t)\alpha_4(u)=\alpha_4(t+u)\alpha_8(tu^{2\theta}),$$
$$\alpha_5(t)\alpha_5(u)=\alpha_5(t+u)\alpha_{12}(tu^{2\theta}),\quad \alpha_6(t)\alpha_6(u)=\alpha_6(t+u)\alpha_{11}(tu^{2\theta}),$$
$$\alpha_i(t)\alpha_i(u)=\alpha_i(t+u)\quad \text{for } i=2,3,7,8,9,10,11,12.$$

The nontrivial commutator relations are as follows, where $[x, y]=x^{-1}y^{-1}xy$.

$[\alpha_1(t), \alpha_3(u)]=\alpha_4(tu)\alpha_5(t^{2\theta+1}u^{2\theta})\alpha_7(t^{2\theta+2}u)\alpha_{11}(t^{4\theta+3}u^{2\theta+1})\alpha_{12}(t^{4\theta+3}u^{2\theta+2})$,
$[\alpha_1(t), \alpha_4(u)]=\alpha_5(tu^{2\theta})\alpha_6(t^{2\theta}u)\alpha_7(t^{2\theta+1}u)\alpha_9(tu^{2\theta+1})\alpha_{10}(t^{2\theta+1}u^{2\theta+1})\alpha_{11}(t^{2\theta+2}u^{2\theta+1})\alpha_{12}(t^{2\theta+1}u^{2\theta+2})$,
$[\alpha_1(t), \alpha_6(u)]=\alpha_7(tu)$,
$[\alpha_1(t), \alpha_8(u)]=\alpha_9(tu)\alpha_{11}(t^{2\theta+2}u)\alpha_{12}(t^{2\theta+1}u^{2\theta})$,
$[\alpha_1(t), \alpha_9(u)]=\alpha_{10}(t^{2\theta}u)\alpha_{11}(t^{2\theta+1}u)\alpha_{12}(tu^{2\theta})$,
$[\alpha_1(t), \alpha_{10}(u)]=\alpha_{11}(tu)$,
$[\alpha_2(t), \alpha_3(u)]=\alpha_5(tu^{2\theta})\alpha_6(tu)\alpha_7(t^{2\theta}u)\alpha_8(tu^{2\theta+1})\alpha_9(t^{2\theta}u^{2\theta+1})$,
$[\alpha_2(t), \alpha_4(u)]=\alpha_7(tu)\alpha_{11}(t^{2\theta}u^{2\theta+1})\alpha_{12}(tu^{2\theta+2})$,
$[\alpha_2(t), \alpha_8(u)]=\alpha_{10}(tu)\alpha_{11}(t^{2\theta}u)\alpha_{12}(tu^{2\theta})$,
$[\alpha_2(t), \alpha_9(u)]=\alpha_{11}(tu)$,
$[\alpha_3(t), \alpha_5(u)]=\alpha_8(tu)$,
$[\alpha_3(t), \alpha_6(u)]=\alpha_8(t^{2\theta}u)\alpha_9(tu^{2\theta})\alpha_{12}(tu^{2\theta+1})$,
$[\alpha_3(t), \alpha_7(u)]=\alpha_9(t^{2\theta}u)\alpha_{10}(tu^{2\theta})$,
$[\alpha_3(t), \alpha_{11}(u)]=\alpha_{12}(tu)$,
$[\alpha_4(t), \alpha_5(u)]=\alpha_9(tu)$,
$[\alpha_4(t), \alpha_7(u)]=\alpha_{10}(t^{2\theta}u)\alpha_{11}(tu^{2\theta})\alpha_{12}(t^{2\theta+1}u)$,
$[\alpha_4(t), \alpha_{10}(u)]=\alpha_{12}(tu)$,
$[\alpha_5(t), \alpha_6(u)]=\alpha_{10}(tu)$,
$[\alpha_5(t), \alpha_7(u)]=\alpha_{11}(tu)$,
$[\alpha_6(t), \alpha_9(u)]=\alpha_{12}(tu)$,
$[\alpha_7(t), \alpha_8(u)]=\alpha_{12}(tu)$.

REMARK. The commutator formulas of $[\alpha_4(t),\alpha_1(u)]$ in [12, Section 2] and $[\alpha_2(\xi),\alpha_3(\eta)]$ in [15, (2.3)] are incorrect. In fact

$$[\alpha_4(t),\ \alpha_1(u)]=\alpha_5(t^{2\theta}u)\alpha_6(tu^{2\theta})\alpha_7(tu^{2\theta+1})\alpha_9(t^{2\theta+1}u)\alpha_{11}(t^{2\theta+1}u^{2\theta+2})\alpha_{12}(t^{2\theta+2}u^{2\theta+1}).$$

Using the notation of [13] we set

$$r=\omega_{34'}\omega_{34}=\alpha_1(1)\alpha_2'(1)\alpha_1(1)^{-1},$$
$$s=\omega_{12'3'4'}\omega_{23'}=\alpha_3(1)\alpha_3'(1)\alpha_3(1).$$

Then $|r|=|s|=2$, $|rs|=8$, and

$$\begin{array}{lll}\alpha_3(t)^r=\alpha_7(t), & \alpha_4(t)^r=\alpha_6(t), & \alpha_5(t)^r=\alpha_5(t),\\ \alpha_8(t)^r=\alpha_{11}(t), & \alpha_9(t)^r=\alpha_{10}(t), & \alpha_{12}(t)^r=\alpha_{12}(t);\\ \alpha_1(t)^s=\alpha_4(t), & \alpha_2(t)^s=\alpha_8(t), & \alpha_5(t)^s=\alpha_6(t),\\ \alpha_7(t)^s=\alpha_9(t), & \alpha_{10}(t)^s=\alpha_{10}(t), & \alpha_{11}(t)^s=\alpha_{12}(t).\end{array}$$

Let H_i and V_i be as in [10, Section 2]. By using [13, (3.16)] we can verify that

$$C_L(H_1)=H_1\times N_1 \quad \text{with} \quad N_1=\langle V_1,\ H_5,\ r\rangle\simeq Sz(q),$$
$$C_L(H_3)=H_3\times N_3 \quad \text{with} \quad N_3=\langle V_3,\ H_{10},\ s\rangle\simeq SL_2(q).$$

III. *Some 2-local subgroups of G*

From now on assume the hypothesis of the theorem. We will continue the notation of Miyamoto [10]. Moreover, let $A_0=W_5A_1$ and $B=W_1W_4B_4$. For $i=1, 3, 4, 5, 6, 7, 9, 10$, note that W_i is the unique Sylow 2-subgroup of M_i containing V_i and so H-invariant. Furthermore $\langle t\rangle W_i\in \mathrm{Syl}_2(C(H_i))$ and $C_Q(H_i)=W_i$. For $i=2, 3, 7, 8, 9, 10, 11, 12$, the map $W_i/V_i\to V_i$; $xV_i\mapsto[t, x]$ is an H-isomorphism. Similarly, for $i=1, 4, 5, 6$, $W_i/V_iZ(W_i)$ and $V_i/Z(V_i)$ are H-isomorphic. By [10, (2.5)],

$$W_3{}^r=W_7,\ W_4{}^r=W_6,\ W_5{}^r=W_5,\ W_8{}^r=W_{11},\ W_9{}^r=W_{10},\ W_{12}{}^r=W_{12};$$
$$W_1{}^s=W_4,\ W_2{}^s=W_8,\ W_5{}^s=W_6,\ W_7{}^s=W_9,\ W_{10}{}^s=W_{10},\ W_{11}{}^s=W_{12}.$$

(3.1) (1) *$Z(Q)=W_{12}$, $Z_2(Q)=B_1$, $Z_3(Q)=B_2$, $C_Q(B_1)=B$, $C_Q(B_2)=B_4$, $C_Q(A_1)=A_0$, and $A'=A_1$.*

(2) *$M_3B\lhd N(B)$ and $M_3\simeq SL_2(q)\times SL_2(q)$.*

PROOF. By (3.9) (2) and (5) of [10], $A_1\lhd AB_4$. Hence $A_1\lhd\langle AB_4,N_1\rangle=M_1A$. [10, (3.9) (4)] gives $C_{M_1A}(A_1)=A_0$ and $Z(A)=W_{12}$. By [10, (3.4) (2)], $[M_1,W_{12}]=1$, so $Z(Q)=W_{12}$. We have $Z_2(Q)\geq B_1$ since $A\leq C(A_1/W_{12})$ and $[W_1,W_{11}]=[W_4,W_{12}]^s=1$. Now $t^{B_1}=\mathscr{I}(tB_1)=tS_1$ and $\mathscr{E}^*(\langle t\rangle B_1)=\{\langle t\rangle S_1,\ B_1\}$, so $N(\langle t\rangle B_1)=B_1N_G(B_1)$. Thus $Z_2(P)=S_1$ forces $Z_2(Q)\cap N(\langle t\rangle B_1)=B_1$. Hence $Z_2(Q)=B_1$. Similarly $[W_1,W_{10}]=[W_4,W_{10}]^s=W_{11}$ and $N(\langle t\rangle B_2)=B_2N_G(B_2)$, so $Z_3(Q)=B_2$. In the proof of [10, (3.10)] it is shown that $C(\langle t\rangle B_4/B_4)=B_4N_G(B_4)$. Then since $C_P(S_2)=S_4$ and [10, (3.8) (3)] give $B_4N_G(B_4)\cap C_Q(B_2)=C_{B_4P}(B_2)=B_4$, $C_Q(B_2)=B_4$. As $W_1=W_4{}^s$, $C_Q(B_1)\geq B$. Moreover $C_{W_3}(B_1)\cap C(t)\leq C_{V_3}(S_1)=1$ implies $C_{W_3}(B_1)=1$ and $|Q:C_Q(B_1)|\geq q^2$. So $C_Q(B_1)=B$.

Let $\tilde{Q}=Q/B_4$. Then $\tilde{Q}\rhd\tilde{W}_4=C_{\tilde{B}}(H_4)$, for $W_4B_4=C_A(Z_2(Q))B_4$. As $\tilde{W}_1=[\tilde{W}_1,\ H_4]$, (1.1) shows $\tilde{B}=\tilde{W}_1\times\tilde{W}_4$, which is elementary abelian. Now $C_{\tilde{B}}(t)=\tilde{S}=[\tilde{B},\ t]$. Applying [10, (1.3)] to the series $1\leq B_2 \lhd B_4 \lhd B$, we have $C(\langle t\rangle B/B)=BN_G(B)=BK$. Let $\overline{N(B)}=N(B)/B$. Then $\overline{N(B)}\cap C(t)=\bar{K}$, so $\bar{N}_3$ is a standard subgroup of $\overline{N(B)}$ and $\langle t\rangle\in \mathrm{Syl}_2(\overline{N(B)}\cap C(\bar{N}_3))$. Now $N(B)\geq\langle W_3, s\rangle=M_3$, so $E(\overline{N(B)})=\bar{M}_3$ by [7] and [6, (1J)], whence $M_3B\lhd N(B)$. Suppose $M_3\simeq SL_2(q^2)$ and let I be a complement of W_3 in $N_{M_3}(W_3)$ containing H_{10}. Then I normalizes $W_3B=Q$ and $H\leq H_3\times I$, so I normalizes $E(C(H_1))=M_1$ and W_1. Now $B_4\lhd\langle Q, s\rangle=M_3B$ and $W_3=[W_3, H_4]$ centralizes $\tilde{W}_4=C_{\tilde{B}}(H_4)$ by (1.1), so [19, (2.7)] shows that $C_{\tilde{B}}(W_3)=\tilde{W}_4$ and I acts transitively on the nonidentity elements of $\tilde{B}_4/\tilde{W}_4\simeq W_1/W_2$. If $M_1\simeq Sz(q)\times Sz(q)$, I normalizes each component of M_1, for $|I|$ is odd. If $M_1\simeq Sp_4(q)$, W_1 has exactly two elementary abelian subgroup of order q^3 and so they are normalized by I. In any case I is not transitive on $(W_1/W_2)^{\#}$, a contradiction. Thus (2) holds.

Let $\bar{Q}=Q/A_1$. [10, (3.8) (3)] gives $B_3\lhd Q$. Then $\bar{W}_7\leq Z(\bar{Q})$ by (1.1), for $\bar{B}_3=\bar{W}_7=C_{\bar{Q}}(H_7)$. Similarly, $\bar{Q}\rhd\overline{C_Q(A_1)}=\bar{A}_0=C_{\bar{Q}}(H_5)$ and $\bar{A}_0\leq Z(\bar{Q})$. Hence $\bar{A}$ is abelian by [10, (3.9)(1)] and the action of N_1. Now $[W_6, W_9]=W_{12}$ and $[W_5, W_7]=[W_6, W_9]^s=W_{11}$, so $B_1\leq A'$ and thus $A'=A_1$ by [10, (3.6) (1)].

By the above lemma we can write $M_3=K_3\times K_3{}^t$ with $K_3\simeq SL_2(q)$. Let v be an element of K_3 such that $vv^t=s$. Set $Y_3=W_3\cap K_3$ and $Z=C_{W_{12}}(K_3)$. Note that H normalizes K_3, Y_3, and Z, for $H\leq H_3\times N_3$. Note also that $Q=W_3B$.

(3.2) $M_3B_1=K_3C_{B_1}(K_3{}^t)\times K_3{}^tC_{B_1}(K_3)$, $K_3C_{B_1}(K_3{}^t)\simeq N_3S_1$, *and* $W_{12}=Z\times Z^t$.

PROOF. Since $Y_3Y_3{}^{st}$ is a 2-group, $C_{B_1}(Y_3)\cap C_{B_1}(Y_3)^{st}\neq 1$. Then $W_{12}\cap W_{12}{}^s=W_{12}\cap W_{11}=1$ implies $C_{B_1}(Y_3)\neq W_{12}$. Thus $C_{B_1}(K_3)\neq 1$, for $K_3=\langle Y_3, Y_3{}^s\rangle$. By [10, (3.5)], N_3 is irreducible on $B_1/S_1\simeq S_1$, so $C_{B_1}(K_3)\cap C_{B_1}(K_3{}^t)=1$. As $C_{M_3B_1}(t)=N_3S_1$, the lemma holds.

Let $D_0=C_A(A_1/Z)$, $Y_i=D_0\cap W_i$ for $i=4$, 6, and $Y_7=Y_3{}^{tr}$. These subgroups are H-invariant. Note that $D_0\lhd M_1A$, for $[M_1, W_{12}]=1$.

(3.3) (1) $D_0=Y_3{}^tY_4Y_6Y_7A_0$, $A/A_0=D_0/A_0\times D_0{}^t/A_0$, $D_0\cap W_3=Y_3{}^t$, $D_0\cap W_7=Y_7$, *and* $|Y_i|=q^3$ *with* $W_i=Y_iY_i{}^t$ *and* $Y_i\cap Y_i{}^t=Z(W_i)$ *for* $i=4$, 6.

(2) $B/B_4=C_{B/B_4}(K_3{}^t)\times C_{B/B_4}(K_3)$, $K_3C_{B/B_4}(K_3{}^t)\simeq N_3S/S_4$, *and* K_3 *centralizes* D_0B_4/B_4.

PROOF. By (3.2), $C_{W_3}(B_1/Z)=Y_3{}^t$. Then as $W_{11}{}^r=W_8$ and $W_3{}^r=W_7$, $C_{W_7}(W_8W_{12}/Z)=Y_7$. Now $A_1=W_8(A_1\cap B_3)$ and $B_3=W_7(A_1\cap B_3)$. Thus $Y_7\leq D_0$ by [10, (3.8) (2)]. Since $Z\cap Z^t=1$ and $[A_1, A]\neq 1$, we have $D_0\neq A$ and by [10, (2.4) (2), (3.9) (1)], $|D_0|=q^4|A_0|$. Moreover $Y_7Y_7{}^t=W_7$, so $D_0D_0{}^t=A$. Thus $A/A_0=D_0/A_0\times D_0{}^t/A_0$, $D_0\cap W_3=Y_3{}^t$, and $D_0\cap W_7=Y_7$.

Let $\widetilde{N(B_4)}=N(B_4)/B_4$. As shown in the proof of (3.1), $\tilde{B}=\tilde{W}_1\times\tilde{W}_4$ and $[W_3, \tilde{W}_4]=1$. Then as in (3.2) we get $\tilde{M}_3\tilde{B}=\tilde{K}_3C_{\tilde{B}}(K_3{}^t)\times\tilde{K}_3{}^tC_{\tilde{B}}(K_3)$, $\tilde{K}_3C_{\tilde{B}}(K_3{}^t)\simeq C_{\tilde{M}_3\tilde{B}}(t)=\tilde{N}_3\tilde{S}$, and $\tilde{W}_4=C_{\tilde{W}_4}(K_3{}^t)\times C_{\tilde{W}_4}(K_3)$. Since $C_Q(B_1/Z)=Y_3{}^tB$, $D_0\leq Y_3{}^tB\cap A\leq Y_3{}^tW_4B_4$. By (3.1), $D_0B_4\leq C(B_2/Z)$, so $[V_4, V_{10}]=V_{12}$ and $Z\cap Z^t=1$ imply $W_4\nleq D_0B_4$. Now $Y_3{}^t\leq D_0\lhd Q$ and every H-invariant proper subgroup of $\tilde{W}_4$ is of order q. Thus $\tilde{D}_0=\tilde{Y}_3{}^tC_{\tilde{W}_4}(K_3)$, proving (2). Since $A\cap B_4=W_6W_7A_0$ and $[W_6, W_8]=1$, $D_0\cap B_4\leq W_6W_7A_0\cap C(W_8W_{12}/Z)=W_6Y_7A_0$ and so $D_0\cap B_4=Y_6Y_7A_0$. Thus $|Y_6/W_{11}|=q$. As $W_6{}^r=W_4$, $Y_6{}^r=Y_4$ and (1) holds.

(3.4) $B_3/B_2=C_{B_3/B_2}(K_3{}^t)\times C_{B_3/B_2}(K_3)$ *and* $K_3C_{B_3/B_2}(K_3{}^t)\simeq N_3S_3/S_2$.

PROOF. Note that $B_3\lhd\langle Q, s\rangle=M_3B$ and $W_9B_2=A_1\cap B_3\lhd Q$, so $W_3=[W_3, H_9]$ centralizes W_9B_2/B_2. Now $W_9{}^s=W_7$ and $W_9B_2\cap W_7B_2=B_2$. Hence as in (3.2) the lemma holds.

(3.5) $Q\in \mathrm{Syl}_2(O^2(G))$.

PROOF. In the proof of [10, (3.11)] it is shown that $\mathscr{I}(tQ)=t^Q$, so $t^Q=t^{N(\langle t\rangle Q)}$ and $N(\langle t\rangle Q)=QN_G(\langle t\rangle Q)$. Thus $\langle t\rangle Q$ is a Sylow 2-subgroup of G and (3.5) holds.

IV. *The Case* $M_1\simeq Sp_4(q)$

In this section we assume that $M_1\simeq Sp_4(q)$. Let J_3 be a complement of Y_3 in $N_{K_3}(Y_3)$ such that $H_{10}\leq J_3J_3{}^t$. Then $J_3{}^v=J_3$. As $Q=W_3B$ and $H_3J_3J_3{}^t$ is an abelian group containing H, J_3 normalizes $C_Q(H_i)=W_i$ for $i=1, 3, 4, 5, 6, 7, 9, 10$. Let $D=D_0\cap D_0{}^v$ and $Y_5=D\cap W_5$. Let $Y_i=Z(D)\cap W_i$ for $i=8, 9, 10, 11$.

(4.1) (1) $D=Y_3{}^tY_4Y_5Y_6Y_7A_1\lhd\langle Q, M_1, K_3, H\rangle$, $\mathscr{E}^*(W_5)=\{Y_5, Y_5{}^t\}$, $A/A_1=D/A_1\times D^t/A_1$, *and* $A/D\simeq D^t/A_1\simeq R/R_1$ *as* HN_1*-modules.*

(2) D/Z *is elementary abelian of order* q^{14}.

(3) $Z(D)=Y_5Y_8Y_9Y_{10}Y_{11}W_{12}$, $Z(D)$ *is elementary abelian of order* q^7, $|Y_i|=q$ *for* $i=8$, 9, 10, 11, *and* $A_0/W_{12}=Z(D)/W_{12}\times Z(D)^t/W_{12}$.

(4) $W_8B_3\cap(W_8B_3)^v=B_3$.

PROOF. By the definition Z is $J_3J_3{}^t$-invariant, so is D_0. The element v normalizes $J_3J_3{}^t$ and H_3, whence $H_3J_3J_3{}^t\leq N(D)$. By (3.1), $B_2\lhd\langle Q, s\rangle=M_3B$ and as $D_0\lhd Q$, $Y_3{}^tB_2\leq D\lhd Q\cap Q^v=Y_3{}^tB$. Set $I=A\cap B_4$. Then $Q\rhd I=W_6W_7A_0$ and $Y_3{}^tI\leq DI\leq D_0I=Y_3{}^tY_4I$. If $DI=Y_3{}^tI$, $[Y_3{}^t, W_1]\leq DI\cap B=I$ and so $[W_3, W_1]\leq I$. But $[V_3,V_1]\nleq I$, a contradiction. Thus $DI=D_0I$, for each H-invariant proper subgroup of W_4/W_8 is of order q. Then as $C_I(H_4)=W_8$, (1.2) gives $C_D(H_4)W_8=C_{D_0I}(H_4)=Y_4$. Now $C_Q(H_4)=W_4$ and each H-invariant proper subgroup of W_8 is of order q, so $C_D(H_4)=D\cap W_4$ and $|D\cap W_8|=q$ or $D\geq W_8$. Suppose $|D\cap W_8|=q$. Then $D\cap W_8=D^t\cap W_8$ or $W_8=(D\cap W_8)\times(D^t\cap W_8)$. If the latter holds, then as $(D\cap W_4)W_8=Y_4$ and $Z(W_4)=W_8$, (3.3) forces $W_4=(D\cap W_4)\times(D^t\cap W_4)$. But $W_4{}^s=W_1\in \mathrm{Syl}_2(M_1)$ and $M_1\simeq Sp_4(q)$, a contradiction. If t normalizes $D\cap W_8$, then $W_4/D\cap W_8$ is abelian, contrary to $W_4'=W_8$. Hence $D\cap W_4=Y_4$, so $Y_3{}^tY_4W_7A_0\leq DW_7A_0\leq D_0W_7$. If $D\leq Y_3{}^tY_4W_7A_0$, $[Y_3{}^t,W_2]\leq D\cap B_4\leq W_7A_0$. As $W_7A_0=B_3A_0\lhd Q$, this gives $[W_3,W_2]\leq W_7A_0$. But $[V_3,V_2]\nleq W_7A_0$, a contradiction. Hence $DW_7A_0=D_0W_7$, for H is irreducible on Y_6/W_{11}. As $W_7A_0\cap C(H_6)=W_{11}$ and $D_0W_7\cap C(H_6)=Y_6$, $D\cap W_6=C_D(H_6)=Y_6$. Then $Y_3{}^tY_4Y_6A_0\leq DA_0\leq D_0$. If $Y_3{}^tY_4Y_6A_0=DA_0$, $B\rhd B_4\cap DA_0=Y_6A_0$ and as $W_6=Y_6Y_6{}^t$, $B\rhd W_6A_0$. But then $[W_6, W_1]\leq W_6A_0$, which conflicts with $[V_1,V_6]=V_7$. Hence $DA_0=D_0$ by the action of H. As $C_{A_0}(H_7)=1$, $D\cap W_7=C_{D_0}(H_7)=Y_7$ by (1.2). We have $D\geq[V_1,V_8]B_2=V_9B_2$ since $B\geq V_1$. Then as $D_0\geq W_9B_2$, (3.4) shows $D\geq W_9B_2$. Therefore $D=Y_3{}^tY_4Y_5Y_6Y_7A_1$. Now $[D, A_1A_1{}^v]\leq Z$ by the definition of D, so $[D, A_1A_1{}^vA_1{}^{vt}]\leq W_{12}$. Moreover $A_1A_1{}^vA_1{}^{vt}\geq B_3\geq W_7$ by (3.4), for $A_1\geq W_9B_2$. As $[V_5,V_7]\nleq W_{12}$, we have $Y_5\neq W_5$. If $Y_5=W_{12}$, $[Y_5{}^t,W_2]\leq D\cap B_4=Y_6Y_7A_1\leq W_6W_7A_1$. By (3.1), $Y_3Y_3{}^t=W_3\leq A\rhd W_6W_7A_1$, so $[W_2,W_3]\leq W_6W_7A_1$. But $[V_2,V_3]\nleq W_6W_7A_1$, a contradiction. Hence $|Y_5|=q^3$ by the action of H.

As $A_1\leq D$, $D\lhd AB=Q$ by (3.1). Thus $D\lhd\langle Q, v\rangle=\langle Q, K_3\rangle$. By (3.4), $W_9W_9{}^vB_2=D\cap B_3$. Let $\overline{N(B_3)}=N(B_3)/B_3$. Then $\bar{B}_4$ is elementary abelian of order q^8 by [10, (3.9)(5)] and $\overline{D\cap B_4}=\bar{Y}_5\bar{Y}_6\bar{W}_8$ is of order q^4. As $\mathscr{E}^*(\langle t\rangle B_3)=\{B_3, \langle t\rangle S_3\}$ and $t^{B_3}=tS_3$, $N(\langle t\rangle B_3)=$

$B_3N_G(S_3)=B_3N_G(S)$ and $\overline{N(B_3)}\cap C(t)=\overline{N_G(S)}$. By [10, (3.9) (4)], $[W_3, W_8]=1$, so K_3 centralizes $\bar{W}_8\cap\bar{W}_8{}^v$. Suppose $\bar{W}_8\cap\bar{W}_8{}^v\neq 1$ and put $\bar{X}=\bar{B}_4\cap C(K_3{}^t)$. Then $\bar{X}$ is M_3-invariant and as $H_{10}\leq M_3$, $\bar{X}\cap\bar{X}^t\leq\bar{B}_4\cap C(H_{10})=1$. Now N_3 is irreducible on $\bar{B}_4\cap C(t)=\bar{S}_4\simeq S_4/S_3$, so the action of N_3 on $\bar{X}\bar{X}^t\cap C(t)\simeq\bar{X}$ gives $\bar{M}_3\bar{B}_4=\bar{K}_3\bar{X}\times\bar{K}_3{}^t\bar{X}^t$ and $\bar{K}_3\bar{X}\simeq\bar{M}_3\bar{B}_4\cap C(t)\simeq N_3S_4/S_3$. Then as $Z(V_3S_4/S_3)=V_8S_3/S_3$, $\bar{W}_8=C_{\bar{X}}(Y_3)\times C_{\bar{X}}(Y_3)^t$ and $\langle\bar{W}_8{}^{K_3}\rangle=\bar{X}C_{\bar{X}}(Y_3)^t$. This is impossible since K_3 normalizes $\overline{D\cap B_4}$. Thus $\bar{W}_8\cap\bar{W}_8{}^v=1$ and $\overline{D\cap B_4}=\bar{W}_8\bar{W}_8{}^v$, so $\langle A_1{}^{K_3}\rangle=D\cap B_4$. Then $D\cap B_4/Z$ is an elementary abelian subgroup of $Z(D/Z)$ by [10, (3.8) (2)]. Note that $W_5=W_1{}^{srs}$. By [6, (2*B*)], there are subgroups U_1 and U_2 of order q such that $W_{12}=U_1U_2$ and $W_{12}-(U_1\cup U_2)$ is the set of square involutions of W_5. As $U_1\cap U_2=1$, U_1 and U_2 are the only subgroups of order q contained in $U_1\cup U_2$. Thus $J_3J_3{}^t$ normalizes U_1 and U_2, so $Z=U_1$ or U_2 by (3.2). As $Y_5\leq D\cap B_4$, Y_5/Z is elementary abelian. Hence $Y_5\in\mathscr{E}^*(W_5)$ by [6, (2*B*) (1), (3)]. As $M_5=M_1{}^{srs}$, the action of t on M_1 shows $\mathscr{E}^*(W_5)=\{Y_5, Y_5{}^t\}$. then as $A'=A_1$, we have $A/A_1=D/A_1\times D^t/A_1$. Now $Y_5/Z\leq Z(D/Z)$, $C_{W_5}(Y_5/Z)=Y_5$, and $D_0=DW_5$. Thus $D=C_A(Y_5A_1/Z)$, which is M_1-invariant since M_1 centralizes A_0/A_1. Since $\mathscr{E}^*(\langle t\rangle A_1)=\{A_1,\ \langle t\rangle R_1\}$ and $t^{A_1}=tR_1$, $N(\langle t\rangle A_1)=A_1N_G(R_1)=A_1N_G(R)$. Hence $D/A_1\simeq(A/A_1)\cap C(t)=RA_1/A_1\simeq R/R_1$ as N_1-modules. Then N_1 is irreducible on $D/Y_5A_1\simeq R/V_5R_1$, so D/Z is elementary abelian. Thus (1) and (2) hold.

As $C_Q(A_1)=A_0$, $A_1\nleq Z(D)\leq D\cap A_0=Y_5A_1$. Moreover, $W_{12}W_{12}{}^v$ is a subgroup of $Y_{11}W_{12}$ of order q^3 by (3.2). By [10, (3.6) (1)], N_1 is irreducible on $A_1/R_1W_{12}\simeq R_1W_{12}/W_{12}$, so $|Z(D)\cap A_1/W_{12}|=q^4$, As $W_{11}\nleq Z(D)$, the action of H gives $W_{12}W_{12}{}^v=Y_{11}W_{12}$ and $W_{11}=Y_{11}\times Y_{11}{}^t$. So $W_8=W_{11}{}^r=Y_8\times Y_8{}^t$. By (4), $Z(D)\nleq A_1$ and the action of H gives $Z(D)A_1=Y_5A_1$. Then as $C_{A_1}(H_5)=W_{12}$, $Z(D)\geq Y_5$ by (1.2) and $Z(D)=Y_5(Z(D)\cap A_1)$. Now $A_1/W_{12}=Z(D)\cap A_1/W_{12}\times Z(D)^t\cap A_1/W_{12}$ by the action of N_1. As $C_Q(H_i)=W_i$ for $i=9$, 10, we get $Z(D)\cap A_1=Y_8Y_9Y_{10}Y_{11}W_{12}$, proving (3).

(4.2) *$Z(Q/D)=W_5W_7D/D$ and $Z(BD/D)$ is a subgroup of $W_2W_5W_7D/D$ of order q^3. Moreover $Z(D)/Z(D)\cap B_3$ and $B_4D/Z(BD$* mod *$D)$ are natural modules for K_3.*

PROOF. As $Z(D)\cap A_1=Y_8(Z(D)\cap B_3)$, Y_3 centralizes Y_8B_3/B_3. Hence by (4.1) (4), $Z(D)B_3/B_3\simeq Z(D)/Z(D)\cap B_3$ is a natural module for K_3. Let $\overline{N(D)}=N(D)/D$. Then $\bar{A}_0\bar{B}_3=\bar{W}_5\bar{W}_7\leq Z(\bar{Q})$. Now $\bar{A}$ is self-centralizing in $\bar{M}_1\bar{A}$ and $C_{P/R_1}(V_1)=V_5V_7R_1/R_1$, so (4.1) (1) implies $Z(\bar{Q})=\bar{W}_5\bar{W}_7$. As $Z(Q/A)=W_2A/A$, $Z(\bar{B})\leq\bar{W}_2(\bar{A}\cap\bar{B})=\bar{W}_4\bar{B}_4$. Then (3.3) (2) gives $Z(\bar{B})\leq\bar{W}_4\bar{B}_4\leq\cap(\bar{W}_4\bar{B}_4)^v=\bar{B}_4$. As K_3 does not normalize $A_0D=D_0$, it does not centralize $Z(\bar{Q})$. So $Z(\bar{Q})\neq Z(\bar{B})$. As $V_1\leq B_4$ and $C_{P/R_1}(V_1)=V_5V_7R_1/R_1$, $\bar{A}\cap Z(\bar{B})=Z(\bar{Q})$ by (4.1) (1). Thus $\bar{W}_6Z(\bar{Q})\neq Z(\bar{B})\bar{W}_6Z(\bar{Q})\leq\bar{B}_4=\bar{W}_2\bar{W}_6Z(\bar{Q})$. If $Z(\bar{B})\bar{W}_6Z(\bar{Q})\geq\bar{W}_2$, then as $\bar{W}_6Z(\bar{Q})\cap C(H_1)=1$, $Z(\bar{B})\geq\bar{W}_2$ by (1.2). But $C_{R/R_1}(V_2)=V_5V_6V_7R_1/R_1$, so $C_{\bar{A}}(V_2)=\bar{A}\cap\bar{B}_4$ by (4.1) (1), a contradiction. Hence $|Z(\bar{B})\cap\bar{W}_2|=q$ by the action of H. As $Y_5{}^s\leq W_6$ and $Y_8{}^s\leq W_2$, we see that $\bar{B}_4=\overline{Z(D)^s}Z(\bar{B})$ and $\bar{B}_4/Z(\bar{B})\simeq Z(D)^sB_3/B_3$ is a natural module for $K_3{}^s=K_3$ by the above.

Denote by U_1 and U_2 the maximal elementary abelian subgroups of W_1. As $\mathscr{I}(tW_1)=t^{W_1}$, it follows from [6, (2.3), (2*B*) (1)] that $U_1{}^t=U_2$. As $HJ_3J_3{}^t$ is of odd order, it normalizes U_1 and U_2. Then $U_1W_4B_4/B_4>C_{B/B_4}(K_3)$ or $C_{B/B_4}(K_3{}^t)$ by (3.3) (2). Replacing U_1 and U_2 if necessary, we may assume that $U_1W_4B_4/B_4>C_{B/B_4}(K_3{}^t)$. Then $U_1W_4B_4/B_4=C_{B/B_4}(K_3{}^t)C_{W_4B_4/B_4}(K_3)$, so K_3 normalizes $U_1W_4B_4$. Since H_1 is fixed-point-free on W_4B_4/B_4, (1.2) shows $U_1B_4/B_4\leq C_{B/B_4}(K_3{}^t)$. Thus K_3 normalizes U_2B_4. We can choose involutions u_1

and u_2 as generators of the Weyl group of M_1 such that $u_1{}^t=u_2$, $(u_1u_2)^2=r$, and u_i normalizes U_i, $i=1$, 2. Set $Y_1=W_1\cap W_1{}^{ru_1}$, $Y_2=W_1\cap W_1{}^{ru_2}$, and $K_i=\langle Y_i, u_i\rangle$ for $i=1$, 2. Then $U_1=Y_2W_2$, $U_2=Y_1W_2$, $K_1{}^t=K_2$, and $K_i\simeq SL_2(q)$ with $Y_i\in \mathrm{Syl}_2(K_i)$. Let J_0 be a complement of W_1 in $N_{M_1}(W_1)$ containing H_5 and set $J_i=K_i\cap J_0$, $i=1,2$. Then u_1 and u_2 normalize J_0. As H_1J_0 is an abelian group containing H and normalizes Q, it normalizes W_i, $1\leq i\leq 12$, Y_1, and Y_2. Set $F_1=U_1(A\cap B)D$ and $F_2=U_2(A\cap B_4)D$. Note that $B_4=W_2(A\cap B_4)\leq F_1\cap F_2$.

(4.3) (1) $C_{A/D}(U_1)=W_5W_7D/D$ *and* $C_{A/D}(U_2)=W_5W_6W_7D/D$.

(2) $F_1\lhd\langle Q, u_1, v\rangle$, $F_2\lhd\langle Q, u_2, v\rangle$, $C(F_1/D)=Z(BD/D)$, $F_1/Z(F_1 \bmod D)$ *is elementary abelian of order* q^4, *and* F_2/D *is elementary abelian of order* q^6.

(3) AF_2/F_2 *is a natural module for* K_2 *and* BF_2/F_2 *is a natural module for* K_3.

PROOF. Note that $F_1=U_1W_4B_4D$ is K_3-invariant and $F_1\lhd\langle Q, v\rangle$, for $Q'\leq F_1$. Let $\overline{N(D)}=N(D)/D$. Then $\bar{F}_1/\bar{B}_4\simeq F_1/B_4D$ is a natural module for K_3. As $C_{\bar{A}}(V_2)=\bar{A}\cap\bar{B}_4$ by (4.1) (1), $\bar{F}_1\cap C(\bar{B}_4)=\bar{B}_4$. Now $Z(\bar{B})\leq\bar{B}_4\leq\bar{F}_1\leq\bar{B}$ and K_3 is irreducible on $\bar{B}_4/Z(\bar{B})$ by (4.2), so $Z(\bar{F}_1)=Z(\bar{B})$. As $V_2\leq U_1$, we get $C_{\bar{A}}(U_1)=\bar{A}\cap Z(\bar{F}_1)=\bar{W}_5\bar{W}_7$. As $V_2\leq U_2$, $Z(\bar{Q})=\bar{W}_5\bar{W}_7\leq C_{\bar{A}}(U_2)\leq\bar{A}\cap\bar{B}_4=\bar{W}_5\bar{W}_6\bar{W}_7$. If $C_{\bar{A}}(U_2)=Z(\bar{Q})$, M_1 normalizes $Z(\bar{Q})$. But N_1 is irreducible on $A/W_5D\simeq R/V_5R_1$ by (4.1) (1). Hence $C_{\bar{A}}(U_2)=\bar{W}_5\bar{W}_6\bar{W}_7$ by the action of H. Then $\bar{F}_2=\bar{U}_2C_{\bar{A}}(U_2)$ is elementary abelian of order q^6 and normalized by u_2. As $F_2=U_2B_4D$ and $\bar{Y}_3\bar{B}=\bar{Q}$, it follows that $F_2\lhd\langle Q, u_2, v\rangle$. Set $\bar{I}/\bar{W}_5\bar{W}_7=(\bar{A}/\bar{W}_5\bar{W}_7)\cap C(U_1)$. Then as $(R/V_5V_7R_1)\cap C(V_2)=V_4V_5V_6V_7R_1/V_5V_7R_1$ and $\bar{Q}\rhd\bar{W}_5\bar{W}_6\bar{W}_7$, (4.1) (1) shows $\bar{W}_5\bar{W}_6\bar{W}_7\leq\bar{I}\leq\overline{A\cap B}=\bar{W}_4\bar{W}_5\bar{W}_6\bar{W}_7$. If $\bar{I}=\bar{W}_5\bar{W}_6\bar{W}_7$, then M_1 normalizes $\bar{I}$, contrary to the action of N_1 on A/W_5D. Thus $\bar{I}=\overline{A\cap B}$, which is u_1-invariant. Hence $F_1\lhd\langle Q, u_1, v\rangle$. If K_2 centralizes AF_2/F_2, then as $Q=AF_2Y_2$, $Q'\leq F_2$. But $(Q/B_4)'=W_4B_4/B_4$ by (3.3) (2). As $W_4\not\leq F_2$, this is a contradiction. Thus AF_2/F_2 is a natural module for K_2 by [6, (1K)]. Finally, BF_2/F_2 is a natural module for K_3 by (3.3) (2).

(4.4) $(u_2v)^3\in F_2$.

PROOF. As $D\leq F_2\leq DB$, $Y_{11}W_{12}\leq Z(F_2)\leq Z(D)\cap B_2$. If $Z(F_2)=Y_{11}W_{12}$, $C(Z(F_2))\geq AB=Q$ by (4.3) (3), contrary to $Z(Q)=W_{12}$. Thus $Z(F_2)=Z(D)\cap B_2$ by the action of H. Then $C_Q(Z(F_2))=F_2$ by (4.3) (2) and (3). It follows from (3.2) that $Y_{11}W_{12}/Z$ is a natural module for K_3. As $J_3{}^v=J_3$, $W_{12}{}^v=Y_{11}Z$ by (1.3). Now K_3 centralizes $Z(F_2)/Y_{11}W_{12}$, so $[Z(F_2)/Z, J_3]=Y_{11}W_{12}/Z$ and $Y_{10}Z/Z=(Z(F_2)/Z)\cap C(J_3)$, which is v-invariant. Thus v centralizes $Y_{10}Z/Z$. If K_2 centralizes $Z(F_2)/W_{12}$, then as $[M_1, W_{12}]=1$, $[K_2, Z(F_2)]=1$. But $C_Q(Z(F_2))=F_2$, a contradiction. Thus $Z(F_2)/W_{12}$ is a natural module for K_2 by [6, (1K)]. Then $[Z(F_2), J_2]=Y_{10}Y_{11}$ and $Y_{10}{}^{u_2}=Y_{11}$ by (1.3) since $J_2{}^{u_2}=J_2$. For $x\in Y_{10}Z/Z$, we have $x^{u_2v}\in W_{12}/Z$, so $x^{u_2vu_2}=x^{u_2v}$ and $x^{u_2vu_2v}=x^{u_2}$. Then as $x^v=x$, $[(u_2v)^3, x]=1$. Similarly $(u_2v)^3$ centralizes $Y_{11}Z/Z$ and W_{12}/Z. Hence $[(u_2v)^3, Z(F_2)]\leq Z$.

K_3 centralizes $Z(D)\cap B_3/Z(F_2)$. By (4.2), J_3 is fixed-point-free on $Z(D)/Z(D)\cap B_3$ and (1.3) shows $Y_5{}^v(Z(D)\cap B_3)=Y_8(Z(D)\cap B_3)$. Then $[Z(D)/Z(F_2), J_3]=Y_5Y_8Z(F_2)/Z(F_2)$ is v-invariant and $Y_5{}^vZ(F_2)=Y_8Z(F_2)$. If K_2 centralizes $Z(D)\cap A_1/Z(F_2)$, $[K_2, Z(D)]\leq Z(F_2)$. By [10, (3.9) (5)], $C(Z(D)/Z(F_2))\geq\langle DB_4, K_2\rangle$. Now $A\cap F_2=A\cap DB_4$ and $A/A\cap F_2$ is a natural module for K_2 by (4.3) (3). But then $A\leq C(Z(D)/Z(F_2))$, contrary to the action of K_3 on $Z(D)/Z(D)\cap B_3$. Thus $Z(D)\cap A_1/Z(F_2)$ is a natural module for K_2 and $Y_8{}^{u_2}Z(F_2)=Y_9Z(F_2)$ by (1.3). J_2 centralizes $Z(D)/Z(D)\cap A_1$, so $(Z(D)/Z(F_2))\cap C(J_2)=Y_5Z(F_2)/Z(F_2)$. Hence

$[u_2, Y_5Z(F_2)]\leq Y_5Z(F_2)\cap A_1=Z(F_2)$. Now as in the first paragraph we have $[(u_2v)^3, Z(D)]\leq Z(F_2)$.

We argue that $F_2'=D\cap B$, $Z(F_2')=Z(D)B_1$, and $[F_2', F_2]=Y_7Y_9{}^tB_2$. By [10, (3.9) (4), (5)], $W_{12}\leq B_4'\leq B_2$. As $Z(D)/W_{12}\simeq(A_0/W_{12})\cap C(t)=V_5R_1W_{12}/W_{12}$ and as $[V_5,V_7]=V_{11}$ and $[V_5, V_6]=V_{10}$, we get $B_2=B_4'\leq F_2'$. By (4.3) (2), $F_2'\leq D\cap B$, so $F_2'(D\cap B_4)=D\cap B$ by the definition of U_2. Now $D/A_1\simeq(A/A_1)\cap C(t)\simeq R/R_1$, so by considering $[V_2,V_3]$ and $[V_2,V_4]$, we have $F_2'A_1\geq D\cap B_4$. Thus $F_2'A_1=D\cap B$. As K_3 centralizes $Z(D)\cap B_3/B_2=Y_9B_2/B_2$, (3.4) shows $Y_9B_2=[Y_3{}^t, W_7]B_2\leq F_2'$. Then $Z(D)\leq F_2'$ by the action of $\langle K_2, K_3\rangle$. As $C_{A_1}(H_7)=1$, $Y_7=D\cap B\cap C(H_7)\leq F_2'$ by (1.2). Then (3.4) shows $Y_7W_9B_2\leq F_2'$. If $[Y_3{}^t, Y_5{}^t]\leq Z(D)W_9B_2$, then as $Y_5\leq Z(D)$, $[Y_3{}^t, Y_5{}^t]\leq Z(D)W_9B_2\cap Z(D)^t=W_9B_2$. Now $Y_3{}^t\leq D$, so $[Y_3{}^t, W_5]\leq W_9B_2$. As $W_9B_2\lhd Q$ and $W_3=Y_3Y_3{}^t$, this implies $[W_3, W_5]\leq W_9B_2$. But $[V_3, V_5]\nleq W_9B_2$, a contradiction. Therefore $F_2'=D\cap B$. By (3.1), $Z(D)B_1\leq Z(F_2')\leq F_2'\cap C(A_1)=Y_5A_1$. As $C_{A_1}(Y_7)=Y_8W_9B_2$, $Z(F_2')\leq Z(D)W_9B_2$. Similarly $Z(D)W_9B_2\cap C(Y_6)=Z(D)B_2$ since $[V_6,V_9]\neq 1$. Moreover $[V_4, V_{10}]\neq 1$ implies $Z(D)B_2\cap C(Y_4)=Z(D)B_1$. Hence $Z(F_2')=Z(D)B_1$. As $Y_2W_4\leq F_1$, (4.1) (1) and (4.3) (2) show $[Y_2, Y_4{}^t]\leq Z(F_1 \bmod D)\cap D^t=Y_5{}^tY_7{}^tA_1$. Then $[Y_1, Y_4]\leq Y_5Y_7A_1$ since $Y_1{}^t=Y_2$. Let $\bar{Q}=Q/Y_5Y_7A_1$. Then $\overline{Z(D)\cap B_4}=\bar{Y}_6\leq Z(\bar{Q})$ and Y_1 centralizes $\bar{Y}_4\bar{Y}_6=\bar{F}_2'\geq[\bar{Y}_1, \bar{Y}_3{}^t]$. For each $a\in(Y_3{}^t)^\#$ the map ϕ_a: $Y_1\to[\bar{Y}_1, \bar{Y}_3{}^t]$; $x\mapsto[\bar{x},\bar{a}]$ is a homomorphism commutable with the action of H_3. As $W_1/W_2=U_1/W_2\times U_2/W_2$, Y_1 and V_1/V_2 are H-isomorphic. Thus H_3 is irreducible on Y_1. By the definition, $[Y_1,Y_3{}^t]\nleq B_4$ and so $[\bar{Y}_1, \bar{a}]\neq 1$. Hence $Y_1\simeq[\bar{Y}_1,\bar{a}]$ as H_3-modules. Now $\bar{Y}_4\simeq V_4/V_8$ and $\bar{Y}_6\simeq V_6/V_{11}$ as H-modules, so it follows from [10, (2.1)] that $[\bar{Y}_1,\bar{a}]=\bar{Y}_4$. Thus $[\bar{Y}_1, \bar{Y}_3{}^t]=\bar{Y}_4$. Since H_1 is irreducible on $\bar{Y}_4$, this implies $[\bar{b}, \bar{Y}_3{}^t]=\bar{Y}_4$ for each $b\in Y_1{}^\#$. So $Y_4\leq[b, Y_3{}^t]Y_5Y_7A_1$. Since $Y_7A_1=(D\cap B_3)A_1\lhd Q$, $Y_5Y_7A_1/A_1\leq Z(Q/A_1)$. Moreover $[D, b, b]\leq A_1$ since $|b|=2$. Thus $[b, Y_4]\leq A_1$, so $[Y_1, Y_4]\leq A_1$. By the action of M_3, $Z(B/B_3)\geq B_4/B_3$. So $[F_2', DB_4]\leq B_3$ by (4.1) (2). Then as $F_2=Y_1DB_4$ and $F_2'=Y_4(D\cap B_4)$, the above shows $[F_2',F_2]\leq A_1B_3$. Thus by (4.1) (4), $[F_2',F_2]\leq D\cap B_3=Y_7W_9B_2$. On the other hand $[W_6,W_9]=W_{12}$, so $Z(F_2)\leq[F_2',F_2]$ by the action of $\langle K_2, K_3\rangle$. Considering $[V_2, V_8]$ and $[V_2, V_9]$, we get $B_2\leq[F_2',F_2]$. (4.3) (1) implies $[Y_2, W_6]\nleq D$, so $[Y_1, Y_6]^t=[Y_2, Y_6{}^t]\nleq A_1$. Thus $[F_2',F_2]A_1=Y_7A_1$. As $C_{A_1}(H_7)=1$, $Y_7=C_{Y_7A_1}(H_7)\leq[F_2',F_2]$ by (1.2). Then $Y_7Y_9{}^tB_2\leq[F_2',F_2]$ by (3.4). If $W_9\leq[F_2',F_2]$, the action of $\langle K_2, K_3\rangle$ forces $Z(D)\leq[F_2',F_2]$, a contradiction. Hence $[F_2',F_2]=Y_7Y_9{}^tB_2$.

As $Z(F_2')\cap[F_2',F_2]=Y_{10}B_1$, $\langle K_2, K_3\rangle$ centralizes $Y_{10}B_1/Z(F_2)$. As $[F_2',F_2]\cap A_1=Y_9{}^tB_2$, K_2 centralizes $[F_2',F_2]/Y_9{}^tB_2$. If K_2 centralizes $Y_9{}^tB_2/Y_{10}B_1$, it centralizes $[F_2',F_2]/Y_{10}B_1$. Note that $[F_2',F_2]/B_2=C_{B_3/B_2}(K_3{}^t)$. Then as $D=Y_3{}^t(D\cap B)$, (4.3)(3) shows $A\leq\langle K_2,D,B\rangle\leq C([F_2',F_2]/B_2)$. But $Y_3\leq A$ and $[F_2',F_2]/B_2$ is a natural module for K_3, a contradiction. Thus $Y_9{}^tB_2/Y_{10}B_1$ is a natural module for K_2. Moreover, K_3 centralizes $B_2/Y_{10}B_1$. Hence arguing as in the first paragraph we obtain $[(u_2v)^3, [F_2',F_2]]\leq Y_{10}B_1$.

We have shown that $(u_2v)^3$ stabilizes $Z(D)\geq Z(F_2)\geq Z\geq 1$ and $[F_2',F_2]\geq Y_{10}B_1\geq Z(F_2)$. Thus $O(\langle(u_2v)^3\rangle)$ centralizes $I=Z(D)[F_2',F_2]$. By (4.1) (1), $(Q/A_0B_3)\cap C(t)=PA_0B_3/A_0B_3$. Then $C_P(V_9S_2)=V_5R_1S_3$ implies $C_Q(W_9B_2)=A_0B_3$, so $C_Q(I)\leq A_0B_3\cap C(Y_8)=Y_7A_0$. Moreover $[V_5,V_7]=V_{11}$, so $(Y_7A_0/W_{12})\cap C(Y_7)=Y_5Y_7A_1/W_{12}$ and $Y_5Y_7A_1\cap C(Y_7)=I$. Thus $C_Q(I)=I$. Now $I\lhd Q\in \mathrm{Syl}_2(O^2(G))$ and $O(O^2(G)\cap C(I))=1$ by the unbalanced group theorem, so $O^2(G)\cap C(I)=I$. Then as $O(\langle(u_2v)^3\rangle)\leq C(I)$, $(u_2v)^3$ is a 2-element. (4.3) (3) and the action of $\langle K_2,K_3\rangle$ on $Z(F_2)/Z$ show $C_Q(Z(F_2)/Z)=F_2$. Let $X=\langle Q,K_2,K_3\rangle$. Then $C_X(Z(F_2)/Z)$ is 2-closed with F_2 the unique Sylow 2-subgroup. As $(u_2v)^3\in C_X(Z(F_2)/Z)$, (4.4) holds.

(4.5) $\langle Q, K_2, K_3\rangle/F_2\simeq PSL_3(q)$.

PROOF. Let $X=\langle Q, K_2, K_3\rangle$ and $\bar{X}=X/F_2$. Then $Q\in \mathrm{Syl}_2(X)$. We have $\bar{Q}\cap\bar{Q}^{u_2}=\bar{A}$ and $\bar{Q}\cap\bar{Q}^{v}=\bar{B}$, so $\bar{Q}^{u_2}\cap\bar{Q}\cap\bar{Q}^{v}=\bar{W}_4$. As $J_2{}^{u_2}=J_2$ and $J_3{}^{v}=J_3$, it follows from (1.3) and (4.3) (3) that $\bar{W}_4{}^{u_2}=\bar{Y}_3$ and $\bar{W}_4{}^{v}=\bar{Y}_2$. Thus $\bar{Q}\cap\bar{Q}^{u_2}\cap\bar{Q}^{vu_2}=\bar{Y}_3$ and $\bar{Q}\cap\bar{Q}^{v}\cap\bar{Q}^{u_2v}=\bar{Y}_2$, so

$$\bar{Q}=\bar{A}(\bar{Q}\cap\bar{Q}^{v}\cap\bar{Q}^{u_2v})=\bar{B}(\bar{Q}\cap\bar{Q}^{u_2}\cap\bar{Q}^{vu_2}). \tag{1}$$

As $W_{12}{}^{u_2v}=Y_{11}Z$ and $Y_{11}{}^{u_2}=Y_{10}$, $Q\cap Q^{u_2vu_2}\leq C_Q(Y_{10})$. By [10, (3.9) (4)], $[W_4,W_{10}]=W_{12}$ and $[W_3,W_{10}]=[W_7,W_9]^r=1$ and as $Y_{10}\leq Z(F_2)$, $C_Q(Y_{10}W_{12}/W_{12})\geq U_2A$. Then as $[V_1,V_{10}]=V_{11}$, $C_Q(Y_{10}W_{12}/W_{12})=U_2A$ by the action of H. As $C_{W_4}(Y_{10})=Y_4$, we get $C_Q(Y_{10})=U_2W_3Y_4B_4$. Thus $\bar{Q}\cap\bar{Q}^{u_2vu_2}\leq\bar{Y}_3$. Now $\bar{Q}^{u_2vu_2}\cap\bar{Y}_3=(\bar{Q}^{u_2}\cap\bar{Y}_2)^{vu_2}$ and $\bar{Q}^{u_2}\cap\bar{Y}_2=\bar{A}\cap\bar{Y}_2=1$. Therefore

$$\bar{Q}\cap\bar{Q}^{u_2vu_2}=\bar{Q}\cap\bar{Q}^{vu_2v}=1 \tag{2}$$

since $\bar{u}_2\bar{v}\bar{u}_2=\bar{v}\bar{u}_2\bar{v}$ by (4.4). Note that J_2 normalizes K_3, for H_1J_0 is an abelian group of odd order containing H_3. As J_3 normalizes $(N_{M_1}(U_2))'=K_2U_2$, it normalizes $N_{K_2U_2}(Q)Q=J_2Q$. Set $E=J_2J_3Q$, which normalizes K_2AF_2 and K_3BF_2. $K_i\simeq SL_2(q)$, $i=2$, 3, so Q acts transitively on $\mathrm{Syl}_2(EK_i)-\{Q\}$. As $N_{EK_i}(Q)=E$, this implies $E=Q(E\cap E^{u_2})=Q(E\cap E^{v})$, $EK_2=E\cup Eu_2E$, and $EK_3=E\cup EvE$. By (1), $\bar{Q}^{u_2}=\bar{A}(\bar{Q}\cap\bar{Q}^{v})^{u_2}$, so $\bar{E}^{u_2}=\bar{Q}^{u_2}(\bar{E}\cap\bar{E}^{v})^{u_2}=\bar{A}(\bar{E}^{u_2}\cap\bar{E}^{vu_2})$. Similarly $\bar{E}^{v}=\bar{B}(\bar{E}^{v}\cap\bar{E}^{u_2v})$. As $\bar{Q}^{vu_2}=(\bar{A}\bar{Y}_2)^{vu_2}=\bar{A}^{vu_2}\bar{Y}_3$ and $\bar{A}\leq\bar{E}\cap\bar{E}^{u_2}$, we get $\bar{E}^{vu_2}=\bar{Q}^{vu_2}(\bar{E}\cap\bar{E}^{u_2})^{vu_2}=\bar{Y}_3(\bar{E}^{vu_2}\cap\bar{E}^{u_2vu_2})$. Similarly $\bar{E}^{u_2v}=\bar{Y}_2(\bar{E}^{u_2v}\cap\bar{E}^{vu_2v})$. Then

$$\begin{aligned}\bar{E}&=\bar{Q}(\bar{E}\cap\bar{E}^{u_2})=\bar{Q}(\bar{E}\cap\bar{A}(\bar{E}^{u_2}\cap\bar{E}^{vu_2}))=\bar{Q}(\bar{E}\cap\bar{E}^{u_2}\cap\bar{E}^{vu_2})\\&=\bar{Q}(\bar{E}\cap\bar{E}^{u_2}\cap\bar{Y}_3(\bar{E}^{vu_2}\cap\bar{E}^{u_2vu_2}))=\bar{Q}(\bar{E}\cap\bar{E}^{u_2}\cap\bar{E}^{vu_2}\cap\bar{E}^{u_2vu_2}).\end{aligned}$$

Thus $\bar{E}=\bar{Q}(\bar{E}\cap\bar{E}^{u_2vu_2})$. By (2), $|\bar{E}\cap\bar{E}^{u_2vu_2}|$ is odd, whence

$$\bar{E}\cap\bar{E}^{u_2vu_2}=\bar{E}\cap\bar{E}^{u_2}\cap\bar{E}^{vu_2}\cap\bar{E}^{u_2vu_2}.$$

By symmetry we have

$$\bar{E}\cap\bar{E}^{vu_2v}=\bar{E}\cap\bar{E}^{v}\cap\bar{E}^{u_2v}\cap\bar{E}^{vu_2v}.$$

Set $\bar{I}=\bigcap_w\bar{E}^{w}$ where w ranges over $\langle u_2, v\rangle$. As $\langle\bar{u}_2,\bar{v}\rangle=\{1,\bar{u}_2,\bar{v},\bar{u}_2\bar{v},\bar{v}\bar{u}_2,\bar{u}_2\bar{v}\bar{u}_2\}$, it follows that $\bar{I}=\bar{E}\cap\bar{E}^{u_2vu_2}=\bar{E}\cap\bar{E}^{vu_2v}$. Set $\bar{N}=\bar{I}\langle\bar{u}_2,\bar{v}\rangle$. Suppose $\bar{Q}=\bar{Q}^{w}$ for some $w\in\langle u_2,v\rangle-F_2$ and choose $x\in\langle u_2,v\rangle$ such that $\bar{w}\bar{x}=\bar{u}_2\bar{v}\bar{u}_2$. Then $\bar{Q}^{x}=\bar{Q}^{wx}$ and so $\bar{Q}\cap\bar{Q}^{x}=1$ by (2), contrary to (1). Hence $\bar{N}\cap\bar{E}=\bar{I}$ and $\bar{N}/\bar{I}\simeq\langle\bar{u}_2,\bar{v}\rangle$. Note that $\bar{u}_2\bar{E}\bar{u}_2\leq\bar{E}\cup\bar{E}\bar{u}_2\bar{E}$ and $\bar{v}\bar{E}\bar{v}\leq\bar{E}\cup\bar{E}\bar{v}\bar{E}$. As $\bar{Q}=\bar{A}\bar{B}$, $\bar{u}_2\bar{Q}\bar{v}=\bar{A}\bar{u}_2\bar{v}\bar{B}$. Then $\bar{u}_2\bar{E}\bar{v}=\bar{u}_2\bar{Q}\bar{I}\bar{v}=\bar{u}_2\bar{Q}\bar{v}\bar{I}\leq\bar{E}\bar{u}_2\bar{v}\bar{E}$. By (1), $\bar{Q}\leq\bar{A}\bar{A}^{v}$ and so $\bar{u}_2\bar{E}\bar{v}\bar{u}_2=\bar{u}_2\bar{Q}\bar{v}\bar{u}_2\bar{I}\leq\bar{A}\bar{u}_2\bar{v}\bar{u}_2\bar{A}\bar{I}\leq\bar{E}\bar{u}_2\bar{v}\bar{u}_2\bar{E}$. Then for $\bar{w}=\bar{v}$ or $\bar{v}\bar{u}_2$, $\bar{u}_2\bar{E}\bar{u}_2\bar{w}\leq\bar{E}\bar{w}\cup\bar{E}\bar{u}_2\bar{E}\bar{w}\leq\bar{E}\bar{w}\cup\bar{E}\bar{u}_2\bar{w}\bar{E}$. Hence $\bar{u}_2\bar{E}\bar{w}\leq\bar{E}\bar{u}_2\bar{w}\bar{E}\cup\bar{E}\bar{w}\bar{E}$ for all $\bar{w}\in\langle\bar{u}_2,\bar{v}\rangle$. By symmetry $\bar{v}\bar{E}\bar{w}\leq\bar{E}\bar{v}\bar{w}\bar{E}\cup\bar{E}\bar{w}\bar{E}$ for all $\bar{w}\in\langle\bar{u}_2,\bar{v}\rangle$.

We have shown that $(\bar{E},\bar{N})$ is a BN-pair of $\bar{X}$ whose Weyl group is dihedral of order 6. Thus $\bar{X}\simeq PSL_3(q)$ by Fong and Seitz[5].

(4.6) $[K_1, K_3]\leq F_1$.

PROOF. Let $X=\langle Q,K_1,K_3\rangle$. In the proof of (4.4) we have shown that $Z(F_2)=Z(D)\cap B_2$. As $F_1\leq DB$, a similar argument gives $Z(F_1)=Z(F_2)$ or $Y_{11}W_{12}$. Then $F_1F_2\geq B$ implies $Z(F_1)=Y_{11}W_{12}$. As $[K_1, W_{12}]=1$, $[K_1,Z(F_1)]=1$ and thus $[(u_1v)^2, Z(F_1)]=1$. Recall that $[F_2', F_2]=Y_7Y_9{}^tB_2$, so K_1 normalizes $[F_2',F_2]^t\cap A_1=Y_9B_2$ and $(Z(D)\cap B_3)B_2=Y_9B_2\lhd X$. As K_3

centralizes B_2/B_1, it centralizes $Y_9B_2/Z(F_1)$ and $[(u_1v)^2, Y_9B_2] \leq Z(F_1)$. Furthermore $[(u_1v)^2, Z(D)B_2] \leq Y_9B_2$ since K_1 centralizes A_0/A_1.

As $Z(F_1/D) \cap U_1D/D = Z(F_1/D) \cap W_2D/D$ and $Z(F_1/D) \cap A/D = Z(Q/D)$, it follows that $[K_1, Z(F_1/D)] = 1$. If K_3 centralizes $Z(F_1/D)/(B_3D/D)$, then as $|B_3D/D| = q$, $[K_3, Z(F_1/D)] = 1$. But $Y_1Y_3F_1 = Q$ and $Z(F_1/D) \neq Z(Q/D)$, a contradiction. Thus $Z(F_1/D)/(B_3D/D)$ is a natural module for K_3.

Let $E = C_Q(Y_9B_2)$. By (3.1) (1), $C_Q(B_2) \leq F_1$, so $E = C_{F_1}(Y_9B_2) \lhd X$. We have $Y_6A_0B_3 \leq E \cap A \leq C_A(B_2) = W_6A_0B_3$. As $[V_6, V_9] \neq 1$, it follows that $E \cap A = Y_6A_0B_3$ and $E \cap D = Y_5Y_6Y_7A_1$. The action of K_3 gives $ED/D = Z(F_1/D)$. As $C_E(H_1) \leq C_{B_4}(H_1) = W_2$, $E = (E \cap W_2)(E \cap A)$. Now $Y_9Y_9{}^t = W_9$ and no nonidentity element of V_2 centralizes V_9. Thus $W_2 = (E \cap W_2) \times (E \cap W_2)^t$. Let $E_0 = E \cap D$. Then $E_0 \leq C_E(Z(D)B_2) \leq E = (E \cap W_2)Y_6A_0B_3$. If $C_E(Z(D)B_2) \cap C(H_1) \neq 1$, $C_E(Z(D)B_2) \geq E \cap W_2$ since H is irreducible on $E \cap W_2$. (1.3) and the action of K_1 and K_3 on $E/E_0 \simeq ED/D = Z(F_1/D)$ show $[E/E_0, J_3] = (E \cap W_2)A_0E_0/E_0$ and $(E \cap W_2)^v E_0 = A_0E_0$. So $A_0 \leq C_E(Z(D)B_2)$, contrary to $C_{W_5}(Y_5) = Y_5$. Thus $C_E(Z(D)B_2) \leq [E, H_1] = E \cap A$, so $C_E(Z(D)B_2) \leq (E \cap A) \cap (E \cap A)^v = Y_5Y_6A_1B_3$. As $C_{W_7}(Y_8) = Y_7$, we get $C_E(Z(D)B_2) = E_0$.

Let $E_1 = \langle E \cap W_2, Y_5{}^t, Y_7{}^t \rangle Z(D)B_2$ and $\bar{E} = E/Z(D)$. Then $E = E_0E_1$, $E_0 \cap E_1 = Z(D)B_2$, and $E_1 \lhd E$ by [10, (3.9) (5)]. By (4.1) (2), $\bar{E}_0 = \bar{Y}_6\bar{Y}_7\bar{Y}_8{}^t\bar{Y}_9{}^t\bar{B}_2$ is elementary abelian of order q^6. Note that $[W_2, W_6] = [W_8, W_5]^s = 1$, $[W_2, W_7] = [W_8, W_9]^s = 1$, and $[W_2, W_5] = [W_{11}, W_4]^{rs} = 1$. Thus $\bar{E}_1 = \overline{(E \cap W_2)}\bar{Y}_5{}^t\bar{Y}_7{}^t\bar{B}_2$ is elementary abelian of order q^5. $[W_5, W_7] = [W_6, W_9]^s = W_{11}$, so the action of H gives $[\bar{Y}_5{}^t, Y_7] = Y_{11}{}^t$, since $[Y_5, Y_7] = 1$ and $Y_5Y_5{}^t = W_5$. If $[Y_5{}^t, Y_6] \leq Z(D)B_1$, then $[Y_5{}^t, Y_6] \leq Z(D)B_1 \cap Z(D)^t = Y_{11}{}^tW_{12}$. As $Y_5 \leq Z(D)$, $[Y_5, Y_6] = 1$. But then $[W_5, W_6] \leq B_1$, contrary to $[V_5, V_6] = V_{10}$. Thus $C_{\bar{E}}(\bar{Y}_5{}^t) = \bar{Y}_8{}^t\bar{Y}_9{}^t\bar{E}_1$. If $[E \cap W_2, Y_9{}^t] = 1$, $[E \cap W_2, W_9] = 1$. But then $W_2 = (E \cap W_2)(E \cap W_2)^t$ centralizes W_9, a contradiction. Thus $[E \cap W_2, Y_9{}^t] = Y_{11}{}^t$, for $[W_2, W_9] = [W_8, W_7]^s = W_{11}$. As shown before $(E \cap W_2)E_0 = A_0{}^vE_0$. Moreover $[A_0{}^v, Z(D)] = Z^v$, so $[E \cap W_2, Y_8] \leq B_1$. If $[E \cap W_2, Y_8{}^t] \leq Z(D)B_1$, then $[E \cap W_2, Y_8{}^t] \leq Z(D)B_1 \cap Z(D)^t \leq B_1$. As $W_8 = Y_8Y_8{}^t$, this implies $[W_2, W_8] \leq B_1$, contrary to $[V_2, V_8] = V_{10}$. Thus $C_{\bar{E}}(\bar{Y}_5{}^t) \cap C(\bar{Y}_8{}^t) = \bar{E}_1$.

Let $I = C_E(O(\langle (u_1v)^2 \rangle))$. By the first paragraph $(u_1v)^2$ stabilizes $Z(D)B_2 > Y_9B_2 > Z(F_1) > 1$, so $I \geq Z(D)B_2$ and $C_E(I) \leq E_0$. As K_1 centralizes E/E_0, $[(u_1v)^2, E] \leq E_0$ and thus $E = E_0I$. Now $C_{\bar{E}}(\bar{E}_1) = \bar{E}_1$ by the above and $\bar{E} = \bar{E}_0\bar{E}_1$. So $\bar{E}_0 \cap C(\bar{I}) = \bar{E}_0 \cap \bar{E}_1 = \bar{B}_2$ and $C_E(I) \leq Z(D)B_2$. Note that $Z(D)B_2 \cap C(W_5) = Y_8Y_9B_2$ and $Y_8Y_9B_2 \cap C(W_7) = Y_9B_2$. As $E_0I \geq \langle W_5, W_7 \rangle$ and $Z(E_0) \geq Z(D)B_2$, we conclude that $C_E(I) = Y_9B_2$. By (3.5), E is a Sylow 2-subgroup of $O^2(G) \cap C(Y_9B_2)$. Moreover $I \lhd E$ by [10, (3.9) (5)] and $O(O^2(G) \cap C(I)) = 1$ by the unbalanced group theorem. Thus $O^2(G) \cap C(I) = Y_9B_2$, so $(u_1v)^2$ is a 2-element. As Q/F_1 is elementary abelian of order q^2, Bender [3] shows $(X/F_1)/O(X/F_1) \simeq SL_2(q) \times SL_2(q)$ and $[K_1, K_3] \leq O(X \bmod F_1)$. Hence $(u_1v)^2 \in F_1$. Now K_1 centralizes AF_1/F_1 and K_3 centralizes F_1F_2/F_1 with $AF_1 = Y_3F_1$ and $F_1F_2 = Y_1F_1$, so the lemma holds.

(4.7) *Let* $G_0 = \langle Q, M_1, M_3 \rangle$. *Then* $G_0 \simeq F_4(q)$.

PROOF. Recall that $\langle K_1, K_2 \rangle = M_1, K_1{}^t = K_2$, and $[K_3, K_3{}^t] = 1$ By (4.5) and (4.6) we can apply Theorem B of Niles [11] to the subgroups $P_1 = K_1F_1A$, $P_2 = P_1{}^t$, $P_3 = K_3Y_3{}^tB$, and $P_4 = P_3{}^t$ and conclude that G_0 has a BN-pair of rank 4. Hence $G_0 \simeq F_4(q)$ by Tits [17].

REMARK. It remains of course to prove that G_0 is normal in G. The first step toward this purpose may be to show that if $t^g \in N(G_0)$ for an element $g\in G$ then $g\in N(G_0)$. Indeed, J normalizes M_1A by [10, (3.9) (1), (3.10)] and K normalizes M_3B by (3.1) and [10, (3.5)], so $C(t)\leq N(G_0)$. Suppose $t^g\in N(G_0)$. Since $C(t)^{(\infty)}=L$, neither t nor t^g lies in $G_0C(G_0)$. Then by [2, (19.5)], $C_{G_0}(t)=C_{G_0}(t^g)=L$ and $t=t^{gx}$ for some $x\in G_0C(G_0)$. Hence $gx\in C(t)$ and we have $g\in N(G_0)$.

V. *The Case* $M_1\simeq Sz(q)\times Sz(q)$

In this section we assume that $M_1\simeq Sz(q)\times Sz(q)$. Let $M_1=K_1\times K_1{}^t$ with $K_1\simeq Sz(q)$ and $Y_1=W_1\cap K_1$. By (3.3), $N_1=C_{M_1}(t)$ is irreducible on $D_0/A_0\simeq C_{A/A_0}(t)\simeq R/V_5R_1$. Since $C_{D_0/A_0}(Y_1)\neq 1$ is $K_1{}^t$-invariant, $K_1{}^t$ centralizes $C_{D_0/A_0}(Y_1)$ or $[Y_1, D_0/A_0]=1$ by [5, (4F)]. In the former the action of K_1 on $C_{D_0/A_0}(K_1{}^t)$ yields $[K_1{}^t, D_0/A_0]=1$ and in the latter $[K_1, D_0/A_0]=1$. Replacing K_1 and $K_1{}^t$ if necessary, we may assume that K_1 centralizes D_0/A_0. Then

(5.1) $M_1A/A_0=K_1{}^tD_0/A_0\times K_1D_0{}^t/A_0$ *and* $K_1{}^tD_0/A_0\simeq N_1R/V_5R_1$.

(5.2) *Let* $D_1=O_2((K_1{}^tD_0)'A_1)$. *Then* $K_1{}^tD_1\lhd M_1A$ *and* $K_1{}^tD_1/A_1$ *is perfect with* $|D_1/A_1|=q^5$.

PROOF. Set $D=D_0\cap D_0{}^v$. Arguing as in the proof of (4.1) we obtain $Q\rhd DA_1=Y_3{}^tY_4Y_6Y_7(DA_1\cap W_5)A_1$ with $|DA_1\cap W_5|=q^3$ and $H\leq N(D)$. Let $\overline{N(A_1)}=N(A_1)/A_1$. Let $\bar{U}$ be a complement of $\overline{DA_1\cap W_5}$ in $\bar{W}_5$. Then $\bar{Y}_1{}^t\bar{D}_0=\bar{Y}_1{}^t\bar{D}\times\bar{U}$ since M_1 centralizes $\bar{W}_5$. By Gaschütz's theorem [9, p. 121] there is a subgroup $\bar{X}$ such that $\bar{K}_1{}^t\bar{D}_0=\bar{X}\times\bar{U}$, so $(\bar{K}_1{}^t\bar{D}_0)'=\bar{X}'$. If $\bar{X}'\cap\bar{W}_5=1$, $\bar{M}_1\bar{A}=\bar{X}'\times\bar{X}'^t\times\bar{W}_5$ by (5.1). But $C_{\bar{M}_1\bar{A}}(t)=\bar{N}_1\bar{R}\simeq N_1R/R_1$, a contradiction. Now $|\bar{X}\cap\bar{W}_5|=q$ and every proper H-invariant subgroup of $\bar{W}_5$ is of order q. Thus $|\bar{X}'\cap\bar{W}_5|=q$. A similar argument shows that $\bar{X}'$ is perfect. By the definition $\bar{X}'=\bar{K}_1{}^t\bar{D}_1$, so the lemma holds.

(5.3) *Let* $D_2=O_2((K_1{}^tD_1)'W_{12})$. *Then* $M_1A/W_{12}=K_1{}^tD_2/W_{12}\times K_1D_2{}^t/W_{12}$ *and* $K_1{}^tD_2/W_{12}\simeq N_1R/V_{12}$.

PROOF. Let D be as in the proof of (5.2). Then $DA_0=D_0>D_1$, so $M_1A\rhd Z(D_1)\cap A_1\geq W_{12}W_{12}{}^v\neq W_{12}$. Since $C_Q(A_1)=A_0$ and N_1 is irreducible on $A_1/R_1W_{12}\simeq R_1W_{12}/W_{12}$ by [10, (3.6) (1)], $|Z(D_1)\cap A_1|=q^6$. Let $\overline{N(W_{12})}=N(W_{12})/W_{12}$. Then $C_{\bar{A}_1}(K_1)\cap C_{\bar{A}_1}(K_1{}^t)=1$ and $K_1{}^t$ acts on $\overline{(Z(D_1)\cap A_1)}\cap C(Y_1)\neq 1$. Hence as in the proof of (5.1), $\bar{M}_1\bar{A}_1=\bar{K}_1{}^tC_{\bar{A}_1}(K_1)\times\bar{K}_1C_{\bar{A}_1}(K_1{}^t)$ and $\bar{K}_1{}^tC_{\bar{A}_1}(K_1)\simeq C_{\bar{M}_1\bar{A}_1}(t)=\bar{N}_1\bar{R}_1$.

Let J_1 be a complement of Y_1 in $N_{K_1}(Y_1)$ such that $H_5\leq J_1J_1{}^t$. Then J_1 centralizes A_0/A_1, so it centralizes D_1/A_1 by (5.1). Moreover $C_{R_1/V_{12}}(H_5)=1$ implies $C_{\bar{A}_1}(K_1)\cap C(J_1{}^t)=1$. Then as $Z(\bar{A})\geq\bar{A}_1$, $\bar{K}_1{}^t\bar{D}_1=\bar{K}_1{}^tC_{\bar{D}_1}(J_1)\times[\bar{D}_1, J_1]$. Since $K_1{}^tD_1/A_1$ is perfect, we get $\bar{K}_1{}^t\bar{D}_2=(\bar{K}_1{}^t\bar{D}_1)'=\bar{K}_1{}^tC_{\bar{D}_1}(J_1)\lhd\bar{M}_1\bar{A}$. Recall that $N(\langle t\rangle B_1)=B_1N_G(B_1)$, so $C_{Q/B_1}(t)=PB_1/B_1$. Then as $Z(P/V_{12})=S_1/V_{12}$, $(Z(\bar{Q})/\bar{B}_1)\cap C(t)=1$ and so $Z(\bar{Q})=\bar{B}_1$. We have $\bar{D}_2\cap\bar{A}_1=C_{\bar{A}_1}(K_1)$, so $Z(\bar{Q})\cap\bar{D}_2\cap\bar{D}_2{}^t\leq C_{\bar{A}_1}(K_1)\cap C_{\bar{A}_1}(K_1{}^t)=1$. As $\bar{D}_2\lhd\bar{Q}$, this implies $\bar{D}_2\cap\bar{D}_2{}^t=1$ and thus $\bar{M}_1\bar{A}=\bar{K}_1{}^t\bar{D}_2\times\bar{K}_1\bar{D}_2{}^t$ with $\bar{K}_1{}^t\bar{D}_2\simeq C_{\bar{M}_1\bar{A}}(t)=\bar{N}_1\bar{R}$.

(5.4) *Let* $I_1/B_4=C_{B/B_4}(K_3)$. *Then* $Y_3{}^tI_1=Y_1{}^tD_2B_4$ *with* $Y_3{}^t\leq D_2$ *and* $Y_1{}^t\leq I_1$.

PROOF. We have $D_0=D_2A_0$, so $D_0B_4=D_2B_4$. (1.2) and $C_{A_0}(H_3)=1$ give $Y_3{}^t=C_{D_0}(H_3)=$

$C_{D_2}(H_3)$. By (3.3) (2), $C_{B/B_4}(Y_3)=I_1W_4/B_4$. As $Y_3\leq D_2{}^t$, (5.3) shows $C_{B/B_4}(Y_3)\geq Y_1{}^tW_4B_4/B_4$. Hence $Y_1{}^t\leq C_{I_1W_4}(H_1)\leq I_1$ by (1.2). Now the lemma follows from (3.3) (2).

(5.5) *Let $D_3=O_2((K_1{}^tD_2)')$. Then $M_1A=K_1{}^tD_3\times K_1D_3{}^t$. Furthermore there is a normal subgroup I_5 of M_3B such that $M_3B=K_3{}^tI_5\times K_3I_5{}^t$ and $Y_1{}^tD_3=Y_3{}^tI_5$.*

PROOF. Let $\overline{N(B_3)}=N(B_3)/B_3$. As $Y_3{}^t\leq D_2$, (5.3) shows $|C_{\bar{B}_4}(Y_3{}^t)|>q^4$. Thus $C_{\bar{B}_4}(K_3{}^t)\neq 1$, for $K_3=\langle Y_3, Y_3{}^s\rangle$. By [10, (3.8)(1)], $C_{\bar{B}_4}(N_3)=1$, so $\bar{B}_4\geq C_{\bar{B}_4}(K_3)\times C_{\bar{B}_4}(K_3{}^t)$. Now N_3 is irreducible on $C_{\bar{B}_4}(t)=\bar{S}_4\simeq S_4/S_3$, whence $\bar{M}_3\bar{B}_4=\bar{K}_3{}^tC_{\bar{B}_4}(K_3)\times\bar{K}_3C_{\bar{B}_4}(K_3{}^t)$ and $\bar{K}_3{}^tC_{\bar{B}_4}(K_3)\simeq C_{\bar{M}_3\bar{B}_4}(t)=\bar{N}_3\bar{S}_4$. Let J_3 be as in the first paragraph of section 4. Then $C_{\bar{B}_4}(K_3{}^t)\cap C(J_3)=1$ since $C_{\bar{S}_4}(H_{10})=1$. Now $Z(\bar{B})\geq\bar{B}_4$, so setting $I_2/B_3=C_{I_1/B_3}(J_3)$ we have $\bar{K}_3{}^t\bar{I}_1=\bar{K}_3{}^t\bar{I}_2\times C_{\bar{B}_4}(K_3{}^t)$. Since $K_3{}^t$ is irreducible both on I_1/B_4 and on $C_{\bar{B}_4}(K_3)$, $(\bar{K}_3{}^t\bar{I}_1)'=\bar{K}_3{}^t\bar{I}_2$ and $M_3B/B_3=K_3{}^tI_2/B_3\times K_3I_2{}^t/B_3$.

Set $I_3/B_2=C_{I_2/B_2}(J_3)$. Then by (3.4) we have $(K_3{}^tI_2/B_2)'=K_3{}^tI_3/B_2$ and $M_3B/B_2=K_3{}^tI_3/B_2\times K_3I_3{}^t/B_2$.

Let J_1 be as in the proof of (5.3). Then $H_1J_1J_1{}^t$ is an abelian group containing $H=H_1H_5$ and normalizes Q, so J_1 normalizes W_i, $1\leq i\leq 12$. Note that J_1 normalizes K_3 since $M_3=E(C(H_3))$. Now $I_1=I_3B_4$, $C_{I_1}(H_1)=Y_1{}^tY_1'$, and $C_{B_4}(H_1)=W_2=Y_1'^tY_1'$. So $C_{I_1}(H_1)=C_{I_3}(H_1)W_2$ by (1.2). Then $Y_1'^t=C_{I_1}(H_1)'\leq C_{I_3}(H_1)$, whence $Y_1{}^tY_1'=C_{I_3}(H_1)\times Y_1'$. Since J_1 is transitive both on $(Y_1/Y_1')^\sharp$ and on $Y_1'^\sharp$, $C_{I_1}(H_1)/Y_1'$ has exactly two $J_1J_1{}^t$-invariant subgroup of order q. Hence $Y_1{}^t=C_{I_3}(H_1)$. By (5.4), $Y_3{}^tI_3\leq Y_1{}^tD_2B_4$, so [10, (3.9) (5)] implies $(Y_3{}^tI_3/B_2)'\leq Y_1{}^tD_2B_2/B_2$. Now $Y_3{}^tI_3/B_2\simeq C_{Q/B_2}(t)\simeq P/S_2$ by the above paragraph and we have $(Y_1{}^tB_2/B_2)(Y_3{}^tI_3/B_2)'=I_3/B_2$, so $I_3\leq Y_1{}^tD_2B_2$. Thus $Y_3{}^tI_3=Y_1{}^tD_2B_2$. Then $Y_3{}^tI_3/B_1=Y_1{}^tD_2B_1/B_1\times(D_2{}^t\cap B_2)B_1/B_1$ by (5.3) and we can write $K_3{}^tI_3/B_1=X/B_1\times(D_2{}^t\cap B_2)B_1/B_1$ for some subgroup X by Gaschütz's theorem since M_3B centralizes B_2/B_1. Setting $I_4=O_2((K_3{}^tI_3)'B_1)$ we have $M_3B/B_1=K_3{}^tI_4/B_1\times K_3I_4{}^t/B_1$.

Set $I_5=C_{I_4}(J_3)$. Then $(K_3{}^tI_4)'=K_3{}^tI_5$ and $M_3B=K_3{}^tI_5\times K_3I_5{}^t$ by (3.2).

Set $Q_1=Y_3{}^tI_5$, so that $Q=Q_1\times Q_1{}^t$ and $Q_1\simeq C_Q(t)=P$. As $Q_1\leq Y_1{}^tD_2B_2$ and B_2 is abelian, $Q_1'\leq Y_1{}^tD_2$. Moreover $Y_1{}^t\leq I_3=I_5B_2$ and $C_{B_2}(H_1)=1$, so $Y_1{}^t\leq I_5$ by (1.2). Thus $Q_1=Y_3{}^tY_1{}^tQ_1'\leq Y_1{}^tD_2$ and $Y_1{}^tD_2=Q_1\times(Q_1{}^t\cap W_{12})$. By Gaschütz's theorem $K_1{}^tD_2=X_1\times(Q_1{}^t\cap W_{12})$ for some subgroup X_1. Hence $M_1A=K_1{}^tD_3\times K_1D_3{}^t$. Now $Q_1\leq Y_1{}^tD_2=Y_1{}^tD_3W_{12}$, so $Q_1'\leq Y_1{}^tD_3$. (1.2) and $D_2=D_3W_{12}$ give $Y_3{}^t=C_{D_2}(H_3)\leq D_3$. Thus $Q_1=Y_1{}^tD_3$ and the lemma is proved.

Let $Q_1=Y_1{}^tD_3=Y_3{}^tI_5$ and $G_1=\langle Q_1, K_1{}^t, K_3{}^t\rangle$. Define an H-homomorphism ϕ: $G_1\rightarrow C=C(t)$ by $x\mapsto xx^t$. Under this homomorphism $K_1{}^tD_3\simeq C_{M_1A}(t)=N_1R$ and $K_3{}^tI_5\simeq C_{M_3B}(t)=N_3S$. Let $U_i=W_i\cap Q_1$, $1\leq i\leq 12$. Then $U_1=Y_1{}^t$, $U_3=Y_3{}^t$, and $W_i=U_i\times U_i{}^t$, $1\leq i\leq 12$. Note that $Z(U_1)=U_2$, $Z(U_4)=U_8$, $Z(U_5)=U_{12}$, $Z(U_6)=U_{11}$, $U_i=C_{D_3}(H_i)$ for $i=3, 4, 5, 6, 7, 9, 10$, and $U_j=C_{I_5}(H_j)$ for $j=1, 4, 5, 6, 7, 9, 10$. By the first paragraph of Section 3,

$$U_3{}^r=U_7,\ U_4{}^r=U_6,\ U_5{}^r=U_5,\ U_9{}^r=U_{10};$$
$$U_1{}^s=U_4,\ U_5{}^s=U_6,\ U_7{}^s=U_9,\ U_{10}{}^s=U_{10}.$$

(5.6) *Let $G_0=\langle Q, M_1, M_3\rangle$. Then $G_0=G_1\times G_1{}^t$ and $G_1\simeq{}^2F_4(q)$.*

PROOF. Since $Q_1\cap Q_1{}^r=D_3$ and $Q_1\cap Q_1{}^s=I_5$,

(1) $$Q_1{}^r\cap Q_1\cap Q_1{}^s=D_3\cap I_5=U_4U_5U_6U_7U_9U_{10}.$$

Transforming this equation by r and s respectively, we have

(2) $$Q_1\cap Q_1^r\cap Q_1^{sr}=U_3U_4U_5U_6U_9U_{10},$$
(3) $$Q_1\cap Q_1^s\cap Q_1^{rs}=U_1U_5U_6U_7U_9U_{10}.$$
By (1) and (2), $Q_1^s\cap Q_1\cap Q_1^r\cap Q_1^{sr}=U_4U_5U_6U_9U_{10}$, so
$$Q_1\cap Q_1^s\cap Q_1^{rs}\cap Q_1^{srs}=U_1U_5U_6U_7U_{10}.$$
By (1) and (3), $Q_1^r\cap Q_1\cap Q_1^s\cap Q_1^{rs}=U_5U_6U_7U_9U_{10}$, so
$$Q_1\cap Q_1^r\cap Q_1^{sr}\cap Q_1^{rsr}=U_3U_4U_5U_9U_{10}.$$
Arguing similarly, we get
(4) $$Q_1\cap Q_1^r\cap Q_1^{sr}\cap Q_1^{rsr}\cap Q_1^{srsr}\cap Q_1^{rsrsr}\cap Q_1^{srsrsr}\cap Q_1^{rsrsrsr}=U_3,$$
(5) $$Q_1\cap Q_1^s\cap Q_1^{rs}\cap Q_1^{srs}\cap Q_1^{rsrs}\cap Q_1^{srsrs}\cap Q_1^{rsrsrs}\cap Q_1^{srsrsrs}=U_1.$$
Since V_1 is transitive on $\mathrm{Syl}_2(N_1)-\{V_1\}$, $r\in V_1^{ra}$ for some $a\in V_1$. Take an involution $u\in K_1^t$ such that $r=uu^t$. Then $u\in U_1^{ra}$, for $r\in W_1^{ra}=U_1^{ra}\times U_1^{rat}$ and $U_1^{ra}\leq K_1^t$. By (4), $U_3\leq D_3^{rsrsrsr}$, so $U_3^s\leq D_3^{(rs)^4}$. By (5), $U_1\leq I_5^{srsrsrs}$, so $U_1^r\leq I_5^{(sr)^4}=I_5^{(rs)^4}$ since $|rs|=8$. Thus $[U_1^r, U_3^{st}]\leq[Q_1, Q_1^t]^{(rs)^4}=1$. Take $b\in U_1$ such that $a=bb^t$. Then as $U_1^r\leq K_1$, $U_1^{ra}=U_1^{rb}$. Since $[b, U_3^{st}]\leq[U_1, K_3]=1$, we have $[U_1^{ra}, U_3^{st}]=1$. Moreover $[U_1^{ra}, I_5^t]\leq[K_1^t, Q_1^t]=1$ and $Q_1^s=U_3^sI_5$. Thus $[u, Q_1^{st}]\leq[U_1^{ra}, Q_1^{st}]=1$. As $Q_1^r=Q_1^u$, this gives $[Q_1^r, Q_1^{st}]=[Q_1, Q_1^{st}]^u\leq[K_3^tI_5, K_3I_5^t]^u=1$. Now $[Q_1, Q_1^t]=[Q_1, Q_1^{st}]=1$ and $[Q_1^r, Q_1^t]\leq[K_1^tD_3, K_1D_3^t]=1$. Hence $[G_1, G_1^t]=1$ and $G_0=G_1*G_1^t\geq L$. Let L_0 denote the image of ϕ and set $X=G_1\cap G_1^t$. Then $C_{G_0}(t)=L_0C_X(t)$. Since $K_1^tD_3$ and $K_3^tI_5$ are perfect, so is G_1. Thus $L_0=L_0'$ and as $C(t)^{(\infty)}=L$, $L_0=L$. The kernel of ϕ is contained in X and the Schur multiplier of ${}^2F_4(q)$ is trivial. Hence (5.6) holds.

REMARK. If one wishes to show that $G_0\triangleleft G$, a key point may be to establish that $N(Q_1)\leq N(G_0)$ (see [14, (14.7)]). We can verify that $Z_4(P)=V_9V_{10}V_{11}V_{12}$, $Z_5(P)=V_7V_8Z_4(P)=R_1S_3$, $\mathscr{E}^*(Z_5(P))=\{R_1, S_3\}$, $C_P(R_1)=V_5R_1$, and $C_P(S_3)=S_3$. Since $Q=Q_1\times Q_1^t$ with $Q\simeq P$, $Z_5(Q)=A_1B_3$ and $\mathscr{E}^*(Z_5(Q))$ consists of four members, two of which are A_1 and B_3 and the remaining two members are interchanged by t. Moreover $|C_Q(A_1)|>|C_Q(E)|$ for each $E\in\mathscr{E}^*(Z_5(Q))-\{A_1\}$. As $Z(Q)=W_{12}$, we get $A=C_Q(A_1/W_{12})\triangleleft N(Q)$. Also $B=C_Q(Z_2(Q))$ by (3.1) (1). It then follows from [10, (3.10)] and (3.1) (2) that $N(Q)=N(\mathrm{A})\cap N(B)\leq N(G_0)$. By [10, (3.1)(2)], $\langle t\rangle\in\mathrm{Syl}_2(C(L))$. So $|C(G_0)|$ is odd since $C(G_0)\triangleleft C(L)\cap N(G_0)$. Note that ${}^2F_4(q)$ has two conjugacy classes of involutions (see [12]) and that the outer automorphism group of ${}^2F_4(q)$ is of odd order. Note also that $N(\langle t\rangle P)=W_{12}N_G(P)$ since $Z(\langle t\rangle P)=\langle t\rangle V_{12}$ and $t^{N(\langle t\rangle P)}=t^{W_{12}}$. Arguing as in [19, (7.22)] we see that if $t\in N(G_0)^g$ for an element $g\in G$ then $g\in N(G_0)$. Hence as in [19, (7.23)] we have $N(Q_1)\leq N(G_0)$.

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