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A REMARK ON THE STANDARD FORM PROBLEM
FOR $^2F_4(2^{2n+1}), n \geq 1$

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In [10] Miyamoto showed that if a finite group $G$ has a standard subgroup $L$ isomorphic to $^2F_4(2^{2n+1}), n \geq 1$, such that $LO(G) \not\approx G$ and a Sylow 2-subgroup of $C_0(L)$ is cyclic, then $O^2(G) \not\approx G$. He obtained much more information. In this paper we proceed to construct a subgroup $G_0$ isomorphic to $F_4(q)$ or $^2F_4(q) \times ^2F_4(q)$, $q = 2^{2n+1}$. More precisely the following theorem is proved.

**Theorem.** Let $G$ be a finite group with $F^*(G)$ simple and suppose $L$ is a standard subgroup of $G$ isomorphic to $^2F_4(q)$, $q = 2^{2n+1} \geq 8$. Assume that a Sylow 2-subgroup of $C_0(L)$ is cyclic and that $L \not\approx G$. Then $O^2(G)$ possesses a subgroup $G_0$ of odd index which is isomorphic to $F_4(q)$ or $^2F_4(q) \times ^2F_4(q)$.

By a further fusion argument it might be possible to show that $G_0$ is normal in $G$ and then $F^*(G) \cong F_4(q)$. But such an effort is no longer important. The simplicity of $F^*(G)$ in the hypothesis of the theorem is necessary since we use, in the proof of the theorem, the unbalanced group theorem to determine the structure of certain 2-local subgroups of $G$. The proof of this fundamental theorem has been completed by the work of many authors. The detailed description of the theorem can be found in Harris [8], Solomon [16], Walter [18].

**Unbalanced Group Theorem.** Let $X$ be a finite group with $F^*(X)$ simple. Assume that $O(N_X(T)) \neq 1$ for some 2-subgroup $T$ of $X$. Then $F^*(X)$ is isomorphic to one of the known finite simple groups.

Our notation is fairly standard. Possible exceptions are the use of the following: For a subset $D$ of a group, $\mathcal{S}(D)$ denotes the set of involutions in $D$ and for a 2-group $P$, $\mathcal{S}^*(P)$ denotes the set of maximal elementary abelian subgroups of $P$.

I. Preliminary Lemmas

(1.1) Let $G$ be a group acting on a group $X$ and let $A$ and $B$ be subgroups of $G$. If $B$ normalizes $C_X(A)$, then $[[A, B], C_X(A)] = 1$.

**Proof.** See [19, (2.4)].

The following lemma is due to Bender [4, 1.1(i'')].
(1.2) Let $P$ be a $p$-group acting on a $p'$-group $K$. Suppose $X^p = X \triangleleft K = XY$ with $Y = Y^p$. Then $C_K(P) = C_X(P)Y(P)$.

The following lemma can be easily verified.

(1.3) Suppose $K \cong SL_2(2^n), n \geq 3$, acts naturally on an elementary abelian group $V$ of order $2^{2n}$. Let $H$ be a subgroup of $K$ which corresponds to a diagonal subgroup of $SL_2(2^n)$ and $w$ an involution of $N_K(H)$. Then $V$ has precisely two $H$-invariant proper subgroups. They are of order $2^n$ and interchanged by $w$.

II. Properties of $^2F_4(2^{2n+1})$

For convenience we summarize some properties of $L = ^2F_4(q), q = 2^{2n+1} \geq 8$, which can be found in Parrott [12], Ree [13], Shinoda [15], or verified by direct computation.

Let $\alpha_i(t), 1 \leq i \leq 12, \text{ and } \theta$ be as in [13]. Then

\[
\begin{align*}
\alpha_1(t) \alpha_1(u) &= \alpha_1(t+u) \alpha_2(tu^2), \\
\alpha_4(t) \alpha_4(u) &= \alpha_4(t+u) \alpha_8(tu^2), \\
\alpha_6(t) \alpha_8(u) &= \alpha_6(t+u) \alpha_{10}(tu^2), \\
\alpha_6(t) \alpha_6(u) &= \alpha_6(t+u) \alpha_{11}(tu^2), \\
\alpha_7(t) \alpha_7(u) &= \alpha_7(t+u) \alpha_{12}(tu^2).
\end{align*}
\]

The nontrivial commutator relations are as follows, where $[x, y] = x^{-1}y^{-1}xy$.

\[
\begin{align*}
\{\alpha_1(t), \alpha_2(u)\} &= \alpha_4(tu) \alpha_6(tu^2) \alpha_7(t^{2+1}u^2) \alpha_{10}(t^{4+3}u^{2+1}) \alpha_{12}(t^{4+3}u^{2+2}), \\
\{\alpha_1(t), \alpha_4(u)\} &= \alpha_5(tu) \alpha_6(tu^2) \alpha_7(t^{2+1}u) \alpha_{10}(t^{4+3}u^{2+1}) \alpha_{11}(t^{2+1}u_{2+2}) \alpha_{12}(t^{2+1}u_{2+2}), \\
\{\alpha_3(t), \alpha_5(u)\} &= \alpha_7(tu) \\
\{\alpha_3(t), \alpha_8(u)\} &= \alpha_9(tu) \alpha_{12}(t^{2+1}u) \\
\{\alpha_3(t), \alpha_9(u)\} &= \alpha_{10}(tu) \alpha_{11}(t^{2+1}u) \\
\{\alpha_3(t), \alpha_{10}(u)\} &= \alpha_{12}(tu). \\
\{\alpha_4(t), \alpha_7(u)\} &= \alpha_9(tu) \alpha_{12}(t^{2+1}u), \\
\{\alpha_4(t), \alpha_9(u)\} &= \alpha_{12}(tu). \\
\{\alpha_5(t), \alpha_6(u)\} &= \alpha_{11}(tu), \\
\{\alpha_5(t), \alpha_8(u)\} &= \alpha_{10}(tu) \alpha_{11}(t^{2+1}u) \\
\{\alpha_5(t), \alpha_9(u)\} &= \alpha_{12}(tu), \\
\{\alpha_5(t), \alpha_{10}(u)\} &= \alpha_{12}(tu). \\
\{\alpha_7(t), \alpha_8(u)\} &= \alpha_{11}(tu), \\
\{\alpha_7(t), \alpha_9(u)\} &= \alpha_{12}(tu), \\
\{\alpha_7(t), \alpha_{10}(u)\} &= \alpha_{12}(tu). \\
\{\alpha_{10}(t), \alpha_8(u)\} &= \alpha_{12}(tu). \\
\{\alpha_{10}(t), \alpha_9(u)\} &= \alpha_{12}(tu). \\
\{\alpha_{10}(t), \alpha_{10}(u)\} &= \alpha_{12}(tu). \\
\{\alpha_{10}(t), \alpha_{11}(u)\} &= \alpha_{12}(tu). \\
\{\alpha_{10}(t), \alpha_{12}(u)\} &= \alpha_{12}(tu). \\
\{\alpha_{11}(t), \alpha_{11}(u)\} &= \alpha_{12}(tu). \\
\{\alpha_{11}(t), \alpha_{12}(u)\} &= \alpha_{12}(tu). \\
\{\alpha_{12}(t), \alpha_8(u)\} &= \alpha_{12}(tu). \\
\{\alpha_{12}(t), \alpha_9(u)\} &= \alpha_{12}(tu). \\
\{\alpha_{12}(t), \alpha_{10}(u)\} &= \alpha_{12}(tu). \\
\{\alpha_{12}(t), \alpha_{11}(u)\} &= \alpha_{12}(tu). \\
\{\alpha_{12}(t), \alpha_{12}(u)\} &= \alpha_{12}(tu).
\end{align*}
\]
REMARK. The commutator formulas of $[\alpha_4(t), \alpha_1(u)]$ in [12, Section 2] and $[\alpha_2(t), \alpha_8(\eta)]$ in [15, (2.3)] are incorrect. In fact

$$[\alpha_4(t), \alpha_1(u)]=\alpha_6(t^2e+1)\alpha_6(tu^{2e})\alpha_6(t^2+1u)\alpha_6(t^2+2u^{2e+1}).$$

Using the notation of [13] we set

$$r=\omega_3\omega_{2e}, s=\omega_{1e}^e \omega_{2e},$$

Then $|r|=s|=2$, $|rs|=8$, and

$$\begin{align*}
\alpha_3(t)^r &= \alpha_3(t), & \alpha_4(t)^r &= \alpha_5(t), & \alpha_6(t)^r &= \alpha_6(t), \\
\alpha_3(t)^s &= \alpha_{10}(t), & \alpha_4(t)^s &= \alpha_4(t), & \alpha_5(t)^s &= \alpha_5(t), \\
\alpha_3(t)^{rs} &= \alpha_9(t), & \alpha_4(t)^{rs} &= \alpha_9(t), & \alpha_5(t)^{rs} &= \alpha_9(t).
\end{align*}$$

Let $H_i$ and $V_i$ be as in [10, Section 2]. By using [13, (3.16)] we can verify that

$$\begin{align*}
C_i(H_i) &= H_i \times N_i \quad \text{with} \quad N_i = \langle V_i, H_5, r \rangle \cong \mathbb{S}_2(q), \\
C_i(H_i) &= H_i \times N_i \quad \text{with} \quad N_i = \langle V_5, H_{10}, s \rangle \cong \mathbb{S}_2(q).
\end{align*}$$

### III. Some 2-local subgroups of $G$

From now on assume the hypothesis of the theorem. We will continue the notation of Miyamoto [10]. Moreover, let $A_0 = W_5 A_1$ and $B = W_4 W_5 B_4$. For $i=1, 3, 5, 6, 7, 9, 10$, note that $W_i$ is the unique Sylow 2-subgroup of $M_i$ containing $V_i$ and so $H$-invariant. Furthermore, $\langle r \rangle W_i \in \text{Syl}_2(C(H_i))$ and $C_H(H_i) = W_i$. For $i=2, 3, 7, 8, 9, 10, 11, 12$, the map $W_i/V_i \to V_i$ is an $H$-isomorphism. Similarly, for $i=1, 4, 5, 6$, $W_i/V_i Z(W_i)$ and $V_i/Z(V_i)$ are $H$-isomorphic. By [10, (2.5)],

$$\begin{align*}
W_i^* &= W_7, & W_4^* &= W_6, & W_5^* &= W_5, & W_6^* &= W_{11}, & W_7^* &= W_{10}, & W_8^* &= W_{12},
W_i^* &= W_4, & W_2^* &= W_6, & W_3^* &= W_9, & W_4^* &= W_{10}, & W_5^* &= W_{11},
W_i^* &= W_6, & W_2^* &= W_6, & W_3^* &= W_9, & W_4^* &= W_{10}, & W_5^* &= W_{11}.
\end{align*}$$

(3.1) (1) $Z(Q) = W_{12}$, $Z_3(Q) = B_1$, $Z_3(Q) = B_2$, $C_4(B_3) = B_4$, $C_4(A_4) = A_0$, and $A' = A_1$.

(2) $M_5 B < N(B) \text{ and } M_5 \cong \mathbb{S}_2(q) \times \mathbb{S}_2(q)$.

PROOF. By (3.9) (2) and (5) of [10], $A_1 \triangleleft AB_4$. Hence $A_1 \triangleleft \langle AB_4, N_1 \rangle = M_1 A$. [10, (3.9) (4)] gives $C_{M_1 A}(A_1) = A_0$ and $Z(A) = W_{12}$. By [10, (3.4) (2)], $[M_1, W_{12}] = 1$, so $Z(Q) = W_{12}$. We have $Z_{\langle r \rangle B_i} \supseteq B_1$ since $A \subseteq C(A_i/W_{12})$ and $[W_i, W_{12}] = [W_4, W_{12}] = 1$. Now $t^g = \langle t \rangle B_4 = t S_1$ and $\mathcal{N}(\langle t \rangle B_1) = \langle t \rangle S_1, B_1$, so $N(\langle t \rangle B_1) = B_1 N_c(B_4)$. Thus $Z_{\langle t \rangle B_4} = S_1$ forces $Z(Q) \subseteq N(\langle t \rangle B_4) = B_1$. Similarly $[W_i, W_{10}] = [W_4, W_{10}] = W_{12}$ and $N(\langle t \rangle B_2) = B_2 N_0(B_2)$, so $Z_2(Q) = B_2$. In the proof of [10, (3.10)] it is shown that $C(\langle t \rangle B_4/B_4) = B_4 N_c(B_4)$. Then since $C_{B_4}(S_2) = S_4$ and [10, (3.8) (3)] give $B_1 N_0(B_4) \cap C(Q) = C_{B_4}(B_4) = B_4$, $C_{B_4}(B_4) = B_4$. As $W_1 = W_4$, $C_{B_4}(B_4) \geq B$. Moreover $C_{W_4}(B_3) \cap C(t) \leq C_{W_4}(S_1) = 1$ implies $C_{W_4}(B_3) = 1$ and $Q: C_{W_4}(B_3) \geq q^2$. So $C_{W_4}(B_3) = B$.

Let $\tilde{Q} = Q/B_4$. Then $\tilde{Q} \cong \tilde{W}_4 = C_{\tilde{B}}(H_4)$, for $W_4 B_3 = C_{\tilde{B}}(Z_4(Q)) B_3$. As $\tilde{W}_1 = [\tilde{W}_1, H_4]$, (1.1) shows $\tilde{B} = \tilde{W}_1 \times \tilde{W}_4$, which is elementary abelian. Now $C_{\tilde{B}}(t) = \tilde{S} = [\tilde{B}, t]$. Applying [10, (1.3)]
to the series \( 1 \leq B_2 \leq B_4 \leq B \), we have \( C(\langle t \rangle|B|B) = BN_B(B) = BK \). Let \( \overline{N(B)} = N(B)/B \). Then \( \overline{N(B)} \cap C(t) = \overline{K} \), so \( \overline{N} \) is a standard subgroup of \( \overline{N(B)} \) and \( \langle t \rangle \subseteq \text{Sym}(\overline{N(B)} \cap C(\overline{N})) \). Now \( \overline{N(B)} \cong \langle W_3, s \rangle = M_3 \), so \( E(\overline{N(B)}) = \overline{M}_3 \) by [7] and [6, (1.1)], whence \( M_3 \leq \overline{N(B)} \). Suppose \( M_3 \cong SL_4(q^2) \) and let \( I \) be a complement of \( W_3 \) in \( N_{\overline{M}_3}(W_3) \) containing \( H_1 = \langle t \rangle \). Then \( I \) normalizes \( W_2 = Q \) and \( H \leq H_2 \times I \), so \( I \) normalizes \( E(C(H_2)) = M_4 \) and \( W_1 \). Now \( B_3 \preceq \langle Q, s \rangle = M_3B \) and \( W_3 = [W_3, H_4] \) centralizes \( \overline{W}_4 = C_B(H_4) \) by (1.1), so [19, (2.7)] shows that \( C_B(W_4) = \overline{W}_4 \) and \( I \) acts transitively on the nonidentity elements of \( \overline{B_4}|\overline{W}_4 = W_1/W_2 \). If \( M_1 \cong Sz(q) \times Sz(q) \), \( I \) normalizes each component of \( M_1 \), for \( |I| \) is odd. If \( M_1 \cong Sp_4(q) \), \( W_1 \) has exactly two elementary abelian subgroup of order \( q^3 \) and they are normalized by \( I \). In any case \( I \) is not transitive on \( (W_1/W_2)^2 \). Thus (2) holds.

Let \( \hat{Q} = Q/A_1 \). [10, (3.8) (3)] gives \( B_2 < Q \). Then \( \overline{W}_7 \leq Z(\hat{Q}) \) by (1.1), for \( \overline{B}_2 = \overline{W}_2 = C_{Z(B)}(H_2) \). Similarly, \( \hat{Q} \cap C_{\hat{Q}}(A_1) = \overline{A}_0 = C_{\overline{Z}(H_2)}(A_1) \). Hence \( \overline{A} \) is abelian by [10, (3.9)(1)] and the action of \( N_3 \). Now \( [W_6, W_8] = W_{12} \) and \( [W_5, W_4] = [W_6, W_3] = W_{11} \), so \( B_1 \leq A' \) and thus \( \overline{A}' = A_1 \) by [10, (3.6) (1)].

By the above lemma we can write \( M_3 = K_3 \times K_3 \) with \( K_3 \cong SL_4(q) \). Let \( v \) be an element of \( K_3 \) such that \( uv = s \). Set \( Y_3 = W_3 \cap K_3 \) and \( Z = C_{\overline{M}_3}(K_3) \). Note that \( H \) normalizes \( K_3, Y_3, \) and \( Z, \) for \( H \leq H_2 \times N_3 \). Note also that \( Q = W_3B \).

(3.2) \( M_3B_1 = K_3C_B(K_3) \times K_3C_B(K_3), K_3C_B(K_3) \cong N_3S_1, \) and \( W_{13} = Z \times Z' \).

\textbf{Proof.} Since \( Y_3 = Y_3' = Y_3'' \) is a 2-group, \( C_{B_1}(Y_3) = C_{B_1}(Y_3') = \overline{1} \). Then \( W_{13} \cap W_{13}' = W_{12} \cap W_{12}' = W_{12} = 1 \) implies \( C_{B_1}(Y_3) \neq W_{12} \). Thus \( C_{B_1}(K_3) \neq 1 \), for \( K_3 = \langle Y_3, Y_3' \rangle \). By [10, (3.5)], \( N_3 \) is irreducible on \( B_1/S_1 \cong S_1 \), so \( C_{B_1}(K_3) \cap C_{B_1}(K_3) \) remains. As \( C_{N_3B_1}(t) = N_3S_1, \) the lemma holds.

Let \( D_0 = C_d(A_1/Z, Y_3 = D_0 \cap W_4 \) for \( i = 4, 6, \) and \( Y_7 = Y_7' \). These subgroups are \( H \)-invariant. Note that \( D_0 \leq M_3A_4 \), for \( [M_1, W_{12}] = 1 \).

(3.3) \( D_0 = Y_4'Y_4Y_6Y_7'Y_6', \) \( A/A_0 = D_0/A_0 \times D_0'/A_0, \) \( D_0 \cap W_3 = Y_3', \) \( D_0 \cap W_7 = Y_7', \) and \( |Y_1| = q^a \) with \( Y_1 = Y_1'Y_1'' \) and \( Y_1'' = Z(W_1) \) for \( i = 4, 6 \).

(2) \( B/B_4 = C_{B/B_4}(K_4) \times C_{B/B_4}(K_4), K_4C_{B/B_4}(K_4) \cap N_3S_1 = 1, \) and \( K_4 \) centralizes \( D_0B_4 \).

\textbf{Proof.} By (3.2), \( C_{W_3}(B_4/Z) = Y_4' \). Then \( W_{13} = W_6 = W_7, \) \( C_{W_3}(W_8 \cap W_{13}) = Y_7', \) \( A_1 = W_6A_1 \cap B_4 \) and \( B_2 = W_4A_1 \). Thus \( Y_7 \leq D_0 \) by [10, (3.8) (2)]. Since \( Z \cap Z' = 1 \) and \( [A_1, A] \neq 1, \) we have \( D_0 = A \) and by [10, (2.4) (2), (3.9) (1)], \( |A| = q^4A_0 \). Moreover \( Y_7, Y_7' = W_2, \) so \( D_0D_0' = A. \) Thus \( A/A_0 = D_0/A_0 \times D_0'/A_0, \) \( D_0 \cap W_3 = Y_3', \) and \( D_0 \cap W_7 = Y_7' \).

Let \( \overline{N(B_3)} = N(B_3)/B_4 \). As shown in the proof of (3.1), \( \overline{B} = \overline{W}_1 \times \overline{W}_4 \) and \( [W_3, \overline{W}_4] = 1 \). Then as in (3.2) we get \( \overline{M}_3B = \overline{K}_3C_B(K_3) \times \overline{K}_3C_B(K_3) \times C_{\overline{M}_3}(t) = \overline{N}_3 \overline{S}, \) and \( \overline{W}_4 = C_{\overline{W}_4}(K_3) \times C_{\overline{W}_4}(K_3). \) Since \( C_{\overline{B}/Z} = Y_3' \cap Z \leq B \), \( D_0 \leq Y_3'Z/B \leq A \leq Y_3'W_3B_4 \). By (3.1), \( B_0B_4 \leq \overline{C_{B}(Z)} = \overline{B}(Z), \) so \( [V_0, V_1] = V_12 \) and \( Z \cap Z' = 1 \) imply \( W_4 \leq D_0B_4 \). Now \( Y_4' \leq D_0 \leq Q \) and every \( H \)-invariant proper subgroup of \( \overline{W}_4 \) is of order \( q^2 \). Thus \( D_0 = Y_4' \cap C_{\overline{W}_4}(K_3), \) proving (2). Since \( A \cap B_4 = W_7A_0 \) and \( W_6A_0 \) and \( W_7A_0 \cap C(W_8 \cap W_{13}) = W_7A_0 \) and so \( W_8A_0 \cap C(W_8 \cap W_{13}) = W_7A_0 \). Thus \( Y_6 \leq W_11 = q^2 \). As \( Y_4 \overline{W}_11 = q \). Hence as in (3.2) the lemma holds.

(3.4) \( B_3B_2 = C_{B_3/B_2}(K_3) \times C_{B_3/B_2}(K_3), \) and \( K_3C_{B_3/B_2}(K_3) \cong N_3S_1/S_2. \)

\textbf{Proof.} Note that \( B_3 \preceq \langle Q, s \rangle = M_3B \) and \( W_8B_3 = A_1 \cap B_3 \preceq Q, \) so \( W_3 = [W_3, H_8] \) centralizes \( W_8B_3/B_2. \) Now \( W_8' = W_7 \) and \( W_8B_3 \cap W_8B_2 = B_2 \). Hence as in (3.2) the lemma holds.
(3.5) \( Q \subseteq \text{Syl}_2(O^*(G)) \).

Proof. In the proof of \([10, (3.11)]\) it is shown that \( \mathcal{S}(tQ) = tQ \), so \( tQ = tQ \) and
\[ N(tQ) = QNC(tQ). \] Thus \( tQ \) is a Sylow 2-subgroup of \( G \) and (3.5) holds.

IV. The Case \( M_1 \cong Sp_4(q) \)

In this section we assume that \( M_1 \cong Sp_4(q) \). Let \( J_3 \) be a complement of \( Y_3 \) in \( N_{X_3}(Y_3) \) such that \( H_{10} \leq J_3 \). Then \( J_3^s = J_3 \). As \( Q = W_2B \) and \( H_{10} \leq J_3 \), it is an abelian group containing \( H_{10} \). \( J_3 \) normalizes \( C_0(H_4) = W_i \) for \( i = 1, 3, 4, 5, 6, 7, 9, 10 \). Let \( D = D_0 \cap D_0^s \) and \( Y_3 = D \cap W_5 \). Let \( Y_i = Z(D) \cap W_i \) for \( i = 8, 9, 10, 11 \).

(4.1) (1) \( D = Y_3^s Y_3^t Y_3^s Y_3^t A \trianglelefteq (Q, M_1, K_3, H) \), \( C^*(W_5) = \{ Y_3, Y_3^t \} \), \( A/A_1 = D/A_1 \times D/A_1^t \), and \( A/D = D^t/A_1 \cong R/R_4 \) as \( HN_1 \)-modules.

(2) \( D/Z \) is elementary abelian of order \( q^4 \).

(3) \( Z(D) = Y_3^s Y_3^t Y_3^{s1} Y_3^{t1} W_{12} \). \( Z(D) \) is elementary abelian of order \( q^4 \), \( |Y_i| = q \) for \( i = 8, 9, 10, 11 \), and \( A_0/W_1 = Z(D)/W_{12} \times Z(D)/W_{12} \).

(4) \( W_5 B_3 \cap (W_5 B_3)^0 = B_3 \).

Proof. By the definition \( Z \) is \( J_3^s \)-invariant, so is \( D_0 \). The element \( \nu \) normalizes \( J_3^s \) and \( H_3 \), whence \( H_3 J_3^s \leq N(D) \). By (3.1), \( B_3 \trianglelefteq (Q, 3) \), \( M_3 B \) and as \( D_0 < Q \), \( Y_3^s B_3 \leq D < Q \) and \( Y_3^s B_3 \). Set \( I = A \cup B_3 \). Then \( Q \trianglelefteq W_5 W_5 A_0 \) and \( Y_3^s I \trianglelefteq D = D J = Y_3^s Y_3^t I \). If \( D = Y_3^s Y_3^t I \), \( [Y_3, I] \leq D \cap B = I \) and so \( [W_5, W_5] \leq I \). But \( [W_5, V_0] \leq I \), a contradiction. Thus \( D = D_0^t \), for each \( H \)-invariant proper subgroup of \( W_4/W_8 \) is of order \( q \). Then as \( C_1(H_4) = W_8 \), (1.2) gives \( C_D(H_4) W_4 = C_{D_0}(H_4) W_4 \). Now \( C_{D_0}(H_4) = W_4 \) and each \( H \)-invariant proper subgroup of \( W_4 \) is of order \( q \), so \( C_D(H_4) = D \cap W_4 \); and \( |D \cap W_5| = q \) or \( q \geq W_8 \). Suppose \( |D \cap W_5| = q \). Then \( D \cap W_5 = D^t \cap W_5 \) or \( W_5 = (D \cap W_5) \times (D^t \cap W_5) \). If the latter holds, then as \( (D \cap W_4) W_4 = Y_4 \) and \( Z(W_4) = W_8 \), (3.3) forces \( W_4 = (D \cap W_4) \times (D^t \cap W_4) \). But \( W_4 = W_3 \leq \text{Syl}_2(M_1) \) and \( M_3 \cong Sp_4(q) \), a contradiction. If \( \nu \) normalizes \( D \cap W_5 \), then \( W_4 \cap D \cap W_5 \) is abelian, contrary to \( W_4 \). Hence \( D \cap W_5 = Y_4 \), so \( Y_4 = Y_4 W_2 A_0 \trianglelefteq D W_2 A_0 \trianglelefteq D_0 W_7 \). If \( D \leq Y_4 Y_4 W_7 A_0 \), \( Y_4 W_2 \leq D \cap B \leq W_7 A_0 \). As \( W_5 A_0 = B_3 \trianglelefteq (Q, 3) \), this gives \( [W_3, W_2] \leq W_7 A_0 \). But \( [V_3, V_3] \leq W_7 A_0 \), a contradiction. Hence \( D W_7 A_0 = D W_7 \), for \( H \) is irreducible on \( Y_6/W_11 \). As \( W_7 A_0 \cap C(H_4) = W_11 \) and \( D_0 W_7 \cap C(H_4) = Y_6 \), \( D \cap W_6 = C_D(H_4) = Y_6 \). Then \( Y_4 Y_4 W_6 A_0 \trianglelefteq D_0 \). If \( Y_4^s Y_4 W_6 A_0 \trianglelefteq D A_0 \), \( B \cap D \cap A_0 \trianglelefteq Y_6 W_6 A_0 \) and as \( W_6 = Y_6^s \), \( B \cap D \cap A_0 \). Then \( W_5 W_6 \cap W_5 W_6 A_0 \), which conflicts with \( [V_3, V_3] = V_7 \). Hence \( D A_0 = D_0 \) by the action of \( H \). As \( C_A(H_2) = Y_6 \cap W_7 = C_D(H_2) = Y_7 \) by (1.2). We have \( D \geq [V_3, V_3] B_3 = V_9 B_3 \), since \( B \geq V_1 \). Then as \( D_0 \leq W_5 B_3 \), (3.4) follows. Therefore \( D \geq W_5 B_2 \). Hence \( D = Y_4^s Y_4 Y_4 Y_7 A_1 \). Now \( [D, A_1 A_1] \leq Z \) by the definition of \( D \), so \( [D, A_1 A_1 A_1] \leq W_12 \). Moreover \( A_1 A_1 A_1 \geq B_2 \geq W_7 \) by (3.4), for \( A_1 \geq W_6 B_2 \). As \( [V_3, V_3] \leq W_4 \), we have \( Y_4 \neq W_5 \). If \( Y_4 = W_12 \), \( [Y_4, V_3] \leq D \cap B = Y_6 Y_7 A_1 \leq W_5 W_7 A_1 \). By (3.1), \( Y_3 Y_5 = W_5 \trianglelefteq D \trianglelefteq W_6 W_7 A_1 \), so \( W_5 \leq W_7 A_1 \). But \( [V_3, V_3] \leq W_6 W_7 A_1 \), a contradiction. Hence \( |Y_3| = q^3 \) by the action of \( H \).

As \( A_1 \leq D, D \cap A B = Q \) by (3.1). Thus \( D \trianglelefteq (Q, 3) \). By (3.4), \( W_6 W_6 A_0 \trianglelefteq D \cap B_3 \). Let \( N(B_3) = N(B_3)/B_3 \). Then \( \widehat{B}_4 \) is elementary abelian of order \( q^3 \) by \([10, (3.9)(5)]\) and \( \overline{D \cap B_4} = Y_5 \). \( \bar{W}_8 \) is of order \( q^4 \). As \( \mathcal{C}^*(\langle t \rangle B_3) = \{ B_3, \langle t \rangle S_3 \} \) and \( t^p = tS_3 \), \( N(\langle t \rangle B_3) = \)
$B_3N_G(S_3) = B_3N_G(S)$ and $N_G(B_3) \cap C(t) = N_G(S)$. By [10, (3.9) (4)], $[W_3, W_3] = 1$, so $K_3$ centralizes $W_3 \cap W_3'$. Suppose $W_3 \cap W_3' \neq 1$ and put $X = B_3 \cap C(K_3')$. Then $X$ is $M_3$-invariant and as $H_4 \leq M_3$, $\bar{X} = X \leq B_3 \cap C(H_4) = 1$. Now $N_X$ is irreducible on $B_3 \cap C(t) = S_3 \simeq S_3/S_3$, so the action of $N_X$ on $\bar{X} = X \leq B_3 \cap C(K_3') = N_S/S_3$. Then as $Z(V_3S_3/S_3) = V_3S_3/S_3$, $\bar{W}_3 = C_X(Y_3) \times C_X(Y_3)$ and $\bar{W}_3' = X \leq X$. This is impossible since $K_3$ normalizes $\bar{D} \cap B_3$. Thus $W_3 \cap W_3' = 1$ and $\bar{D} \cap B_3 = \bar{W}_3 \bar{W}_3'$, so $\langle A \rangle \bar{X}' = DB_3$. Then $D \cap B_3/Z$ is an elementary abelian subgroup of $Z(D/Z)$ by [10, (3.8) (2)].

Note that $W_3 = W_3'$. By [6, (2B)], there are subgroups $U_1$ and $U_2$ of order $q$ such that $W_12 = U_1U_2$ and $W_12 - (U_1 \cup U_2)$ is the set of square involutions of $W_3$. As $U_1 \cup U_2 = 1$, $U_1$ and $U_2$ are the only subgroups of order $q$ contained in $U_1 \cup U_2$. Thus $J_3'G'$ normalizes $U_1$ and $U_2$, so $Z = U_1$ or $U_2$ by (3.2). As $Y \leq D \cap B_4$, $Y/Z$ is elementary abelian. Hence $Y \in E^*(W_3)$ by [6, (2B) (1), (3)]. As $M_3 = M_3'$, the action of $t$ on $M_3$ shows $E^*(W_3) = \langle Y_3, Y_3' \rangle$. Then as $A' = A_1$, we have $A/\langle A \rangle = D \cap A_1 \times D/A_1$. Now $Y_3/Z \leq Z(D/Z)$, $C_{Y_3}(Y_3/Z) = Y_3$, and $D_0 = D_0$. Thus $D = C_\lambda(Y_3A_1/Z)$, which is $M_1$-invariant since $M_1$ centralizes $A_0/A_1$. Since $E^*(A_1) = \{A_1, <tR_1 \rangle \}$ and $t_1 = tR_1$, $N_1 < tR_1 = A_1N(C(R_1)) = A_1N(C(R_1))$. Hence $D/A_1 \in A_1 < tR_1 \rangle \cap C(t) = RA_1/A_1 \in R/R_1$ as $N_1$-modules. Then $N_1$ is irreducible on $D/Y_3A_1 \leq R/V_3R_1$, so $D/Z$ is elementary abelian. Thus (1) and (2) hold.

As $C_{Y_3}(A_1) = A_0$, $A_1 \leq Z(D \cap D) = Y_3A_1$. Moreover, $W_12W_12' = Z(D \cap D)$ is a subgroup of $Y_1W_12$ of order $q^3$ by (3.2). By [10, (3.6) (1)], $N_1$ is irreducible on $A_1/\langle A \rangle$, so $W_12W_12' = W_12W_12'$. Then $Z(D) \leq Z(D \cap A_1) = Y_3A_1/Z$, which is $M_1$-invariant since $M_1$ centralizes $A_0/A_1$. Since $E^*(A_1) = \{A_1, <tR_1 \rangle \}$ and $t_1 = tR_1$, $N_1 < tR_1 = A_1N(C(R_1)) = A_1N(C(R_1))$. Hence $D/A_1 \in A_1 < tR_1 \rangle \cap C(t) = RA_1/A_1 \in R/R_1$ as $N_1$-modules. Then $N_1$ is irreducible on $Y_3A_1 \leq R/V_3R_1$, so $D/Z$ is elementary abelian. Thus (1) and (2) hold.


Proof. As $Z(D) \cap A_1 = Y_3(Z(D \cap B_3)$, $Y_3$ centralizes $Y_3B_3B_3$. Hence by (4.1) (4), $Z(D)B_3B_3 = Z(D) \cap B_3$ is a natural module for $K_3$. Let $N(D) = N(D)/D$. Then $A_0B_3 = B_3 \cap W_3 \leq Z(D)$. Now $\overline{A} = \overline{A}_1 = \overline{A}_1 \leq \overline{A}_1 \leq \overline{A}$ and $C_{Y_3}(V_3) = V_3V_3R_1$ so (4.1) implies $Z(D) \leq V_3W_3V_3$. As $Z(D) = W_3A/\langle A \rangle$, $Z(D) \leq W_3A/\langle A \rangle = W_3B_4$. Then (3.2) gives $Z(D) \leq W_3B_4$. $K_3$ does not normalize $A_0D = D_0$, so it does not centralize $Z(D)$. So $V_3 \leq B_3$. As $V_3 < B_3$ and $C_{Y_3}(V_3) = V_3V_3R_1$, $\overline{A}_1 \leq \overline{Z(D)}$. If (4.1) (1), then $W_3 \leq Z(D) \leq Z(D)W_3 \leq Z(D)$. If (4.1) (1), then $Z(D) \leq Z(D)W_3 \leq Z(D)W_3 \leq Z(D)W_3$ by the action of $C_4$. Hence $Z(D) \leq Z(D)W_3 = Z(D)$. Since $Z(D) \leq Z(D)W_3 = Z(D)W_3$, we see that $B_4 = Z(D)Z(D) \leq DB_3B_3$ is a natural module for $K_3' = K_3$ by the above.

Denote by $U_1$ and $U_2$ the maximal elementary abelian subgroups of $W_1$. As $S(tW_1) = tW_1$, it follows from [6, (2.3), (2B) (1)] that $U_1 = U_2$. As $H_4 \leq J_4'$ is of odd order, it normalizes $U_1$ and $U_2$. Then $U_1W_3B_3B_3 > C_{N_3}(K_3)$ or $C_{N_3}(K_3')$ by (3.3) (2). Replacing $U_1$ and $U_2$ if necessary, we may assume that $U_1W_3B_3B_3 > C_{N_3}(K_3)$. Then $U_1W_3B_3B_3 = C_{N_3}(K_3)C_{N_3}(K_3')$, so $K_3$ normalizes $U_1W_3B_3$. Since $H_4$ is fixed-point-free on $W_3B_3$ and (1.2) shows $U_1W_3B_3 \leq C_{N_3}(K_3')$. Thus $K_3$ normalizes $U_2B_3$. We can choose involutions $u_1$
and \( u_2 \) as generators of the Weyl group of \( M_1 \) such that \( u_i^t = u_2 \), \((u_1u_2)^2 = r \), and \( u_i \) normalizes \( U_i \), \( i = 1, 2 \). Set \( Y_1 = W_1 \cap W_1^{r=1} \), \( Y_2 = W_1 \cap W_1^{r=2} \), and \( K_i = \langle Y_i, u_i \rangle \) for \( i = 1, 2 \). Then \( U_1 = Y_2 W_2 \), \( U_2 = Y_1 W_2 \), \( K_1 = K_2 \), and \( K_i \cong \mathbb{Z}_2 \langle q \rangle \) with \( Y_i \in \mathfrak{S}_1(K_i) \). Let \( J_0 \) be a complement of \( W_2 \) in \( N_{M_1}(W_1) \) containing \( H \) and set \( J_i = K_i \cap J_0 \), \( i = 1, 2 \). Then \( u_1 \) and \( u_2 \) normalize \( J_0 \). As \( HJ_0 \) is an abelian group containing \( H \) and normalizes \( Q \), it normalizes \( W_4 \), \( 1 \leq i \leq 12 \), \( Y_1 \), and \( Y_2 \). Set \( F_1 = U_i(A \cap B)D \) and \( F_2 = U_i(A \cap B_2)D \). Note that \( B_4 = W_d(A \cap B_2) \leq F_1 \cap F_2 \).

(4.3) (1) \( C_{A1}(U_1) = W_5 W_2^1D/D \) and \( C_{A1}(U_2) = W_5 W_6 W_7^1D/D \).

(2) \( F_1 \cong \langle Q, u_1, v \rangle \), \( F_2 \cong \langle Q, u_2, v \rangle \), \( C(F_1/D) = Z(BD/D) \), \( F_1/Z(F_1 \text{ mod } D) \) is elementary order \( g^4 \), and \( F_2/D \) is elementary abelian of order \( g^6 \).

(3) \( AF_2/F_2 \) is a natural module for \( K_2 \) and \( BF_2/F_2 \) is a natural module for \( K_2 \).

**Proof.** Note that \( F_1 = U_iW_4B_4D \) is \( K_5 \)-invariant and \( F_1 \cong \langle Q, v \rangle \), for \( Q' \leq F_1 \). Let \( D(N(D)) = D(N(D)/D) \). Then \( F_1 \cap B_4 \cong F_1 B_4 \) is a natural module for \( K_5 \). As \( C_{K_5}(V_3) = \overrightarrow{A} \cap \overrightarrow{B} \) by (4.1) (1), \( F_1 \cap C(B_3) = \overrightarrow{B} \). Now \( Z(\overrightarrow{B}) \neq \overrightarrow{B} \) and \( \overrightarrow{B} \) is irreducible on \( \overrightarrow{B} \) by (4.2), so \( Z(\overrightarrow{B}) = Z(\overrightarrow{B}) \). As \( V_1 \leq U_1 \), we get \( C_{K_5}(U_1) = \overrightarrow{A} \cap (Z(\overrightarrow{F}_1) = \overrightarrow{W}_3 \overrightarrow{W}_7 \). As \( V_2 \leq U_2 \), \( Z(Q) = \overrightarrow{W}_5 \overrightarrow{W}_7 \). If \( C_{K_5}(U_2) = Z(Q) \), \( M_1 \) normalizes \( Z(Q) \). But \( N_1 \) is irreducible on \( A/W_2 \), \( D_r = R/V_2 \), \( R \). Then \( F_1 \cong U_iC_{K_5}(U_2) \) is elementary abelian of order \( g^4 \) and normalized by \( u_2 \). As \( F_2 = U_2 B_2 D \) and \( \overrightarrow{Y}_2 \overrightarrow{B} = \overrightarrow{Q} \), it follows that \( F_2 \cong \langle Q, u_2, v \rangle \). Set \( \overrightarrow{I} \) \( \overrightarrow{W}_3 \overrightarrow{W}_7 = (\overrightarrow{A} \cap \overrightarrow{W}_3 \overrightarrow{W}_7) \cap C(U_1) \). Then \( \overrightarrow{I} \) \( \overrightarrow{W}_3 \overrightarrow{W}_7 \leq \overrightarrow{I} \leq \overrightarrow{A} \cap \overrightarrow{B} = \overrightarrow{W}_4 \overrightarrow{W}_4 \overrightarrow{W}_7 \). If \( \overrightarrow{I} \leq \overrightarrow{W}_4 \overrightarrow{W}_4 \overrightarrow{W}_7 \), then \( M_1 \) is a natural module on \( A/W_2 \). Thus \( \overrightarrow{I} \leq \overrightarrow{A} \cap \overrightarrow{B} \), which is \( u_1 \)-invariant. Hence \( F_1 \cong \langle Q, u_1, v \rangle \). As \( K_2 \) centralizes \( AF_2/F_2 \), then as \( Q = AF_2 Y_2, Q' \leq F_2 \). But \( (Q/B_3) = W_4 B_4 D_4 \) by (3.3) (2). As \( W_4 \leq F_2 \), this is a contradiction. Thus \( AF_2/F_2 \) is a natural module for \( K_2 \) by (6, (1K)). Finally, \( BF_2/F_2 \) is a natural module for \( K_3 \) by (3.3) (2).

(4.4) \( (u_4 V)^3 \in F_2 \).

**Proof.** As \( D \leq F_2 = D B \), \( Y_1 W_12 \leq Z(F_2) \leq Z(D) \cap B_2 \). If \( Z(F_2) = Y_1 W_12 \), \( C(Z(F_2)) \geq AB \), \( AB = B \). Thus \( F_2 \cong \langle Q \rangle \). Then \( F_2 \leq D \), \( D \) is a natural module for \( K_2 \). As \( J_3 = J_9, W_12 = Y_12, Z \). Now \( K_3 \) centralizes \( Z(F_2) \) \( Y_11 W_12 \), so \( [Z(F_2)/Z, J_3] = Y_11 W_12/Z \) and \( Y_11 W_12-Z = (Z(F_2)/Z) \cap C(J_3) \), which is \( \gamma \)-invariant. Thus \( \gamma \) centralizes \( Z_10Z/Z \). If \( K_2 \) centralizes \( Z(F_2)/W_12 \), then as \( [M_1, W_12] = K_2, Z(F_2)] \). But \( C_0(Z(F_2)) = F_2 \), a contradiction. Thus \( Z(F_2)/W_12 \) is a natural module for \( K_2 \) by (6, (1K)). Then \( [Z(F_2), J_3] = Y_10 Y_11 \) and \( Y_{10} y_{12} = Y_11 \). By (1.3) since \( J_2 y_{12} = J_2 \). For \( x_1 Y_{10} Z, x \), \( x^* = x \), \( [u_2 V]^3 \in \overrightarrow{W}_2 \). Similarly \( (u_4 V)^3 \) centralizes \( Y_11 Z \) \( W_12/Z \). Hence \( [(u_4 V)^3 \in F_2] \).

\( K_3 \) centralizes \( Z(D) \cap B \cap Z(F_3) \). By (4.2), \( J_2 \) is fixed-point-free on \( Z(D) \cap B \) and \( L_3 \) shows \( Y_3 \in \langle Z(D) \cap B \rangle = Y_3 \in \langle Z(D) \cap B \rangle \). Then \( Z(D)/Z(F_3), J_3 = Y_3 Y_3 Z(F_3)/Z(F_3) \) is \( \gamma \)-invariant and \( Y_3 Z(F_3) = Y_3 Z(F_3) \). If \( K_2 \) centralizes \( Z(D) \cap A_1/Z(F_3), [K_2, Z(D)] \leq Z(F_2) \). By (10, (3.9) (5)), \( C(Z(D)/Z(F_3)) \geq \langle DB_4 \rangle \). Now \( A \cap F_3 = A \cap DB_4 \), so \( A \cap F_3 \) is a natural module for \( K_2 \) by (4.3) (3). But then \( A \leq C(Z(D)/Z(F_3)) \), contrary to the action of \( K_2 \) on \( Z(D)/Z(D) \cap B \). Thus \( Z(D)/Z(D) \cap B_3 \) is a natural module for \( K_2 \) and \( Y^* \in Z(D)/Z(F_2) = Y_3 Z(F_2)/Z(F_2) \). Hence
We have shown that \((\mu_2)^3\) stabilizes \(Z(D) \geq Z(F_2) \geq Z \geq 1\) and \([F'_2,F_2] \geq Y_{10}B_1 \geq Z(F_2)\). Thus \(O((\mu_2)^3)\) centralizes \(I=Z(D)[F'_2,F_2]\). By (4.1) (1), \((Q/A_0B_2)_t \cap C(I)=PA_0B_2/A_0B_2\). Then \(C_{R}(V_9S_2)=V_9R_2S_2 \leq A_0B_2\), so \(C_{Q}(l) \leq A_0B_2 \cap C(Y_9)=Y_9A_0\). Moreover \([V_9,F'_2] \leq V_{11}\), so \((Y_7A_0/W_{12}) \cap C(Y_9)=Y_9Y_{7}A_0/W_{12}\) and \(Y_7A_0 \cap C(Y_9)=I\). Thus \(C_{Q}(l) = I\). Now \(I \leq Q \leq SL_2(O^2(G))\) and \(O^2(G) \cap C(I)=I\) by the unbalanced group theorem, so \(O^2(G) \cap C(I)=I\). Then as \(O((\mu_2)^3) \leq C(I), (\mu_2)^3\) is a 2-element. (4.3) (3) and the action of \(K_9,K_3\) on \(Z(F_2)/Z\) show \(C_{Q}(Z(F_2)/Z)=F_2\). Let \(X=\langle Q, K_9, K_3 \rangle\). Then \(C_X(Z(F_2)/Z)\) is 2-closed with \(F_2\) the unique Sylow 2-subgroup. As \((\mu_2)^3 \in C_X(Z(F_2)/Z), (4.4)\) holds.
(4.5) \( \langle Q, K_2, K_3 \rangle | F_6^0 = PSL_2(q) \).

**Proof.** Let \( X = \langle Q, K_2, K_3 \rangle \) and \( \bar{X} = X/F_6 \). Then \( Q \subseteq \text{Syl}_2(X) \). We have \( \bar{Q} \cap \bar{Q}^{u_2} = \bar{A} \) and \( \bar{Q} \cap \bar{Q}^{u} = \bar{B} \), so \( \bar{Q}^{u_2} \cap \bar{Q} \cap \bar{Q}^{u} = \bar{W}_4 \). As \( J_2 = J_3 \) and \( J_3 = J_3 \), it follows from (1.3) and (4.3) (3) that \( \bar{W}_4^{u_2} = \bar{Y}_3 \) and \( \bar{W}_4^u = \bar{Y}_2 \). Thus \( \bar{Q} \cap \bar{Q}^{u_2} \cap \bar{Q}^{u_2} = \bar{Y}_3 \) and \( \bar{Q} \cap \bar{Q} \cap \bar{Q}^{u_2} = \bar{Y}_2 \), so

\[
\bar{Q} = \bar{A} (\bar{Q} \cap \bar{Q}^{u_2} \cap \bar{Q}^{u_2}) = \bar{B} (\bar{Q} \cap \bar{Q}^2 \cap \bar{Q}^{u_2}).
\]

As \( W_4^{u_2} = Y_1 \) and \( Y_1^{u_2} = Y_1 \), \( Q \cap Q^{u_2} \leq C_0(Y_1) \). By [10, (3.9) (4)], \( [W_4, W_{10}] = W_{12} \) and \( [W_2, W_{10}] = [W_7, W_8] = 1 \) and as \( Y_1 \leq Z(F_2) \), \( C_0(Y_1) \leq W_{12} \). Then \( \bar{Q} \cap \bar{Q}^{u_2} \leq \bar{Y}_3 \). Now \( \bar{Y}_3 \) is the stabilizer of \( \bar{Y}_3 \) and \( \bar{Q} \cap \bar{Q} \cap \bar{Y}_3 = \bar{Y}_2 \), so \( \bar{Q} \cap \bar{Q} \cap \bar{Q}^{u_2} = \bar{Y}_2 \). Therefore

\[
\bar{Q} \cap \bar{Q}^{u_2} = \bar{Q} \cap \bar{Q}^{u_2} = 1
\]

since \( \bar{u}_2 \bar{u}_2 = \bar{v} \bar{u}_2 \bar{v} \) by (4.4). Note that \( J_2 \) normalizes \( K_3 \), as \( Q \) is an abelian group of odd order containing \( K_3 \). As \( J_2 \) normalizes \( N_{F_1}(U_2) = K_2 U_2 \), it normalizes \( N_{K_2}(U_2) = J_2 Q \). Set \( E = J_2 Q \), which normalizes \( K_2, K_2 B_2 \). Let \( E \) be a transitive on \( \text{Syl}_2(EK_2) \) \( (Q) \), then \( E = Q \cap (E \cap E^u) \). As \( N_{K_2}(Q) = E \), this implies \( E = Q \cap (E \cap E^u) \). \( E = E \cap E^u \), and \( E \cap E^u = E \cap (E \cap E^u) = \bar{A} (\bar{E} \cap \bar{E}^{u_2}). \) Similarly \( E^u = \bar{B} (\bar{E} \cap \bar{E}^{u_2}). \) As \( \bar{E}^{u_2} = (\bar{A} \bar{Y}_3)^{u_2} = \bar{A} \bar{Y}_3 \) and \( \bar{A} \leq \bar{E} \), we get \( \bar{E}^{u_2} = \bar{E}^{u_2} (\bar{E} \cap \bar{E}^{u_2}) = \bar{Y}_3 (\bar{E} \cap \bar{E}^{u_2} \cap \bar{E}^{u_2}). \) Similarly \( \bar{E}^{u_2} = \bar{E}^{u_2} (\bar{E} \cap \bar{E}^{u_2} \cap \bar{E}^{u_2}). \)

Thus \( \bar{E} = \bar{E} (\bar{E} \cap \bar{E}^{u_2} \cap \bar{E}^{u_2}). \) (2) \( | \bar{E} \cap \bar{E}^{u_2} | \) is odd, whence \( \bar{E} \cap \bar{E}^{u_2} = \bar{E} \cap \bar{E}^{u_2} \),

By symmetry we have

\[
\bar{E} \cap \bar{E}^{u_2} = \bar{E} \cap \bar{E}^{u_2} \cap \bar{E}^{u_2}.
\]

Set \( I = \cap \bar{I} \) where \( w \) ranges over \( \langle u_2 \rangle \). As \( \langle \bar{u}_2, \bar{v} \rangle = \{ 1, \bar{u}_2, \bar{v}, \bar{u}_2 \bar{v}, \bar{u}_2, \bar{v} \} \), it follows that

\[
I = \bar{E} \cap \bar{E}^{u_2} = \bar{E} \cap \bar{E}^{u_2}.
\]

Set \( \bar{N} = I \langle \bar{u}_2, \bar{v} \rangle \). Suppose \( \bar{Q} = \bar{Q}^{w} \) for some \( w \in \langle u_2 \rangle \) such that \( x \in \langle u_2 \rangle \) such that \( \bar{w} \bar{x} = \bar{u}_2 \bar{v} \). Then \( \bar{Q}^{w} = \bar{Q}^{x} = \bar{Q} \) and so \( \bar{Q} \cap \bar{Q}^{w} = 1 \) by (2), contrary to (1). Hence \( \bar{N} \cap \bar{E} = \bar{I} \cap \bar{I} \). Hence \( \bar{N} \cap \bar{E} = I \) and \( \bar{I} \cap \bar{I} \). Note that \( \bar{u}_2 \bar{E} \bar{v} \leq \bar{E} \bar{u}_2 \bar{E} \) and \( \bar{v} \bar{E} \bar{v} \leq \bar{E} \bar{v} \bar{E} \). As \( \bar{Q} = \bar{A} \bar{B} \), \( \bar{u}_2 \bar{v} \bar{A} \bar{v} \bar{A} \neq \bar{A} \bar{u}_2 \bar{v} \bar{A} \). Then \( \bar{u}_2 \bar{v} \bar{A} \bar{v} \bar{A} \leq \bar{E} \bar{u}_2 \bar{E} \bar{v} \bar{E} \). By (1), \( \bar{Q} \leq \bar{A} \bar{A} \), so \( \bar{u}_2 \bar{E} \bar{u}_2 \bar{v} \bar{A} \bar{v} \bar{A} \leq \bar{E} \bar{u}_2 \bar{E} \bar{v} \bar{E} \). Hence \( \bar{u}_2 \bar{E} \bar{u}_2 \bar{v} \bar{A} \bar{v} \bar{A} \leq \bar{E} \bar{u}_2 \bar{E} \bar{v} \bar{E} \) for all \( w \in \langle u_2, v \rangle \). By symmetry \( \bar{v} \bar{E} \leq \bar{E} \bar{v} \bar{E} \) for all \( w \in \langle u_2, v \rangle \).

We have shown that \( (\bar{E}, \bar{N}) \) is a BN-pair of \( \bar{X} \) whose Weyl group is dihedral of order 6. Thus \( \bar{X} \cong PSL_2(q) \) by Fong and Seitz[5].

(4.6) \( [K_1, K_2] \leq F_1 \).

**Proof.** Let \( X = \langle Q, K_1, K_2 \rangle \). In the proof of (4.4) we have shown that \( Z(F_2) = Z(D) \cap B_2 \). As \( F_1 \leq DB_2 \), a similar argument gives \( Z(F_2) = Z(F_3) \) or \( Y_1 \cap Y_1 \). Then \( F_2 \geq B \). Hence \( Y_1 \cap Y_1 \). As \( [K_1, Y_1] = 1 \), \( [K_1, Z(F_2)] = 1 \) and thus \( \langle u_2 \rangle \), \( Z(F_3) = 1 \). Recall that \( [F_2, F_2] = Y_7 Y_9 B_2 \), so \( K_1 \) normalizes \( [F_2', F_2] \cap A_1 = Y_8 B_2 \) and \( \langle D \cap B_2 \rangle B_2 = Y_8 B_2 \). As \( K_3 \)
centralizes $B_1/B_1$, it centralizes $Y_2B_2/Z(F_1)$ and $[u_1v]^2, Y_1B_2 \leq Z(F_1)$. Furthermore $[u_1v]^2, Z(D)B_2 \leq Y_2B_2$ since $K_3$ centralizes $A_i/A_i$.

As $Z(F_1)/D \cap U_1/D = Z(F_1)/D \cap W_2/D$ and $Z(F_1)/D \cap U_1/D = Z(Q/D)$, it follows that $[K_1, Z(F_1)/D] = 1$. If $K_3$ centralizes $Z(F_1)/D(B_2/D)$, then $[B_2/D] = q, [K_3, Z(F_1)/D] = 1$. But $Y_1Y_3B_1 = Q$ and $Z(F_1)/D \neq Z(Q/D)$, a contradiction. Thus $Z(F_1)/D(B_2/D)$ is a natural module for $K_3$.

Let $E = C_G(Y_2B_2)$. By (3.1) (1), $C_G(Y_2B_2) \leq E \leq G$, so $E = C_{F_i}(Y_2B_2) \triangleleft G$. We have $Y_2A_0B_3 \leq E \cap A \leq C_B(B_2) = W_0A_0B_3$. As $[V_0, V_0] \neq 1$, it follows that $E \cap A = Y_2A_0B_3$ and $E \cap D = Y_2Y_3A_4$. The action of $K_3$ gives $ED/D = Z(F_1)/D$. As $C_{B}(H_1) \leq C_{B}(H_2) = W_0$, $E = (E \cap W_2)(E \cap A)$. Now $Y_2Y_3 = W_0$ and no nonidentity element of $V_2$ centralizes $V_0$. Thus $W_1 = (E \cap W_3) \times (E \cap W_2)^e$. Let $E_0 = E \cap D$. Then $E_0 \leq C_{E}(Z(D)B_2) \leq (E \cap W_3)A_0B_3$. If $C_{B}(Z(D)A_2) \cap C(H_1) \neq 1$, $C_{E}(Z(D)B_2) \geq E \cap W_2$ since $H$ is irreducible on $E \cap W_2$. (1.3) and the action of $K_1$ and $K_3$ on $E/E_0 = Y_2ED/D = Z(F_1)/D$ show $[E/E_0, J_2] = (E \cap W_3)A_0E_0/E_0$ and $(E \cap W_3)^eE_0 = A_0E_0$. So $A_0 \leq C_{E}(Z(D)B_2)$, contrary to $C_{W_2}(Y_2B_2) = Y_2$. Thus $C_{B}(Z(D)B_2) \leq [E, H_1] = E \cap A$, so $C_{B}(Z(D)B_2) \leq (E \cap A) \cap (E \cap A)^e = Y_2Y_3A_4B_3$. As $C_{W_2}(Y_2B_2) = Y_2$, we get $C_{B}(Z(D)B_2) = E_0$.

Let $E_1 = \langle E \cap W_2, Y_2, Y_2 \rangle Z(D)B_2$ and $E = E/Z(D)$. Then $E = E_0E_1, E_0 \cap E_1 = Z(D)B_2$, and $E_1 \triangleleft E$ by [10, (3.9) (5)]. By (4.1) (2), $E_0 = \overline{Y}_2 \overline{Y}_2 \overline{Y}_2 \overline{Y}_2 \overline{B}_2$ is elementary abelian of order $q^6$. Note that $[Y_2, W_3] = [W_3, W_3] = 1, [W_3, W_3] = 1$, and $[W_3, W_3] = [W_{11}, W_4]^{e} = 1$. Thus $E_0 = (E \cap W_2) \overline{Y}_2 \overline{Y}_2 \overline{B}_2$ is elementary abelian of order $q^6$. $[W_2, W_2] = [W_2, W_2] = W_1$, so the action of $H$ gives $[\overline{Y}_2, Y_2] = Y_1^e$, since $[Y_2, Y_2] = 1$ and $Y_2Y_2^e = W_0$. If $[Y_2, Y_2] \leq Z(D)B_2$, then $[Y_2, Y_2] \leq Z(D)B_2 \cap Z(D)B_2 = Y_1^eY_1^e$ since $Y_2 \leq Z(D)B_2, Y_2, Y_2 = 1$. But then $W_2 = W_2 \cap U_2$, contrary to $[W_2, W_2] = -Y_1^e$. Thus $C_{E}(\overline{Y}_2) \cap C(\overline{Y}_2) = \overline{Y}_2$. If $[V_0, W_6] = -Y_1^e$, then $[W_2, W_6] = -Y_1^e$. As shown before $(E \cap W_2)E_0 = A_0^eE_0$. Moreover $[A_0^e, Z(D)] = Z_3^e$, so $[E \cap W_2, Y_2] = 1$. If $[E \cap W_2, Y_2] \leq Z(D)B_2$, then $[E \cap W_2, Y_2] \leq Z(D)B_2 \cap Z(D)B_2$. As $W_2 = Y_2Y_2^e$, this implies $[W_2, W_2] \leq Y_2 \leq Y_2$. Thus $C_{E}(\overline{Y}_2) \cap C(\overline{Y}_2) = \overline{Y}_2$.

Let $I = C_G(O(\langle u_1v \rangle))$. By the first paragraph $(u_1v)^2$ stabilizes $Z(D)B_2 \geq Y_2B_2 \geq Z(F_1), I \geq Z(D)B_2$ and $C_E(I) \leq E_0$. As $K_1$ centralizes $E_0E_0, [u_1v]^2, E \leq E_0$ and thus $E \leq E_0$. Now $C_E(I) = E_0$ by the above and $E = E_0E_0$. So $E_0 \cap C(I) = E_0 \cap E_0 = \overline{E}_0$. Let $E_0 \cap C(I) = \overline{E}_0$. Note that $Z(D)B_2 \cap C(W_2) = Y_2Y_2B_2$ and $Y_2Y_2B_2 \cap C(W_2) = Y_2B_2$. As $E_i \geq \langle W_2, W_2 \rangle$ and $Z(E_0) \geq Z(D)B_2$, we conclude that $C(I) = Y_2B_2$. By (3.5), $E$ is a Sylow 2-subgroup of $O_2(G) \cap C(Y_2B_2)$. Moreover $I \leq E$ by [10, (3.9) (5)] and $O(2^G) \cap C(I) = 1$ by the unbalanced group theorem. Thus $O(2^G) \cap C(I) = Y_2B_2$, so $(u_1v)^2$ is a 2-element. As $Q/F_1$ is elementary abelian of order $q^2$, Bender [3] shows $(X/F_1)O(X/F_1) \simeq SL_d(q) \times SL_d(q)$ and $[K_1, K_2] \leq O(X \mod F_1)$. Hence $(u_1v)^2 \in F_1$. Now $K_1$ centralizes $AF_1/F_1$ and $K_3$ centralizes $F_2F_2/F_1$ with $AF_1 = Y_2F_1$ and $F_2F_2 = Y_2F_1$, so the lemma holds.

(4.7) Let $G_0 = \langle Q, M_1, M_2 \rangle$. Then $G_0 \simeq F_4(q)$.

**Proof.** Recall that $[K_1, K_2] = M_1, K_1 = K_2$, and $[K_2, K_3] = 1$. By (4.5) and (4.6) we can apply Theorem B of Niles [11] to the subgroups $P_1 = K_1F_1A, P_2 = P_1, P_3 = K_3F_2B$, and $P_4 = P_3$ and conclude that $G_0$ has a BN-pair of rank 4. Hence $G_0 \simeq F_4(q)$ by Tits [17].

**Remark.** It remains of course to prove that $G_0$ is normal in $G$. The first step toward
this purpose may be to show that if \( t \in N(G_0) \) for an element \( g \in G \) then \( g \in N(G_0) \). Indeed, \( J \) normalizes \( M_1A \) by \([10, 3.9 (1), 3.10]\) and \( K \) normalizes \( M_2B \) by \((3.1)\) and \([10, 3.5]\), so \( C(t) \leq N(G_0) \). Suppose \( t \in N(G_0) \). Since \( C(t)^{\infty} = L \), neither \( t \) nor \( t^* \) lies in \( G_0C(G_0) \). Then by \([2, 19.5]\), \( C_{G_0}(t) = C_{G_0}(t^*) = L \) and \( t = t^* \) for some \( x \in G_0C(G_0) \). Hence \( gx \in C(t) \) and we have \( g \in N(G_0) \).

V. The Case \( M_1 = Sz(q) \times Sz(q) \)

In this section we assume that \( M_1 = Sz(q) \times Sz(q) \). Let \( M_1 = K_1 \times K_1^t \) with \( K_1 = Sz(q) \) and \( Y_1 = W_1 \cap K_1 \). By \((3.3)\), \( N_1 = C_{M_1}(t) \) is irreducible on \( D_0/A_0 \cong C_{M_1}(t) \cong R/V_5R_1 \). Since \( C_{D_0/A_0}(Y_1) \neq 1 \) is \( K_1^t \)-invariant, \( K_1^t \) centralizes \( C_{D_0/A_0}(Y_1) \) or \( [Y_1, D_0/A_0] = 1 \) by \([5, (4F)]\). In the former the action of \( K_1 \) on \( C_{D_0/A_0}(K_1^t) \) yields \([K_1^t, D_0/A_0] = 1 \) and in the latter \([K_1, D_0/A_0] = 1 \). Replacing \( K_1 \) and \( K_1^t \) if necessary, we may assume that \( K_1 \) centralizes \( D_0/A_0 \). Then

\[
(5.1) \quad M_1A|A_0 = K_1^tD_0/A_0 \times K_1D_0^t/A_0 \quad \text{and} \quad K_1^tD_0/A_0 = N_1R/V_5R_1.
\]

(5.2) Let \( D_1 = O_2((K_1^tD_0)')A_1 \). Then \( K_1^tD_1 \triangleleft M_1A \) and \( K_1^tD_1/A_1 \) is perfect with \( |D_1/A_1| = q^3 \).

PROOF. Set \( D = D_0 \cap D_0'^t \). Arguing as in the proof of \((4.1)\) we obtain \( \Gamma \triangleleft DA_1 = Y_3^tY_4Y_5(DA_1 \cap W_5)A_1 \), with \( |DA_1 \cap W_5| = q^3 \) and \( H \leq N(D) \). Let \( \tilde{N}(A_1) = N(A_1)/A_1 \). Let \( \tilde{U} \) be a complement of \( DA_1 \cap W_5 \) in \( \tilde{W}_5 \). Then \( \tilde{U} \tilde{D}_0 = \tilde{Y} \tilde{D}_0 \tilde{U} \) since \( M_1 \) centralizes \( \tilde{W}_5 \). By Gaschütz’s theorem \([9, p. 121]\) there is a subgroup \( \tilde{X} \) such that \( \tilde{K}_1 \tilde{D}_0 = \tilde{X} \times \tilde{U} \), so \( \tilde{K}_1 \tilde{D}_0 = \tilde{X}^t \). If \( \tilde{X}^t \cap \tilde{W}_5 = 1 \), \( \tilde{M}_1A = \tilde{X} \times \tilde{X}^t \times \tilde{W}_5 \) by \((5.1)\). But \( C_{\tilde{M}_1A}(t) = N_1R \approx N_1R/R_1 \), a contradiction. Now \( |\tilde{X}^t \cap \tilde{W}_5| = q^3 \) and every proper \( H \)-invariant subgroup of \( \tilde{W}_5 \) is of order \( q^3 \). Thus \( |\tilde{X}^t \cap \tilde{W}_5| = q^3 \). A similar argument shows that \( \tilde{X}^t \) is perfect. By the definition \( \tilde{X}^t = \tilde{K}_1 \tilde{D}_0 \), the lemma holds.

(5.3) Let \( D_2 = O_4((K_1^tD_1)')W_{12} \). Then \( M_1A|W_{12} = K_1^tD_2W_{12} = K_1D_1^tW_{12} \) and \( K_1^tD_1^tW_{12} = N_1R/V_5R_{12} \).

PROOF. Let \( D \) be as in the proof of \((5.2)\). Then \( DA_0 = D_0 \supset D_1 \), so \( M_1A \supset Z(D_1) \supset A_1 \geq W_{12}W_{12} = W_{12} \). Since \( C_{D_0}(A_1) = A_0 \) and \( N_1 \) is irreducible on \( A_1/R_1W_{12} \cong R_1W_{12}/W_{12} \) by \([10, (3.6) (1)]\), \( |Z(D_1) \cap A_1| = q^6 \). Let \( N(W_{12}) = N(W_{12})/W_{12} \). Then \( C_{\tilde{M}_1A}(K_3) \cap C_{\tilde{M}_1A}(K_4)^t = 1 \) and \( K^t \) acts on \( Z(D_1) \cap A_1 \cap C(Y_1) \neq 1 \). Hence as in the proof of \((5.1)\), \( \tilde{M}_1A = \tilde{K}_1^tC_{\tilde{M}_1A}(K_1) \times \tilde{K}_1^tC_{\tilde{M}_1A}(K_1^t) \) in \( \tilde{M}_1A = \tilde{K}_1^tC_{\tilde{M}_1A}(K_1) \). Let \( J_1 \) be a complement of \( Y_1 \) in \( N_{K_1}(Y_1) \) such that \( H \leq J_1 \). Then \( J_1 \) centralizes \( A_1/A_0 \), so it centralizes \( D_1/A_1 \) by \((5.1)\). Moreover \( C_{B_{11}/V_{11}}(H) = 1 \) implies \( C_{\tilde{M}_1A}(K_4)^t \cap C(J_1^t) = 1 \).

Then as \( Z(\tilde{A}_1) \geq \tilde{A}_1, \tilde{K}_1^t\tilde{D}_1 = \tilde{K}_1^tC_{\tilde{D}_1}(J_1) \times [\tilde{D}_1, J_1] \). Since \( K_1^tD_1/A_1 \) is perfect, we get \( \tilde{K}_1^t\tilde{D}_2 = (\tilde{K}_1^t\tilde{D}_1)^t = \tilde{K}_1^tC_{\tilde{D}_1}(J_1) \times \tilde{M}_1A = \tilde{K}_1^t\tilde{D}_2 \times \tilde{K}_1^t\tilde{D}_2 \) with \( \tilde{K}_1^t\tilde{D}_2 \cong C_{\tilde{M}_1A}(t) = N_1R \).

(5.4) Let \( J_1/B_4 = C_{B_4}(K_3) \). Then \( Y_3^tJ_1 = Y_3^tD_2B_4 \) with \( Y_3^t \leq D_2 \) and \( Y_3^t \leq I_1 \).

PROOF. We have \( D_0 = D_2A_0 \), so \( D_0B_4 = D_2B_4 \). \((1.2)\) and \( C_{A_0}(H_3) = 1 \) give \( Y_3^t = C_{D_0}(H_3) = \)
$C_{D_2}(H_3)$. By (3.3) (2), $C_{B\mid B_4}(Y_3)=I_1W_2/B_4$. As $Y_3\leq D_2$, (5.3) shows $C_{B\mid B_4}(Y_3)\geq Y_4W_4B_4/B_4$. Hence $Y_4^t \leq C_{I\mid W_4}(H_4) \leq I_1$ by (1.2). Now the lemma follows from (3.3) (2).

(5.5) Let $D_3=O_5((K_t', D_3'))$. Then $M_3A=K_t' D_3 \times K_3 D_3'$. Furthermore there is a normal subgroup $I_6$ of $M_3B$ such that $M_3B=K_t I_6 \times K_3 I_6'$ and $Y_4^t D_3=I_6^t D_3'$.

PROOF. Let $\overline{N}(B_3)=N(B_3)/B_3$. As $Y_3 \leq D_3$, (5.3) shows $|C_{B\mid B_3}(Y_3)|>q^4$. Thus $C_{B\mid B_3}(K_t')\neq 1$, for $K_t=\langle Y_3, Y_3' \rangle$. By [10, (3.8)](1), $C_{B\mid B_3}(N_3)=1$, so $B_3\leq C_{B\mid B_3}(K_t') \times C_{B\mid B_3}(K_t)$. Now $N_3$ is irreducible on $C_{B\mid B_3}(t)=\mathcal{S}_3 \times S_4/S_4$, whence $\overline{M}(B_3)=\mathcal{K}_3C_{B\mid B_3}(K_t') \times K_3C_{B\mid B_3}(K_t)$ and $\mathcal{K}_3C_{B\mid B_3}(K_t)=C_{B\mid B_3}(t)=\mathcal{N}_3S_4$. Let $J_4$ be as in the first paragraph of section 4. Then $C_{B\mid B_3}(K_t') \cap C(J_4)=1$ since $C_{B\mid B_3}(H_3)=1$. Now $Z(\mathcal{B}) \geq \mathcal{B}$, so setting $I_6/B_3=C_{I\mid W_4}(J_3)$ we have $\mathcal{K}_3 I_6=\mathcal{K}_3 I_6' \times C_{B\mid B_3}(K_t')$. Since $K_t'$ is irreducible both on $I_6/B_3$ and on $C_{B\mid B_3}(K_t), (K_t' I_6)'=K_t' I_6$ and $M_3B/B_3=K_t' I_6 \times K_3 I_6'/B_3$.

Set $I_6/B_3=C_{I\mid W_4}(J_3)$. Then by (3.4) we have $(K_t' I_6/B_3)'=K_t' I_6/B_2$ and $M_3B/B_3=K_t' I_6 \times K_3 I_6'/B_2$.

Let $J_1$ be as in the proof of (5.3). Then $H_1 J_1 J_t^t$ is an abelian group containing $H=H_1 H_5$ and normalizes $Q$, so $J_1$ normalizes $W_4$, 1 $\leq i \leq 12$. Note that $J_1$ normalizes $K_3$ since $M_3B=E(C(H_3))$. Now $I_1=I_2B_4, C_t(H_3)=Y_2 Y_1$, and $C_{B\mid B_3}(H_3)=W_2 Y_2 Y_1^t$. So $C_{t_1}(H_3)=C_{t_1}(H_3) W_2$ by (1.2). Then $Y_2 Y_1= C_{t_1}(H_3) Y_1= C_t(H_3) Y_1$ and $Y_1 Y_1= C_{t_1}(H_3) Y_1$. \(Y_2^t \leq C_{t_1}(H_3) Y_1 \leq C_{t_1}(H_3) Y_1 \), where $Y_1 Y_1= C_{t_1}(H_3) Y_1 Y_1$. Since $J_1$ is transitive both on $(Y_1,Y_1')^t$ and on $Y_1^t$, $C_{t_1}(H_3) Y_1$ has exactly two $J_1 J_t^t$-invariant subgroup of order $q$. Hence $Y_1 Y_1= C_{t_1}(H_3) Y_1$.

By (5.4), $Y_4^t I_2 Y_1 D_2 B_4$, so [10, (3.9)] (5) implies $(Y_2 Y_2^t Y_2^t)' \leq Y_2 Y_2^t Y_2^t B_2$. Now $Y_2 Y_2^t B_2 \subseteq C_{B\mid B_3}(t) \cap P/\mathcal{S}_4$ by the above paragraph and we have $(Y_2 Y_2^t Y_2^t)' \leq Y_2 Y_2^t B_2$, so $I_2 \leq Y_2 Y_2^t B_2$. Thus $Y_2 Y_2^t B_2= Y_2 Y_2^t B_2$. Then $Y_2 Y_2^t B_2= Y_2 Y_2^t B_2$ and $Y_2 Y_2^t B_2= (D_2 \cap B_2) B_2$. By (5.3) and we can write $K_3 Y_2 Y_2^t B_2= X B_2 \cap (D_2 \cap B_2) B_2$. For some subgroup $X$ by Gaschütz's theorem since $M_3B$ centralizes $B_2/B_3$. Setting $I_4=O_5((K_t' I_6'/B_3)$ we have $M_3B/B_3=K_3 I_6'/B_3 \times K_3 I_6'/B_3$.

Set $I_5= C_{t_1}(J_3)$. Then $(K_t' I_6)'=K_t' I_6$ and $M_3B=B_3 \times K_3 I_6 \times K_3 I_6'$ by (3.2).

Set $Q_1=Y_2 Y_2^t I_5$, so that $Q_1=Q_1 \times Q_1^t$ and $Q_1=C_0(t)=P$. As $Q_1 \leq Y_2 Y_2^t D_2 B_2$ and $B_2$ is abelian, $Q_1 \leq Y_2 Y_2^t D_2$. Moreover $Y_2 Y_2^t \leq I_5 B_2$ and $C_{B\mid B_3}(H_3)=1$, so $Y_2 Y_2^t \leq I_5$ by (1.2). Thus $Q_1=Y_2 Y_2^t Q_1 \leq Y_2 Y_2^t D_2$ and $Y_2 Y_2^t Q_1 \leq (Q_1 \cap W_4)$. By Gaschütz's theorem $K_t' D_2=K_3 Y_2 Y_2^t (Q_1 \cap W_4)$ for some subgroup $X$. Hence $M_3A=K_t' D_2 \times K_3 D_3$. Now $Q_1 \leq Y_2 Y_2^t D_2$, so $Q_1 \leq Y_2 Y_2^t D_2$. (1.2) and $D_2= D_2 W_4$ give $Y_1^t \leq C_{t_6}(H_4) \leq D_3$. Thus $Q_1 \leq Y_2 Y_2^t D_2$ and the lemma is proved.

Let $Q_1=Y_1 D_3= Y_1 Y_5$ and $G_1=\langle Q_1, K_t', K_3' \rangle$. Define an $H$-homomorphism $\phi: G_1 \rightarrow C=C(t)$ by $x \mapsto x t$. Under this homomorphism $K_t' D_3= C_{K_t'A(t)}= I_1 R$ and $K_3' \leq C_{K_3'B(t)}=N_3 S$. Let $U_i=W_i \cap Q_1$, 1 $\leq i \leq 12$. Then $U_1=Y_1 t^t, U_2=Y_2^t$, and $W_i=U_i \times U_i^t, 1 \leq i \leq 12$. Note that $Z(U_4)=U_6, Z(U_5)=U_8, Z(U_6)=U_11, Z(U_8)=U_11, U_i=C_{t_6}(H_4)$ for $i=3, 4, 5, 6, 7, 9, 10$, and $U_i= C_{t_6}(H_4)$ for $j=1, 4, 5, 6, 7, 9, 10$. By the first paragraph of Section 3, $U_5^t=U_7, U_5^t=U_6, U_5^t=U_6, U_5^t=U_6, U_5^t=U_6, U_5^t=U_6, U_5^t=U_6, U_5^t=U_6$.

(5.6) Let $G_0=\langle Q, M_3, M_2 \rangle$. Then $G_0=G_1 \times G_1$ and $G_1 \cong E_4(q)$. 

PROOF. Since $Q_1 \cap Q_1^t= D_3$ and $Q_1 \cap Q_1^t= I_5$, 

\(Q_1 \cap Q_1^t \cap Q_1^t \cap D_1= U_4 U_5 U_6 U_7 U_8 U_9 U_{10}\).

Transforming this equation by $r$ and $s$ respectively, we have
A REMARK ON THE STANDARD FORM PROBLEM FOR $\mathfrak{g}F_4(2^{a_1+1})$, \(n \geq 1\)

(2) \(Q_1 \cap Q_1^* \cap Q_1^{**} = U_3 U_4 U_5 U_6 U_7 U_8 U_9\).

(3) \(Q_1 \cap Q_1^* \cap Q_1^{***} = U_3 U_4 U_5 U_6 U_7 U_8 U_9\).

By (1) and (2), \(Q_1^* \cap Q_1 \cap Q_1^* \cap Q_1^{**} = U_4 U_5 U_6 U_7 U_8 U_9\). So

\(Q_1 \cap Q_1^* \cap Q_1^{**} = U_3 U_4 U_5 U_6 U_7 U_8 U_9\)\).

By (1) and (3), \(Q_1^* \cap Q_1 \cap Q_1^* \cap Q_1^{***} = U_4 U_5 U_6 U_7 U_8 U_9\). So

\(Q_1 \cap Q_1^* \cap Q_1^{***} = U_3 U_4 U_5 U_6 U_7 U_8 U_9\).

Arguing similarly, we get

(4) \(Q_1 \cap Q_1^* \cap Q_1^{**} \cap Q_1^{**} \cap Q_1^{****} \cap Q_1^{**} \cap Q_1^{****} \cap Q_1^{**} = U_9\).

(5) \(Q_1 \cap Q_1^* \cap Q_1^{**} \cap Q_1^{**} \cap Q_1^{****} \cap Q_1^{**} \cap Q_1^{****} \cap Q_1^{**} = U_9\).

Since \(V_1\) is transitive on \(\text{Syl}_2(N_1) - \{V_1\}\), \(r \in V_1^{rs}\) for some \(a \in V_1\). Take an involution \(u \in K_1\) such that \(r = ua^t\). Then \(u \in U_1^{rs}\), for \(r \in W_1^{rs} = U_1^{rs} \times U_1^{rs}\) and \(U_1^{rs} \subseteq K_1\). By (4), \(U_3 \subseteq D_9^{rsrs}\). By (5), \(U_1 \subseteq D_9^{rsrs}\), so \(U_1 \subseteq D_9^{rsrs}\) since \(|rs| = 8\). Thus \([U_1, U_3] = [Q_1, Q_1]\) is trivial. Take \(b \in U_1\) such that \(a = bb^t\). Then as \(U_1^{rs} \subseteq K_1\), \(U_1^{rs} = U_1^{rs}\).

Since \(b, U_3^{rs} \subseteq [U_1, K_3] = 1\), we have \([U_1, U_3] = 1\). Moreover \([U_1^{rs}, I_5] \subseteq [K_1, Q_1]\) and \(Q_1 = U_3^{rs}\). Thus \(u, Q_1^{rs} \subseteq [U_1^{rs}, Q_1^{rs}] = 1\). As \(Q_1^* = Q_1\), this gives \([Q_1^*, Q_1^{rs}] = [Q_1, Q_1^{rs}] \subseteq [K_3, I_5, K_3, I_5] \subseteq 1\). Now \([Q_1, Q_1^{rs}] = 1\) and \([Q_1, Q_1] = 1\) since \([K_1, D_9, K_1, D_9] = 1\). Hence \(G_1, G_1^{rs} \subseteq 1\) and \(G_1 = G_1 \ast G_1^{rs} \subseteq L\). Let \(L_0\) denote the image of \(\phi\) and set \(X = G_1 \cap G_1^{rs}\). Then \(C_0(t) = L_0C_x(t)\). Since \(K_3D_9\) and \(K_3I_5\) are perfect, so is \(G_1\). Thus \(L_0 = L_0\) and \(C(t)^{oo} = L, L_0 = L\). The kernel of \(\phi\) is contained in \(X\) and the Schur multiplier of \(\mathfrak{g}F_4(q)\) is trivial. Hence (5.6) holds.

**Remark.** If one wishes to show that \(G_0 \leq G\), a key point may be to establish that \(N(Q_1) \leq N(G_0)\) (see [14, (14.7)]). We can verify that \(Z_4(P) = V_9 V_{10} V_{11} V_{12}\), \(Z_4(P) = V_9 V_8 Z_4(P) = R_1 S_4\), \(Z_4^*(Z_4(P)) = \{R_1, S_4\}, C_P(R_1) = V_{11} R_1\), and \(C_P(S_4) = S_4\). Since \(Q = Q_1 \times Q_1^t\) with \(Q \simeq P, Z_4(P) = A_1 B_3\) and \(Z_4^*(Z_4(P))\) consists of four members, two of which are \(A_1\) and \(B_3\) and the remaining two members are interchanged by \(t\). Moreover \(|C_0(A_1)| > |C_0(E)|\) for each \(E \in Z_4^*(Z_4(P)) = \{A_1\}\). As \(Z(Q) = W_{12}\), we get \(A = C_0(A_1) W_{12} < N(Q)\). Also \(B = C_0(Z_4(P))\) by (3.1) (1). It then follows from [10, (3.10)] and (3.1) (2) that \(N(Q) = N(A) \cap N(B) \leq N(G_0)\). By [10, (3.1) (2)] \(\langle t \rangle \in \text{Syl}_2(C(L)\). So \(|C(G_0)|\) is odd since \(C(G_0) < C(L) < N(G_0)\). Note that \(\mathfrak{g}F_4(q)\) has two conjugacy classes of involutions (see [12]) and that the outer automorphism group of \(\mathfrak{g}F_4(q)\) is of odd order. Note also that \(N(\langle t \rangle P) = W_{12} N_0(P)\) since \(Z(\langle t \rangle P) = \langle t \rangle V_{12}\) and \(t^{N(G_0)} = t P\). Arguing as in [19, (7.22)] we see that if \(t \in N(G_0)^t\) for an element \(g \in G\) then \(g \in N(G_0)\). Hence as in [19, (7.23)] we have \(N(Q) \leq N(G_0)\).

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REFERENCES


[15] Shinoda, K., "A characterization of odd order extensions of the Ree groups 2F_4(q)," *Journal of the Faculty of Science, University of Tokyo*, Section IA 22 (1975), 79–102.


[19] Yamada, H., "Finite groups with a standard subgroup isomorphic to G_2(2^n)," *Journal of the Faculty of Science, University of Tokyo*, Section IA 26 (1979), 1–52.