Moral-Hazard Premium

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March 12, 2014

Abstract

This paper provides an explicit asset-pricing formula in a continuous-time general-equilibrium exchange economy in the presence of moral hazard. Specifically, it solves an optimal consumption/wealth allocation problem of a representative lender in financial markets under regular market risk and rare-event risk when an endowment process is subject to a firm manager's moral hazard. Consequently, it shows that, under the moral-hazard problem, a positive premium is stipulated on a riskless rate in market equilibrium – call it a moral-hazard premium – due to the necessity to give the manager an incentive to avoid his opportunistic misbehavior.

Keywords: moral hazard, asset prices, rare-event risk, regular market risk, riskless rate.
JEL Classification Codes: D51, D82, G12.

1 Introduction

It is well known that moral hazard is one of the most serious problems in finance. There exists a huge literature on moral hazard in corporate finance (e.g. Tirole (2006, Section 3.2)). In a typical moral-hazard model, the probability of success of a firm's project depends on its manager's hidden effort. An investor then needs to give the manager an incentive to avoid his opportunistic misbehavior. Thus the moral hazard distorts optimal risk sharing and allocation. The distortion should then influence the valuation of not only the firm but also all other financial assets, because it twists the investor's pricing kernel as well as the consumption/wealth allocation.

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Surprisingly, however, there have been few studies of such valuation of moral hazard in asset pricing and financial engineering (e.g. fixed-income investment, the term structure of interest rates, corporate risk management, and actuarial insurance). A notable exception is Ou-Yang (2005), who studies equilibrium asset pricing in the presence of moral hazard on the assumption of the exponential utility function and an exogenous constant riskless rate. Still, Ou-Yang (2005, p.1283) himself discusses the importance of incorporating more general utility functions such as the power utility function. In addition, the assumption of the exogenous riskless rate seems to have limited its applicability to financial practices. Thus, there has been a wide gap in the research of moral hazard between corporate finance and asset pricing/financial engineering.

The purpose of this paper is to fill the gap by (1) providing an explicit asset-pricing formula on the assumption of the power utility function and an endogenous riskless rate in the presence of moral hazard and (2) making clear the structural effect of moral hazard on asset prices, especially on riskless rate. Specifically, we incorporate moral hazard into a continuous-time general-equilibrium exchange economy.[1] This paper solves an optimal consumption/wealth allocation problem of a representative lender in financial markets under two types of risk, i.e., regular market risk and rare-event risk, when an endowment process is subject to a firm manager's moral hazard.

More specifically, there exist not only a representative investor but also a representative firm (i.e., firm manager) over $[0, T]$.[2] The firm produces a single non-storable consumption good over time. No productive resources are utilized: the production is an endowment. However, the firm can control the probability measure by incurring effort costs. The endowment process is then subject to a dynamic moral-hazard problem. The lender controls ex post optimally a consumption/wealth allocation while having access to financial markets, designing ex ante optimally a contract with the firm so as for the manager not to expect better rewards from his opportunistic misbehavior.

Still, this paper is not a generalization of Ou-Yang (2005), but rather is complementary to it. Mathematically, the investor's optimization problem is subject to two constraints on states: an incentive constraint provoked by moral hazard and the firm's participation constraint. As Yong and Zhou (1999, p.155) point out, it is very hard, in general, to solve stochastic-control problems with such state constraints. This model is no exception, i.e., it is hard to obtain a general solution

[1]As to standard general-equilibrium exchange economies, see e.g. Lucas (1978), Breeden (1979), Cox et al. (1985), Dana and Jeanblanc (2007, Ch.7).

[2]We could construct a model of heterogeneous firms and then solve an aggregation problem in the presence of moral hazard. That will be our next research topic.

not only analytically but also numerically. It implies that some approximations are necessary for explicit valuation of moral hazard in financial practices. To obtain an explicit (closed-form) solution in this paper, we restrict the contract form to a stationary linear contract, which means that it is linear (i.e., proportional to the production) at a constant (i.e, time-independent) rate of change, in exchange for generalizing the utility function into the power utility one and endogenizing the riskless rate, as compared to Ou-Yang (2005).

This paper consequently provides an explicit asset-pricing formula under the moral-hazard problem and obtains equilibrium state prices (in particular, equilibrium riskless rate and market prices of diffusive risk and jump risk). It then makes clear the structural effects of moral hazard on equilibrium asset prices under the two types of risk. Most notably, it shows that, under the moral-hazard problem, a positive premium is stipulated on a riskless rate in market equilibrium – call it a moral-hazard premium – because the lender demands compensation for a loss caused due to the necessity to give the firm an incentive to avoid his opportunistic misbehavior. The premium is time-varying, due to the optimal dynamic behavior of the firm's effort. It implies that the risk-free rate puzzle, explored first by Weil (1989), is more serious in the presence of moral hazard.[3]

Moreover, the explicit result could be used as a benchmark for approximating numerically general solutions under more general contract forms, e.g. via the Taylor expansion method, in future. We can conjecture that the linearity of the contract is not quite restrictive in optimum because the whole system of equations is linear in this model. Based on the conjecture, the equilibrium market prices of diffusive risk and jump risk obtained under the stationary linear contract hold true for the more general contract forms as well. So does the premium. However, the stationarity (i.e., the constant sharing ratio) may not be optimal for them. If the sharing ratio of the firm is decreasing (increasing) in time, then the moral-hazard premium is raised (lowered), because the marginal utility of the investor is decreasing (increasing).

This paper is most closely related to Cvitanić and Zhang (2007) in the rapidly growing literature on continuous-time moral hazard.[4] Both papers study optimal contracting on the assumption of more general utility functions than the exponential one.[5] However, this paper extends Cvitanić and Zhang (2007)'s bilateral optimal contracting model into a general-equilibrium model where the

[3]Note, however, that Weil (1989) assumes recursive utility in contrast to our paper.

[4]E.g., Holmström and Milgrom (1987), Schättler and Sung (1993), Sannikov (2008). As to a survey of the literature, see the book of Cvitanić and Zhang (2013).

[5]Cvitanić and Zhang (2007) examine the problem of adverse selection as well.

lender has access to the financial markets.[6]

This paper also has several technological departures from Cvitanić and Zhang (2007). To draw asset-pricing implications, we assume that the consumption takes place all over $[0, T]$, whereas they assume that the players consume only at the terminal point T. Also, this paper assumes not only Brownian motions as regular market risk but also Poisson processes as rare-event risk, whereas they assume only a Brownian motion.

Furthermore, more importantly, this model assumes that the firm controls directly the probability measure, in the spirit of standard discrete-time moral-hazard models (e.g., Tirole (2007, Section 3.2)). In contrast, Cvitanić and Zhang (2007) presume that the agent controls the drift rate based on the information set generated only by a history of the endowment, neither by a history of his own observable true shocks nor of his efforts.[7] In other words, in their paper, the agent controls the drift rate while continuing to forget how he has controlled it and has suffered the true shocks until then. The presumption seems irrelevant in practice. This model avoids the irrelevance by assuming the firm's control of the probability measure.

Also, this paper formulates the effort cost as relative entropy. The relative entropy is a measure of statistical discrimination between the original measure and the controlled probability measure, and stands for a 'distance' between the two measures.[8] It has been lately used as a cost of controlling the probability measure in economics and finance.[9] The relative entropy is a good measure to gauge the firm's effort cost. This formulation is tractable as there exists the rare-event risk as well.[10]

The rest of this paper is organized as follows. Next section defines the environment of this model. Section 3 gives a formal representation of the firm's and the lender's optimization. Section 4 defines market equilibrium and characterizes it in a closed form explicitly. Section 5 examines equilibrium asset prices and draws asset-pricing implications. Final section concludes.

[6] Ou-Yang (2005) extends the optimal-contracting model with the exponential utility function (e.g., Holmström and Milgrom (1987), Schättler and Sung (1993)) to an asset-pricing model.

[7] Cvitanić and Zhang (2007) solve the optimal contracting problem in the weak formulation on the assumption of the firm's drift control. This paper solves it in the weak formulation, but assumes the firm's control of the probability measure. On the other hand, the paper of Nakamura and Takaoka (2013) solves the problem in the strong formulation, where the information set is generated by a history of an agent's efforts as well as a history of true shocks, although it restricts the contract function to being almost everywhere differentiable.

[8] See e.g. Cover and Thomas (2006, p.18) in the statistics literature.

[9] See e.g. Hansen and Sargent (2007), Hansen et al. (2006), Sims (2003) in economics. Also, in mathematical finance, Delbaen et al. (2002) use it as a penalty in hedging contingent claims.

[10] For more details, see e.g. a companion work of this current paper, namely Misumi et al. (2013).

2 Environment

2.1 Players and filtered probability space

We consider a dynamic stochastic economy with two players: a representative firm manager (simply called a firm) and a representative investor on a time interval $[0, T]$ for a finite time $T > 0$. The firm and the investor are indexed by player 1 and player 2, respectively. For convenience, we will use female pronouns for the investor, and male ones for the firm.

Define a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}(t)\}_{0\le t\le T}, \mathbb{P})$. $\{B_1(t), \cdots, B_n(t)\}_{0\le t\le T}$ are n independent one-dimensional standard $\mathbb{F}$-Brownian motions on the probability space, i.e., for any t, s satisfying $0 \le t \le s$, $B_j(s) - B_j(t)$ is independent of $\mathcal{F}(t)$ and $B_j(0) = 0$ for $j = 1, \cdots, n$. $\{N_1(t), \cdots, N_m(t)\}_{0\le t\le T}$ are m independent Poisson processes, each of which is characterized by its intensity $\lambda_i > 0$ for $i = 1, \cdots, m$. Let the compensated Poisson process be denoted by $M_i(t) := N_i(t) - \lambda_i t$, which is a $\mathbb{P}$-martingale. The Poisson processes are independent of $\{B_j(t); j = 1, \cdots, n\}_{0\le t\le T}$ as well. The filtration $\mathbb{F}$ is generated by $\{B_j(t); j = 1, \cdots, n\}_{0\le t\le T}$ and $\{N_i(t); i = 1, \cdots, m\}_{0\le t\le T}$. For notational convenience, we may also write an n dimensional process $B(t) := (B_1(t), \cdots, B_n(t))^\top$, an m dimensional process $N(t) := (N_1(t), \cdots, N_m(t))^\top$, and an m dimensional process $M(t) := (M_1(t), \cdots, M_m(t))^\top$.

Define a measure $\mathbb{Q}$ that is absolutely continuous w.r.t. $\mathbb{P}$, written as $\mathbb{Q} \ll \mathbb{P}$, i.e., $\mathbb{P}(A) = 0$ implies $\mathbb{Q}(A) = 0$ for $A \in \mathcal{F}$. Define also the Radon-Nikodym derivative process as

$$Z(t) := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}(t)} = \mathbb{E}_t^{\mathbb{P}}[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}].$$

By the Martingale Representation Theorem (cf. Theorem 5.43 of Medvegyev (2007)), there exist $\mathbb{F}$-predicable processes θ_j and $\alpha_i \ge -1$, where $\int_0^T (\theta_j(t))^2 \,\mathrm{d}t < \infty$ and $\int_0^T \alpha_i(t)\,\mathrm{d}t < \infty$, for all $i = 1, \cdots, m$ and all $j = 1, \cdots, n$ such that

$$\mathrm{d}Z(t) = Z(t_-)\Big\{\sum_{j=1}^{n} \theta_j(t)\,\mathrm{d}B_j(t) + \sum_{i=1}^{m} \alpha_i(t)\,\mathrm{d}M_i(t)\Big\}. \tag{2.1}$$

Note that, once $Z(\tau) = 0$ at some time τ due to a jump, $Z(t) = 0$ for $t \ge \tau$. And, for each $j = 1, \cdots, n$,

$$\widetilde{B}_j(t) := B_j(t) - \int_0^t \theta_j(s)\,\mathrm{d}s \tag{2.2}$$

is a $\mathbb{Q}$-Brownian motion, and for each $i = 1, \cdots, m$,

$$\widetilde{M}_i(t) := N_i(t) - \int_0^t \widetilde{\lambda}_i(s)\, \mathrm{d}s \tag{2.3}$$

is a $\mathbb{Q}$-(local) martingale where $\widetilde{\lambda}_i(s) := \lambda_i\{\alpha_i(s) + 1\}$ (cf. Theorem 41 of Protter (2010, Ch.III)). Note that $\widetilde{B}_j(t)$ and $\widetilde{M}_i(t)$ (or $N_i(t)$) are uncorrelated instantaneously for any i, j, i.e., the quadratic variations $\mathrm{d}\widetilde{B}_j(t) \cdot \mathrm{d}N_i(t) = 0$ and $\mathrm{d}\widetilde{B}_j(t) \cdot \mathrm{d}\widetilde{M}_i(t) = 0$ for any i, j, but are not necessarily independent under $\mathbb{Q}$, whereas $B_j(t)$ and $M_i(t)$ (or $N_i(t)$) are independent under $\mathbb{P}$ for any i, j.

Applying the Itô's formula to Eq.(2.1) (see e.g. T3 Theorem of Brémaud (1981, p.166) and Theorem 11.6.9 of Shreve (2004, p.503)),

$$\begin{aligned} Z(t) \quad = \quad & \prod_{j=1}^{n} \exp\left\{ \int_0^t \theta_j(s)\, \mathrm{d}B_j(s) - \frac{1}{2} \int_0^t (\theta_j(s))^2\, \mathrm{d}s \right\} \cdot \\ & \prod_{i=1}^{m} \exp\left\{ \sum_{0<s\le t} \log\left(\frac{\widetilde{\lambda}_i(s)}{\lambda_i} \right) \Delta N_i(s) + \int_0^t (\lambda_i - \widetilde{\lambda}_i(s))\, \mathrm{d}s \right\}. \end{aligned} \tag{2.4}$$

2.2 Production

The firm produces a single non-storable consumption good over time. No productive resources are utilized: the production is an endowment. The endowment process, denoted by X, is characterized by the following stochastic differential equation (SDE):

$$\begin{aligned} \mathrm{d}X(t) \quad &= \quad X(t_-)\, \mathrm{d}G(t) \\ &:= \quad X(t_-)\Big(\mu^G\, \mathrm{d}t + \sum_{j=1}^{n} \sigma_j^G\, \mathrm{d}B_j(t) + \sum_{i=1}^{m} z_i^G\, \mathrm{d}M_i(t)\Big), \quad X(0) = x_0 > 0 \end{aligned} \tag{2.5}$$

where $\mathrm{d}G$ denotes the rate of endowment growth (i.e., GDP growth), $\mu^G, \sigma_j^G, z_i^G\, \forall\, i, j$ are constants, $\sigma_j^G > 0\, \forall\, j$, $-1 < z_i^G < 0\, \forall\, i$, and $z_{i_1}^G \neq z_{i_2}^G$ if $i_1 \neq i_2$. We may also write an n dimensional row vector $\sigma^G := (\sigma_1^G, \cdots, \sigma_n^G)$ and an m dimensional row vector $z^G := (z_1^G, \cdots, z_m^G)$. In financial terms, $\{B_j;\, j = 1, \cdots, n\}$ stands for regular market (i.e., diffusive) risk and $\{N_i;\, i = 1, \cdots, m\}$ stands for rare-event (i.e., jump) risk. For each $i = 1, \cdots, m$, z_i^G stands for the jump size of N_i, and $-1 < z_i^G < 0$ means that the jump causes a loss. $\sum_{i=1}^m z_i^G N_i$ can be interpreted as a mixed Poisson process with its intensity $\sum_{i=1}^m \lambda_i$, for which process $\frac{\lambda_i}{\sum_{i=1}^m \lambda_i}$ stands for the probability of having the jump size z_i^G when a jump occurs.

2.3 Moral hazard

We assume that the firm can control the probability measure so as to maximize his own expected payoff, in the spirit of the standard moral-hazard literature in corporate finance (see e.g. Tirole (2006, Section 3.2)). More specifically, $\mathbb{P}$ is the original probability measure, that is, the measure when the firm would not control it – we may also call it the reference measure. The firm changes the probability measure from $\mathbb{P}$ into $\mathbb{Q}$ such that $\mathbb{Q} \ll \mathbb{P}$.[11] Assume that $\mathbb{P}$ is the public information, and that the investor knows the fact that $\mathbb{Q}$ is absolutely continuous w.r.t $\mathbb{P}$, but cannot observe $\mathbb{Q}$ directly, i.e., $\mathbb{Q}$ is the private information of the firm.

We also assume that the firm incurs a utility cost when controlling the probability measure. The cost is represented by relative entropy, denoted by $H(\mathbb{Q}||\mathbb{P})$,[12] which is defined as:

$$\begin{aligned} H(\mathbb{Q}\,||\,\mathbb{P}) \quad &:= \quad \mathbb{E}^{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\left(\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\mathbf{1}_{\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}>0\}}\right] \\ &= \quad \mathbb{E}^{\mathbb{Q}}\left[\left(\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\mathbf{1}_{\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}>0\}}\right] = \mathbb{E}^{\mathbb{Q}}\left[\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right]. \end{aligned}$$

Assume that $H(\mathbb{Q}\,||\,\mathbb{P}) < \infty$.[13] Roughly speaking, the relative entropy is a measure of the distance between the probability measures $\mathbb{P}$ and $\mathbb{Q}$.[14] From a statistical viewpoint, it represents a measure of the type-I error of rejecting the true probability measure $\mathbb{Q}$ and, instead, assuming $\mathbb{P}$ incorrectly. That is, it stands for the statistical inefficiency of assuming that the probability measure is $\mathbb{P}$ when the true measure is $\mathbb{Q}$. A low level of the relative entropy means that $\mathbb{Q}$ and $\mathbb{P}$ are not so distant as to significantly discriminate $\mathbb{P}$ against $\mathbb{Q}$. Thus, in this model, the relative entropy means how far the true probability measure $\mathbb{Q}$ is distorted from the reference measure $\mathbb{P}$. The effort cost impedes the firm's adopting the probability measure far away from $\mathbb{P}$. As we will show below, due to this

[11] Under the absolute-continuity restriction, zero probability is necessarily assigned, under $\mathbb{Q}$, to the state to which zero probability is assigned under $\mathbb{P}$. In other words, this model does not look at the states that are supposed not to occur under the reference measure $\mathbb{P}$.

[12] This formulation of moral hazard is borrowed basically from a companion work of this paper, namely Misumi et al. (2013).

[13] Note that this finiteness assumption is imposed for removing the indeterminacy of the firm's optimal expected utility defined in Eq.(3.1) below.

[14] The relative entropy is always non-negative and is zero if and only if $\mathbb{Q} = \mathbb{P}$. Strictly speaking, it is not a true distance because neither the symmetry nor the triangle inequality is satisfied. However, it is well known that it is useful to regard the relative entropy as a distance between two probability measures. See e.g. Cover and Thomas (2006, p.18) in the statistics literature. The relative entropy has been lately used as a cost of controlling probability measures in the economics literature. See e.g. Hansen and Sargent (2007), Hansen et al. (2006), Sims (2003). Also, Delbaen et al. (2002) use it as a penalty in hedging contingent claims in the finance literature.

cost, the investor can infer the true probability measure $\mathbb{Q}$, although she cannot observe it directly.

In this framework, from Eq.(2.2), Eq.(2.3) and Eq.(2.4), we can characterize the relative entropy by using θ_j and α_i for all i, j as follows:

$$
\begin{aligned}
H(\mathbb{Q}\,||\,\mathbb{P}) &= \mathbb{E}^{\mathbb{Q}}\left[\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right] = \mathbb{E}^{\mathbb{Q}}\left[\begin{array}{c}\sum_{j=1}^{n}\int_0^T\theta_j(s)\,\mathrm{d}B(s) - \frac{1}{2}\int_0^T(\theta_j(s))^2\,\mathrm{d}s+ \\ \sum_{i=1}^{m}\int_0^T\left\{\widetilde{\lambda}_i(s)\log\left(\frac{\widetilde{\lambda}_i(s)}{\lambda_i}\right)+(\lambda_i-\widetilde{\lambda}_i(s))\right\}\mathrm{d}s\end{array}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\sum_{j=1}^{n}\int_0^T\frac{(\theta_j(s))^2}{2}\,\mathrm{d}s + \sum_{i=1}^{m}\int_0^T\left\{\widetilde{\lambda}_i(s)\log\left(\frac{\widetilde{\lambda}_i(s)}{\lambda_i}\right)+(\lambda_i-\widetilde{\lambda}_i(s))\right\}\mathrm{d}s\right] \quad (2.6)
\end{aligned}
$$

Recall that $\frac{\widetilde{\lambda}_i(s)}{\lambda_i} = \alpha_i(s)+1$. If there are no jump terms, $H(\mathbb{Q}\,||\,\mathbb{P}) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{j=1}^{n}\frac{1}{2}\int_0^T(\theta_j(s))^2\,\mathrm{d}s\right]$. The second term inside the expectation of Eq.(2.6) corresponds to the jump terms.

There is a caveat. In contrast to Cvitanić and Zhang (2007), this paper does not assume that the firm controls directly the drift rate θ_j and the jump intensity $\widetilde{\lambda}_i$ for each i, j. θ_j and α_i for each i, j are adapted to $\mathbb{F}$, not to the filtration generated by the controlled $\widetilde{B}_j(t)$ and $\widetilde{M}_i(t)$ for all i, j, in the weak formulation.[15] If we assume that the firm controls θ_j and α_i for each i, j, then the controls would be undertaken based on the information set generated only by X, which continues to lose the information of a history of the controls and the true shocks over time. It would seem irrelevant in practice.

2.4 Consumption good

The players consume the consumption good at each instant over $[0, T]$ by sharing the produced consumption good. Let $S = \{S(t)\}_{0\leq t\leq T}$ and $C = \{C(t)\}_{0\leq t\leq T}$ denote the consumption processes of the firm and the investor, respectively. Let $\mathcal{H}$ denote the Hilbert space of all $\mathbf{R}$-valued adapted processes Y such that $\mathbb{E}^{\mathbb{P}}\left[\int_0^T Y(t)^2\,\mathrm{d}t + Y(T)^2\right] < \infty$ with the inner product $(Y|Z) := \mathbb{E}^{\mathbb{P}}\left[\int_0^T Y(t)Z(t)\,\mathrm{d}t + Y(T)Z(T)\right]$ for $Y, Z \in \mathcal{H}$. $\mathcal{H}_+$ denotes the set of all non-negative processes in $\mathcal{H}$. C, S are in $\mathcal{H}_+$. Also, the endowment process X characterized by Eq.(2.5) is in $\mathcal{H}_+$. Due to the non-storability, the terminal stock levels of the consumptions are zeros, i.e., $C(T) = 0, S(T) = 0$.

The good is shared according to terms of a contract – call S a contract or a payment rule. S is the firm's consumption, while C is the investor's one out of her allocation $X - S$. We will see below

[15] Nakamura and Takaoka (2013) generalize the information set so as to be generated by a history of an agent's efforts as well as a history of true shocks. That paper then solves the optimization problem in the strong formulation, rather than in the weak one.

that, by designing the payment structure properly, the lender can control indirectly the probability measure $\mathbb{Q}$ that is controlled directly by the firm. The investor offers a menu of contract payoffs $\{S(t)\}_{0\le t\le T}$ to the firm ex ante, and the firm then decides whether or not to accept it at time 0. We will specify the detailed form of the contract below.

2.5 Financial markets

Financial markets are accessible only to the investor, not to the firm. There are two types of financial securities in the markets: a riskless asset and d risky assets. The return process of the riskless asset with its price $\{P_0(t)\}_{0\le t\le T}$ is characterized by:

$$\frac{\mathrm{d}P_0(t)}{P_0(t)} = r(t)\,\mathrm{d}t$$

where $r(t)$ is the riskless rate and is determined endogenously in the markets. This model assumes that $r(t)$ can be negative and, in addition, $\int_0^T |r(t)|\,\mathrm{d}t < \infty\, a.s.$.

The excess returns of the risky assets, denoted by d-dimensional $\mathrm{d}R$, are characterized by:

$$\mathrm{d}R(t) = \mu^R\,\mathrm{d}t + \sum_{j=1}^{n} \sigma_j^R\,\mathrm{d}B_j(t) + \sum_{i=1}^{m} z_i^R\,\mathrm{d}M_i(t)$$

where the elements of $\mu^R \in \mathbf{R}^d$, $\sigma_j^R \in \mathbf{R}^d$ and $z_i^R \in \mathbf{R}^d\,\forall\, i,j$ are constants. In particular, the elements of $\sigma_j^R > 0\,\forall\, j$, and the elements of $z_i^R > -1\,\forall\, i$. We may also write a $(d\times n)$-matrix $\sigma^R := (\sigma_1^R,\cdots,\sigma_n^R)$ and a $(d\times m)$-matrix $z^R := (z_1^R,\cdots,z_m^R)$.

The investor has initial funds $W(0) = w_0 = 0$ at time 0 and makes a financial portfolio among the financial securities dynamically over $[0,T]$. This model presumes that the representative investor can take non-zero positions of the financial securities in off-equilibrium, but takes zero positions in equilibrium. Also, to make the price process well-defined, this model assumes that the wealth process can be negative, as in Cox et al. (1985).[16] The portfolio ratio on the risky assets is denoted by an $\mathbf{R}^d$-valued predictable process β such that:

Definition 2.1 *Define the set of β as $\mathcal{B}$ such that $\mathcal{B}$ is the set of $\mathbb{F}$-predictable processes β satisfying $\int_0^T |\beta(t)|^2\,\mathrm{d}t < +\infty\, a.s.$ and $\beta(t)^\top z_i^R \ge -1,\ \forall\, i$.*

[16] In contrast to Cox et al. (1985), there exist negative proportional jumps in this model. When a jump occurs for negative wealth, the wealth-level improves. This seems inconceivable. However, such negative wealth never takes places in equilibrium.

The residual is invested in the riskless asset. Thus the wealth process W is characterized by

$$\mathrm{d}W(t) = W(t_-)r(t)\,\mathrm{d}t + W(t_-)\beta(t)^\top\,\mathrm{d}R(t) + \Big(X(t) - C(t) - S(t)\Big)\,\mathrm{d}t, \quad W(0) = w_0. \tag{2.7}$$

The wealth is storable whereas the consumption good is non-storable. Thus the last term $\Big(X(t) - C(t) - S(t)\Big)$ on the right-hand side of Eq.(2.7) is the *current* value of the saving of the consumption good in the *future* spot market. It means that the term can be negative: the investor can borrow from the markets. In equilibrium, however, all the consumption good is used up, and thus the equilibrium saving is zero in the financial markets.

Since μ^R, σ^R, z^R are constants and $\int_0^T |\beta(t)|^2\,\mathrm{d}t < +\infty\, a.s.$ due to Definition 2.1,

$$\int_0^T |\beta(t)^\top \mu^R|\,\mathrm{d}t < +\infty, \quad \int_0^T |\beta(t)^\top \sigma^R|^2\,\mathrm{d}t < +\infty, \text{ and } \int_0^T |\beta(t)^\top z_i^R|\,\mathrm{d}t < +\infty\,\forall i,\ a.s..$$

Also,

$$\begin{aligned}
\beta(t)^\top\,\mathrm{d}R(t) &= \sum_{k=1}^d \beta_k(t)\Big(\mu_k^R\,\mathrm{d}t + \sum_{j=1}^n \sigma_{k,j}^R\,\mathrm{d}B_j(t) + \sum_{i=1}^m z_{k,i}^R\,\mathrm{d}M_i(t)\Big) \\
&= \sum_{k=1}^d \beta_k(t)\mu_k^R\,\mathrm{d}t + \sum_{j=1}^n \Big\{\sum_{k=1}^d \beta_k(t)\sigma_{k,j}^R\Big\}\,\mathrm{d}B_j(t) + \sum_{i=1}^m \Big\{\sum_{k=1}^d \beta_k(t) z_{k,i}^R\Big\}\,\mathrm{d}M_i(t).
\end{aligned}$$

The condition $\beta(t)^\top z_i^R \geq -1\,\forall\, i$ ensures that the wealth does not turn into opposite signs by jumps.

2.6 Utility

The two players rank their own consumption/wealth processes via time-separable utility of consumption, characterized by a common constant instantaneous discount factor $\delta > 0$ and different instantaneous utility forms of consumption/wealth. Let $f_k : \mathbf{R}_{++} \to \mathbf{R} \cup \{-\infty\}$ for $k \in \{1, 2\}$ denote player k's instantaneous utility function of his or her own consumption. In particular, we set $f_1(x) = a \log x$ for $x > 0$ and $f_2(x) = \frac{x^{1-\gamma}}{1-\gamma}$ for $x > 0$ where a, γ are constants and $0 < \gamma < 1$. The investor's power utility with $0 < \gamma < 1$ means that she is less risk-averse than the firm. Obviously, for $k = 1, 2$, the utility function f_k is non-decreasing and concave, and is continuously differentiable on its effective domain denoted by $\operatorname{dom} f_k := \{x \in \mathbf{R} \,|\, f_k(x) > -\infty\}$. Also, the investor enjoys a linear utility of the terminal wealth $W(T)$ whereas the firm does not receive any utility of wealth.[17]

[17]If the financial markets are accessible to the firm, the moral-hazard problem would influence the firm's utility via the financial markets as well. With such accessibility, the firm might make

To define the players' performance criteria well, we first define mathematical regularities for the control triples $(S, C, \beta) \in \mathcal{H}_+ \times \mathcal{H}_+ \times \mathcal{B}$:

Definition 2.2 *Define $\mathcal{A}$ as the set of the control triples $(S, C, \beta) \in \mathcal{H}_+ \times \mathcal{H}_+ \times \mathcal{B}$ such that*

(i) $0 < S(t) \leq X(t) \; \forall \, [0, T)$ *a.s. and* $S(T) = 0$,

(ii) *S is of the Markovian form $S(t) := \tilde{s}(t, X(t))$ for some deterministic function $\tilde{s}$. In particular, we restrict the contract form as follows: for some constant $s \in (0, 1) \in \mathbf{R}_{++}$, $S(t) = sX(t) \, \forall \, t \in [0, T)$ a.s.. Call it a stationary linear payment rule. Note that it does not depend on W.*

(iii) $\mathbb{P}[\int_0^T e^{-\delta u} \log S(u) \, \mathrm{d}u > -\infty] > 0$,

(iv) $0 < C(t) \leq X(t) \; \forall \, [0, T)$ *a.s. and* $C(T) = 0$,

(v) *C is of the Markovian form $C(t) := c(t, X(t), W(t))$ for some deterministic function c,*

(vi) *β is of the Markovian form $\beta(t) := b(t, X(t), W(t))$ for some deterministic function b,*

(vii) *The wealth process W, which is generated via Eq.(2.7), satisfies $\mathbb{E}^{\mathbb{P}}[(W(T))^2] < \infty$.*

Due to the assumption of the Markovian controls, we avoid the measurability problem of the control functions in dynamic programing (see e.g. Pham (2009, p.42)).

The players' performance criteria are written as follows. For a controlled probability measure $\mathbb{Q}$ and for a consumption process $S(t) > 0$ for $0 \leq t \leq T$, let $U_1(S; \mathbb{Q})$ denote the firm's expected discounted utility, net of the effort cost:

$$U_1(S; \mathbb{Q}) = \mathbb{E}^{\mathbb{Q}}[\int_0^T e^{-\delta u} f_1(S(u)) \, \mathrm{d}u] - H(\mathbb{Q} \, || \, \mathbb{P}). \tag{2.8}$$

He is exogenously given a reservation utility, denoted by a constant $\rho \in \mathbf{R}$, at time 0. If the investor offers to the firm any lower utility than the reservation utility ρ ex ante, the firm would not take the offer. To make the contractual relationship viable, we assume

Assumption 2.1 $\rho < \log \mathbb{E}^{\mathbb{P}} \left[e^{\int_0^T e^{-\delta u} a \log X(u) \, \mathrm{d}u} \right]$.

---

higher efforts than otherwise, because, by doing so, he would receive higher financial returns.

As we will discuss in detail below, if this assumption is not satisfied, the firm has no incentive to participate in the contact. Let us check the integrability of U_1. The process X, characterized by Eq.(2.5), is integrable. So is S due to Definition 2.2 (i). By the concavity of f_1, $U_1 < +\infty$. On the other hand, Definition 2.2 (iii) does not always ensure $U_1 > -\infty$. Still, since the firm reserves $\rho \in \mathbf{R}$ as an outside option in this model, U_1 is well-defined in equilibrium.

Next, we define the investor's performance criterion. For a given $\mathbb{Q}$, let $U_2(C, W(T); \mathbb{Q})$ denote the investor's expected discounted utility. I.e.,

$$\begin{aligned} U_2(C, W(T); \mathbb{Q}) &= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-\delta u} f_2(C(u))\,\mathrm{d}u + e^{-\delta T} W(T)\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\left\{\int_0^T e^{-\delta u} f_2(C(u))\,\mathrm{d}u + e^{-\delta T} W(T)\right\}\right]. \end{aligned}$$

Let us check the integrability of U_2. Due to the integrability of X and Definition 2.2 (iv), $U_2 < +\infty$. In addition, since $f_2 \geq 0$, $U_2 \geq 0$. Hence, U_2 is well-defined.

3 Optimization

3.1 Firm's optimization

For a contact $S \in \mathcal{S}$, we define the firm's optimal expected utility under the controlled probability measure $\mathbb{Q}$, denoted by V_1, as:

$$V_1 := \sup_{\substack{\mathbb{Q} \ll \mathbb{P} \\ H(\mathbb{Q} \| \mathbb{P}) < \infty}} U_1(S; \mathbb{Q}). \tag{3.1}$$

We obtain the following lemma:

Lemma 3.1 *For $S \in \mathcal{S}$,*

$$V_1 = \log \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}\right]. \tag{3.2}$$

The maximizer, denoted by $\mathbb{Q}^*$, is then characterized by an L_2-process:

$$\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} = \frac{e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}}{\mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}\right]} = e^{-V_1} e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}. \tag{3.3}$$

This result and its variants are known in the fields of operations research and mathematical finance: for a literature review, see e.g. the first remark in Section 1 of Delbaen et al. (2002). For the sake of completeness, we present a proof.

Proof: See appendix.

The investor cannot directly observe the true probability measure $\mathbb{Q}$, but can verify the optimal $\mathbb{Q}^*$ by designing the contract to satisfy Eq.(3.3). Thus the investor can implement the optimal $\mathbb{Q}^*$ by controlling S through Eq.(3.3). To ensure that the firm participates in the contract, the investor provides him with no lower utility than his reservation utility, i.e.,

$$V_1 \geq \rho. \tag{3.4}$$

We call it the participation constraint. From Eq.(3.3) and Eq.(3.4),

$$\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} = e^{-V_1} e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u} \leq e^{-\rho} e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u} \tag{3.5}$$

with equality if the participation constraint is binding (i.e., $V_1 = \rho$). Eq.(3.5) plays an incentive constraint in the investor's optimization, as shown in the next subsection.

3.2 Investor's optimization

We define the investor's optimization subject to the firm's optimization. Specifically, taking as given the firm's optimization characterized by Lemma 3.1, the investor optimizes his utility while controlling $\mathbb{Q}$ indirectly by giving the firm the incentive Eq.(3.5). In addition, the investor's optimization is subject to the firm's participation constraint Eq.(3.4), i.e., the lender gives the firm no lower utility than his reservation utility ρ. Let $\{V_2(t)\}_{0 \leq t \leq T}$ denote the process of the investor's optimal expected discounted utility. We formulate the investor's optimization problem with respect to $(S, C, \beta) \in \mathcal{A}$ as follows:

$$\begin{aligned}
V_2(0) &= \sup_{(S,C,\beta)\in\mathcal{A}} U_2\left(C, W(T)\,;\,\mathbb{Q}^*\right) && (3.6)\\
\text{s.t.} & \\
\mathrm{d}W(t) &= W(t_-)r(t)\,\mathrm{d}t + W(t_-)\beta(t)^\top\,\mathrm{d}R(t) + \Big(X(t) - C(t) - S(t)\Big)\,\mathrm{d}t,\ W(0) = w_0,\\
\mathrm{d}R(t) &= \mu^R\,\mathrm{d}t + \sum_{j=1}^{n}\sigma_j^R\,\mathrm{d}B_j(t) + \sum_{i=1}^{m} z_i^R\,\mathrm{d}M_i(t),\\
\mathrm{d}X(t) &= X(t_-)\Big(\mu^G\,\mathrm{d}t + \sum_{j=1}^{n}\sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m} z_i^G\,\mathrm{d}M_i(t)\Big),\ X(0) = x_0,\\
V_1 &= \log\mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a\log S(u)\,\mathrm{d}u}\right] \geq \rho, && \text{(from Eq.(3.4))}\\
\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} &= e^{-V_1} e^{\int_0^T e^{-\delta u} a\log S(u)\,\mathrm{d}u} && \text{(from Eq.(3.5))}
\end{aligned}$$

where $U_2\left(C, W(T)\,;\,\mathbb{Q}^*\right) = \mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T e^{-\delta u}\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T}W(T)\right]$. Note that S is designed ex ante, whereas, by contrast, the pair $(C,\,\beta)$ is controlled ex post. Due to Eq.(3.5), the investor can take her expectation under $\mathbb{Q}^*$. Due to Definition 2.2, the integrability is ensured in Eq.(3.6).

Definition 3.1 *A control triple $(S, C, \beta) \in \mathcal{A}$ is said to be optimal for Eq.*(3.6) *if $(S, C, \beta) \in \mathcal{A}$ is the maximizer of Eq.*(3.6)*, if any.*

From Eq.(3.6), we define the Lagrangian as follows:

$$\begin{aligned}
&\sup_{(S,C,\beta)\in\mathcal{A}} \mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T e^{-\delta u}\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T}W(T) + \chi\right] && (3.7)\\
= &\sup_{(S,C,\beta)\in\mathcal{A}} e^{-V_1}\mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a\log S(u)\,\mathrm{d}u}\left(\int_0^T e^{-\delta u}\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T}W(T) + \chi\right)\right]\\
\text{s.t.}\quad & \mathrm{d}W(t) = W(t_-)r(t)\,\mathrm{d}t + W(t_-)\beta(t)^\top\,\mathrm{d}R(t) + \Big(X(t) - C(t) - S(t)\Big)\,\mathrm{d}t, \quad W(0) = w_0,\\
& \mathrm{d}R(t) = \mu^R\,\mathrm{d}t + \sum_{j=1}^{n}\sigma_j^R\,\mathrm{d}B_j(t) + \sum_{i=1}^{m} z_i^R\,\mathrm{d}M_i(t),\\
& \mathrm{d}X(t) = X(t_-)\Big(\mu^G\,\mathrm{d}t + \sum_{j=1}^{n}\sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m} z_i^G\,\mathrm{d}M_i(t)\Big), \quad X(0) = x_0
\end{aligned}$$

where χ denotes the Lagrangian multiplier associated with the participation constraint Eq.(3.4).

4 Market equilibrium

4.1 Definition

We define market equilibrium, in which a control triple (S, C, β) is optimal for Eq.(3.6) and the markets of the good and the financial assets are cleared. We focus on time-consistent market equilibrium, in that the investor voluntarily commits herself ex post to the ex-ante designed contract S.

Definition 4.1 *A control triple $(S, C, \beta) \in \mathcal{A}$ is said to be in (time-consistent) market equilibrium if the following conditions are satisfied:*

(i) *The control triple $(S, C, \beta) \in \mathcal{A}$ is optimal for Eq.*(3.6)*,*

(ii) *For all t, $C(t) = X(t) - S(t)$,*

(iii) *For all t, all the elements of $\beta(t) = 0$,*

4.2 Characterization

It is hard to obtain directly a closed-form solution to the investor's optimization problem, because it is subject to two constraints on states: the participation constraint Eq.(3.4) and the incentive constraint Eq.(3.5). More specifically, due to the incentive constraint Eq.(3.5), the firm's utility is embedded in the lender's utility (i.e., as in the term e^{-V_1}) while, at the same time, due to the participation constraint Eq.(3.4), the Lagrangian multiplier χ is involved in Eq.(3.7).[18]

To characterize the solution in this paper, we examine a modified version of market equilibrium, instead of looking directly at the time-consistent one. The modification consists of two steps. First, we assume that the investor provides the firm with a contract that promises the expected utility equivalent to his reservation utility *ex ante*, but does not need to commit herself to the promise *ex post*. Precisely, after making the contract with the firm, the lender can revise it *ex ante* (i.e. before starting the allocation): Eq.(3.5) holds equality *ex ante*, whereas Eq.(3.4) is not imposed ex post. As the second step, we check whether or not the solution at the first step satisfies the participation constraint. If not, the solution is adjusted to the constraint by controlling the sharing ratio s.

We can characterize the modified equilibrium explicitly in a closed form. This is because, on the second line of Eq.(3.7), the term e^{-V_1} is replaced with a constant $e^{-\rho}$ and the Lagrangian multiplier χ is removed. We call the modified equilibrium time-inconsistent market equilibrium.

[18] As to a general discussion of such mathematical difficulty, see e.g. Yong and Zhou (1999, p.155).

We give the formal definition of the time-inconsistent market equilibrium. Condition (i) of Definition 4.1 is replaced by Condition (i)' as follows:

Definition 4.2 *A control triple $(S, C, \beta) \in \mathcal{A}$ is said to be in time-inconsistent market equilibrium if the following conditions are satisfied:*

(i)' *The control triple $(S, C, \beta) \in \mathcal{A}$ is optimal for Eq.(3.6) except that (1) Eq.(3.5) holds equality ex ante and (2) the participation constraint Eq.(3.4) is not imposed ex post,*

(ii) *For all t, $C(t) = X(t) - S(t)$,*

(iii) *For all t, all the elements of $\beta(t) = 0$.*

The lender's optimal utility in the time-inconsistent market equilibrium, denoted by a process $\{\widehat{V}_2(t)\}_{0 \le t \le T}$, is:

$$
\begin{aligned}
\widehat{V}_2(0) \;&:=\; \sup_{(S,C,\beta)\in\mathcal{A}} \mathbb{E}^{\mathbb{P}}\left[\frac{e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}}{\mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right]} \left(\int_0^T e^{-\delta u} \frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T} W(T) \right) \right] \\
&=\; \sup_{(S,C,\beta)\in\mathcal{A}} e^{-\rho} e^{\int_0^T e^{-\delta u} a \log s\,\mathrm{d}u} \mathbb{E}^{\mathbb{P}} \left[\begin{array}{l} \int_0^T e^{-\delta u} \mathbb{E}_u^{\mathbb{P}}[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}] \frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u \\ + e^{-\delta T} e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u} W(T) \end{array} \right] \\
&=\; \sup_{(S,C,\beta)\in\mathcal{A}} e^{-\rho + \frac{(1-e^{-\delta T}) a \log s}{\delta}} \mathbb{E}^{\mathbb{P}} \left[\begin{array}{l} \int_0^T e^{-\delta u} \mathbb{E}_u^{\mathbb{P}}[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}] \frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u \\ + e^{-\delta T} e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u} W(T) \end{array} \right] \qquad (4.1) \\
\text{s.t. } \mathrm{d}W(t) \;&=\; W(t_-) r(t)\,\mathrm{d}t + W(t_-)\beta(t)^\top \mathrm{d}R(t) + \Big(X(t) - C(t) - S(t) \Big)\,\mathrm{d}t, \quad W(0) = w_0, \\
\mathrm{d}R(t) \;&=\; \mu^R\,\mathrm{d}t + \sum_{j=1}^n \sigma_j^R\,\mathrm{d}B_j(t) + \sum_{i=1}^m z_i^R\,\mathrm{d}M_i(t), \\
\mathrm{d}X(t) \;&=\; X(t_-)\Big(\mu^G\,\mathrm{d}t + \sum_{j=1}^n \sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^m z_i^G\,\mathrm{d}M_i(t) \Big), \quad X(0) = x_0.
\end{aligned}
$$

As defined above, in the time-inconsistent market equilibrium, the participation constraint is satisfied ex ante. Let $\hat{s}$ denote the sharing rule s when the participation constraint binds ex ante. Because $\rho < \log \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right]$ due to Assumption 2.1, there exists $\hat{s}$ such that

$$
\begin{aligned}
\rho \;&=\; \frac{(1-e^{-\delta T}) a \log \hat{s}}{\delta} + \log \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right]. \\
\text{I.e.,} \quad \hat{s} \;&=\; \exp\left\{ \frac{\delta}{a(1-e^{-\delta T})} \left(\rho - \log \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right] \right) \right\}. \qquad (4.2)
\end{aligned}
$$

$\log \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right]$ stands for the firm's optimal utility in the case that he would receive the whole production over time. If Assumption 2.1 is not satisfied, the firm would not obtain a level of the reservation utility under the contract even if he receives the whole production over time. Thus Assumption 2.1 is a necessary condition for making the contractual relationship viable.

We characterize the process of the distorted probability measure. Define a time-t distortion of the probability measure caused by the endowment process X all over $[0, T]$ as

$$Y(t) := \mathbb{E}_t^{\mathbb{P}}\left[e^{a\int_0^T e^{-\delta u}\log X(u)\,\mathrm{d}u}\right]. \tag{4.3}$$

We obtain the following lemma:

Lemma 4.1

$$Y(t) = \exp\left(\begin{array}{l} a\left(\begin{array}{l} \int_0^t e^{-\delta u}\log X(u)\,\mathrm{d}u + \frac{e^{-\delta t}-e^{-\delta T}}{\delta}\log X(t) \\ +e^{-\delta t}\left(\mu^G - \frac{1}{2}\sum_{j=1}^n(\sigma_j^G)^2 - \sum_{i=1}^m \lambda_i\right)\frac{1-e^{-\delta(T-t)}(1+\delta(T-t))}{\delta^2} \end{array}\right) \\ +\sum_{j=1}^n \frac{(ae^{-\delta t}\sigma_j^G)^2}{2}\frac{1}{\delta^2}\int_0^{T-t}(e^{-\delta u}-e^{-\delta(T-u)})^2\,\mathrm{d}u \\ +\sum_{i=1}^m \lambda_i(T-t)(-1+e^{-\frac{e^{-\delta(T-t)}}{\delta}\log(1+z_i^G)}\frac{1}{T-t}\int_0^{T-t} e^{\frac{e^{-\delta u}}{\delta}\log(1+z_i^G)}\,\mathrm{d}u) \end{array}\right).$$

Proof : See appendix.

Plugging $Y(0) = \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right]$ into Eq.(4.2),

$$\hat{s} = \exp\left\{\begin{array}{l} \frac{\delta}{a(1-e^{-\delta T})}\times \\ \left(\rho - \left(\begin{array}{l} a\left(\begin{array}{l} \frac{1-e^{-\delta T}}{\delta}\log x_0 \\ +\left(\mu^G - \frac{1}{2}\sum_{j=1}^n(\sigma_j^G)^2 - \sum_{i=1}^m \lambda_i\right)\frac{1-e^{-\delta T}(1+\delta T)}{\delta^2} \end{array}\right) \\ +\sum_{j=1}^n \frac{(a\sigma_j^G)^2}{2}\frac{1}{\delta^2}\int_0^T(e^{-\delta u}-e^{-\delta T})^2\,\mathrm{d}u \\ +\sum_{i=1}^m \lambda_i T(-1+e^{-\frac{e^{-\delta T}}{\delta}\log(1+z_i^G)}\frac{1}{T}\int_0^T e^{\frac{e^{-\delta u}}{\delta}\log(1+z_i^G)}\,\mathrm{d}u) \end{array}\right)\right) \end{array}\right\}. \tag{4.4}$$

We next characterize the stochastic differential of $Y(t)$. Note that $Y(t)$ is a martingale.

Lemma 4.2 *The martingale $Y(t)$ satisfies*

$$\mathrm{d}Y(t) = Y(t_-)\left\{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\sum_{j=1}^n \sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^m\left((1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}} - 1\right)\mathrm{d}M_i(t)\right\}.$$

Proof: See appendix.

Note that it is because of the assumption of the firm's log utility function that we have obtained such explicit characterization of Y in Lemma 4.1 and Lemma 4.2.

We solve Problem (4.1) by using Y as well as X, W as state variables and applying a verification theorem for the optimal control problem with the classical Hamilton-Jacobi-Bellman (HJB) equation (see e.g. Øksendal and Sulem (2007, Theorem 3.1, p.46), Pham (2009, Theorem 3.5.2, p.47)). For a twice continuously differentiable function $h(t, X(t), Y(t), W(t)) \in C^{1,2,2,2}$, define the generator of $(t, X(t), Y(t), W(t))$ as:

$$
\begin{aligned}
& \mathcal{L}^{C,\beta} h(t, X(t), Y(t), W(t)) \qquad (4.5) \\
:= \ & h_t + \mu^G X(t) h_x + \Big(rW(t) + \beta(t)^\top \mu^R W(t) + \big((1-s)X(t) - C(t) \big) \Big) h_w \\
+ \ & \frac{1}{2} \sum_{j=1}^{n} \left\{ (\sigma_j^G)^2 X(t)^2 h_{xx} + (\beta(t)^\top \sigma_j^R)^2 W(t)^2 h_{ww} + (a \frac{e^{-\delta t} - e^{-\delta T}}{\delta} \sigma_j^G)^2 Y(t)^2 h_{yy} \right\} \\
+ \ & \sum_{j=1}^{n} \left\{ \begin{array}{l} (\beta(t)^\top \sigma_j^R \sigma_j^G) X(t) W(t) h_{xw} + (a \frac{e^{-\delta t} - e^{-\delta T}}{\delta} (\sigma_j^G)^2) X(t) Y(t) h_{xy} \\ + (a \frac{e^{-\delta t} - e^{-\delta T}}{\delta} \beta(t)^\top \sigma_j^R \sigma_j^G) Y(t) W(t) h_{yw} \end{array} \right\} \\
+ \ & \sum_{i=1}^{m} \lambda_i \left\{ \begin{array}{l} h(t, (1+z_i^G) X(t), (1+z_i^G)^{a \frac{e^{-\delta t} - e^{-\delta T}}{\delta}} Y(t), (1+\beta(t)^\top z_i^R) W(t)) \\ -h(t, X(t), Y(t), W(t)) \end{array} \right\}
\end{aligned}
$$

where $h_t := \frac{\partial h}{\partial t}$, $h_x := \frac{\partial h}{\partial X(t)}$, $h_y := \frac{\partial h}{\partial Y(t)}$, $h_w := \frac{\partial h}{\partial W(t)}$, $h_{xx} := \frac{\partial^2 h}{\partial X(t)^2}$, $h_{yy} := \frac{\partial^2 h}{\partial Y(t)^2}$, $h_{ww} := \frac{\partial^2 h}{\partial W(t)^2}$, $h_{xy} := \frac{\partial^2 h}{\partial X(t) \partial Y(t)}$, $h_{yw} := \frac{\partial^2 h}{\partial Y(t) \partial W(t)}$ and $h_{xw} := \frac{\partial^2 h}{\partial X(t) \partial W(t)}$ that are continuous. Note that, in Eq.(4.5), s is fixed and is not necessarily equal to the ex-ante designed $\hat{s}$, because the lender can break the promise $\hat{s}$ in the time-inconsistent market equilibrium.

For notational convenience, using Eq.(4.4), define a constant K as

$$
K := \frac{1}{Y(0)} = \frac{1}{\mathbb{E}^{\mathbb{P}} \left[e^{\int_0^T e^{-\delta u} a \log X(u) \, du} \right]} = e^{-\rho + \frac{(1-e^{-\delta T}) a \log \hat{s}}{\delta}}.
$$

By the verification theorem, if, for $J(t, X(t), Y(t), W(t)) \in C^{1,2,2,2}$, the HJB equation

$$
0 = \sup_{C(t), \beta(t)} \left(K Y(t) \frac{C(t)^{1-\gamma}}{1-\gamma} + \mathcal{L}^{C,\beta} J(t, X(t), Y(t), W(t)) \right) \qquad (4.6)
$$

holds under some regularities for a fixed s, then $\widehat{V}_2(t) = J(t, X(t), Y(t), W(t))$. It can then be shown that $J(t, X(t), Y(t), W(t))$ is strictly increasing and concave in $X(t), Y(t), W(t)$ (see e.g.

Jarrow and Rosenfeld (1984, p.341)[19]).

From the HJB equation (4.6), we obtain a necessary and sufficient condition for optimality:

$$
\begin{aligned}
&[C(t)] \quad K\,Y(t)C(t)^{-\gamma} = J_w > 0, && (4.7)\\
&[\beta(t)] \quad W(t)J_w(\mu^R)^\top + W(t)^2 J_{ww}\beta(t)^\top\sigma^R(\sigma^R)^\top + W(t)X(t)J_{xw}\sigma^G(\sigma^R)^\top \\
&\qquad\quad +W(t)Y(t)J_{wy}\sigma^G(\sigma^R)^\top a\frac{e^{-\delta t}-e^{-\delta T}}{\delta} + W(t)J_w\lambda^\top(z^R)^\top = 0. && (4.8)
\end{aligned}
$$

Now, we characterize the equilibrium solution. First, we look at the optimal β. Dividing both sides of Eq.(4.8) by $W(t)$ and taking the limit $W(t) \to 0$ due to the market clearing condition, we characterize the excess return as follows:

$$
\mu^R + z^R\lambda = -\sigma^R(\sigma^G)^\top \frac{X(t)J_{xw} + Y(t)J_{wy}a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}}{J_w}. \tag{4.9}
$$

Second, we look at the optimal consumption/wealth. From Eq.(4.6), we see that the lower s is better for the lender so long as the participation constraint is satisfied. Therefore, in optimum, the participation constraint is binding, i.e., the optimal s is equivalent to the sharing ratio $\hat{s}$ characterized by Eq.(4.4). Letting the optimal sharing ratio be denoted by s^*,

$$
s^* = \hat{s}. \tag{4.10}
$$

Accordingly, in the remaining, we call both types of market equilibrium simply 'market equilibrium' in common. In the market equilibrium, the equilibrium consumption processes of the investor and the firm, denoted by C^* and S^* respectively, are written as: using s^* defined by Eq.(4.10),

$$
\begin{aligned}
C^*(t) &= (1-s^*)X(t), && (4.11)\\
S^*(t) &= s^*X(t). && (4.12)
\end{aligned}
$$

Also, the equilibrium wealth, denoted by W^*, is: $W^*(t) = 0\,\forall\,t$ a.s..

We characterize the investor's value function in the market equilibrium in a closed form. Since $W^*(t) = 0\,\forall\,t$ a.s., it is sufficient to consider that the value function is $J(t, X(t), Y(t), 0)$ in the

[19]The proof is identical to a proof contained in Cox et al. (1985, Lemma 2).

market equilibrium. As to the generator Eq.(4.5) for $h = J$, the J_w-term in the market equilibrium is

$$\left(r \cdot 0 + \beta(t)^\top \mu^R \cdot 0 + \Big((1-s^*)X(t) - C^*(t)\Big)\right) J_w(t, X(t), Y(t), 0) = 0.$$

Let the value function be denoted by $J^*(t, X(t), Y(t)) := J(t, X(t), Y(t), 0)$. From Eq.(4.6), for $x = X(t)$, $y = Y(t)$,

$$\begin{aligned}
0 &= K(1-s^*)^{1-\gamma} y \frac{x^{1-\gamma}}{1-\gamma} + J_t^* + \mu^G x J_x^* \\
&+ \frac{1}{2}\sum_{j=1}^{n}\left\{(\sigma_j^G)^2 x^2 J_{xx}^* + (a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\sigma_j^G)^2 y^2 J_{yy}^*\right\} + \sum_{j=1}^{n} a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}(\sigma_j^G)^2 xy J_{xy}^* \\
&+ \sum_{i=1}^{m}\lambda_i\left\{J^*(t, (1+z_i^G)x, (1+z_i^G)^{a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}} y) - J^*(t,x,y)\right\}.
\end{aligned}$$

We try $J^*(t,x,y) = py\frac{x^{1-\gamma}}{1-\gamma}$ for some deterministic function of time $p = p(t)$ with $p(T) = 0$. Thus,

$$p' + pL(t) = -K(1-s^*)^{1-\gamma}; \quad p(T) = 0$$

$$\begin{aligned}
\text{where} \quad L(t) &:= (1-\gamma)\mu^G - (1-\gamma)\sigma^G(\sigma^G)^\top\left(\frac{\gamma}{2} - a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\right) \\
&\quad + \sum_{i=1}^{m}\lambda_i\left\{(1+z_i^G)^{(1-\gamma)+a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}} - 1\right\}.
\end{aligned}$$

Hence, for some constant C and for $L(t) \neq 0$,

$$p(t) = Ce^{-\int L(t)\,\mathrm{d}t} - \frac{K(1-s^*)^{1-\gamma}}{L(t)}$$

$$\begin{aligned}
\text{where} \int L(t)\,\mathrm{d}t &= \left((1-\gamma)\mu^G - (1-\gamma)\sigma^G(\sigma^G)^\top\left(\frac{\gamma}{2} + \frac{a}{\delta}e^{-\delta T}\right) - \sum_{i=1}^{m}\lambda_i\right)t \\
&\quad -(1-\gamma)\sigma^G(\sigma^G)^\top\frac{a}{\delta^2}e^{-\delta t} \\
&\quad - \frac{1}{\left((2-\gamma) + a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\right) a e^{-\delta t}}\sum_{i=1}^{m}\lambda_i(1+z_i^G)^{(2-\gamma)+a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}}
\end{aligned}$$

From $p(T) = 0$,

$$p(t) = \frac{K(1-s^*)^{1-\gamma}}{L(t)}\left(\frac{L(t)\exp\left\{-\int L(t)\,\mathrm{d}t\right\}}{L(T)\exp\left\{-\left(\int L(t)\,\mathrm{d}t\right)_{t=T}\right\}} - 1\right)$$

$$\text{where} \qquad L(T) = (1-\gamma)\mu^G - \frac{\gamma(1-\gamma)}{2}\sigma^G(\sigma^G)^\top + \sum_{i=1}^m \lambda_i \left\{(1+z_i^G)^{1-\gamma} - 1\right\},$$

$$\left(\int L(t)\,\mathrm{d}t\right)_{t=T} = \left((1-\gamma)\mu^G - (1-\gamma)\sigma^G(\sigma^G)^\top\left(\frac{\gamma}{2} + \frac{a}{\delta}e^{-\delta T}\right) - \sum_{i=1}^m \lambda_i\right)T$$
$$-(1-\gamma)\sigma^G(\sigma^G)^\top\frac{a}{\delta^2}e^{-\delta T} - \frac{1}{(2-\gamma)ae^{-\delta T}}\sum_{i=1}^m \lambda_i(1+z_i^G)^{2-\gamma}.$$

Thus we obtain the value function $J^*(t,x,y)$ explicitly. Also, from Eq.(4.9),

$$\mu^R + z^R\lambda = -\sigma^R(\sigma^G)^\top\left(-\gamma + a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\right).$$

5 Equilibrium asset prices

We characterize the behavior of asset prices in the market equilibrium. Since the assets are traded ex post (i.e., after the contract is made), we consider the equilibrium behavior of asset prices by taking as given the optimally designed contract S^* defined by Eq.(4.12). For notational convenience, define the process of the investor's consumption/terminal wealth process $\phi \in \mathcal{H}_+$ as

$$\phi(t) := \begin{cases} C(t) & \text{for } 0 \le t < T \\ W(T) & \text{for } t = T \end{cases}$$

Since S^* defined by Eq.(4.12) is independent of W and β, the consumption/wealth process ϕ is determined uniquely for each $(S^*, C, \beta) \in \mathcal{A}$ and is thus well-defined in $\mathcal{A}$ with S^* given. Let $\Phi(S^*)$ denote the set of (ϕ, β) that corresponds to $(S^*, C, \beta) \in \mathcal{A}$.

In the following, we will first formalize a state price, which is the price of a security that agrees to pay one unit of the consumption/wealth ϕ if a particular time-path ω occurs at a particular time t in the future and to pay zero unit of it otherwise. Second, we will find optimal controls $(\phi, \beta) \in \Phi(S^*)$ – call them ex-post optimal controls– and then obtain an equilibrium state price by imposing the market-clearing conditions. Finally, we will obtain a riskless rate and market prices of risk and jump risk in equilibrium.[20]

[20] This method is similar to the one in Skiadas (2007). However, there exists two main departures from that. First, we explore an SDE method, rather than a backward SDE method. Second, this method deals with jumps.

5.1 State prices

We give a formal representation of state prices in the financial markets. Define a process $\Pi \in \mathcal{H}_+$ for some n dimensional process η and m dimensional process ξ as:

$$\mathrm{d}\Pi(t) = \Pi(t_-)\Big(- r(t)\,\mathrm{d}t - \eta(t)^\top \,\mathrm{d}B(t) - \xi(t)^\top \,\mathrm{d}M(t)\Big), \quad \Pi(0) = 1 \tag{5.1}$$

where η stands for the market price of (diffusive) risk and ξ stands for the market price of jump risk. Define an $(m \times m)$-matrix I^ξ in which its diagonal element $x_{ii} = \xi_i$ for all $i = 1, \cdots, m$ and the other elements $x_{ij} = 0$ for $i, j = 1, \cdots, m$ and $i \neq j$.

Definition 5.1 Π *is said to be a state price at a pair* $(\phi, \beta) \in \Phi(S^*)$ *if, for the pair* $(\phi, \beta) \in \Phi(S^*)$ *and for any* $h \in \mathcal{H}$ *such that* $(\phi + h, \beta') \in \Phi(S^*)$, $(\Pi|h) \leq 0$.

We put an assumption on the market opportunities as follows.

Assumption 5.1 *For the market prices of risk and jump risk* η, ξ, *the law of one price holds in the markets. That is,*

$$\mu^R = \sigma^R \eta + z^R I^\xi \lambda.$$

From Eq.(2.7),

$$\begin{aligned}
\mathrm{d}W(t) &= -\Big(C(t) + S^*(t) - X(t) - W(t_-)r(t) - W(t_-)\beta(t)^\top \mu^R\Big)\,\mathrm{d}t \\
&\quad + W(t_-)\beta(t)^\top \left(\sigma^R \,\mathrm{d}B(t) + z^R \,\mathrm{d}M(t)\right), \quad W(0) = w_0.
\end{aligned}$$

Thus, by the Itô's formula,

$$\begin{aligned}
\mathrm{d}(\Pi(t)W(t)) &= W(t_-)\,\mathrm{d}\Pi(t) + \Pi(t_-)\,\mathrm{d}W(t) + \mathrm{d}\Pi(t)\,\mathrm{d}W(t) \\
&= \left\{ \begin{array}{l} -\Pi(t_-)\Big(C(t) + S^*(t) - X(t)\Big) \\ +\Pi(t_-)W(t_-)\beta(t)^\top \mu^R - \Pi(t_-)W(t_-)\beta(t)^\top \sigma^R \eta(t) \end{array} \right\} \mathrm{d}t \\
&\quad -\Pi(t_-)W(t_-)\beta(t)^\top z^R I^\xi(t)\,\mathrm{d}N(t) + \cdots \mathrm{d}B(t) + \cdots \mathrm{d}M(t).
\end{aligned}$$

Assumption 5.2 *For a pair $(\phi, \beta) \in \Phi(S^*)$,*

$$\mathbb{E}^{\mathbb{P}}[\sup_t \Pi(t) W(t)] < \infty.$$

Lemma 5.1 *Fix S^* defined by Eq.(4.12). Under Assumption 5.1, Π is a state price at the pair $(\phi, \beta) \in \Phi(S^*)$ satisfying Assumption 5.2.*

Proof: See appendix.

Note that β does not influence the state price, except via $W(T)$ indirectly.

5.2 Optimal ex-post consumption/wealth allocation and equilibrium state price

Under the contract S^* defined by Eq.(4.12), the investor controls her consumption/wealth ex post optimally while trading the assets. Note that we will show shortly below that this ex-post equilibrium is equivalent to the market equilibrium obtained in Section 4.

Let $\widehat{\mathbb{Q}}^*$ denote the optimal $\mathbb{Q}$ under $S = S^*$ when Eq.(3.4) binds. From Eq.(3.5),

$$\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}} = e^{-\rho} e^{\int_0^T e^{-\delta u} a \log S^*(u)\,\mathrm{d}u} = e^{-\rho} e^{\frac{1-e^{-\delta T}}{\delta} a \log s^*} e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}.$$

From Eq.(2.9), let $\{\widehat{U}_2(t)\}_{0 \le t \le T}$ denote the investor's utility process when Eq.(3.4) binds:

$$\widehat{U}_2(t) := e^{\delta t} \mathbb{E}_t^{\mathbb{P}} \left[\int_t^T e^{-\delta u} \mathbb{E}_u^{\mathbb{P}}[\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}}] \frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T} \frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}} W(T) \right]$$

At $t = 0$, $\widehat{U}_2(0) = U_2(C, W(T); \widehat{\mathbb{Q}}^*)$ from Eq.(2.9). Also, at $t = T$, $\widehat{U}_2(T) = W(T)$. Since $e^{-\delta t}\widehat{U}_2(t) + \int_0^t e^{-\delta u} \mathbb{E}_u^{\mathbb{P}}[\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}}] \frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u = \mathbb{E}_t^{\mathbb{P}} \left[\int_0^T e^{-\delta u} \mathbb{E}_u^{\mathbb{P}}[\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}}] \frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T} \frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}} W(T) \right]$, it is a martingale. By the Martingale Representation Theorem (cf. Theorem 5.43 of Medvegyev (2007)), there exist $\mathbb{F}$-predicable processes Θ_j and $\Xi_i \ge -1$, where $\int_0^T (\Theta_j(t))^2\,\mathrm{d}t < \infty$ and $\int_0^T \Xi_i(t)\,\mathrm{d}t < \infty$, for all $i = 1, \cdots, m$ and all $j = 1, \cdots, n$ such that

$$\begin{aligned} & e^{-\delta t}\widehat{U}_2(t) + \int_0^t e^{-\delta u} \mathbb{E}_u^{\mathbb{P}}[\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}}] \frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u \\ = \; & \mathbb{E}^{\mathbb{P}} \left[\int_0^T e^{-\delta u} \mathbb{E}_u^{\mathbb{P}}[\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}}] \frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T} \frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}} W(T) \right] \\ & + \int_0^T \sum_{j=1}^n \Theta_j(u)\,\mathrm{d}B_j(u) + \int_0^T \sum_{i=1}^m \Xi_i(u)\,\mathrm{d}M_i(u). \end{aligned}$$

Thus, correspondingly, there exist $\mathbb{F}$-predicable processes Σ and Γ such that

$$-\delta\,\widehat{U}_2(t)\,\mathrm{d}t + \mathrm{d}\widehat{U}_2(t) + \mathbb{E}_t^{\mathbb{P}}[\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}}]\frac{C(t)^{1-\gamma}}{1-\gamma}\,\mathrm{d}t = \Sigma(t)\,\mathrm{d}B(t) + \Gamma(t)\,\mathrm{d}M(t).$$

$$\text{I.e.,}\qquad \begin{aligned} \mathrm{d}\widehat{U}_2(t) &= -\widehat{F}(t,\phi(t),\widehat{U}_2(t);\widehat{\mathbb{Q}}^*)\,\mathrm{d}t + \Sigma(t)\,\mathrm{d}B(t) + \Gamma(t)\,\mathrm{d}M(t), \\ \widehat{U}_2(T) &= \widehat{F}(T,\phi(T),\widehat{U}_2(T);\,\widehat{\mathbb{Q}}^*) \end{aligned}$$

where

$$\widehat{F}(t,\phi(t),\widehat{U}_2(t);\widehat{\mathbb{Q}}^*) := \begin{cases} e^{-\rho}e^{\frac{1-e^{-\delta T}}{\delta}a\log s^*}Y(t)\frac{C(t)^{1-\gamma}}{1-\gamma} - \delta\widehat{U}_2(t) & \text{for } 0 \le t < T, \\ e^{-\rho}e^{\frac{1-e^{-\delta T}}{\delta}a\log s^*}Y(T)W(T) & \text{for } t = T. \end{cases}$$

Recall that $Y(t) = \mathbb{E}_t^{\mathbb{P}}[e^{\int_0^T e^{-\delta u}a\log X(u)\,\mathrm{d}u}]$. Define $\widehat{F}_u(t) := \frac{\partial\widehat{F}(t)}{\partial\widehat{U}_2(t)}$, $\widehat{F}_c(t) := \frac{\partial\widehat{F}(t)}{\partial C(t)}$, and $\widehat{F}_{cc}(t) := \frac{\partial^2\widehat{F}(t)}{\partial C(t)^2}$. We can confirm that $\widehat{F}$ is concave in ϕ and $\widehat{U}_2$. Define $\mathcal{E}(t) := e^{-\delta t}$ and

$$\Lambda(t) := \mathcal{E}(t)\widehat{F}_\phi(t) := \begin{cases} \mathcal{E}(t)\widehat{F}_c(t) & \text{for } t \in [0,T), \\ \mathcal{E}(T)e^{-\rho}e^{\frac{1-e^{-\delta T}}{\delta}a\log s^*}Y(T) & \text{for } t = T. \end{cases}$$

where $\widehat{F}_c(t) = e^{-\rho}e^{\frac{1-e^{-\delta T}}{\delta}a\log s^*}Y(t)C(t)^{-\gamma}$.

Lemma 5.2 *Fix S^* defined by Eq.(4.12). For $(\phi,\beta) \in \Phi(S^*)$ and for any $h \in \mathcal{H}$ such that $(\phi+h,\beta') \in \Phi(S^*)$,*

$$\widehat{U}_2(\phi+h,S^*) \le \widehat{U}_2(\phi,S^*) + (\Lambda|h).$$

Proof: See appendix.

We obtain our first main result:

Proposition 5.1 *Under Assumption 5.1 and Assumption 5.2, for the optimal contract S^* defined by Eq.(4.12), $\Pi = \Lambda = \mathcal{E}\widehat{F}_\phi$ is a state price at:*

$$\begin{aligned} \phi(t) &= \begin{cases} C^*(t) & \textit{for } 0 \le t < T, \\ 0 & \textit{for } t = T \end{cases} \\ \beta(t) &= \;\; 0 \qquad\;\; \textit{for } 0 \le t \le T \end{aligned}$$

where C^* is defined by Eq.(4.11)*.*

Proof: See appendix.

Let us look at how this result is related to the market equilibrium obtained in Section 4. In the environment of Subsection 4.2, from Eq.(4.7), due to the market-clearing condition,

$$\widehat{F}_c(t)\Big|_{C(t)=C^*(t)} = e^{-\rho} e^{\frac{1-e^{-\delta T}}{\delta} a \log s^*} Y(t) C^*(t)^{-\gamma} = J_w$$

Thus, the result in Proposition 5.1 is equivalent to the market equilibrium obtained in Section 4.

Accordingly, we obtain a state price in the market equilibrium – call it an equilibrium state price. Define $\Lambda^* := \mathcal{E} F_c^*$ where $F_c^*(t) := \widehat{F}_c(t)\Big|_{C(t)=C^*(t)} = e^{-\rho} e^{\frac{1-e^{-\delta T}}{\delta} a \log s^*} Y(t) C^*(t)^{-\gamma} = e^{-\rho} e^{\frac{1-e^{-\delta T}}{\delta} a \log s^*} (1-s^*)^{-\gamma} Y(t) X(t)^{-\gamma}$. The equilibrium state price is written as

$$\Pi(t) = \Lambda^*(t) = \mathcal{E}(t) F_c^*(t) \quad \text{for } 0 \leq t < T. \tag{5.2}$$

In it, $e^{-\delta t} X(t)^{-\gamma}$ corresponds to the equilibrium state price in the absence of moral hazard. Also, $Y(t)$ is the stochastic part other than $e^{-\delta t} X(t)^{-\gamma}$. Note that $\Pi(T) = \Lambda(T) = \mathcal{E}(T) F_\phi^*(T)$.

5.3 Riskless rate and market prices of risk and jump risk in equilibrium

We characterize explicitly a riskless rate and market prices of risk and jump risk in the market equilibrium. Recalling Eq.(5.1), from Eq.(5.2),

$$\begin{aligned} \frac{\mathrm{d}\Pi(t)}{\Pi(t_-)} &= -r(t)\,\mathrm{d}t - \eta(t)^\top \mathrm{d}B(t) - \xi(t)^\top \mathrm{d}M(t) \\ = \frac{\mathrm{d}\Lambda^*(t)}{\Lambda^*(t_-)} &= -\delta\,\mathrm{d}t + \frac{\mathrm{d}F_c^*(t)}{F_c^*(t_-)}, \qquad \text{with} \quad \Pi(0) = \Lambda(0) = 1. \end{aligned} \tag{5.3}$$

Now, solving $\frac{\mathrm{d}F_c^*(t)}{F_c^*(t_-)}$ in Eq.(5.3), we obtain our second main result:

Proposition 5.2 *In the environment of Proposition 5.1, the equilibrium riskless rate and the equilibrium market prices of risk and jump risk in the absence of moral hazard, denoted by r^s, η^s, ξ^s*

respectively, are constants:

$$\begin{aligned} r^s &= \delta + \gamma\mu^G - \frac{\gamma(\gamma+1)}{2}\sum_{j=1}^{n}(\sigma_j^G)^2 + \sum_{i=1}^{m}\left\{1-(1+z_i^G)^{-\gamma} - \gamma z_i^G\right\}\lambda_i, \\ \eta_j^s &= \gamma\sigma_j^G \quad \text{for } j=1,\cdots,n, \\ \xi_i^s &= 1-(1+z_i^G)^{-\gamma} \quad \text{for } i=1,\cdots,m. \end{aligned} \tag{5.4}$$

On the other hand, the equilibrium riskless rate and the equilibrium market prices of risk and jump risk in the presence of moral hazard are time-varying:

$$\begin{aligned} r(t) &= r^s + \left(\begin{array}{l} \left(\gamma a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\right)\sum_{j=1}^{n}(\sigma_j^G)^2 \\ -\sum_{i=1}^{m}\left(1-(1+z_i^G)^{-\gamma}\right)\left(1-(1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}}\right)\lambda_i \end{array}\right) \\ &= \delta + \gamma\mu^G + \left(\gamma a\frac{e^{-\delta t}-e^{-\delta T}}{\delta} - \frac{\gamma(\gamma+1)}{2}\right)\sum_{j=1}^{n}(\sigma_j^G)^2 \\ &\quad + \sum_{i=1}^{m}\left\{\left(1-(1+z_i^G)^{-\gamma}\right)(1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}} - \gamma z_i^G\right\}\lambda_i, \\ \eta_j(t) &= \eta_j^s - a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\sigma_j^G = \left(\gamma - a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\right)\sigma_j^G \text{ for } j=1,\cdots,n, \\ \xi_i(t) &= \xi_i^s + (1+z_i^G)^{-\gamma}\left(1-(1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}}\right) = 1-(1+z_i^G)^{-\gamma+a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}} \text{ for } i=1,\cdots,m. \end{aligned}$$

Proof: From Eq.(5.2),

$$\frac{\mathrm{d}\Lambda^*(t)}{\Lambda^*(t_-)} = -\delta\,\mathrm{d}t + \frac{\mathrm{d}(X(t)^{-\gamma})}{X(t_-)^{-\gamma}} + \frac{\mathrm{d}Y(t)}{Y(t_-)} + \frac{\mathrm{d}Y(t)\,\mathrm{d}(X(t)^{-\gamma})}{Y(t_-)X(t_-)^{-\gamma}}. \tag{5.5}$$

Firstly, by Itô's formula,

$$\begin{aligned} \frac{\mathrm{d}(X(t)^{-\gamma})}{X(t_-)^{-\gamma}} &= \left(-\gamma\mu^G + \frac{\gamma(\gamma+1)}{2}\sum_{j=1}^{n}(\sigma_j^G)^2 + \sum_{i=1}^{m}\left\{(1+z_i^G)^{-\gamma} - 1 + \gamma z_i^G\right\}\lambda_i\right)\mathrm{d}t \\ &\quad -\gamma\sum_{j=1}^{n}\sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m}\left\{(1+z_i^G)^{-\gamma}-1\right\}\mathrm{d}M_i(t). \end{aligned} \tag{5.6}$$

Suppose that there is no moral hazard. Under the restriction of the stationary linear payment rule, $\frac{\mathrm{d}\left(C(t)^{-\gamma}\right)}{C(t_-)^{-\gamma}} = \frac{\mathrm{d}\left(X(t)^{-\gamma}\right)}{X(t_-)^{-\gamma}}$ in the market equilibrium.[21] Therefore, the first two terms on Eq.(5.5)

[21] If we relax the restriction, then the equilibrium consumption in the presence of moral hazard could be different form the one in the absence of moral hazard.

correspond to the equilibrium state price in the absence of moral hazard, denoted by Λ^s:

$$\begin{aligned}
\frac{\mathrm{d}\Lambda^s(t)}{\Lambda^s(t_-)} &= -\delta\,\mathrm{d}t + \frac{\mathrm{d}(X(t)^{-\gamma})}{X(t_-)^{-\gamma}} \\
&= -\delta\,\mathrm{d}t + \left(-\gamma\mu^G + \frac{\gamma(\gamma+1)}{2}\sum_{j=1}^{n}(\sigma_j^G)^2 + \sum_{i=1}^{m}\left\{(1+z_i^G)^{-\gamma} - 1 + \gamma z_i^G\right\}\lambda_i \right)\mathrm{d}t \\
&\quad -\gamma\sum_{j=1}^{n}\sigma_j^G\,\mathrm{d}B_j(t) - \sum_{i=1}^{m}\left\{1-(1+z_i^G)^{-\gamma}\right\}\mathrm{d}M_i(t).
\end{aligned} \tag{5.7}$$

From Eq.(5.3),

$$\begin{aligned}
r^s &= \delta + \gamma\mu^G - \frac{\gamma(\gamma+1)}{2}\sum_{j=1}^{n}(\sigma_j^G)^2 + \sum_{i=1}^{m}\left\{1-(1+z_i^G)^{-\gamma} - \gamma z_i^G\right\}\lambda_i, \\
\eta_j^s &= \gamma\sigma_j^G \quad \text{for } j = 1,\cdots,n, \\
\xi_i^s &= 1-(1+z_i^G)^{-\gamma} \quad \text{for } i = 1,\cdots,m.
\end{aligned}$$

In other words, these represent the equilibrium pricing behavior driven only by the aggregate shock $\frac{\mathrm{d}\left(X(t)^{-\gamma}\right)}{X(t_-)^{-\gamma}}$, as in the standard general-equilibrium exchange economies.

On the other hand, in this model, the equilibrium pricing behavior is influenced by moral hazard as well, due to the necessity to make the firm avoid his opportunistic misbehavior. The third and fourth terms on Eq.(5.5) stand for the effect of the moral hazard on the dynamics of the equilibrium state price. Recall from Lemma 4.2 that

$$\frac{\mathrm{d}Y(t)}{Y(t_-)} = a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\sum_{j=1}^{n}\sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m}\{(1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}} - 1\}\,\mathrm{d}M_i(t). \tag{5.8}$$

From Eq.(5.7) and Eq.(5.8),

$$\begin{aligned}
\frac{\mathrm{d}\Lambda^*(t)}{\Lambda^*(t_-)} &= \frac{\mathrm{d}\Lambda^s(t)}{\Lambda^s(t_-)} + \left(\begin{array}{l} -\left(\gamma a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\right)\sum_{j=1}^{n}(\sigma_j^G)^2 \\ +\sum_{i=1}^{m}\left((1+z_i^G)^{-\gamma}-1\right)\left((1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}} - 1\right)\lambda_i \end{array} \right)\mathrm{d}t \\
&\quad + a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\sum_{j=1}^{n}\sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m}(1+z_i^G)^{-\gamma}\left((1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}} - 1\right)\mathrm{d}M_i(t).
\end{aligned} \tag{5.9}$$

From Eq.(5.3) and Eq.(5.9), the market prices of risk and jump risk are:

$$\begin{aligned}
\eta_j(t) &= \eta_j^s - a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\sigma_j^G \\
&= \left(\gamma - a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\right)\sigma_j^G \quad \text{for } j = 1, \cdots, n, \\
\xi_i(t) &= \xi_i^s + (1+z_i^G)^{-\gamma}\left(1 - (1+z_i^G)^{a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}}\right) \\
&= 1 - (1+z_i^G)^{-\gamma + a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}} \quad \text{for } i = 1, \cdots, m.
\end{aligned}$$

Also, with regard to the riskless rate,

$$\begin{aligned}
r(t) &= r^s + \left(\begin{array}{l} \left(\gamma a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\right)\sum_{j=1}^n (\sigma_j^G)^2 \\ -\sum_{i=1}^m \left(1 - (1+z_i^G)^{-\gamma}\right)\left(1 - (1+z_i^G)^{a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}}\right)\lambda_i \end{array} \right) \\
&= \delta + \gamma\mu^G + \left(\gamma a\frac{e^{-\delta t} - e^{-\delta T}}{\delta} - \frac{\gamma(\gamma+1)}{2}\right)\sum_{j=1}^n (\sigma_j^G)^2 \\
&\quad + \sum_{i=1}^m \left\{\left(1 - (1+z_i^G)^{-\gamma}\right)(1+z_i^G)^{a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}} - \gamma z_i^G\right\}\lambda_i.
\end{aligned}$$

□

We call $(r(t) - r^s)$ a moral-hazard premium on the equilibrium riskless rate:

$$\begin{aligned}
r(t) - r^s &= \left(\gamma a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\right)\sum_{j=1}^n (\sigma_j^G)^2 \\
&\quad - \sum_{i=1}^m \left(1 - (1+z_i^G)^{-\gamma}\right)\left(1 - (1+z_i^G)^{a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}}\right)\lambda_i.
\end{aligned} \tag{5.10}$$

Also,

$$\begin{aligned}
\eta_j(t) - \eta_j^s &= -a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\sigma_j^G \le 0 \\
&\quad \text{with equality if } t = T \quad \text{for } j = 1, \cdots, n, \\
\xi_i(t) - \xi_i^s &= (1+z_i^G)^{-\gamma}\left(1 - (1+z_i^G)^{a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}}\right) \ge 0 \\
&\quad \text{with equality if } t = T \quad \text{for } i = 1, \cdots, m.
\end{aligned}$$

are distortions of the market prices of risk and jump risk caused by moral hazard, respectively.

We draw asset-pricing implications of moral hazard from Proposition 5.2. First, from Eq.(5.5) and Eq.(5.7), the market prices of risk and jump risk are distorted, as compared to the case of no moral hazard, only through $\frac{\mathrm{d}Y(t)}{Y(t_-)}$. More essentially, inside of $Y(t)$, $e^{a\log X(t)\int_t^T e^{-\delta u}\,\mathrm{d}u} = X(t)^{\alpha(t)}$, where $\alpha(t) = a\int_t^T e^{-\delta u}\,\mathrm{d}u = a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}$, plays a pivotal role in distorting the market prices of risk and jump risk. Recall that $Y(t)$ represents a time-t distortion of the probability measure caused by the endowment process X all over $[0,T]$. In parallel with $Y(t)$, $X(t)^{\alpha(t)}$ represents a time-t distortion of the probability measure caused by the time-t endowment. From Eq.(5.8), the distortion of the market price of risk is the diffusion coefficient of the stochastic differential of the distortion of the probability measure. While the market price of risk in the absence of moral hazard (i.e., $\gamma\sigma_j^G$ for $j = 1,\cdots,n$) is positive because the lender is risk-averse, the distortion of the market price of risk is negative because a positive (negative) diffusive shock improves (decreases) the probability measure. The distortion of the probability measure works as a hedge against the diffusive shock. Thus the moral hazard reduces the market price of risk. However, as $t \to T$, the effect of the reduction is diminishing to zero because $\alpha(t) \downarrow 0$.

On the other hand, the distortion of the market price of jump risk consists of two parts: (1) the jump coefficient of the distorted probability measure and (2) the jump coefficient of the quadratic covariation of the distorted probability measure and $X(t)^{-\gamma}$. The distortion of the market price of jump risk is positive because a negative jump shock decreases the probability measure. Thus the moral hazard raises the market price of jump risk, as compared to the one in the absence of moral hazard (i.e., $1-(1+z_i^G)^{-\gamma} < 0$ for $i = 1,\cdots,m$). However, again, as $t \to T$, the effect is diminishing because $\alpha(t) \downarrow 0$.

Second, the moral-hazard premium is positive, since both of the two terms are positive on the right-hand side of Eq.(5.10). The positivity of the premium holds true for either the regular market risk or the rare-event risk. We investigate the moral-hazard premium in more detail. With regard to the first term, since the distortion of the probability measure caused by moral hazard is a hedge against the diffusive shock, it reduces the diffusive-risk premium by $\left(\gamma a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\right)\sum_{j=1}^n(\sigma_j^G)^2$. Thus it raises the riskless rate by the amount, i.e., the first term is positive. It is diminishing as $t \to T$, in parallel with the distortion of the market price of risk.

On the other hand, the second term represents the compensation for the market-valued jump risk distorted by moral hazard $(\xi_i(t)-\xi_i^s)\,\mathrm{d}M_i$ for all $i = 1,\cdots,m$. The distortion of the market-valued jump risk aggravates average profitability in the financial markets by its covariance (adjusted by the size of $(1+z_i^G)^{-\gamma}$) with the original jump risk $\xi_i^s\,\mathrm{d}M_i$. The second term then makes up for the loss of the profitability. Since $-1 < z_i^G < 0$, $1-(1+z_i^G)^{-\gamma} < 0$ for all $i = 1,\cdots,m$. Thus the second term is positive and raises the jump-risk premium as compared to the one in the absence of moral hazard (i.e., $\sum_{i=1}^{m}\left\{(1+z_i^G)^{-\gamma}-1\right\}\lambda_i$).[22] Again, it is diminishing as $t \to T$.

These results give new insights into the risk-free rate puzzle, which was explored first by Weil (1989). From Eq.(5.4), in the absence of moral hazard, large negative jumps (i.e., low z^G) tend to mitigate the puzzle by lowering the riskless rate r^s. However, from Eq.(5.10), moral hazard raises the riskless rate. It implies that the risk-free puzzle is more serious under moral hazard.

5.4 Discussion of directions for future research

In this section, we end up with discussing directions for future research. We imposed two crucial simplifications on this model for obtaining the explicit (closed-form) solution. First, we assumed the CRRA/log utility function. It is more general than the exponential utility function, but is still time- and state-separable. It would be desirable to have more general utility forms such as stochastic differential utility, habit formation, and ambiguity aversion. In particular, as in Weil (1989), the risk-free rate puzzle should be investigated in a model with time-nonseparable utility.

Second, we assumed the stationary linear contract. This assumption is restrictive, in the context of the literature on optimal contracting under moral hazard (see e.g., Cvitanić and Zhang (2007)). We can conjecture that the linearity of the contract is not quite restrictive in optimum because the whole system of equations is linear in this model. Based on the conjecture, the above-obtained equilibrium market prices of risk and jump risk hold true for more general contract forms as well. So does the above-obtained premium.

However, the stationarity (i.e., the constant sharing ratio) may not be optimal for them, due to the finiteness of the time period. When the optimal contract is time-dependent, the moral-hazard premium can be decreased or increased as compared to the case of the stationary linear payment rule. For example, if the sharing ratio of the firm is decreasing (increasing) in time, then the moral-hazard premium is raised (lowered), because the marginal utility of the investor is decreasing (increasing). Still, when it is difficult to obtain any explicit general solutions under more general contract forms, the result obtained under the stationary linear contract could be used as a benchmark for approximating numerically the general solutions, e.g. via the Taylor expansion method, in future.

[22]For positive jumps (i.e., $z_i^G > 0$ for some i), the second term could be negative. As the positive jumps are a reward for the investor, the negative return on the investment would be demanded.

6 Concluding remarks

This paper provides an explicit asset-pricing formula under the moral-hazard problem and obtains equilibrium state prices (in particular, equilibrium riskless rate and market prices of diffusive risk and jump risk). It thus makes clear the structural effect of moral hazard on equilibrium asset prices. Notably, it shows that, under the moral-hazard problem, a positive premium is stipulated on a riskless rate in market equilibrium – call it a moral-hazard premium – because the lender demands compensation for a loss caused due to the necessity to make the firm manager avoid his opportunistic misbehavior. This result implies that the risk-free rate puzzle, explored first by Weil (1989), is more serious in the presence of moral hazard.

This paper imposes two simplifications on the model: (1) the CRRA utility and (2) the stationary linear payment rule. The risk-free rate puzzle might be, at least partly, resolved by assuming more general utility forms like stochastic differential utility and habit formation (see e.g. Nakamura et al. (2009)). Second, this paper assumes the stationary linear payment rule for obtaining the explicit solution. The assumption is restrictive, but the explicit solution obtained under the restriction could be used as a good benchmark for future numerical approximations, e.g. via the Taylor expansion method.

Appendix

A Proof of Lemma 3.1

Taking exponential of $\mathbb{E}^{\mathbb{Q}}[\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u] - H(\mathbb{Q}\,||\,\mathbb{P})$,

$$
\begin{aligned}
e^{\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u\right] - H(\mathbb{Q}\,||\,\mathbb{P})} &= e^{\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u - \log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right]} \\
&\leq \mathbb{E}^{\mathbb{Q}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u - \log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}}\right] \quad \text{(by Jensen's inequality)} \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u} \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\right] = \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u} \mathbf{1}_{\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} > 0\}}\right] \\
&\leq \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}\right]
\end{aligned}
$$

with equality if and only if $\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u - \log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ is a constant. Therefore,

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \frac{e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}}{\mathbb{E}^{\mathbb{P}}\Big[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}\Big]}.$$

Thus $\mathbb{Q}^*$ is obtained. $\square$

B Proof of Lemma 4.1

From Eq.(2.5), $\mathrm{d}X(t) = X(t_-)\Big(\mu^G\,\mathrm{d}t + \sum_{j=1}^n \sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^m z_i^G\,\mathrm{d}M_i(t)\Big)$, $X(0) = x_0 > 0$. Hence,

$$\begin{aligned}
X(t) &= x_0 e^{\mu^G t + \sum_{j=1}^n \Big(\sigma_j^G B_j(t) - \frac{(\sigma_j^G)^2}{2} t\Big) - \sum_{i=1}^m z_i^G \lambda_i t} \prod_{i=1}^m \prod_{0\le s\le t} (1 + z_i^G \Delta N_i(s)), \\
\text{i.e., } \log \frac{X(t)}{x_0} &= \mu^G t + \sum_{j=1}^n \Big(\sigma_j^G B_j(t) - \frac{(\sigma_j^G)^2}{2} t\Big) - \sum_{i=1}^m z_i^G \lambda_i t + \sum_{i=1}^m \sum_{0\le s\le t} \log(1 + z_i^G \Delta N_i(s)) \\
&= \Big(\mu^G - \frac{(\sigma_j^G)^2}{2} - \sum_{i=1}^m z_i^G \lambda_i\Big) t + \sum_{j=1}^n \sigma_j^G B_j(t) + \sum_{i=1}^m \Big(\log(1 + z_i^G)\Big) N_i(t). \qquad \text{(B.1)}
\end{aligned}$$

Substituting Eq.(B.1) into Eq.(4.3), we examine each term on the right-hand side of Eq.(B.1). Firstly, the t-term is simply solved. Secondly, we look at the B_j-term ($j = 1, \cdots, n$). In particular, for a constant c, we solve $\mathbb{E}^{\mathbb{P}}[e^{c\int_0^T e^{-\delta t} B_j(t)\,\mathrm{d}t}]$. By Itô's formula,

$$-\frac{e^{-\delta T}}{\delta} B_j(T) = \int_0^T e^{-\delta t} B_j(t)\,\mathrm{d}t - \int_0^T \frac{e^{-\delta t}}{\delta}\,\mathrm{d}B_j(t) \quad (\because \mathrm{d}B_j(t)\,\mathrm{d}t = 0).$$

$$\begin{aligned}
\text{I.e., } \int_0^T e^{-\delta t} B_j(t)\,\mathrm{d}t &= -\frac{e^{-\delta T}}{\delta} B_j(T) + \frac{1}{\delta}\int_0^T e^{-\delta t}\,\mathrm{d}B_j(t) \\
&= -\frac{e^{-\delta T}}{\delta} \int_0^T \mathrm{d}B_j(t) + \frac{1}{\delta}\int_0^T e^{-\delta t}\,\mathrm{d}B_j(t) \\
&= \frac{1}{\delta}\int_0^T \Big(e^{-\delta t} - e^{-\delta T}\Big)\,\mathrm{d}B_j(t) \qquad \sim \quad \mathcal{N}\Big(0, \frac{1}{\delta^2}\int_0^T \Big(e^{-\delta t} - e^{-\delta T}\Big)^2\,\mathrm{d}t\Big)
\end{aligned}$$

where, for some $\mu \in \mathbf{R}$ and $\sigma > 0$, $\mathcal{N}(\mu, \sigma^2)$ denotes a normal distribution with mean μ and variance σ^2. Therefore,

$$\mathbb{E}^{\mathbb{P}}[e^{c\int_0^T e^{-\delta t} B_j(t)\,\mathrm{d}t}] = e^{\frac{c^2}{2}\frac{1}{\delta^2}\int_0^T \left(e^{-\delta t} - e^{-\delta T}\right)^2\,\mathrm{d}t}. \qquad \text{(B.2)}$$

Lastly, we look at the N_i-term ($i = 1, \cdots, m$). In particular, for a constant c, we solve $\mathbb{E}^{\mathbb{P}}[e^{c\int_0^T e^{-\delta t} N_i(t)\,\mathrm{d}t}]$. Still, before solving it directly, we make preparations for it. Let the k-th jump time of N_i be denoted by τ_k. Consider the case of $N_i(T) = p$ for some integer $p \geq 1$. We can choose distinct $2p$ points of time $(s_1,\, s_2,\, \cdots,\, s_p)$ and $(t_1,\, t_2,\, \cdots,\, t_p)$ satisfying

$$0 \leq s_1 < t_1 < s_2 < t_2 < \cdots < s_p < t_p < T.$$

We then take a joint probability conditional on $N_i(T) = p$ as follows:

$$\begin{aligned}
&\mathbb{P}[s_k < \tau_k < t_k,\, k = 1, \cdots p \,|\, N_i(T) = p\,] \\
= \;& \mathbb{P}[N_i(s_1) = 0,\, N_i(t_1) = 1,\, N_i(s_2) = 1,\, \cdots,\, N_i(s_p) = p-1,\, N_i(t_p) = p \,|\, N_i(T) = p\,] \\
= \;& \frac{1}{\frac{(\lambda_i T)^p}{p!} e^{-\lambda_i T}} \left(\begin{array}{c} e^{-\lambda_i s_1} \lambda_i (t_1 - s_1) e^{-\lambda_i (t_1 - s_1)} e^{-\lambda_i (s_2 - t_1)} \lambda_i (t_2 - s_2) e^{-\lambda_i (t_2 - s_2)} \\ \cdots e^{-\lambda_i (s_p - t_p)} \lambda_i (t_p - s_p) e^{-\lambda_i (T - s_p)} \end{array} \right) \\
= \;& p! \prod_{k=1}^{p} \left(\frac{t_k - s_k}{T} \right).
\end{aligned} \tag{B.3}$$

As a brief remark, we give an intuitive explanation to the conditional joint probability. As $t_k \to s_k$ for all k, the corresponding joint probability density conditional on $N_i(T) = p$ is $\frac{p!}{T^p}$. As it is well known, the volume of the domain $\{(t_1, \cdots, t_p) \in \mathbf{R}^p \,|\, 0 < t_1 < t_2 < \cdots < t_p < T\}$ is $\frac{T^p}{p!}$. Hence, the asymptotic joint probability density is equivalent to a uniform distribution over the volume.

On the other hand, we take p independent real-valued random variables $U_1, U_2, \cdots, U_p$, each of which is defined on $[0, T]$ with a uniform distribution. For each scenario $\omega \in \Omega$, let the p realizations be arranged in ascending order as $R_1, R_2, \cdots, R_p$, i.e., $R_1 \leq R_2 \leq \cdots \leq R_p$. Note that the equalities hold with probability zero. Thus, applying Eq.(B.3) to this case,

$$\mathbb{P}[s_k < R_k < t_k,\, k = 1, \cdots p\,] = p! \prod_{k=1}^{p} \left(\frac{t_k - s_k}{T} \right).$$

Since $\mathbb{P}[s_k < R_k < t_k,\, k = 1, \cdots p\,] = p!\,\mathbb{P}[s_k < U_k < t_k,\, k = 1, \cdots p\,]$,

$$\mathbb{P}[s_k < U_k < t_k,\, k = 1, \cdots p\,] = \prod_{k=1}^{p} \left(\frac{t_k - s_k}{T} \right). \tag{B.4}$$

Now, using Eq.(B.4) and the terminology used during its derivation, we solve $\mathbb{E}^{\mathbb{P}}[e^{c\int_0^T e^{-\delta t} N_i(t)\,\mathrm{d}t}]$

for a constant c. Taking expectation conditional on $N_i(T) = p$,

$$\mathbb{E}^{\mathbb{P}}[e^{c\int_0^T e^{-\delta t}N_i(t)\,\mathrm{d}t}] = \sum_{p=0}^{\infty} \mathbb{P}[\,N_i(T) = p\,]\,\mathbb{E}^{\mathbb{P}}[e^{c\int_0^T e^{-\delta t}N_i(t)\,\mathrm{d}t}\,|\,N_i(T) = p\,].$$

Focusing on $\int_0^T e^{-\delta t}N_i(t)\,\mathrm{d}t$ in the equation,

$$\begin{aligned}
\int_0^T e^{-\delta t}N_i(t)\,\mathrm{d}t &= 0\cdot\frac{1-e^{-\delta\tau_1}}{\delta} + 1\cdot\frac{e^{-\delta\tau_1}-e^{-\delta\tau_2}}{\delta} + \cdots \\
&\quad +(p-1)\cdot\frac{e^{-\delta\tau_{p-1}}-e^{-\delta\tau_p}}{\delta} + p\cdot\frac{e^{-\delta\tau_p}-e^{-\delta T}}{\delta} \\
&= -p\frac{e^{-\delta T}}{\delta} + \sum_{k=1}^{p}\frac{e^{-\delta\tau_k}}{\delta} \\
&\overset{(d)}{=} -p\frac{e^{-\delta T}}{\delta} + \sum_{k=1}^{p}\frac{e^{-\delta R_k}}{\delta} = -p\frac{e^{-\delta T}}{\delta} + \sum_{k=1}^{p}\frac{e^{-\delta U_k}}{\delta}
\end{aligned}$$

where the equality $\overset{(d)}{=}$ stands for equality in probability distribution. By Eq.(B.4),

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[e^{c\int_0^T e^{-\delta t}N_i(t)\,\mathrm{d}t}] &= \sum_{p=0}^{\infty}\frac{(\lambda_i T)^p}{p!}e^{-\lambda_i T}\times e^{-pc\frac{e^{-\delta T}}{\delta}}\,\mathbb{E}^{\mathbb{P}}[e^{c\sum_{k=1}^{p}\frac{e^{-\delta U_k}}{\delta}}] \\
&= \sum_{p=0}^{\infty}\frac{(\lambda_i T)^p}{p!}e^{-\lambda_i T}\times e^{-pc\frac{e^{-\delta T}}{\delta}}\prod_{k=1}^{p}\mathbb{E}^{\mathbb{P}}[e^{c\frac{e^{-\delta U_k}}{\delta}}] \\
&= \sum_{p=0}^{\infty}\frac{(\lambda_i T)^p}{p!}e^{-\lambda_i T}\times e^{-pc\frac{e^{-\delta T}}{\delta}}\left(\frac{1}{T}\int_0^T e^{\frac{c}{\delta}e^{-\delta u}}\,\mathrm{d}u\right)^p \\
&= \exp\left\{\lambda_i T\left(-1 + e^{-c\frac{e^{-\delta T}}{\delta}}\frac{1}{T}\int_0^T e^{\frac{c}{\delta}e^{-\delta u}}\,\mathrm{d}u\right)\right\}. \qquad \text{(B.5)}
\end{aligned}$$

where $\int_0^T e^{\frac{c}{\delta}e^{-\delta u}}\,\mathrm{d}u$ is called an exponential integral.[23]

Finally, we solve $\mathbb{E}_t^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u}a\log(X(u))\,\mathrm{d}u}\right]$ explicitly. From Eq.(B.1),

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u}a\log(X(u))\,\mathrm{d}u}\right] \\
=\;& e^{a\int_0^t e^{-\delta u}\log(X(u))\,\mathrm{d}u}\,\mathbb{E}\left[e^{ae^{-\delta t}\int_0^{T-t} e^{-\delta u}a\log(\widetilde{X}(u))\,\mathrm{d}u}\,\middle|\,\widetilde{X}(0) = X(t)\right]
\end{aligned}$$

[23]For the reference, we look at the case of $\delta = 0$, although $\delta = 0$ is not assumed in this paper. By taking the same procedures,

$$\mathbb{E}^{\mathbb{P}}[e^{c\int_0^T N_i(t)\,\mathrm{d}t}] = \exp\left\{\lambda_i T\left(-1 + \frac{e^{cT}-1}{cT}\right)\right\}.$$

where $\log \widetilde{X}(u) = \log X(0) + (\mu^G - \frac{1}{2}\sum_{j=1}^n (\sigma_j^G)^2 - \sum_{i=1}^m \lambda_i)u + \sum_{j=1}^n \sigma_j^G B_j(u) + \sum_{i=1}^m \log(1 + z_i^G)N_i(u)$. Using Eq.(B.2) and Eq.(B.5),

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log(X(u))\,\mathrm{d}u}\right] \\
= \ &\exp\left(\begin{array}{l}
a\int_0^t e^{-\delta u}\log(X(u))\,\mathrm{d}u + a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\log X(t) \\
\ +ae^{-\delta t}(\mu^G - \frac{1}{2}\sum_{j=1}^n(\sigma_j^G)^2 - \sum_{i=1}^m \lambda_i)\int_0^{T-t} e^{-\delta u} u\,\mathrm{d}u \\
+\sum_{j=1}^n \frac{(ae^{-\delta t}\sigma_j^G)^2}{2}\frac{1}{\delta^2}\int_0^{T-t}(e^{-\delta u} - e^{-\delta(T-u)})^2\,\mathrm{d}u \\
+\sum_{i=1}^m \lambda_i (T-t)(-1 + e^{-\frac{e^{-\delta(T-t)}}{\delta}\log(1+z_i^G)}\frac{1}{T-t}\int_0^{T-t} e^{\frac{e^{-\delta u}}{\delta}\log(1+z_i^G)}\,\mathrm{d}u)
\end{array}\right)
\end{aligned}$$

By integration by parts, $\int_0^{T-t} e^{-\delta u} u\,\mathrm{d}u = \frac{1-\exp\{-\delta(T-t)(1+\delta(T-t))\}}{\delta^2}$. □

C Proof of Lemma 4.2

Recalling Eq.(2.5),

$$\mathrm{d}X(t) = X(t_-)\Big(\mu^G\,\mathrm{d}t + \sum_{j=1}^n \sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^m z_i^G\,\mathrm{d}M_i(t)\Big), \quad X(0) = x_0 > 0.$$

Thus,

$$\begin{cases}
\mathrm{d}X^c(t) = X(t_-)\Big(\mu^G\,\mathrm{d}t + \sum_{j=1}^n \sigma_j^G\,\mathrm{d}B_j(t) - \sum_{i=1}^m z_i^G \lambda_i\,\mathrm{d}t\Big), \\
\Delta X(t) = X(t_-)\sum_{i=1}^m z_i^G \Delta N_i(t).
\end{cases}$$

When a jump of N_i occurs at time t,

$$\Delta X(t) = X(t_-)z_i^G, \quad \text{i.e.,} \quad X(t) = X(t_-)(1+z_i^G). \tag{C.1}$$

For a twice continuously differentiable function $f(x,t)$, by Itô's formula,

$$\begin{aligned}
f(X(t),t) \ &= \ f(X(0),0) + \int_0^t \frac{\partial f}{\partial x}(X(u_-),u)\,\mathrm{d}X^c(u) + \int_0^t \frac{\partial f}{\partial t}(X(u_-),u)\,\mathrm{d}u \\
&\quad + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(X(u_-),u)(\mathrm{d}X^c(u))^2 + \sum_{0<u\le t}\Delta f(X(u),u).
\end{aligned}$$

First, set $f(x,t) = \log x$. $\frac{\partial f}{\partial x} = \frac{1}{x}$, $\frac{\partial^2 f}{\partial x^2} = -\frac{1}{x^2}$, and $\frac{\partial f}{\partial t} = 0$. Taking logarithms on both sides of Eq.(C.1),

$$\log X(t) = \log X(t_-) + \log(1 + z_i^G) \quad \text{i.e.,} \quad \Delta \log X(t) = \log(1 + z_i^G).$$

Thus, for $u > t$,

$$\begin{aligned} \log X(u) &= \log X(t) + \sum_{j=1}^{n} \sigma_j^G (B_j(u) - B_j(t)) + \Big(\mu^G - \frac{1}{2}\sum_{j=1}^{n} (\sigma_j^G)^2 - \sum_{i=1}^{m} z_i^G \lambda_i\Big)(u - t) \\ &\quad + \sum_{i=1}^{m} \log(1 + z_i^G)\{N_i(u) - N_i(t)\}. \end{aligned}$$

Hence, $\log X(u) - \log X(t)$ is independent of $\mathcal{F}_t$. On the other hand,

$$\begin{aligned} Y(t) &= \mathbb{E}_t^{\mathbb{P}}\left[e^{a\int_0^T e^{-\delta u} \log X(u)\,\mathrm{d}u}\right] \\ &= e^{a\int_0^t e^{-\delta u} \log X(u)\,\mathrm{d}u}\, e^{a(\log X(t))\int_t^T e^{-\delta u}\,\mathrm{d}u}\, \mathbb{E}_t^{\mathbb{P}}\left[e^{a\int_t^T e^{-\delta u}\{\log X(u) - \log X(t)\}\,\mathrm{d}u}\right]. \end{aligned}$$

There, the second factor is $X(t)^{\alpha(t)}$, where $\alpha(t) := a\int_t^T e^{-\delta u}\,\mathrm{d}u = a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}$. Also, the third factor is equivalent to the unconditional expected value, and thus is a deterministic function that is differentiable with respect to t. Therefore, the stochastic differential of $\widetilde{Y}(t) := \frac{Y(t)}{X(t)^{\alpha(t)}}$ takes the form of $\mathrm{d}\widetilde{Y}(t) = \cdots \mathrm{d}t$. That is, there exist neither $\mathrm{d}B_j(t)$-terms nor $\mathrm{d}N_i(t)$-terms in it.

Next, we look at the dynamics of $X(t)^{\alpha(t)}$. Set $f(x,t) = x^{\alpha(t)}$ and apply Itô's formula to it. Note that $\frac{\partial f}{\partial x} = \alpha(t)x^{\alpha(t)-1}$. Since, as shown above, $X(t) = X(t_-)(1 + z_i^G)$ when a jump of N_i occurs at time t,

$$X(t)^{\alpha(t)} = X(t_-)^{\alpha(t)}(1 + z_i^G)^{\alpha(t)}, \quad \text{i.e.,} \quad \Delta X(t)^{\alpha(t)} = X(t_-)^{\alpha(t)}\{(1 + z_i^G)^{\alpha(t)} - 1\}.$$

Hence,

$$\begin{aligned} \mathrm{d}\Big(X(t)^{\alpha(t)}\Big) &= \alpha(t)X(t_-)^{\alpha(t)-1}X(t_-)\Big(\sum_{j=1}^{n} \sigma_j^G\,\mathrm{d}B_j(t) + \cdots \mathrm{d}t\Big) \\ &\quad + \cdots \mathrm{d}t + \cdots \mathrm{d}t + X(t_-)^{\alpha(t)} \sum_{i=1}^{m} \{(1 + z_i^G)^{\alpha(t)} - 1\}\,\mathrm{d}N_i(t). \end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{d}Y(t) &= \mathrm{d}\left(\widetilde{Y}(t)X(t)^{\alpha(t)}\right) \\
&= \widetilde{Y}(t_-)\,\mathrm{d}\left(X(t)^{\alpha(t)}\right) + X(t_-)^{\alpha(t)}\,\mathrm{d}\widetilde{Y}(t) + \mathrm{d}\widetilde{Y}(t)\,\mathrm{d}\left(X(t)^{\alpha(t)}\right) \\
&= \widetilde{Y}(t_-)X(t_-)^{\alpha(t)}\Big(\alpha(t)\sum_{j=1}^{n}\sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m}\{(1+z_i^G)^{\alpha(t)}-1\}\,\mathrm{d}M_i(t) + \cdots \mathrm{d}t\Big) \\
&\quad + \cdots \mathrm{d}t + 0 \\
&= Y(t_-)\Big(\alpha(t)\sum_{j=1}^{n}\sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m}\{(1+z_i^G)^{\alpha(t)}-1\}\,\mathrm{d}M_i(t)\Big) + \cdots \mathrm{d}t.
\end{aligned}$$

Since Y is a martingale, the total sum of the dt-terms is zero. The result is obtained. □

D Proof of Lemma 5.1

For $(\phi, \beta) \in \Phi(S^*)$ satisfying Assumption 5.2, due to Assumption 5.1, by Lebesgue's dominated convergence theorem,

$$\begin{aligned}
W(t) &= \mathbb{E}_t^{\mathbb{P}}\left[\int_t^T \begin{pmatrix} \frac{\Pi(s)}{\Pi(t)}\Big(C(s)+S^*(s)-X(s)\Big) \\ +\frac{\Pi(s)}{\Pi(t)}W(s)\beta(s)^\top\Big(\mu^R - \sigma^R\eta(s) - z^R I^\xi(s)\lambda\Big) \end{pmatrix} \mathrm{d}s + \frac{\Pi(T)}{\Pi(t)}W(T)\right] \\
&= \mathbb{E}_t^{\mathbb{P}}\left[\int_t^T \frac{\Pi(s)}{\Pi(t)}\Big(C(s)+S^*(s)-X(s)\Big)\,\mathrm{d}s + \frac{\Pi(T)}{\Pi(t)}W(T)\right].
\end{aligned}$$

$$\begin{aligned}
\text{I.e., } w_0 &= \Pi(0)W(0) \\
&= \mathbb{E}^{\mathbb{P}}\left[\int_0^T \Pi(u)(C(u)+S^*(u)-X(u))\,\mathrm{d}u + \Pi(T)W(T)\right] = (\Pi|\phi + S^* - X).
\end{aligned}$$

On the other hand, for any $h \in \mathcal{H}$ such that $\phi + h \in \Phi(S^*)$, by Fatou's lemma,

$$\Pi(0)W(0) + (\Pi|X - S^*) \geq (\Pi|\phi + h).$$

Therefore, $(\Pi|h) \leq 0$. □

E Proof of Lemma 5.2

Define

$$\begin{aligned}
\Delta_U &:= \widehat{U}_2(\phi+h,S^*)-\widehat{U}_2(\phi,S^*),\\
\Delta_\Sigma &:= \Sigma(\phi+h,S^*)-\Sigma(\phi,S^*),\\
\Delta_\Gamma &:= \Gamma(\phi+h,S^*)-\Gamma(\phi,S^*).
\end{aligned}$$

Also, define

$$\Delta_F(t) := \begin{cases} \widehat{F}(\phi,S^*,\widehat{U}_2)+\widehat{F}_c h-\delta\Delta_U-\widehat{F}(\phi+h,S^*,\widehat{U}_2+\Delta_U) & \text{for } 0\le t<T,\\ 0 & \text{for } t=T. \end{cases}$$

Due to the concavity of $\widehat{F}$, $\Delta_F(t)\ge 0\,\forall\, t$. Recalling $\mathrm{d}\widehat{U}_2=-\widehat{F}(\phi,S^*,\widehat{U}_2)\,\mathrm{d}t+\Sigma\,\mathrm{d}B+\Gamma\,\mathrm{d}M$; $\widehat{U}_2(T)=\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}}W(T)$,

$$\begin{aligned}
\mathrm{d}\Delta_U &= \mathrm{d}\widehat{U}_2(\phi+h,S^*)-\mathrm{d}\widehat{U}_2(\phi,S^*)\\
&= -(\widehat{F}_c h-\delta\Delta_U-\Delta_F)\,\mathrm{d}t+\Delta_\Sigma\,\mathrm{d}B+\Delta_\Gamma\,\mathrm{d}M;\\
\Delta_U(T) &= \widehat{F}_\phi(T)h(T)-\Delta_F(T)=\widehat{F}_\phi(T)h(T).
\end{aligned}$$

Since $\mathcal{E}$ is deterministic (i.e., $\mathrm{d}\mathcal{E}(t)=-\delta\,\mathrm{d}t$),

$$\begin{aligned}
\mathrm{d}(\mathcal{E}\Delta_U) &= (-\mathcal{E}\widehat{F}_c h+\mathcal{E}\Delta_F)\,\mathrm{d}t+\cdots\mathrm{d}B+\cdots\mathrm{d}M;\\
& \mathcal{E}(T)\Delta_U(T)=\mathcal{E}(T)\widehat{F}_\phi(T)h(T)-\mathcal{E}(T)\Delta_F(T)=\mathcal{E}(T)\widehat{F}_\phi(T)h(T).
\end{aligned}$$

By Lebesgue's dominated convergence theorem,

$$\begin{aligned}
\Delta_U(0) &= \mathbb{E}^{\mathbb{P}}\left[\int_0^T(\mathcal{E}\widehat{F}_c h-\mathcal{E}\Delta_F)\,\mathrm{d}t+\Big(\mathcal{E}(T)\widehat{F}_\phi(T)h(T)-\mathcal{E}(T)\Delta_F(T)\Big)\right]\\
&= (\mathcal{E}\widehat{F}_\phi|h)-(\mathcal{E}|\Delta_F).
\end{aligned}$$

Since $\Delta_F(t)\ge 0\,\forall\, t$, $\widehat{U}_2(\phi+h,S^*)-\widehat{U}_2(\phi,S^*)\le(\Lambda|h)$. $\square$

F Proof of Proposition 5.1

By Lemma 5.1 and Lemma 5.2, for $(\phi, \beta) \in \mathcal{C}(S^*)$ satisfying Assumption 5.2 and for any $h \in \mathcal{H}$ such that $(\phi + h, \beta') \in \mathcal{C}(S^*)$,

$$\Delta_U(0) = \widehat{U}_2(\phi + h, S^*) - \widehat{U}_2(\phi, S^*) \leq (\Pi|h) \leq 0.$$

Hence, at such (ϕ, β), Π is a state price and the utility is maximized. Since the optimal C is uniquely determined for the given s^* in the market equilibrium where $W(t) = 0$ for all t, the equilibrium C is equal to C^* defined in Eq.(4.11). Obviously, such (ϕ, β) is in $\mathcal{C}(S^*)$. □

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