Political Competition and Preference Aggregation in Representative Democracy

by

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Abstract

This dissertation consists of four essays related to political competition and preference aggregation in representative democracy.

The first essay presents a model of two-party electoral competition where only parties know the cost of a public project. In the election, the parties choose platforms from two alternatives: implementing the project or not. In this setting, the parties' choices of platforms influence voters' behavior not only through voters' preferences, but also through formation of their expectations on the cost. Each party's objective is to maximize vote. The model has two types of perfect Bayesian equilibrium (PBE) outcomes: in the first, both parties always choose the same fixed policy irrespective of the cost; in the second, one party always chooses not to implement the project while the other party switches the platform around a unique threshold of the cost. To examine which of the equilibria are plausible, we introduce a perturbation concept which we call "conjectural perturbation." We show that only the PBE outcomes of the second type are robust to all conjectural perturbations. This is in contrast with the typical conclusion from Downsian models that two parties behave identically in equilibrium.

The second essay modifies the model in the first essay by assuming that each party only observes a private signal of the cost. Each party now infers from its private signal what signal the opponent has received. Despite this additional structure, we show that the model's PBE outcomes are similar to those in the previous chapter. In particular, there are PBE outocmes in which one party always announces not to implement the

project while the other party switches the platform around a unique threshold of its private signal.

In the third essay, we consider a multi-dimensional probabilistic voting model where each party maximizes its expected vote share. In contrast to existing models, we allow the parties to hold different prior beliefs about the profile of voters' ideal policies. We first observe that (at least in part) due to the discontinuity of the party's payoffs, a Nash equilibrium rarely exists even in a one-dimensional policy space. We then characterize "approximate equilibria" (i.e., a platform pair at which the parties almost best respond to each other). The condition characterizing approximate equilibria is quite weaker than that characterizing Nash equilibria. In particular, for one-dimensional policy spaces, approximate equilibria exist generally. Nonetheless, we show that if the policy space has multiple dimensions, and if the parties' beliefs are close enough, then an approximate equilibrium typically fails to exist. Thus, the existence of an approximate political equilibrium requires that the parties' beliefs differ sufficiently.

In the fourth essay, we study the range of social preference relations over policies that a legislature with at most three parties can realize through majority voting. Here, a party means a group of representatives with identical transitive preferences over policies. By limiting attention to three-party systems, we try to understand the minimal effect arising from the absence of a single-majority party. We show that this range contains all social preference relations that involve no "closed path" consisting of 3-cycles. This condition reveals a set of social preference relations that can be realized by some three-party legislature, but cannot be realized by any legislature with a single-party majority.

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Contents

Chapter 1

Introduction

1.1 Background of the thesis

Majority voting is the primary method of political decision in a democratic society. In such a society, referendums where citizens directly vote for policies are relatively rare. More often, a policy realizes via congressional (and presidential) elections, followed by legislative voting. In elections, parties or candidates announce their platforms, which may constrain their post-election legislative behaviors. Observing the platforms, citizens make their voting decisions. The elected legislature makes final decisions on what policies to implement. What platforms do parties choose? Is there any relationship between the chosen policies and citizens' preferences? Can legislative voting with a limited number of parties substitute for referendums?

Consider a group of voters with preferences over a set of policies. A "core" policy refers to a policy that defeats all other policies in pairwise majority comparisons. The well-known "Median Voter Theorem (MVT)" (Black, 1958) shows a case where the existence of a unique core policy is guaranteed. The theorem says that if all policies can be aligned on a straight line and all voters' preferences are single-peaked on the line, then the median of the peaks, if it is unique, is the unique core policy. For example, suppose a legislature must determine the rate of some tax levied on citizens.

The legislature comprises three parties each having about 1/3 of the seats. Each party, representing some group of citizens, has a single-peaked preference over possible tax rates. In this case, the theorem says that the middle one among the best tax rates for the three parties is the core policy; once the legislature arrives at this policy, it cannot reject the policy by proposing any other alternative.

In fact, a more stronger conclusion has been drawn regarding majority decision in single-peaked environments. That is, under the same hypothesis as in the MVT, the binary relation formed by pairwise majority comparisons of policies is transitive (Arrow, 2012[1]). The majority preference relation in such cases is therefore "rational" in the standard sense of economics. The existence of a core policy in the MVT can be understood as a consequence of the rationality since a rational preference always has a maximal element.

The MVT also gives some insights into behaviors of parties (or candidates) in elections. Downs (1957) constructed a model of party competition that would become a prototype of many subsequent models. In his model, two parties compete in an election. Each party wants to maximize its vote share. The parties choose their platform policies from a one-dimensional policy space.[2] Each citizen has a single-peaked preference over policies, and votes for the party with the preferred platform. Then it follows from the definition of a core policy and the MVT that at the equilibrium of this game, both parties will choose the core policy, namely, the median of the citizens' ideal points. This provides a logical background for the argument that in a two-party system the parties tend to be similar to each other, both converging to a central point in the distribution of citizens' political positions.

However, in contrast to this "policy convergence" result, empirical data suggests that parties or candidates often adopt different platforms. This gap between theory and reality led some researchers to develop alternative models. Wittman (1983), Calvert

[1]The first edition was published in 1951.

[2]Downs assumed that the parties choose ideological positions rather than policies. For convenience, we interpret his model as concerning policy choices.

(1985), and Roemer (1994) analyzed models where parties have different preferences over policies and are uncertain about voters' preferences.[3] They show that at any equilibrium of such a model (if it exists) the parties choose different platforms. In particular, Roemer (1994) provides a natural case with a uni-dimensional policy space for which an equilibrium exists for sure. Roemer (2001) also constructs a model where each party comprises factions with different motivations, but we will come back to this later. Palfrey (1984) considers a uni-dimensional model of two-party competition with potential third party entry. Although the parties in his model are interested only in their vote shares, an equilibrium exists at which the established parties take distinct positions. Our first two essays (Chapters 2 and 3) are another attempt to account for platform divergences in real elections.

So far we have limited our attention to single-peaked environments where policies are ordered on a line and each voter has a preference peak on the line. We have seen that in this case a core policy exists, and hence an equilibrium of the Downsian model also exists. What occurs in other political settings?

The so-called "Paradox of Voting" illustrates a simple situation where majority voting lacks both transitivity and a core policy. Consider a group of three voters with preferences over three policies, $A \succ_1 B \succ_1 C$, $B \succ_2 C \succ_2 A$, and $C \succ_3 A \succ_3 B$. Pairwise majority comparisons of policies constitute a cycle: A beats B, B beats C, and C beats A. Moreover, since any policy is majority-dominated by another policy, there is no core policy. Arrow (2012) extended this observation far beyond majority voting. He specified properties of the majority rule that seem reasonable to require for a general social decision rule. His "General Possibility Theorem" shows that not only the majority rule, but any social decision rule satisfying these seemingly minimal requirements is necessarily intransitive.

For multi-dimensional policy spaces, the existence of a core policy is extremely

[3] Wittman and Calvert consider "candidates" instead of "parties." Throughout, we stick to the term "parties" only to avoid unnecessary complication.

rare even if we restrict voters' preferences to some reasonable class. A policy in such a space may be interpreted as specifying positions on various political issues, or as specifying redistribution of income or wealth among various socio-economic classes of citizens. Plott (1967) assumes that voters have differentiable utility functions, and characterizes core policies by some "symmetry" condition in terms of voters' preferences. His condition indicates that if there are two or more dimensions, a core point exists only in exceptional cases.[4] This implies that when the policy space has multiple dimensions, the standard Downsian model typically lacks a political equilibrium.

The nonexistence result directly extends to some "probabilistic voting model" in which parties only have probabilistic beliefs about voters' preferences. In particular, suppose that each voter's preference over policies is characterized by an ideal policy. Suppose furthermore that the parties have the same prior belief about the profile of voters' ideal policies, and want to maximize their expected vote shares.[5] Then the resulting model may be viewed as a deterministic Downsian model by simply interpreting the expected distribution of voters' ideal policies as a deterministic distribution. Therefore, again, typically no equilibrium exists for a multi-dimensional policy space. A natural question at this point is whether this result depends critically on the assumption that the parties have exactly the same prior, because it is this assumption that allows us to exploit the analogy with the deterministic model. Our third essay (Chapter 4) studies this question.

The Paradox of Voting and Arrow's Possibility Theorem tell us that the choice patterns made by an individual and by a group of individuals are generally quite different. In particular, the Paradox shows that the range of possible majority preference relations over three alternatives induced by three voters (each with a transitive preference)

[4]Rubinstein (1979) proves, under the assumption of continuous preferences, but without any assumption on the dimensionality, that the set of profiles of voters' preferences for which a core policy exists is very "small" (in a topological sense).

[5]This type of model is studied by Duggan (2006) for the uni-dimensional case. Another kind of probabilistic voting model was developed by Hinich, Ledyard, and Ordeshook (1972) and Lindbeck and Weibull (1993) in which voters have partisan biases that are unobservable for parties.

includes cyclic relations, and thus is broader than those induced by one or two voters. Several researchers have generalized the Paradox by looking at how the range of majority preference relations changes with the numbers of voters and of alternatives. McGarvey (1953) showed that any (transitive and complete) relation over n alternatives is realized by some set of $n(n-1)$ voters. Stearns (1959) and Erdös and Moser (1964) provided estimates for the minimum number of voters required to realize any relation on n alternatives for large n, which have the form $cn/\log n$ for some constant c.

We can interpret these observations as indicating that the majority decision of the entire society (i.e., referendums) may be quire different from that of any legislature with a limited number of representatives. That is, there is only limited possibility for a legislature to "represent" the society even if it can freely choose the composition of the legislature. Note that here our notion of "representation" is in the very narrow sense that legislative voting realizes a given social preference relation over policies. While the above studies initiated by McGarvey do not assume any bound on the number of voters, a typical situation in the real politics is that each representative in the legislature belongs to some party, and there are only a limited number of parties. The results of these studies imply that in such cases, when the number of policies is sufficiently large, not all preference relations over policies can be realized. Given a fixed number of parties, what type of preference relations are realizable through legislative voting? Our fourth essay (Chapter 5) explores this problem, focusing on the case where at most three parties may form.

1.2 Structure of the thesis

Chapter 2 presents a model of electoral competition where only parties know the cost of a public project. In the election, the parties choose platforms from two alternatives: implementing the project or not. In this setting, the parties' choices of platforms influence voters' behavior not only through voters' preferences, but also through formation of their expectations about the cost. Each party's objective is to maximize vote. The model has two types of perfect Bayesian equilibrium (PBE) outcomes: in the first, both parties always choose the same fixed policy irrespective of the cost; in the second, one party always chooses not to implement the project while the other party switches the platform around a unique threshold of the cost. To examine which of the equilibria are plausible, we introduce a perturbation concept which we call "conjectural perturbation." We show that only the PBE outcomes of the second type are robust to all conjectural perturbations. This is in contrast with the typical conclusion from Downsian models that the two parties behave identically in equilibrium.

Chapter 3 modifies the model in the previous chapter by assuming that each party only observes a private signal of the cost. As a result, each party now infers from its private signal what signal the opponent has received. Despite this additional structure, we show that the PBE outcomes in this extended model are similar to those in the previous model. In particular, there are PBE outcomes in which one party always announces not to implement the project while the other party switches the platform around a unique threshold of its private signal.

Some researchers have already studied models of electoral competition in which party platforms play the role of a signal for voters, and derived equilibria with policy divergence. Kartik and McAfee (2007) consider a model where voters do not know whether a candidate has a "character" or not. Callaner (2008) assumes that candidates commit to their platforms but after the election they will choose the level of effort for implementing the policy; voters do not know whether a candidate will exert high effort or not. Both models have an equilibrium at which candidates offer distinct platforms, due to either distinct policy preferences or private information (or randomized choices) of the two candidates. In our model, the parties play quite different strategies, even when they have no exogenous differences.

In Chapter 4, we consider a multi-dimensional probabilistic voting model where each party maximizes its expected vote share. In contrast to existing models, we allow the parties to hold different prior beliefs about the profile of voters' ideal policies. We first observe that (at least in part) due to the discontinuity of the parties' payoffs, a Nash equilibrium rarely exists even if the policy space is one-dimensional. We then characterize "approximate equilibria" (i.e., a platform pair at which the parties almost best respond to each other). The condition characterizing approximate equilibria is quite weaker than that characterizing Nash equilibria. In particular, for one-dimensional policy spaces, approximate equilibria exist generally. Nonetheless, we show that if the policy space has multiple dimensions, and if the parties' beliefs are close enough, then an approximate equilibrium typically fails to exist. Thus, the existence of an approximate political equilibrium requires that the parties' beliefs differ sufficiently.

There are some recent papers that address similar problems. Banks and Duggan (2006) show that if a deterministic Downsian model has no (Nash) equilibrium, which is the typical case in a multi-dimensional setting, then introduction of slight uncertainty about voters' preferences where the parties have the same belief does not create an equilibrium. Duggan and Fey (2005) analyze a deterministic model with policy-motivated candidates and a multi-dimensional policy space. They show that if the model has no equilibrium, which is the typical case when the number of dimensions is more than two,[6] then small amounts of probabilistic voting and of office motivation do not generate an equilibrium. In this essay, we focus on the effects of slight differences between the parties' beliefs, rather than the effect of slight uncertainty with a common belief.

Our main finding is a negative result on the existence of political equilibrium, and does not answer the important question of under what assumptions a multi-dimensional model has a political equilibrium. Roemer (2001) takes a novel approach to this question. He assumes that a party consists of "factions" with different objectives: "opportunists" want to win elections; "reformists" want to maximize their payoffs from the

[6]For two-dimensional policy spaces, they identify a class of equilibria that are not necessarily the convergence to a core point, and show that this class of equilibria is robust to small perturbations of the model.

realized policy; and "militants" want to announce their most preferred policy. He introduces a concept called "party-unanimity Nash equilibrium (PUNE)" where no party has unanimous agreement among its factions to change the platform. This formulation is in contrast with Downs' (1957) and Wittman's (1983) models that regard a party as a team with a single goal. Roemer shows that in some two-dimensional cases, PUNEs exist.

In Chapter 5, we study the range of social preference relations over policies that a legislature with at most three parties can realize through majority voting. Here, a party means a group of representatives with identical transitive preferences over policies. By limiting attention to three-party systems, we try to understand the minimal effect arising from the absence of a single-majority party. We show that this range contains all social preference relations that involve no "closed path" consisting of 3-cycles. This condition identifies a set of social preference relations that can be realized by some three-party legislature, but cannot be realized by any legislature with a single-party majority.

Recently, Brandt, Harrenstein, Kardel, Seedig (2013) provided a full characterization of social preference relations compatible with some three-party system. Although their characterization is logically stronger, our sufficient condition may be useful as it is easy to check.

Chapter 2

Downsian political competition with asymmetric information: possibility of policy divergence

2.1 Introduction

The Median Voter Theorem (MVT), a well-known theoretical result in the classical Downsian model of political competition, states that under some natural assumptions, two office-seeking political parties will announce the same platform: the median voter's ideal policy (Downs, 1957). Despite the wide acceptance of the model, the inconsistency between the conclusion of MVT and the real political phenomena is pointed out frequently. Most empirical studies observe policy divergences, rather than convergence, between the parties. In this paper, we construct a model to explain why policy divergences emerge in real politics.

We focus on the fact that in actual elections, voters are unfamiliar with data necessary for evaluating policies, while parties obtain richer knowledge, for example, through research activities. In such cases, voters infer useful information from the observed party platforms. For example, consider an election in which the main issue is fiscal policy. Two competing parties choose their platform policies on the amount of fiscal deficit. Voters' preferences over these policies depend on the costs of the deficit, such as future tax increases and the possibility of a fiscal crisis. The parties are better informed of the costs than the voters. Since the parties formulate their policies on the basis of their information, the voters infer the costs from the announced platforms in order to determine which policy to support.

As another example, when income redistribution is at issue, the extent to which taxation on income affects efficiency of the economy by lowering labor incentives depends on the average income elasticity ϵ_l of labor. Parties know the value of ϵ_l, whereas voters only have the information on its prior distribution. Suppose that party 1's strategy has a higher threshold of ϵ_l at which it switches its policy from progressive tax to regressive tax. Now, suppose that parties 1 and 2 announce progressive and regressive tax policies as their respective platforms. Voters will then infer that ϵ_l is somewhere between the two parties' thresholds.

In this paper, we present a simple Downsian election model with two alternative policies: implementing a public project or not. If the project is implemented, each voter incurs a per capita cost of c. However, voters do not perfectly know the cost, and only have a prior expectation that the project is quite costly. Two parties observe the cost c and then simultaneously announce their platforms. Observing these platforms, each voter votes for a party. Each party is motivated to maximize vote. In this setting, the two parties' choices of platforms influence voters' behavior not only through voters' preferences but also through the formation of expectation about the cost.

In this political game with asymmetric information, we identify the class of perfect Bayesian equilibria in which the parties' strategies satisfy a monotonicity condition. We interpret monotonicity as the requirement that a party's strategy be not only optimal in terms of vote but also compatible with the incentives of expert members who hold the information of the cost. There are two types of PBE outcome: in the first, both parties announce the same fixed platform a irrespective of the cost (there are two such outcomes corresponding to the choices of platform a); in the second, one party always chooses not to implement the project, while the other party switches the platform around a unique threshold of the cost (there are two such outcomes corresponding to the choices of the party playing each role).

We then discuss which of the equilibrium outcomes are plausible. We introduce slight perturbations of the model which we call "conjectural perturbations," and examine whether the equilibria are robust to them. Under a conjectural perturbation, each party has a small chance of making a wrong conjecture about the electorate's response to the parties' platforms. We assume that the probability of such a mistake in conjectures is relatively high for platform pairs that almost never realize in equilibrium. Our use of conjectural perturbation is motivated by the fact that while the PBE concept assumes that each party correctly expects voters' actions after any possible platform pairs, the party has no opportunity to learn voters' actions after out-of-equilibrium platform pairs. The concept of self-confirming equilibrium of Fudenberg and Levine (1993) has a similar motivation. Robustness to a conjectural perturbation is also similar in spirit to the notion of stability against mistake probabilities as formulated by Selten (1975), in that both make all platform pairs realize with positive probabilities thereby enabling voters to update their beliefs about the cost by Bayes' rule.

We show that the two PBE outcomes at which one party is irresponsive to the cost level while the other party switches the platform around a threshold are robust to *all* conjectural perturbations. The other two PBE outcomes at which the parties' platforms always coincide are fragile if, for instance, either party tends to overestimate the electorate's support for the project especially when the true cost is very low. This result is in contrast with the conclusion of policy convergence in MVT and more widely with the typical conclusion from many Downsian models that two parties or candidates play identical strategies.

Many authors have studied incomplete information in two-party political games. Schultz (1996) and Heidhues and Lagerlöf (2003) already consider models with a basic structure similar to out model where parties are better informed than voters about the effectiveness of alternative policies. Martinelli (2001) studies a model where both parties and voters receive private information about the state of the world. Banks (1990) and Callander and Wilkie (2007) allow for the possibility of candidates' lies and study a situation where voters infer from the announced platforms the policies that the candidates will actually implement when elected to the office. Callander (2008) assumes that the candidates commit to their platforms but after the election they will choose the level of effort for implementing the policy. Kartik and McAfee (2007) construct a model in which voters do not know whether a candidate has "character" or not (i.e., whether he is of the type who commits to a particular platform or he is just acting strategically) but infer it from the observed platforms.

There are also papers that study policy divergences in elections. Wittman (1983), Calvert (1985), and Roemer (1994) show that if the parties are interested in realizing different policies but imperfectly know the profile of voters' preferences, they adopt different platforms. Palfrey (1984) considers a model with vote-maximizing parties in which there is threat of third-party entry. He shows that the model has an equilibrium at which the two established parties take different positions.

Except for Palfrey's work, most of previous studies have explained policy divergences as a result of differences in the parties or candidates' characteristics such as policy preferences or private information. Our result does not depend on such exogenous differences between the parties, although we will also allow one party to have some advantage over the other.

In Section 2.2, we construct the model. In Section 2.3, we characterize the PBEs of the model. In Section 2.4, we study robustness of the PBE outcomes to conjectural perturbations. Finally, in Section 2.5, we conclude.

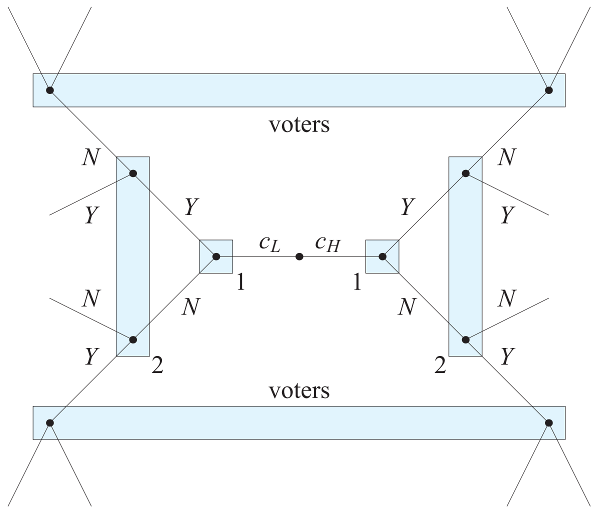

Figure 2.1: The game tree with only two cost levels.

2.2 Model

We consider two-party electoral competition. Each party can choose a platform from two alternatives, "implement a public project" (Y) or "not implement the project" (N). If the project is implemented, each voter incurs a per capita cost of c. To decide on his vote, the voter needs the information of c. However, this information is asymmetric between the parties and voters. Only the parties know c.

The model is a signaling game (see Figure 2.1). First, the (per capita) cost c randomly realizes according to a prior distribution p. Observing c, parties 1 and 2 simultaneously announce platforms a_1 and a_2 from $\{Y, N\}$. Observing the pair (a_1, a_2) of announced platforms, but without observing c, each voter votes for either party. We assume that the prior p is an absolutely continuous distribution with support $C = [\underline{c}, \overline{c}]$.

Each voter k has a valuation v_k of the project, and a *non-policy preference relation* $\succsim_k$ between the two parties. Let F be the population distribution function of valuations v_k (i.e., $F(v)$ is the fraction of voters k with $v_k \leq v$). We assume that F is continuous and has support C. Thus there is a continuum of voters. For each $\gamma \in [0, 1]$, let Q_γ

denote the γ-quantile of F, i.e., the point such that $F(Q_\gamma) = \gamma$. Non-policy preferences $\succsim_k$ are determined by exogenous characteristics of the parties that are unrelated to their positions regarding the public project, such as their fixed positions on other political issues. For each party i, we call a voter k an *i-partisan* if $i \succ_k j$, $j \neq i$; and we call him a *nonpartisan* if $1 \sim_k 2$. We denote by α the proportion of nonpartisans. For each party $i(\neq j)$, let ϕ_i denote the proportion of i-partisans among all partisans; thus $\phi_1 + \phi_2 = 1$. Throughout the paper, we assume that

$$\alpha > 0 \text{ and } Q_{(1-\alpha)\phi_i+\alpha} < E(c) \text{ for } i = 1, 2. \tag{2.1}$$

For example, this is satisfied in the following situation: ex ante, a majority of voters prefer not to implement the project (i.e., $Q_{1/2} < E(c)$); the abstention rate α is positive but sufficiently small; and each ϕ_i is sufficiently close to $1/2$.

Each voter compares the two parties lexicographically, first according to his policy preference, and second according to his non-policy preference. If the parties announce distinct platforms $(a_i, a_j) = (Y, N)$, then conditional on c voter k prefers to vote for party i if and only if $v_k > c$.[1] If the parties announce the same platform ($a_1 = a_2$), only partisans participate in voting and nonpartisans abstain; in this case, $(1-\alpha)\phi_i$ of voters vote for party i while α of voters abstain.

Each party i's payoff is the number of votes that it obtains. An *aggregate voting plan* is a vector $\pi = (\pi_i(a_1, a_2))_{i=1,2,(a_1,a_2)\in\{Y,N\}^2}$ which specifies the proportion $\pi_i(a_1, a_2)$ of voters who will vote for party i after the platforms (a_1, a_2) are announced. The above assumptions imply that: if $a_1 \neq a_2$ then $\pi_1(a_1, a_2) + \pi_2(a_1, a_2) = 1$; and if $a_1 = a_2$ then $\pi_i(a_1, a_2) = (1-\alpha)\phi_i$ for each party i. Thus, with abuse of notation, we also refer to a two-dimensional vector $\pi = (\pi_1(Y, N), \pi_1(N, Y)) \in [0, 1]^2$ as an aggregate voting plan. Let $\Pi = [0, 1]^2$ be the set of such vectors.

A (pure) strategy for a party i is a function $\sigma_i : C \to \{Y, N\}$ that assigns to each cost

[1]Since the distribution function F of voters' values is continuous, the measure of voters with $v_k = c$ is zero.

level c a platform. A strategy for a voter k is a function $\tau_k : \{(Y,N),(N,Y)\} \to \{1,2\}$ that assigns to each pair of distinct platforms $a_1 \neq a_2$ a party. The value $\tau_k(a_1,a_2)$ represents the party for which the voter votes when (a_1,a_2) are announced. A belief system for a voter k is a pair $\beta_k = (\beta_k(\cdot \mid a_1,a_2))_{a_1 \neq a_2}$ of distributions on C, where $\beta_k(\cdot \mid a_1,a_2)$ represents the belief about c that the voter holds after observing the platform pair (a_1,a_2).[2] Given a profile $\tau = (\tau_k)$ of voters' strategies, we denote by π^τ the aggregate voting plan induced by the profile.

A strategy σ_i for party i is called *monotonic* if for any two cost levels $c < c'$, $\sigma_i(c) = N$ implies $\sigma_i(c') = N$. Each monotonic strategy has a threshold $x_i \in C$ such that

$$\sigma_i(c) = \begin{cases} Y & \text{if } c < x_i \\ N & \text{if } c > x_i. \end{cases}$$

We identify all monotonic strategies with the same threshold x_i and call them simply *the monotonic strategy with threshold* x_i. Note that the two completely pooling strategies, "always choose Y" and "always choose N," belong to the class of monotonic strategies.

In this paper, we analyze perfect Bayesian equilibria (PBEs) of the model in which: (i) each party plays a monotonic strategy; and (ii) voters have the same belief system.

Condition (i) may be interpreted as the requirement that each party's equilibrium strategy be not only optimal in terms of vote but also compatible with its members' incentives. Suppose that a party i employs a group E_i of expert members in which each $k \in E_i$ knows the true cost c and has some personal valuation v_k of the project. Given a best-response strategy σ_i in terms of vote, the party devices some mechanism to elicit information about c from these members so that it can always choose the platform $a_i = \sigma_i(c)$ specified by the strategy. Assume, however, that each member $k \in E_i$ wants the party to choose as its platform the the policy he prefers (i.e., Y if $v_k > c$ and N if

[2] We abbreviate the voter's belief and action when the platforms converge, because we have specified them exogenously.

Table 2.1: The platform choice game G_β.

	Y	N
Y	$(1-\alpha)\phi_1, (1-\alpha)\phi_2$	$1 - F(E_\beta(c\mid Y, N)), F(E_\beta(c\mid Y, N))$
N	$F(E_\beta(c\mid N, Y)), 1 - F(E_\beta(c\mid N, Y))$	$(1-\alpha)\phi_1, (1-\alpha)\phi_2$

$v_k < c$).[3] A well-known result in implementation theory implies that the party can find such a mechanism only if the strategy σ_i is monotonic.[4] On the other hand, whether a given monotonic strategy σ_i is implemented by some mechanism depends on the valuation profile $(v_k)_{k\in E_i}$ of the experts. We assume that given any monotonic strategy σ_i, the party can appoint an appropriate group E_i of experts to implement σ_i.[5]

We impose condition (ii) only for analytical convenience. If all voters have the same belief system β, then given any pair of distinct platforms $(a_i, a_j) = (Y, N)$, the electorate is divided into the groups of supporters for the two parties in the following simple manner: voters k with $v_k > E_\beta(c\mid a_i, a_j)$ vote for party i and voters with $v_k < E_\beta(c\mid a_i, a_j)$ vote for party j, where $E_\beta(\cdot\mid a_i, a_j)$ denotes expectation with respect to belief $\beta(\cdot\mid a_i, a_j)$. While this assumption restricts voters' equilibrium behaviors, it does not restrict the set of the strategy pairs for the parties that may arise in equilibrium.

To summarize, our basic solution concept is defined as follows.

Definition. A combination $(\sigma_1, \sigma_2, \tau, \beta)$ of strategies and a common belief system is called a *perfect Bayesian equilibrium (PBE)* if the following E1-E4 hold.

E1: For each i, σ_i is a monotonic strategy.

E2: For each i and each c, $a_i = \sigma_i(c)$ maximizes $\pi_i^\tau(a_i, \sigma_j(c))$.

[3]Thus we exclude the possibility that a member cares about the effect of the platform choice on the voting outcome.

[4]σ_i may be seen as a social choice function for the group E_i where the only variable parameter for the preference profile is $c \in [0, 1]$ and the possible outcomes are $\{Y, N\}$. As well known, σ_i is implementable in Nash equilibrium only if it is a Maskin monotonic social choice function. This is equivalent to that σ_i is a monotonic strategy.

[5]Of course, if σ_i is the monotonic strategy with threshold x_i, then choosing $E_i = \{k\}$ with $v_k = x_i$ suffices for this purpose.

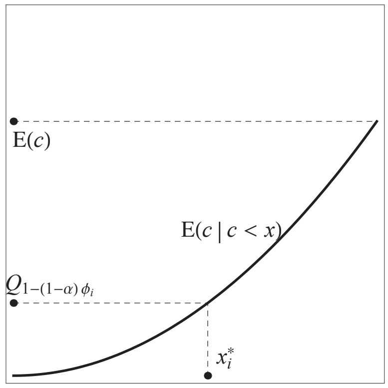

Figure 2.2: Point x_i^*.

E3: For each voter k and each pair $(a_i, a_j) = (Y, N)$ of distinct platforms, $\tau_k(a_i, a_j) = i$ if and only if $E_\beta(c| \, a_i, a_j) < v_k$.

E4: For each $(a_i, a_j) = (Y, N)$, the belief $\beta(\cdot| \, a_i, a_j)$ is obtained by Bayes-updating the prior belief p whenever (a_i, a_j) is realizable under the strategies (σ_1, σ_2).

Given a common belief system β, E3 implies that for the platform pair $(a_i, a_j) = (Y, N)$, $\pi_i^\tau(a_i, a_j) = 1 - F(E_\beta(c| \, a_i, a_j))$. Hence, the belief system β induces the *platform choice game* between parties 1 and 2, denoted G_β, as summarized in Table 2.1.

2.3 Perfect Bayesian equilibrium

We now characterize PBEs of the model. For each party i, let x_i^* denote the point in C satisfying

$$E(c| \, c < x_i^*) = Q_{1-(1-\alpha)\phi_i}. \tag{2.2}$$

See Figure 2.2. Since $1 - (1-\alpha)\phi_i = (1-\alpha)\phi_j + \alpha$ for $j \neq i$, assumption (2.1), together with the continuity and support assumptions on p and F, implies that x_i^* is a unique interior point in C.

Proposition 2.1. *All PBE outcomes are as follows:*

1. *Both parties always choose the same fixed platform $a \in \{Y, N\}$ irrespective of the cost c, and each party i obtains a vote share of $(1-\alpha)\phi_i$.*[6]

2. *Party i always chooses N while party $j(\neq i)$ plays the monotonic strategy with threshold x_j^*. If $c < x_j^*$, party i obtains a vote share of $(1-\alpha)\phi_i + \alpha$ while party j obtains $(1-\alpha)\phi_j$; if $c > x_j^*$, the parties obtain $(1-\alpha)\phi_i$ and $(1-\alpha)\phi_j$, respectively.*[7]

Proof. We first show that each strategy pair (σ_1, σ_2) for the parties described in the proposition consists in a PBE. Note that the strategy pair is a PBE with voters' belief system β if and only if every platform pair (a_1, a_2) that is realizable under (σ_1, σ_2) is a (Nash) equilibrium of the platform choice game G_β of Table 2.1.

Both parties always choose platform a, $a = N, Y$. If both parties always announce N, E4 imposes no constraints on β. (N, N) is an equilibrium of G_β if each belief $\beta(\cdot|\, a_1, a_2)$, $a_1 \neq a_2$, has a sufficiently high expected cost. The proof for $a = Y$ is similar.

Party i always chooses N, and party j plays the strategy with threshold x_j^*. We only consider the strategy pair for $(i, j) = (1, 2)$. If the parties play these strategies then the platform pair (N, Y) realizes when $c < x_2^*$, and the platform pair (N, N) realizes when $c > x_2^*$. Voters' belief system β satisfies E4 if and only if $\beta(\cdot|\, N, Y)$ is the conditional distribution of c given $c < x_2^*$. Take such a belief system β and consider the game G_β. By the definition (2.2) of x_2^*, we have $\pi_1^\tau(N, Y) = F(E_\beta(c|\, N, Y)) = (1-\alpha)\phi_1 + \alpha$, and hence $\pi_1^\tau(N, Y) > (1-\alpha)\phi_1 = \pi_1^\tau(Y, Y)$. Recalling that $\phi_1 + \phi_2 = 1$, we also have

$$\pi_2^\tau(N, Y) = \pi_2^\tau(N, N) = (1-\alpha)\phi_2. \tag{2.3}$$

Thus (N, Y) is an equilibrium of G_β. Finally, (N, N) is also an equilibrium of G_β if we choose $\beta(\cdot|\, Y, N)$ so that $E_\beta(c|\, Y, N) \geq Q_{(1-\alpha)\phi_2+\alpha}$.

[6]There are two such outcomes corresponding to the choices of $a \in \{Y, N\}$.
[7]There are two such outcomes corresponding to the choices of $i \in \{1, 2\}$.

Next we show that there is no other strategy pairs for the parties that consist in a PBE. Note that in order for a strategy pair of the form (σ_1, σ_2) =(always N, threshold x_2) with an interior point x_2 in C to be a PBE with β, condition (2.3) is necessary, and it implies that $x_2 = x_2^*$. Hence there is no other PBE in which the parties play strategies of this form. Similarly, there is no other equilibrium strategy pair having the same form as (threshold x_1^*, always N). It only remains to check the following three classes of strategy pairs.

Each party i plays the strategy with an interior threshold x_i. Without loss of generality consider any (x_1, x_2) with $\underline{c} < x_1 \leq x_2 < \overline{c}$. Suppose voters have a belief system β. If the parties play these strategies, they both announce Y when $c < x_1$ and they both announce N when $c > x_2$. Suppose that (Y, Y) is an equilibrium of the game G_β played when $c < x_1$. Party 1's deviation to N is unprofitable, i.e., $\pi_1^\tau(Y, Y) = (1 - \alpha)\phi_1 \geq \pi_1^\tau(N, Y)$. However, since $\alpha > 0$, this implies that $\pi_2^\tau(N, N) = (1 - \alpha)\phi_2 = (1 - \alpha)(1 - \phi_1) < 1 - (1 - \alpha)\phi_1 \leq 1 - \pi_1^\tau(N, Y) = \pi_2^\tau(N, Y)$. That is, when $c > x_2$, party 2's choice of platform N is not optimal.

Party i plays the strategy with an interior threshold x_i and party j always chooses Y. We only consider such a strategy pair for $(i, j) = (1, 2)$. The platform pair (N, Y) realizes when $c > x_1$. E4 implies that $E_\beta(c| N, Y) = E(c| c > x_1)$. From the assumption (2.1), we have $E(c| c > x_1) > E(c) > Q_{(1-\alpha)\phi_1+\alpha} > Q_{(1-\alpha)\phi_1}$. Thus, when $c < x_1$, party 1 can obtain a higher vote share by deviating to N.

Party i always chooses N while party j always chooses Y. Consider the strategy pair for $(i, j) = (1, 2)$. The platform pair (N, Y) realizes irrespective of c. E4 implies that $\beta(\cdot| N, Y)$ coincides with the prior belief p, and hence, by (2.1), we have $E_\beta(c| N, Y) = E(c) > Q_{(1-\alpha)\phi_2}$. Thus party 2 can obtain a higher vote share by deviating to N. □

2.4 Robustness to imperfect conjectures

We now discuss which of the PBE outcomes are plausible. The notion of PBE leaves a great degree of freedom in voters' beliefs after the announcement of unexpected platform pairs. Yet the optimality of a party's strategy depends critically on voters' reactions to such out-of-equilibrium platform pairs. One difficulty in interpreting the PBE concept is that the parties have no opportunity to learn the electorate's out-of-equilibrium responses, although they are assumed to conjecture these responses correctly. Motivated by this fact, we attempt to incorporate into out model a small possibility of wrong conjectures about the voting outcome due to limited chances of learning. We argue that a plausible equilibrium should not be disturbed discontinuously by such a modification of the model.

We define a *conjectural perturbation* to describe a situation where each party has a small chance of making a wrong conjecture about the electorate's aggregate behavior. Suppose that voters' strategies τ induces the "true" aggregate voting plan π^τ. For any $a_1 \neq a_2$, each party i randomly and independently forms a *conjecture* $\pi_1^i(a_1, a_2)$ about the actual vote share $\pi_1^\tau(a_1, a_2)$,[8] from past experiences of similar situations in elections. We assume that $\pi_1^i(Y, N)$ and $\pi_1^i(N, Y)$ are also independent. Given the cost c, the conjecture $\pi_1^i(a_1, a_2)$ correctly coincides with $\pi_1^\tau(a_1, a_2)$ with probability $1 - \eta_i(c, r(a_1, a_2))$, where $r(a_1, a_2)$ is the *realizability* of (a_1, a_2), i.e., the frequency with which the platform pair appears. Otherwise the conjecture is either too low, say $\pi_1^i(a_1, a_2) = 0$, or too high, say $\pi_1^i(a_1, a_2) = 1$, each with probability $\eta_i(c, r(a_1, a_2))/2$.[9] We assume that $\eta_i(c, r(a_1, a_2))$ is decreasing in $r(a_1, a_2)$ and, in particular, the probability is relatively very high when $r(a_1, a_2) = 0$.

Definition. For $\epsilon > 0$, an *(ϵ-)conjectural perturbation* is a pair $\eta = (\eta_i)_{i=1,2}$ of func-

[8] Recall that if $a_1 \neq a_2$, party 2's vote share is given by $\pi_2(a_1, a_2) = 1 - \pi_1(a_1, a_2)$.

[9] For simplicity, we will stick with the assumption that the support of $\pi_1^i(a_1, a_2)$ is $\{0, \pi_1^\tau(a_1, a_2), 1\}$. The following arguments will hold even under a weaker assumption (e.g., that the lowest and highest values of π^i are more intermediate than 0 and 1). It will be also understood that the assumption that the extreme values 0 and 1 have equal probabilities is not essential for our results.

tions $\eta_i : C \times [0,1] \to (0,\epsilon)$ with the following property:

P1: $\eta_i(c, r(a_1, a_2))$ is jointly continuous, and differentiable with respect to $r(a_1, a_2)$ with $\frac{\partial \eta_i}{\partial r(a_1,a_2)} < 0$.

P2: If $r(a_1, a_2) > \epsilon$, then $\sup_c \eta_i(c, r(a_1, a_2)) < \epsilon \inf_c \eta_i(c, 0)$ and $|\frac{\partial \eta_i(c, r(a_1,a_2))}{\partial r(a_1,a_2)}| < \epsilon$.

In any ϵ-conjectural perturbation, the probability of a wrong conjecture is positive but less than ϵ. P1 requires some well-behavedness condition and that the probability of a wrong conjecture about voters' aggregate response $\pi_i^\tau(a_1, a_2)$ to platforms (a_1, a_2) decreases with the frequency of (a_1, a_2). P2 further says that an error in conjecture on $\pi_i^\tau(a_1, a_2)$ occurs mostly when the platform pair (a_1, a_2) is extremely unlikely, and its probability is almost constant and close to zero when the platform pair appears with probability more than ϵ.

Let $R = \{r = (r(a_1, a_2))_{a_1 \neq a_2} \in [0,1]^2 : r(Y,N) + r(N,Y) \leq 1\}$ be the set of *realizability vectors*. Any $r \in R$ represents a possible probability that each pair of distinct platforms realizes.

A *response strategy* for party i is a function ρ_i that assigns to each $(c, \pi^i) \in C \times \Pi$ a platform $\rho_i(c, \pi^i) \in \{Y, N\}$. A response strategy describes a way in which the party chooses a platform given the observed cost c and the expected voting pattern π^i. As η_i describes a distribution of π^i (which depends on the true aggregate voting plan π^τ as well as (c, r)), any combination $(c, \rho_i, \eta_i, \pi^\tau, r)$ induces the probability distribution $\sigma_i(\cdot \mid c, \rho_i, \eta_i, \pi^\tau, r)$ over $\{Y, N\}$ of party i's platform conditional on c. We call the function that assigns to each c the distribution $\sigma_i(\cdot \mid c, \rho_i, \eta_i, \pi^\tau, r)$ the *mixed strategy induced by* $(\rho_i, \eta_i, \pi^\tau, r)$.

Definition. An *equilibrium under conjectural perturbation* η is a combination $(\rho_1, \rho_2, \tau, \beta, r)$ of response strategies for the parties, strategies for voters, a common belief system, and a realizability vector that satisfies E3, E4, and the following conditions:

E1': Each ρ_i is a response strategy.

E2': For each party $i(\neq j)$ and each $(c, \pi^i) \in C \times \Pi$, $a_i = \rho_i(c, \pi^i)$ maximizes $\sum_{a_j=Y,N} \sigma_j(a_j | c, \rho_j, \eta_j, \pi^\tau, r)\pi_i^i(a_i, a_j)$.

E5': For each $a_1 \neq a_2$, $r(a_1, a_2) = \int \prod_{i=1,2} \sigma_i(a_i | c, \rho_i, \eta_i, \pi^\tau, r)p(dc)$.

E1' and E2' say that given any c each party i maximizes expected vote, where the opponent's platform choice depends on its randomly formed conjecture π^j about voters' behavior. Recall that, although π^j mostly coincides with the true voting pattern π^τ induced by voters' strategies τ, the expectation is less precise for a less often realized platform pair; hence the parties' behaviors depend also on the realizability vector r. E5' requires that this r be consistent with the actual frequencies with which the parties announce each platform.

Definition. A PBE $(\sigma_1, \sigma_2, \tau, \beta)$ is called *robust* to a sequence $(\eta^n)_{n=1}^\infty$ of ϵ^n-conjectural perturbations with $\epsilon^n \to 0$ if there exists a sequence $(\rho_1^n, \rho_2^n, \tau^n, \beta^n, r^n)_{n=1}^\infty$ of equilibria under the perturbations such that for each party i, $\sigma_i(\cdot | \cdot, \rho_i^n, \eta_i^n, \pi^{\tau^n}, r^n)$ converges pointwise to the strategy σ_i and voters' strategies τ^n converge to τ. A PBE outcome is called *robust* to a perturbation sequence if some PBE that induces the outcome is robust to the sequence.

Our notion of robustness is similar in spirit to the notion of stability against mistake probabilities as formulated by Selten (1975). Indeed we may view those platform choices by a party that have only small probabilities as its mistakes. Taking the possibility of such errors into account, the opponent maximizes expected vote. However, while in contrast to Selten's formulation, the error probabilities in conjectural perturbations are explicitly related to the equilibrium strategies for the parties and voters. In particular, as we have assumed in P1, the parties are more likely to make wrong conjectures about the consequence of a platform pair if it realizes less frequently.

To state our result, we strengthen the assumption (2.1) by adding the following:

$$x_i^* > Q_{1-(1-\alpha)\phi_j} \text{ for each } i \neq j. \tag{2.4}$$

This is satisfied if, for any given α, ϕ_i is sufficiently close to $1/2$, since the the definition (2.2) already implies that $x_i^* > Q_{1-(1-\alpha)\phi_i}$.

Proposition 2.2. *For each $i \neq j$, the PBE outcome in which party i always chooses N and the other party plays the monotonic strategy with threshold x_j^* is robust to any sequence of conjectural perturbations.*

Proof. See Appendix. □

Here we sketch the proof of the proposition for $(i, j) = (1, 2)$. Note that for the strategy pair (σ_1, σ_2) = (always N, threshold x_2^*), the platform pair $(a_1, a_2) = (Y, N)$ is out-of-equilibrium. There are two key observations in the proof: (i) When the parties' mixed strategies approach (σ_1, σ_2), after observing (Y, N) voters expect that the cost c is so high that party 1's deviation to Y becomes indeed unprofitable; (ii) under any conjectural perturbation, when party 2 plays a mixed strategy close to the strategy σ_2, party 1's choice of Y is more likely roughly when $c > x_2^*$ than when $c < x_2^*$.

We can show (i) as follows: As the strategies approach $(\sigma_i)_{i=1,2}$, platforms (Y, N) appear almost only when party 1 makes an error while party 2 follows the original strategy; this occurs almost only if $c > x_2^*$. Since voters infer this, by (2.4), $E_\beta(c| Y, N) > x_2^* > Q_{1-(1-\alpha)\phi_1}$; hence $\pi_1^\tau(Y, N) < (1-\alpha)\phi_1 = \pi_1^\tau(N, N)$. Fact (ii) follows from property P2 of perturbation. As the platform pair (Y, N) almost never realizes, party 1 makes a wrong conjecture about $\pi_1^\tau(Y, N)$ much more often than about $\pi_1^\tau(N, Y)$. Consequently, when party 2 mostly announces N, i.e., when $c > x_2^*$, party 1 is more likely to choose Y based on a wrong conjecture.

Given (i), it only remains to ensure that there exist party 2's strategy near σ_2 that is best response even when it takes into account the possibility of errors by the opponent. This is done as follows. As party 2's threshold[10] x_2 decreases slightly from x_2^*, $E_\beta(c|, N, Y)$ becomes lower than $Q_{1-(1-\alpha)\phi_2}$. For such x_2, when $c < x_2$, as far as party 2's conjecture is correct, the choice of Y becomes a strictly better response to party 1, which mostly chooses N. On the other hand, it follows from (ii) that when $c > x_2$, it is relatively more likely that party 1 announces Y driven by a wrong conjecture; hence in this case, in terms of expected vote, party 2's choice of N is still a strictly better response.

Proposition 2.3. *The two PBE outcomes where both parties always announce the same fixed platform is not robust to sequences of conjectural perturbations in which either party is sufficiently more likely to make a wrong conjecture about the aggregate voting plan when c is extremely low or high.*

Proof. We show the statement for the PBE outcome with platform N. For small $\epsilon > 0$, let η be a conjectural perturbation. Suppose that there exists an equilibrium under η that is close to the situation of the PBE outcome. At the equilibrium, party 1 always chooses N with probability close 1. Then the choice of N is optimal for party 2 in the platform game only if its conjecture π^2 satisfies approximately that $\pi_2^2(N, Y) = 1 - \pi_1^2(N, Y) \leq (1 - \alpha)\phi_2$. Conversely, Party 2 chooses Y if $\pi_1^2(N, Y)$ is too low, which occurs probability $\eta_2(c, r(N, Y))/2$. By assumption the probability of party 2's choice of Y is close to 0; hence at the true π^τ it is indeed optimal for the party to choose N. It thus follows that party 2 chooses Y with probability of the same order as $\eta_2(c, r(N, Y))/2$. Thus, conditional on $(a_1, a_2) = (N, Y)$ voters' belief is approximately given by $\beta(dc|\, N, Y) \sim \eta_2(c, r(N, Y))p(dc)/\int \eta_2(c', r(N, Y))p(dc')$. Hence if $\eta_2(c, r(N, Y))$ is sufficiently higher for lower cost levels c, then $E_\beta(c|\, N, Y)$ is small enough. This contradicts the fact that for the true aggregate voting plan π^τ, a deviation to Y is unprofitable. □

[10]In the perturbed situation, we may think of a threshold as a cost level until which the party announces Y with probability close to 1.

For example, consider the PBE outcome at which both parties always announce N. Suppose that party 1 tends to overestimate the electorate's support for Y especially when the true cost c is very low. This may be the case if the party believes that when the cost is extremely low or high, voters somehow understand it. In this case, voters will infer from party 1's announcement of Y that the cost is very low; consequently party 1 deviates to Y.

It is easy to establish that all four PBE outcomes are trembling-hand perfect in the sense that for some small mistake probability for each player, there exists a nearby equilibrium. However, as Proposition 2.3 suggests, the two completely pooling PBE outcomes are fragile to those sequences in which some party's mistake probabilities are relatively high at extreme levels of cost.

2.5 Conclusion

We analyzed a model of electoral competition in which parties are better informed than voters about the cost of a public project. We showed that there are two types of PBE outcomes: in the first, both parties always choose the same fixed platform; in the second, one party always chooses not to implement the project while the other party switches the platform around a threshold of the cost. To examine which of the equilibria are plausible, we introduced the notion of conjectural perturbation. We showed that only the PBE outcomes of the second type are robust to all conjectural perturbations, while the outcomes of the first type are fragile to some class of conjectural perturbations.

We have assumed that both parties have complete information of the cost. In the next chapter, we will investigate the case where each party receives a noisy private information. Our analysis has been also limited to the case with two available policies. It is not obvious how the result in this paper extends to the case with three or more alternatives. This issue may be addressed in future research.

2.6 Appendix: Proof of Proposition 2.2

Proof. We only show the statement for $(i, j) = (1, 2)$. Let η be an ϵ-conjectural perturbation for small $\epsilon > 0$. Suppose that both parties are playing some response strategies. For convenience, we take an interior point x_2 in C, and describe the situation as if the "correct" platform choice for party 1 given any c is N, and that for party 2 is the same as the choice specified by the monotonic strategy with an interior threshold x_2 (i.e., Y for $c < x_2$ and N for $c > x_2$). The underlying response strategy for party i then induces the "wrong" platform choice with some probability $\delta_i(c, r)$ given the cost c and the realizability vector r.

We construct the error probabilities as follows, and show that these probabilities (and hence the underlying response strategies) consist in an equilibrium:

$$\delta_1(c, r) = \begin{cases} \frac{\eta_1(c, r(N, Y))}{2} & \text{if } c < x_2 \\ \frac{\eta_1(c, r(Y, N))}{2} & \text{if } c > x_2 \end{cases}$$

and

$$\delta_2(c, r) = \begin{cases} \frac{\eta_2(c, r(N, Y))}{2} & \text{if } c < x_2 \\ \frac{\eta_2(c, r(N, Y))}{2} + \frac{(1-\eta_2(c, r(N, Y)))\eta_2(c, r(Y, N))}{2} & \text{if } c > x_2. \end{cases}$$

The requirement of E5' is written as follows:

$$\begin{aligned} r(Y, N) &= \int_{c<x_2} \delta_1(c, r)\delta_2(c, r)p(dc) + \int_{c>x_2} \delta_1(c, r)(1 - \delta_2(c, r))p(dc) \\ r(N, Y) &= \int_{c<x_2} (1 - \delta_1(c, r))(1 - \delta_2(c, r))p(dc) + \int_{c>x_2} (1 - \delta_1(c, r))\delta_2(c, r)p(dc). \end{aligned} \tag{2.5}$$

Since δ_i's are continuous in r, by Brouwer's fixed point theorem, (2.5) has a solution r. Since all δ_i's are close to zero (less than $\epsilon/2$), the above equations indicate roughly that $r(Y, N) \sim 0$, $r(N, Y) \sim p\{c < x_2\} > \epsilon$. Property P2 of the perturbation then implies that $\eta_i(c,\cdot)$'s are almost constantly decreasing around $r(N,Y)$ while they are relatively sharply decreasing around $r(Y,N)$. Thus we may rewrite (2.5) approximately as follows: $r(Y,N) = \int_{c>x_2} \frac{\eta_1(c,r(Y,N))}{2} p(dc)$ and $r(N,Y) = \int_{c<x_2}(1 - \frac{\eta_1(c,r(N,Y))+\eta_2(c,r(N,Y))}{2})p(dc) + \int_{c>x_2} \frac{\eta_2(c,r(Y,N))+\eta_2(c,r(N,Y))}{2} p(dc)$. Since the right-hand side of the first equation is relatively sharply decreasing in $r(Y,N)$, we observe that it has a unique solution $r(Y,N)$ that may depend only very weakly on $r(N,Y)$. Taking this $r(Y,N)$ as given, the right-hand side of the second equation is almost constant since $\eta_i(c,r(N,Y))$ are almost irresponsive to $r(N,Y)$; hence its solution is also unique. Therefore the solution r to (2.5) is unique, for a given x_2. We write r^* for this unique solution, which implicitly depends on x_2. Since the right-hand sides in (2.5) are continuous in r and x_2, the vector r^* continuously changes with x_2.

Let

$$\overline{\pi} = (1-\alpha)\phi_1.$$

Each party makes an error when its correct platform (i.e., party 1's choice of N for all c, and party 2's choice of Y for $c < x_2$ and N for $c > x_2$) is not optimal in the platform choice game, given the opponent's error probability. That is, the optimality condition E2' for party 1 is satisfied if $\delta_1(c,r^*)$ equals the probability of

$$\begin{cases} (1-\delta_2(c,r^*))(\pi_1^1(N,Y)-\overline{\pi}) \leq \delta_2(c,r^*)(\pi_1^1(Y,N)-\overline{\pi}) & \text{if } c < x_2 \\ \delta_2(c,r^*)(\pi_1^1(N,Y)-\overline{\pi}) \leq (1-\delta_2(c,r^*))(\pi_1^1(Y,N)-\overline{\pi}) & \text{if } c > x_2; \end{cases}$$

similarly E2' for party 2 is satisfied if $\delta_2(c,r^*)$ equals the probability of

$$\begin{cases} (1-\delta_1(c,r^*))(\pi_1^2(N,Y)-\overline{\pi}-\alpha) \geq \delta_1(c,r^*)(\pi_1^2(Y,N)-\overline{\pi}-\alpha) & \text{if } c < x_2 \\ (1-\delta_1(c,r^*))(\pi_1^2(N,Y)-\overline{\pi}-\alpha) \leq \delta_1(c,r^*)(\pi_1^2(Y,N)-\overline{\pi}-\alpha) & \text{if } c > x_2, \end{cases}$$

where $\pi^i = (\pi_1^i(Y,N), \pi_1^i(N,Y))$ is party i's random conjecture about the aggregate

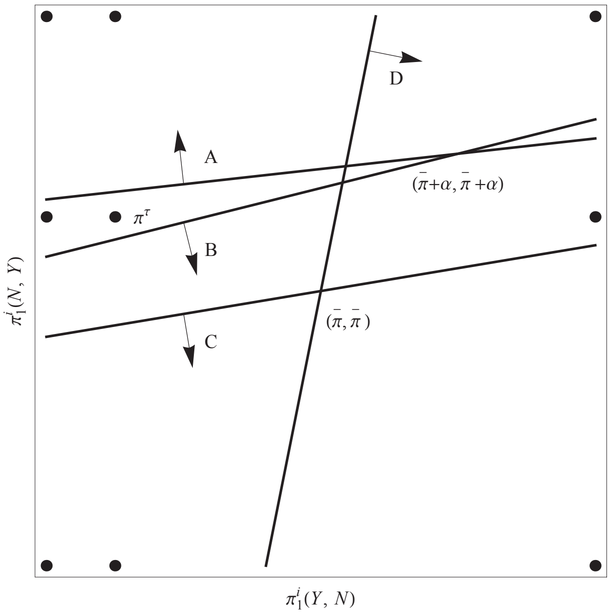

Figure 2.3: The solutions sets to the inequalities for π^i. The slopes of the lines are: (A) $\frac{\delta_1(c_L,r^*)}{1-\delta_1(c_L,r^*)} \sim \frac{\eta_1(c_L,r^*(N,Y))}{2}$; (B) $\frac{\delta_1(c_H,r^*)}{1-\delta_1(c_H,r^*)} \sim \frac{\eta_1(c_H,r^*(Y,N))}{2}$; (C) $\frac{\delta_2(c_L,r^*)}{1-\delta_2(c_L,r^*)} \sim \frac{\eta_2(c_L,r^*(N,Y))}{2}$; (D) $\frac{1-\delta_2(c_H,r^*)}{\delta_2(c_H,r^*)} \sim \frac{\eta_2(c_H,r^*(N,Y))+\eta_2(c_H,r^*(Y,N))}{2}$.

voting plan π^τ.[11] Figure 2.3 illustrates possible configurations of the solution sets to these inequalities, for two representative cost levels $c_L < x_2 < c_H$, where we have not yet determined the location of π^τ.

Note that $\sup_{c<x_2} \delta_1(c,r^*) \leq \sup_{c\in C} \eta_1(c,r^*(N,Y))/2 =: s_{NY}$ and $\inf_{c>x_2} \delta_1(c,r^*) \geq \inf_{c\in C} \eta_1(c,r^*(Y,N))/2 =: s_{YN}$. The inequality for π^2 at $c < x_2$ implies that $\pi_1^2(N,Y) - (\bar{\pi}+\alpha) \geq -\frac{\delta_1(c,r^*)}{1-\delta_1(c,r^*)}(\bar{\pi}+\alpha) \geq -\frac{s_{NY}}{1-s_{NY}}(\bar{\pi}+\alpha)$. The inequality for π^2 at $c > x_2$ implies that, if $\pi_1^2(Y,N) \leq \bar{\pi}$, then $\pi_1^2(N,Y) - (\bar{\pi}+\alpha) \leq -\frac{\delta_1(c,r^*)}{1-\delta_1(c,r^*)}\alpha \leq -\frac{s_{YN}}{1-s_{YN}}\alpha$. Recall that P2 implies that s_{NY}/s_{YN} is very close to 0. Hence we have $-\frac{s_{YN}}{1-s_{YN}}\alpha < -\frac{s_{NY}}{1-s_{NY}}(\bar{\pi}+\alpha)$. Thus, letting

$$\underline{q} := (\bar{\pi}+\alpha) - \frac{s_{YN}}{1-s_{YN}}\alpha > \bar{\pi} \text{ and } \bar{q} := (\bar{\pi}+\alpha) - \frac{s_{NY}}{1-s_{NY}}(\bar{\pi}+\alpha) < \bar{\pi}+\alpha,$$

[11]To express the condition for party 2 in terms of π_1^2 and $\bar{\pi}$, we used the fact that $\pi_1^2(a_1,a_2)+\pi_2^2(a_1,a_2) = 1$ for $a_1 \neq a_2$ and $\phi_1 + \phi_2 = 1$.

the inequality for π^2 at any c excludes the non-empty region

$$\Pi^* = (0, \overline{\pi}) \times (\underline{q}, \overline{q}).$$

In Figure 2.3, Π^* is a subregion of $(0, \overline{\pi}) \times (\overline{\pi}, \overline{\pi} + \alpha)$ that is sandwiched by lines A and B regardless of the choice of (c_L, c_H). It is easy to see also that the inequalities for π^1 (indicated by lines C and D) at any c exclude the region $(0, \overline{\pi}) \times (\overline{\pi}, \overline{\pi} + \alpha)$. Thus, none of the above inequalities for π^i, $i = 1, 2$, has a solution in Π^*. Since η_1 is continuous and r^* is continuous in x_2, by the Maximum Theorem, the boundaries $(\underline{q}, \overline{q})$ are continuous in x_2.

We now claim that for a given x_2, voters' belief system satisfies approximately that $E_\beta(c|\, N, Y) \sim E(c|\, c < x_2)$ and $E_\beta(c|\, Y, N) > x_2$, and that both expectations are continuous in x_2. First, $E_\beta(c|\, N, Y) \sim E(c|\, c < x_2)$ follows immediately from the fact that the parties announce platforms $(a_1, a_2) = (N, Y)$ with probability close to 1 when $c < x_2$, but with probability close to 0 when $c > x_2$. On the other hand, if $c < x_2$ then $(a_1, a_2) = (Y, N)$ arises only as a result of both parties' errors, while if $c > x_2$ then only party 1's mistake suffices to generate (Y, N). Comparing the orders of the likelihoods of these two cases as $\epsilon \to 0$, we obtain $E_\beta(c|\, Y, N) > x_2$. Finally, note that the expectations $E_\beta(c|\, a_1, a_2)$ depend on x_2 also via the error probabilities $\delta_i(c, r^*)$ since r^* depends on x_2. Since r^* is continuous in x_2 and δ_i are continuous, the expectations are indeed continuous in x_2.

As x_2 moves from $\underline{c}$ to $\overline{c}$, $E(c|\, c < x_2)$ continuously increases from $\underline{c}$ to $E(c) > Q_{\overline{\pi}+\alpha}$ where the inequality is due to (2.1). It thus follows that $\pi_1^\tau(N, Y) = F(E_\beta(c|\, N, Y)) \sim F(E(c|\, c < x_2))$ also continuously changes approximately from 0 to $F(E(c))$. Since $\underline{q} < \overline{q} < \overline{\pi} + \alpha$ and the boundaries $(\underline{q}, \overline{q})$ are continuous in x_2, there exists x_2 for which $\pi_1^\tau(N, Y) \in (\underline{q}, \overline{q})$. As $\epsilon \to 0$, both $\underline{q}$ and $\overline{q}$ converge to the point $\overline{\pi} + \alpha$; hence $\pi_1^\tau(N, Y) \sim F(E(c|\, c < x_2)) \to \overline{\pi} + \alpha$, and from the definition (2.2) of x_2^* we have $x_2 \to x_2^*$. By (2.4), for sufficiently small ϵ, $E_\beta(c|\, Y, N) > x_2 > Q_{1-\overline{\pi}}$. Thus we also have $\pi_1^\tau(Y,N) \sim 1 - F(E(c|\, c > x_2)) < \overline{\pi}$. Therefore, for the above x_2, we have $\pi^\tau \in \Pi^*$.

Given these x_2 and τ, the solution sets to the inequalities for π^i are exactly as in Figure 2.3 for any two cost levels $c_L < x_2 < c_H$. In the figure, the response strategies ρ_i for the parties that underlie the error probabilities δ_i are defined as follows: for any $c \in \{c_L, c_H\}$, party i chooses the platform according to the correct strategy (i.e., "always N" for party 1 and "threshold x_2" for party 2) if and only if its conjecture π^i falls *outside* the solution set of the inequality determined by δ_j, $j \neq i$, at the cost level c. By identifying those points in the support $\{0, \pi_1^\tau(Y,N), 1\} \times \{0, \pi_1^\tau(N,Y), 1\}$ of the variable π^i that lie in each solution set,[12] we confirm that for such τ, the error probabilities $(\delta_i)_{i=1,2}$ indeed satisfy the equilibrium condition E2'. □

[12]See Figure 2.3. For example, the solution set of the inequality for π^2 at c_L shares with the support of π^2 the following four points: $\{(0,0), (\pi_1^\tau(Y,N), 0), (1,0), (1, \pi_1^\tau(N,Y))\}$. The probability of $\{(0,0), (\pi_1^\tau(Y,N), 0), (1,0)\}$ is $\eta_2(c_L, r^*(N,Y))/2$ and the probability of $\{(1, \pi_1^\tau(N,Y))\}$ is $(1 - \eta_2(c_L, r^*(N,Y)))\eta_2(c_L, r^*(Y,N))/2$; the sum equals the definition of $\delta_2(c_L, r^*)$.

Chapter 3

Privately informed parties and policy divergence

3.1 Introduction

Elections often involve uncertainty about the effects of alternative policies. In such situations, parties may take advantage of expertise offered by private think tanks or government officials, whereas most voters only have publicly accessible information such as that provided by the mass media.

In Chapter 2 we constructed a model of political competition where two vote-maximizing parties (that is, "Downsian" parties) have complete information about the cost of a public project, while voters have incomplete information. In this setting we showed that there exist two perfect Bayesian equilibrium (PBE) outcomes that are robust to certain perturbations. At the equilibria the parties play different strategies: one party always announces not to implement the project while the other party switches its platform around some threshold of the cost. We contrasted this result with standard Downsian models where parties behave identically in equilibrium.

This chapter modifies the previous model by assuming that each party only observes a noisy private signal of the cost. An interpretation of the model is that parties have different sources of policy-relevant information which are unavailable to voters. For example, the Democratic and the Republican parties in the United States rely on different think tanks for policy research.

As in the previous model, there is informational asymmetry between the parties and the electorate. Hence party platforms may transmit useful information of the cost to voters. Unlike the previous model, however, there is now also informational asymmetry between the parties. A new feature of the model is therefore that each party infers from its private signal what signal the opponent has received.

We are particularly interested in whether this additional structure overturns the results in the preceding chapter. We show that in fact the PBE outcomes in the model with privately informed parties are similar to those in the previous model. That is, there are two types of PBE outcomes: in the first, both parties always announce the same fixed platform; in the second, one party always announces not to implement the project while the other switches the platform around a unique threshold of its *signal*. Thus, it is still true that although both parties are purely motivated to maximize vote, they play quite different strategies at an equilibrium. We do not attempt to refine the four PBE outcomes using a perturbation notion like the one we adopted in the previous chapter, as it seems a quite complicated task.

There are other papers that study political competition between privately informed parties. Among them, a study by Heidhues and Lagerlöf (2003) presents a model closest to ours. The only difference between their and our settings is that they assume a binary signal space, while we assume a continuous signal space. In their model, at the only pure-strategy equilibria that satisfy a condition called "symmetric voting," the parties choose the same fixed policy irrespective of their private signals. In contrast, our model possesses equilibria satisfying symmetric voting at which the parties play different pure strategies. For more detailed literature review, see Section 2.1.

The rest of this chapter is organized as follows. In Section 3.2, we construct the model of political competition with privately informed parties. In Section 3.3, we study PBEs of the model. In Section 3.4, we conclude.

3.2 Model

The model in this chapter has the same basic structure as the model in the previous chapter (see Section 2.2). The only difference is that now each party does not perfectly know the cost of the public project, and instead observes a noisy private signal. The game proceeds as follows: (i) the cost c realizes; (ii) each party i receives a private signal s_i of the cost; (iii) knowing only its own signal s_i, each party i announces a platform a_i; and (iv) after observing the announced platforms (a_1, a_2), each voter votes for a party. Except the addition of stage (ii), we maintain all the notations and the assumptions in Chapter 2.

As in the previous chapter, the cost c has an (absolutely continuous) prior distribution p with support $C = [\underline{c}, \overline{c}]$. We assume that the signals s_i have the following properties:

S1: Conditional on cost c, signals s_i, $i = 1, 2$, are independent and distributed according to densities $\psi_i(\cdot|\, c)$ with support C.

S2: ψ_i satisfies the *strict monotone likelihood ratio property (SMLRP)*, i.e., for any $s_i, s_i', c, c' \in C$, $[s_i' > s_i,\, c' > c]$ implies $[\psi_i(s_i'|\, c')/\psi_i(s_i'|\, c) > \psi_i(s_i|\, c')/\psi_i(s_i|\, c)]$.

S2 says roughly that a higher signal indicates a higher cost. (See Milgrom (1981) for implications of SMLRP.) It is stronger than the assumption that if $c < c'$ then $\psi_i(\cdot|\, c')$ first-order stochastically dominates $\psi_i(\cdot|\, c)$. Let $\Psi_i(\cdot|\, c)$ denote the distribution function of density $\psi_i(\cdot|\, c)$. For each party $i (\neq j)$, let $\psi_{ij}(s_i|\, s_j)$ be the density of signal s_i conditional on signal s_j, and $\Psi_{ij}(\cdot|\, s_j)$ the distribution function of $\psi_{ij}(\cdot|\, s_j)$. In addition to (2.1), we assume the following:

$$E(c|\, s_i = \underline{c}) < Q_{(1-\alpha)\phi_i+\alpha} \text{ for } i = 1, 2. \tag{3.1}$$

That is, the expected cost conditional on the lowest signal for any party is sufficiently low.

A *perfect Bayesian equilibrium (PBE)* is defined in the same manner as in the previous chapter, with the only exception that we now must define a strategy for a party as a function of its private signal. A *strategy* for party i is a function $\sigma_i : C \to \{Y, N\}$ that assigns to each signal $s_i \in C$ a platform $\sigma_i(s_i) \in \{Y, N\}$.[1] We still impose the monotonicity condition E1 on equilibrium strategies for the parties. Each monotonic strategy σ_i has a *threshold* $x_i \in C$ such that

$$\sigma_i(s_i) = \begin{cases} Y & \text{if } s_i < x_i \\ N & \text{if } s_i > x_i \end{cases}$$

Since by property S2 the signals are positively correlated with the cost c, we can still interpret monotonicity as the requirement that the equilibrium strategy for a party i be not only optimal in terms of vote but also compatible with the incentives of some "expert members" who provide the information s_i for the party.

3.3 Perfect Bayesian equilibrium

For each party i, define y_i^* to be the unique point in C satisfying

$$E(c|\, s_i < y_i^*) = Q_{1-(1-\alpha)\phi_i},$$

an analogue of x_i^* in the previous model. It is easy to see that, by the SMLRP of the signal, $E(c|\, s_i < x)$ is continuously increasing in x. Hence by (3.1) the point y_i^* is a unique interior point in C.

Proposition 3.1. *All PBE outcomes are as follows:*

1. *Both parties always choose the same fixed platform $a \in \{Y, N\}$ irrespective of their signals, and each party i obtains a vote share of* $(1-\alpha)\phi_i$.[2]

[1]Since the support of s_i is C, we define σ_i as a function on C.
[2]There are two such outcomes corresponding to the choices of $a \in \{Y, N\}$.

2. *Party i always chooses N while party j($\neq$ i) plays the monotonic strategy with threshold y_j^*. If $s_j < y_j^*$, party i obtains a vote share of $(1-\alpha)\phi_i + \alpha$ while party j obtains $(1-\alpha)\phi_j$; if $s_j > y_j^*$, the parties obtain $(1-\alpha)\phi_i$ and $(1-\alpha)\phi_j$, respectively.*[3]

Proof. The two outcomes of the first type are PBE outcomes. If both parties always announce the same platform, say N, then voters' beliefs given any pair of distinct platforms are arbitrary. The parties' strategies of always choosing N are therefore PBE with any belief system β in which $\beta(\cdot|\, a_1, a_2)$ has a sufficiently high expected cost for any $a_1 \neq a_2$.

The two outcomes of the second type are PBE outcomes. We only consider the strategy pair $(\sigma_1, \sigma_2) = $ (always N, threshold y_2^*). Given that party 2 switches the platform around the signal $s_2 = y_2^*$, party 1's choice of N is optimal for any s_1 if and only if for any s_1,

$$\Psi_{21}(y_2^*|\, s_1)\pi_1^\tau(N,Y) + (1-\Psi_{21}(y_2^*|\, s_1))\bar{\pi} \geq \Psi_{21}(y_2^*|\, s_1)\bar{\pi} + (1-\Psi_{21}(y_2^*|\, s_1))\pi_1^\tau(Y,N),$$

where $\bar{\pi}$ is the fraction of 1-partisans in the electorate:

$$\bar{\pi} = (1-\alpha)\phi_1.$$

The above condition is satisfied for all s_1 if

$$\pi_1^\tau(N,Y) \geq \bar{\pi} \geq \pi_1^\tau(Y,N).$$

Now, given that party 1 always chooses N, party 2's choice of Y when $s_2 < y_2^*$ is optimal if $\pi_2^\tau(N,Y) \geq (1-\alpha)\phi_2$, i.e., if $\pi_1^\tau(N,Y) \leq \bar{\pi} + \alpha$; similarly its choice of N when $s_2 > y_2^*$ is optimal if $\pi_1^\tau(N,Y) \geq \bar{\pi} + \alpha$. That is, optimality of party 2's strategy

[3]There are two such outcomes corresponding to the choices of $i \in \{1, 2\}$.

is equivalent to that $\pi_1^\tau(N,Y) = \bar{\pi} + \alpha$. Thus the strategies for the parties consist in a PBE if and only if voters' strategies τ satisfy

$$\pi_1^\tau(Y,N) \leq \bar{\pi} < \bar{\pi} + \alpha = \pi_1^\tau(N,Y).$$

Since (Y,N) is not realizable, we can choose $\beta(\cdot|\,Y,N)$ to have a sufficiently high expected cost so that the first inequality is satisfied. The last equation $\pi_1^\tau(N,Y) = \bar{\pi}+\alpha$ holds if and only if $E_\beta(c|\,N,Y) = E(c|\,s_2 > y_2^*) = Q_{\bar{\pi}+\alpha} = Q_{1-(1-\alpha)\phi_2}$, which is satisfied by the definition of y_2^*.

Now we show that *no strategy pair in which each party i plays a strategy with an interior threshold x_i consists in a PBE.* The proofs for other remaining strategy pairs are similar as in the proof of Proposition 2.1 in the previous model, and we thus omit them. Given that party 2 switches the platform around $s_2 = x_2$, party 1's choice of Y when $s_1 < x_1$ is optimal if and only if for any $s_1 < x_1$,

$$\Psi_{21}(x_2|\,s_1)\bar{\pi} + (1 - \Psi_{21}(x_2|\,s_1))\pi_1^\tau(Y,N) \geq \Psi_{21}(x_2|\,s_1)\pi_1^\tau(N,Y) + (1 - \Psi_{21}(x_2|\,s_1))\bar{\pi},$$

which is simplified to that

$$(1 - \Psi_{21}(x_2|\,s_1))(\pi_1^\tau(Y,N) - \bar{\pi}) \geq \Psi_{21}(x_2|\,s_1)(\pi_1^\tau(N,Y) - \bar{\pi}) \text{ for all } s_1 < x_1.$$

Similarly, party 1's choice of N when $s_1 > x_1$ is optimal if and only if for any $s_1 > x_1$,

$$(1 - \Psi_{21}(x_2|\,s_1))(\pi_1^\tau(Y,N) - \bar{\pi}) \leq \Psi_{21}(x_2|\,s_1)(\pi_1^\tau(N,Y) - \bar{\pi}) \text{ for all } s_1 > x_1.$$

By continuity of distributions, the two conditions imply that π^τ locates on the line L^* defined by

$$(1 - \Psi_{21}(x_2|\,x_1))(\pi_1^\tau(Y,N) - \bar{\pi}) = \Psi_{21}(x_2|\,x_1)(\pi_1^\tau(N,Y) - \bar{\pi}).$$

Since the signals s_i are positively correlated, $\Psi_{21}(x_2|\, s_1)$ is decreasing in s_1. Thus the above inequality at $s_1 < x_1$ is satisfied only by those $\pi^\tau \in L^*$ with

$$\pi_1^\tau(N,Y) \leq \overline{\pi}.$$

By similar considerations, optimality of party 2's strategy is expressed as follows:

$$\Psi_{12}(x_1|\, s_2)(\pi_1^\tau(Y,N) - \overline{\pi} - \alpha) \geq (1 - \Psi_{12}(x_1|\, s_2))(\pi_1^\tau(N,Y) - \overline{\pi} - \alpha) \text{ for all } s_2 < x_2$$

$$\Psi_{12}(x_1|\, s_2)(\pi_1^\tau(Y,N) - \overline{\pi} - \alpha) \leq (1 - \Psi_{12}(x_1|\, s_2))(\pi_1^\tau(N,Y) - \overline{\pi} - \alpha) \text{ for all } s_2 > x_2.$$

This implies that π^τ locates on the line L^{**} defined by

$$\Psi_{12}(x_1|\, x_2)(\pi_1^\tau(Y,N) - \overline{\pi} - \alpha) = (1 - \Psi_{12}(x_1|\, s_2))(\pi_1^\tau(N,Y) - \overline{\pi} - \alpha).$$

Since $\Psi_{12}(x_1|\, s_2)$ is decreasing in s_2, the above inequality at $s_2 < x_2$ is satisfied only by those $\pi^\tau \in L^{**}$ with

$$\pi_1^\tau(N,Y) \geq \overline{\pi} + \alpha.$$

This is incompatible with the optimality condition for party 1. □

3.4 Conclusion

In this chapter, we extended the model of the previous chapter to the case where the parties have noisy private information of the state. Despite this difference, we showed that the extended model has four PBE outcomes similar to those in the previous model. In particular, a PBE exists at which the parties play quite different strategies. An important next step in this research would be to refine the PBEs by some analog of conjectural perturbation which we introduced in chapter 2.

Chapter 4

Multi-dimensional political competition with non-common beliefs

4.1 Introduction

Studies of electoral competition often assume that parties' (or candidates') beliefs about the electorate's preference distribution coincide. In standard "Downsian models" (Downs, 1957), two parties know the preference distribution. In "probabilistic voting models" where parties are uncertain about the preference distribution, the parties often hold the same probabilistic belief. One interesting question would be whether results obtained for these models change drastically as the parties' beliefs slightly diverge.

In this chapter, we explore the above question, focusing on a particular probabilistic voting model. In the model, two parties have to choose their platforms in an election. Each voter has an ideal policy, and votes for a party whose platform is closer to the ideal policy. Each party has a probabilistic belief about the profile of ideal policies, and seeks to maximize expected vote share. The parties' beliefs are common knowledge.

When the policy space is one-dimensional and the parties' beliefs are identical, our model is a special case of Duggan's (2006) model. In this case, Duggan shows that there exists a unique equilibrium; at the equilibrium, both parties locate at the median of the average of marginal distributions of voters' ideal policies.[1]

When the policy space has multiple dimensions and the parties' beliefs are identical, it is known that a (pure strategy) equilibrium rarely exists.[2] More precisely, in the general case with d dimensions, due to our specification of the shapes of voters' preferences, the above characterization for the one-dimensional case simply extends as follows: if an equilibrium exists, it is unique; at the equilibrium, both parties locate at the "median (in all directions)" of the average distribution of ideal policies (i.e., the policy such that every hyperplane passing through it divides the policy space into half-spaces with equal masses).[3] For $d \geq 2$, the existence of a median is rare, and so is the existence of an equilibrium.

We investigate whether the above characterization of equilibria of the model is robust to slight differences in beliefs. In particular, does the restrictiveness of the existence of equilibria in multi-dimensional policy spaces depend heavily on the assumption that the parties have the same belief? Of course, without a common belief, it is possible that each party simultaneously expects that given the opponent's platform, its platform will achieve a vote share above $1/2$. One might speculate, furthermore, that it is possible that each party simultaneously expects a vote share very close to the maximum possible given the opponent's platform, even when the model is very close to the one with a common belief where no equilibrium exists.

After describing the model formally, in Section 4.4 we first check that in the extended model where the parties may have different beliefs, the characterization of equilibria becomes more severe. The non-existence stems, at least in part, from the fact that the parties' payoff functions are discontinuous and, as a result, their best response correspondences are generically empty. In such situations, it is plausible that each party makes do with a platform that will achieve a sufficiently high expected vote share.

[1]Duggan (2006) proves that this is a unique mixed strategy equilibrium.

[2]Duggan's (2007) existence result for a voting model with a continuum of voters implies that in our model, when the parties have a common belief, a mixed strategy equilibrium exists. In this paper we focus on pure strategies of the parties.

[3]See Plott (1967) and Calvert (1985) for related results.

This observation leads us to study “ϵ-equilibria,” where an ϵ-equilibrium is defined to be a platform pair for which each party’s expected vote share lies in the ϵ-range below the maximum that the party can attain given the opponent’s platform.[4] Indeed, as we see in Section 4.5, even when the model has no exact equilibrium, it may have an ϵ-equilibrium for any $\epsilon > 0$.

However, in Section 4.6, we demonstrate that as the beliefs of the parties converge to a common belief, all “limit equilibria,” defined to be limits of ϵ-equilibria as $\epsilon \to 0$, must converge to the median of average distributions for the common belief. As a corollary, we show that if a model with a common belief has no equilibrium, then for some $\epsilon > 0$ and for any pair of the parties’ beliefs that are sufficiently close to the common belief, no ϵ-equilibrium exists.

4.2 Related literature

There are works that address the issue of robustness of deterministic voting models to slight uncertainty about voting behavior. Banks and Duggan (2006) provide several robustness results for equilibria in probabilistic voting models and the deterministic Downsian model, both with plurality-maximizing candidates. In particular, they show that if the Downsian model has no equilibrium, then introduction of slight uncertainty does not create an equilibrium.[5] Duggan and Fey (2005) consider a model with policy-motivated candidates. They show that when the policy space has more than two dimensions, the model has an equilibrium only under some restrictive condition. Furthermore, if the model has no equilibrium, the non-existence persists under small amounts of probabilistic voting and of office motivation. In both papers, the candidates are assumed to have the same belief about voters’ preferences.

Bernhardt, Duggan, and Squintani (2007) construct a one-dimensional model in

[4] I thank Yasuhiro Shirata for suggesting me to study ϵ-equilibria.

[5] Banks and Duggan (2006) study a probabilistic voting model that is different from the one studied in this paper. Moreover, they also provide a similar robustness result that allow for mixed strategy equilibria.

which each candidate observes a private signal of the median ideal policy, and maximizes the probability of winning the election. They show that while a pure strategy Bayesian equilibrium may not exist, a mixed strategy Bayesian equilibrium exists generally. They also demonstrate that equilibria of two symmetric-information versions of the model, namely, the Downsian model and the probabilistic voting model, are robust to a small amount of private information. Informational asymmetry and mixed strategies, which feature their analysis, are not considered in our work.

Bade (2011) presents a multi-dimensional probabilistic voting model where parties are uncertainty averse. In the model, each party has multiple beliefs about the electorate's preference distribution.[6] Given the opponent's platform, each party chooses a platform that maximizes the minimum possible expected vote share, where the minimum is taken over the party's set of beliefs. She shows that under some conditions on the parties' belief sets, a pure strategy equilibrium exists. The existence results are valid even if the parties have different sets of beliefs. Thus, the assumption of uncertainty averse parties provides a possible way out of the non-existence of equilibria in multi-dimensional policy spaces.

4.3 Model

We consider a model of electoral competition between two parties (parties 1 and 2). A finite number (m) of voters participate in voting. Without knowing the exact preferences of voters, the parties simultaneously choose their electoral platforms. Comparing the two platforms, each voter decides his vote.

The model is a triple (X, P, Q).

X is a compact convex subset of $\mathbb{R}^d$ with non-empty interior, called the *policy space*.

[6]Bade (2011) considers a richer class of beliefs than considered in our model. A belief in her model is defined over preference profiles of voters in which a voter's preferences are characterized not only by the voter's ideal policy but also by the shapes of his indifference curves.

Each voter i votes according to the *Euclidean preference* with an *ideal policy* $v_i = (v_{i1}, \ldots, v_{id}) \in X$: if the two parties' platforms are x and y with $\|x - v_i\| < \|y - v_i\|$, voter i votes for the party with platform x.

P and Q are two prior joint distributions of the ideal policies $(v_1, \ldots, v_m)$ defined on $\mathbb{R}^{dm}$, called the *beliefs* of parties 1 and 2, respectively, with the following properties:

(a) P and Q are absolutely continuous relative to Lebesgue measure on $\mathbb{R}^{dm}$.

(b) For each i, the supports of P_i and Q_i, the marginal distributions of voter i's ideal policies, are X.

Each party seeks to maximize expected vote share. For each pair (x, y) of platforms of parties 1 and 2, let $\pi_P(x, y)$ denote party 1's expected vote share, and let $\pi_Q(y, x)$ denote party 2's expected vote share. We assume that

$$\pi_P(x, y) = \pi_Q(y, x) = 1/2 \text{ whenever } x = y. \tag{4.1}$$

The *political game* is defined to be the game played by the two parties with a common strategy space X and payoff functions π_P and π_Q.

4.4 Definitions of equilibrium and median

A *Nash equilibrium* of the political game is defined as a pure strategy Nash equilibrium in the usual sense.

Given $\epsilon > 0$, a party 1's platform x is called an *ϵ-best response* to party 2's platform y if $\pi_P(x, y) \geq \sup_{z \in X} \pi_P(z, y) - \epsilon$. Party 2's ϵ-best responses are similarly defined. A platform pair (x, y) is called an *ϵ-equilibrium* if the parties' platforms are best responses to each other.

A platform pair (x, y) is called a *limit equilibrium* if there exists a sequence (x^n, y^n, ϵ^n) in $X \times X \times (0, 1)$ such that (x^n, y^n) is an ϵ^n-equilibrium for every n, $(x^n, y^n) \to (x, y)$, and $\epsilon^n \downarrow 0$.

A limit equilibrium (x, y) is said to have a *direction* $s \in S$, where $S = \{t \in \mathbb{R}^d : \|t\| = 1\}$, if there exists a sequence (x^n, y^n, ϵ^n) for which, in addition to the above properties, the following condition holds:

$$x^n \neq y^n \text{ for every } n \text{ and } \|x^n - y^n\|^{-1}(x^n - y^n) \to s.$$

For $z \in \mathbb{R}^n$ and $s \in \mathbb{R}^n \setminus \{0\}$, let $H_{z,s}$ denote the closed half-space

$$H_{z,s} = \{x \in \mathbb{R}^d \mid s \cdot x \geq s \cdot z\}.$$

For an absolutely continuous distribution Λ on $\mathbb{R}^d$ with convex support, a point z is called a *median in all directions* of Λ if

$$\Lambda(H_{z,s}) = 1/2 \text{ for every } s \in S. \tag{4.2}$$

Such a distribution Λ has at most one median in all directions.

4.5 Basic properties of expected vote share functions

In the following lemma, we collect key properties of expected vote share function π_P. (Of course, π_Q has the same properties.)

Given a belief P, let $\overline{P}$ denote the average of the marginal distributions $P_1, \ldots, P_m$:

$$\overline{P} = \frac{1}{m}(P_1 + \ldots + P_m).$$

Under our assumptions (a) and (b) on the belief, $\overline{P}$ is an absolutely continuous distribution on $\mathbb{R}^d$ with support X.

Lemma 4.1. *The expected vote share function π_P has the following properties:*

(i) If $x \neq y$*, then* $\pi_P(x, y) = \overline{P}(H_{(x+y)/2, x-y})$*.*

(ii) π_P *is continuous on* $\{(x, y) : x \neq y\}$*.*

(iii) For any $s \in S$ *and* $z \in X$ *such that* $z + \alpha s \in X$ *for some* $\alpha > 0$*, as* $\alpha \downarrow 0$*,* $\pi_P(z + \alpha s, z)$ *strictly increases and converges to* $\overline{P}(H_{z,s})$*.*

(iv) For any $y \in X$*,* $\sup_{x \in X} \pi_P(x, y) = \max_{s \in S} \overline{P}(H_{y,s})$*.*

(v) $\sup_{x \in X} \pi_P(x, y)$ *is a continuous function of* y*.*

Proof. (i) Voter i strictly prefers x to y if and only if $v_i \in H_{(x+y)/2, x-y}$. He is indifferent with probability 0 by the absolute continuity of P. Hence voter i votes for candidate 1 with probability $P_i(H_{(x+y)/2, x-y})$. Thus candidate 1's expected vote share is $E_P(\sharp\{i : v_i \in H_{(x+y)/2, x-y}\}/m) = \overline{P}(H_{(x+y)/2, x-y})$.

(ii) This follows directly from the expression (i) and the absolute continuity of P (and hence of $\overline{P}$).

(iii) Since $\overline{P}$ has convex support, with the expression (i), $\pi_P(z+\alpha s, z) = \overline{P}(H_{z+(\alpha/2)s, s})$ strictly increases as $\alpha \downarrow 0$. The limit is $\overline{P}(H_{z,s})$ by absolute continuity.

(iv) The expression $\sup_{x \in X} \pi_P(x, y) = \max_{s \in S} \overline{P}(H_{y,s})$ follows from (iii). (See Caplin and Nalebuff (1988) for a more detailed proof.)

(v) By absolute continuity, $\overline{P}(H_{y,s})$ is continuous in (y, s), where s is a variable in the compact set S. The Maximum Theorem shows that $\max_{s \in S} \overline{P}(H_{y,s})$ is continuous in y. □

Part (i) says that for $x \neq y$, party 1's expected vote share is the expected fraction of ideal points in the halfspace comprising those points that are closer to x than to y. Part (iii) says that locating closer to the opponent's platform strictly increases the party's expected vote share. Thus, the maximum possible expected vote share is achieved by choosing an appropriate direction from which the party approaches the opponent's location (part (iv)).

4.6 Nash equilibrium

We begin with the following observation about the best responses for the parties.

Lemma 4.2. *Party 1's platform x is a best response to party 2's platform y if and only if $x = y$ and y is a median in all directions of $\overline{P}$.*

Proof. "If." By assumption (4.1), $\pi_P(y, y) = 1/2$. By Lemma 4.1 (iv) and the definition of a median in all direction, $1/2$ is candidate 1's maximum payoff against y.

"Only if." First suppose $x \neq y$. By Lemma 4.1 (iii), $\pi_P(y + \alpha(x - y), y) > \pi_P(x, y)$ for small $\alpha > 0$, and thus x is not a best response to y. Suppose now that $x = y$ but y is not a median in all directions of $\overline{P}$. Then for some direction s, $\overline{P}(H_{y,s}) > 1/2$. By continuity (Lemma 4.1 (ii)), for small $\alpha > 0$, $\pi_P(x + \alpha s, y) > 1/2 = \pi_P(x, y)$. Thus $x = y$ is not a best response to y. □

The following theorem is a direct consequence of the above lemma.

Theorem 4.1. (x, y) *is a Nash equilibrium if and only if $x = y =: z$ and z is a median in all directions of both $\overline{P}$ and $\overline{Q}$.*[7]

Thus, the existence of a Nash equilibrium is rare. It requires not only that both parties' beliefs have a median in all directions but also that these two medians coincide. In particular, even when the policy space is unidimensional, no Nash equilibrium exists if the two beliefs have different medians.

4.7 ϵ-Equilibria and limit equilibira

We now turn to the analysis of ϵ- and limit equilibria. We first provide basic facts about limit equilibria and ϵ-equilibrium sequences converging to them.

[7]This is similar to Duggan's (2006) characterization of equilibria in a one-dimensional model where voters' preferences are not restricted to Euclidean preferences. Clearly, our characterization depends on the assumption of Euclidean preferences.

Lemma 4.3.

(i) If (x, y) is a limit equilibrium, then $x = y$.

(ii) Let (x, y) be a limit equilibrium that is not a Nash equilibrium. Let (x^n, y^n, ϵ^n) be a sequence as in the definition of limit equilibrium. Then $x^n \neq y^n$ eventually.

(iii) If (x, y) is a Nash equilibrium, then it is a limit equilibrium with any direction.

Proof. (i) Suppose $x \neq y$. Let (x^n, y^n, ϵ^n) be a sequence as in the definition of limit equilibrium. Given (x^n, y^n), by moving straight toward y^n, party 1 can increase his payoff by approximately $\overline{P}(H_{y^n, x^n - y^n}) - \overline{P}(H_{(x^n+y^n)/2, x^n - y^n})$, which, by absolute continuity, converges to $\overline{P}(H_{y, x-y}) - \overline{P}(H_{(x+y)/2, x-y}) > 0$. Thus (x^n, y^n) is not an ϵ^n-equilibrium for all large n. Hence (x, y) is not a limit equilibrium.

(ii) By (i), $x = y =: z$. By Proposition 4.1, z is not a median in all directions of $\overline{P}$. Thus for some s, $\overline{P}(H_{z,s}) > 1/2$. Suppose $x^{n_i} = y^{n_i}$ for some subsequence. Given (x^{n_i}, y^{n_i}), by deviating slightly in the direction s, party 1 can increase his payoff by approximately $\overline{P}(H_{y^{n_i}, s}) - 1/2$, which converges to $\overline{P}(H_{z,s}) - 1/2 > 0$. Thus (x^{n_i}, y^{n_i}) is not an ϵ^{n_i}-equilibrium for large i, a contradiction.

(iii) By Proposition 4.1, $x = y =: z$ and z is a median in all directions of both $\overline{P}$ and $\overline{Q}$. Clearly, z is an interior point of X. Fix any $s \in S^{d-1}$. Consider a platform pair of the form $(z + \alpha s, z - \alpha s)$ with $\alpha > 0$. Given $(z + \alpha s, z - \alpha s)$, party 1's gain from deviation is bounded by $\max_{t \in S^{d-1}} \overline{P}(H_{z - \alpha s, t}) - \overline{P}(H_{z,s}) = \max_{t \in S^{d-1}} \overline{P}(H_{z - \alpha s, t}) - 1/2$, which, by Lemma 4.1 (v), converges to 0 as $\alpha \to 0$. Thus there is a sequence of ϵ-equilibria of the form $(z + \alpha s, z - \alpha s)$ converging to (z, z). □

The following theorem provides a full characterization of limit equilibria.

Theorem 4.2. *A platform pair (x, y) is a limit equilibrium with direction s if and only if $x = y =: z$ and*

$$\overline{P}(H_{z,s}) = \max_{t \in S^{d-1}} \overline{P}(H_{z,t}) \text{ and } \overline{Q}(H_{z,s}) = \min_{t \in S^{d-1}} \overline{Q}(H_{z,t}).$$

Proof. "If." Fix $\epsilon > 0$. As in the proof of Lemma 4.3 (iii), consider a platform pair of the form $(z+\alpha s, z-\alpha s)$. Party 1's maximum gain from deviation is $\overline{P}(H_{z-\alpha s,s})-\overline{P}(H_{z,s})$, which converges to 0 as $\alpha \to 0$. Thus there is a sequence of ϵ-equilibria of this form converging to (z,z).

"Only if." By Lemma 4.3 (i), we know that $x = y =: z$. Let (x^n, y^n) be a sequence of ϵ^n-equilibria with $(x^n, y^n) \to (z,z)$ and $\epsilon^n \downarrow 0$. By Lemma 4.3 (ii) and (iii), we only have to consider the case where $x^n \neq y^n$ eventually. Since x^n is an ϵ^n-best response to y^n,

$$\max_{s\in S^{d-1}} \overline{P}(H_{y^n,s}) - \overline{P}(H_{(x^n+y^n)/2,\|x^n-y^n\|^{-1}(x^n-y^n)}) \leq \epsilon^n.$$

Suppose $\|x^n - y^n\|^{-1}(x^n - y^n) \to s$ for some $s \in S^{d-1}$, and let $n \to \infty$ on both sides of the inequality. By the continuity of $\overline{P}(H_{a,t})$ in (a,t) and Lemma 4.1 (iv), we have $\max_{t\in S^{d-1}} \overline{P}(H_{z,t}) = \overline{P}(H_{z,s})$. In the same way, we can show that $\min_{t\in S^{d-1}} \overline{Q}(H_{z,t}) = \overline{Q}(H_{z,s})$. □

Intuitively, at any ϵ-equilibrium for small $\epsilon > 0$ each party locates very close to the opponent. Thus, at any limit equilibrium the parties choose the same policy z. If z is a median in all directions, the condition in the theorem holds trivially. Otherwise, when ϵ is close to zero, both parties almost best respond to z by adopting slightly different platforms. This is possible if and only if the condition of the theorem holds.

In contrast with the characterization of Nash equilibrium, the condition in Theorem 4.2 for limit equilibrium does not even mention a median in all directions. Yet, it follows that if some party's belief has a median in all direction, then it is a limit equilibrium that both parties locate at this median. Moreover, in the case of a unidimensional policy space, a limit equilibrium exists generally.

Corollary 4.1. *(i) If z is a median in all directions of either $\overline{P}$ or $\overline{Q}$, then (z,z) is a limit equilibrium.*

(ii) If the policy space is unidimensional, then (z,z) is a limit equilibrium if and only if z lies between the medians of $\overline{P}$ and $\overline{Q}$.

Proof. (i) Suppose $\overline{P}$ has a median in all directions z. Let $s \in S$ be any direction that minimizes $\overline{Q}(H_{z,s})$. Then s also maximizes $\overline{P}(H_{z,s})$, and Theorem 4.2 establishes that (z,z) is a limit equilibrium.

(ii) If the medians of $\overline{P}$ and $\overline{Q}$ satisfy $\mu_{\overline{P}} \geq \mu_{\overline{Q}}$, then for any $z \in [\mu_{\overline{Q}}, \mu_{\overline{P}}]$, $s = 1$ maximizes $\overline{P}(H_{z,s})$ and minimizes $\overline{Q}(H_{z,s})$, and Theorem 4.2 establishes that (z,z) is a limit equilibrium. □

4.8 Robustness of non-existence of equilibrium

Theorem 4.2 fully characterizes the set of limit equilibria. But it is still unclear whether this set is generally non-empty or not. In this section, we study this question, focusing on the case where the parties' beliefs are very close.

The following is a merely technical lemma.

Lemma 4.4. *Let (Λ^n) be a sequence of distributions on $\mathbb{R}^d$ converging weakly to an absolutely continuous distribution Λ. Let (z^n, s^n) be a sequence in $\mathbb{R}^d \times S^{d-1}$ converging to (z,s). Then $\Lambda^n(H_{z^n,s^n}) \to \Lambda(H_{z,s})$.*

Proof. See Appendix. □

We now prove that if the parties have a common belief with no median in all directions, then for some $\epsilon > 0$, any slight divergence of the parties' beliefs around the initial common belief does not generate an ϵ-equilibrium.

Theorem 4.3. *Let F be a belief for which $\overline{F}$ does not have a median in all directions. Then, there exists $\epsilon > 0$ such that for any sequence (P^n, Q^n) of belief pairs in which P^n and Q^n converge weakly to F, for some N and for all $n > N$, the political game with beliefs (P^n, Q^n) has no ϵ-equilibrium.*

Proof. Suppose the contrary. Then for every natural number k, there exists a sequence (P_k^n, Q_k^n) of belief pairs such that P_k^n and Q_k^n converge weakly to F as $n \to \infty$, and a k^{-1}-equilibrium (x_k^n, y_k^n) exists for the belief pair (P_k^n, Q_k^n) for all n. Obviously we may assume that $x_k^n \neq y_k^n$ for all n. Since X and S are compact, for every k, without loss of generality we assume that $(x_k^n, y_k^n) \to (x_k, y_k)$ for some (x_k, y_k) and $s_k^n := (x_k^n - y_k^n)/\|x_k^n - y_k^n\| \to s_k$ for some $s_k \in S$.

Now, consider the k^{-1}-best response conditions for the strategy pair (x_k^n, y_k^n) in the game with beliefs (P_k^n, Q_k^n):

$$\overline{P}_k^n(H_{(x_k^n+y_k^n)/2, s_k^n}) \geq \max_{t \in S} \overline{P}_k^n(H_{y_k^n, t}) - k^{-1}$$
$$\overline{Q}_k^n(H_{(x_k^n+y_k^n)/2, -s_k^n}) \geq \max_{t \in S} \overline{Q}_k^n(H_{x_k^n, t}) - k^{-1}.$$

By Lemma 4.4 and Maximum Theorem, as $n \to \infty$ we have

$$\overline{F}(H_{(x_k+y_k)/2, s_k}) \geq \max_{t \in S} \overline{F}(H_{y_k, t}) - k^{-1}$$
$$\overline{F}(H_{(x_k+y_k)/2, -s_k}) \geq \max_{t \in S} \overline{F}(H_{x_k, t}) - k^{-1}.$$

Take a subsequence (x_{k_i}, y_{k_i}) of (x_k, y_k) such that $(x_{k_i}, y_{k_i}) \to (x, y)$ for some (x, y) and $s_{k_i} \to s$ for some $s \in S$. As $i \to \infty$ in the above inequalities, we have

$$\overline{F}(H_{(x+y)/2, s}) \geq \max_{t \in S} \overline{F}(H_{y, t}) \text{ and } \overline{F}(H_{(x+y)/2, -s}) \geq \max_{t \in S} \overline{F}(H_{x, t}).$$

By Lemma 4.1 (iii), this means that $x = y$, which in turn implies that x $(= y)$ is a median in all directions of $\overline{F}$. □

Intuitively, if for any $\epsilon > 0$ there is a sequence of belief pairs converging to (F, F) along which an ϵ-equilibrium $(x_\epsilon^n, y_\epsilon^n)$ exists, then we can find a convergent subsequence (with respect to n and then to ϵ). The limit (x, y) turns out to be a median in all directions of $\overline{F}$.

4.9 Conclusion

We have constructed a multidimensional model of political competition where parties may have different probabilistic beliefs about voters' ideal policies. Our primary interest has been in whether the well-known nonexistence of equilibrium is a consequence of the special assumption that the parties have completely the same belief. We have shown that at least when the parties have sufficiently similar beliefs, the non-existence is still a typical case. This is true even if we allow for approximate equilibria at which the parties imperfectly best respond to each other. An essential assumption in our model is that each party maximizes the expected vote share. We do not know whether the conclusion of this chapter would be valid even if we instead assumed that the parties care other variables such as the winning probabilities. This question may be addressed in future research.

4.10 Appendix: Proof of Lemma 4.1

Write $E^n = H_{z^n,s^n}$ and $E = H_{z,s}$. Let $F^N = \bigcap_{n \geq N} E^n$. we have for all $n \geq N$, $\Lambda^n(E^n) \geq \Lambda^n(F^N)$. Since $\Lambda^n \to \Lambda$ weakly,

$$\liminf_n \Lambda^n(E^n) \geq \Lambda(F^N).$$

Since $F^N \uparrow \liminf_n E^n$ as $N \to \infty$,

$$\liminf_n \Lambda^n(E^n) \geq \lim_N \Lambda(F^N) = \Lambda(\liminf_n E^n).$$

Take any $x \in E^\circ := \{y : s \cdot y > s \cdot z\}$. Then clearly $x \in E^n = \{y : s^n \cdot y \geq s^n \cdot z^n\}$ eventually. Thus $E^\circ \subset \liminf_n E^n$. By absolute continuity, $\Lambda(E^\circ) = \Lambda(E)$. Hence $\liminf_n \Lambda^n(E^n) \geq \Lambda(E)$. Similarly, we can prove that $\limsup_n \Lambda^n(E^n) \leq \Lambda(E)$ by noting that $\limsup_n E^n \subset E$. $\square$

Chapter 5

On the realizability of social preferences in three-party legislatures

5.1 Introduction

A widely held view about parliamentary systems is that while a single majority party makes decisions more quickly, a multi-party system achieves more precise representation of society. But the scope of such an advantage that a multi-party system has is rarely discussed.

In this chapter we limit the notion of representation of society to the sense that a legislature realizes a given social preference relation over alternative policies. A social preference relation is possibly intransitive. For example, it may represent cyclic social majority preferences as illustrated by the well-known "Paradox of Voting." Thus a legislature that perfectly represents the society may induce cyclic decisions when it deals with multiple agenda.

We focus on legislatures with at most three parties where policy is chosen through majority voting. Each party is assumed to have transitive preferences. We study the range of social preferences such a legislature can realize. By limiting attention to three-party systems, we try to understand the minimal effect arising from the absence of a single-party majority. As we will see below, it has been shown that without any constraint on the number of parties, representation of society is always possible. But it seems unrealistic to expect that arbitrarily many parties may form.[1]

We provide a sufficient condition for a social preference relation to be compatible with some three-party system. The condition says that the social preference relation has no "closed path of 3-cycles." This reveals a region of social preference relations that is induced by some three-party legislature, but cannot be induced by any legislature with a single-party majority.

The topic of this chapter is closely related to the well-known Paradox of Voting. The Paradox states that three or more voters may induce cyclic majority preferences over a set of alternatives even when each voter has transitive preferences. Thus the social majority preference relation may be intransitive.[2] An intransitive social preference relation cannot be realized if the legislature consists of at most two parties with transitive preferences (except under a special tie-breaking rule). Three is therefore the minimum number of parties that can realize an intransitive social preference relation.

McGarvey (1953) demonstrates that the Paradox of Voting can be formulated in a more general way. He shows that any preference relation over n alternatives is generated by some set of voters with transitive preferences. Stearns (1959) provides a lower bound for the minimum number of voters necessary to induce all preference relations over n alternatives. The lower bound is of the form $c_1 n/\log n$, with some constant c_1. Erdös and Moser (1964) provide a sharp upper bound of the form $c_2 n/\log n$.

These results have two implications for our study. On the one hand, if we wish to count all possible social majority preference relations as potential social preference relations (and if the population of society is large enough relative to the number of alternatives), we should admit all preference relations. This is why we put no a priori

[1]Lijphart (1994) reports the effective numbers of parties for 70 electoral systems in 27 countries between 1945 and 1990. The average effective number of parliamentary parties over these systems is 3.4.

[2]DeMeyer and Plott (1970), Gehrlein and Fishburn (1975), Jones, Radcliff, Taber, and Timpone (1995), and Tangian (2000) compute the probability that the majority voting by a random set of voters induces a cycle.

restriction on the social preference relation. On the other hand, when the number n of policies is sufficiently large, there exists a preference relation that cannot be induced by any set of three parties.

A recent paper by Brandt, Harrenstein, Kardel, and Seedig (2013) provides a full characterization of preference relations that are realized by some three parties (voters, in their framework). Based on Dushnik and Miller's (1941) characterization of partial orders having dimension ≤ 2, they show that a preference relation is compatible with some three parties if and only if it is partitioned ("arc-disjointly") into two sub-relations such that one is a partial order and the other is an acyclic relation which is a reorientation of some partial order. Although our result is only a sufficient condition, it may be useful as it is easy to check.

Alon, Brightwell, Kierstead, Kostochka, and Winkler (2006) provide a necessary condition for a preference relation to be compatible with some three-party system.[3] A "dominating set" is defined as a subset A of policies such that for each policy x, some policy $a \in A$ is majority-preferred to x. Alon et al. show that every three-party majority preference relation over policies has a dominating set containing at most three policies.

After describing the model (Section 5.2), we characterize the relation between a social preference relation and a three-party system that realizes it (Sections 5.3 and 5.4). We then show a sufficient condition for a social preference relation to be realized by some three-party system (Section 5.5). The proof of the sufficient condition provides a procedure to construct a three-party system from a given social preference relation.

5.2 Model

Consider a society in which policy is chosen through majority voting in a legislature. The society has a collective preference relation over the set of policies. It seeks to send a set of representatives to the legislature in such a way that they will realize policy

[3] Their original result is written in different (graph-theoretic) terminology.

choices in accordance with the social preference relation. Suppose that the society can only choose a set of representatives that is partitioned into at most three parties.

Let X be a finite set of policies.

The *social preference relation* is a complete and asymmetric binary relation $\succ_S$ on X. An example is the majority preference relation derived for individuals' preferences.

A *3-party system* is a triple $\succ_P = (\succ_1, \succ_2, \succ_3)$ in which each party i has a complete, asymmetric, and transitive preference relation $\succ_i$ over X. A 3-party system represents a possible composition of a legislature, where the parties together occupy all seats, and no single party has a majority of seats. (Thus a 3-party system can be interpreted simply as a profile of three voters each having one vote.)

A 3-party system $\succ_P$ is said to *represent* the social preference relation $\succ_S$ if the majority preference relation generated by $\succ_P$ coincides with $\succ_S$ (i.e., if for every pair $\{x, y\} \subset X$ of distinct policies,[4] $x \succ_i y$ for two or three parties $i \in \{1, 2, 3\}$ if and only if $x \succ_S y$). The social preference relation $\succ_S$ is called *3-party representable* if some 3-party system represents $\succ_S$.

Example 5.1. (*The Paradox of Voting*) Let $X = \{a, b, c\}$, and consider the cyclic social preference relation $a \succ_S b \succ_S c \succ_S a$. Define a 3-party system $\succ_P$ as follows: $a \succ_1 b \succ_1 c$ and $a \succ_1 c$; $b \succ_2 c \succ_2 a$ and $b \succ_2 a$; and $c \succ_3 a \succ_3 b$ and $c \succ_3 b$. Then $\succ_P$ represents $\succ_S$. $\square$

Remark 5.1. A legislature with a single-party majority (e.g., a 2-party system without majority ties) can be expressed as a 3-party system. For example, a 2-party system $(\succ_1, \succ_2)$ where party 1 is a majority in the legislature is defined as the 3-party system $(\succ_1, \succ_1, \succ_2)$. $\square$

[4]Throughout the chapter, an expression of the form "$\{x, y\}$" always represents an unordered pair of distinct elements.

5.3 3-party representation and labeling

Given a preference relation $\succ_i$ for party i, the *disagreement set* $D(\succ_i, \succ_S)$ is defined as the set of policy pairs on which the party and the society disagree:

$$D(\succ_i, \succ_S) = \{\{x, y\} \subset X : (x \succ_i y \text{ and } y \succ_S x) \text{ or } (y \succ_i x \text{ and } x \succ_S y)\}.$$

Given a 3-party system $\succ_P$, let $D(\succ_P, \succ_S)$ denote the family of disagreement sets:

$$D(\succ_P, \succ_S) = (D(\succ_i, \succ_S))_{i=1,2,3}.$$

A *labeling* of the social preference relation $\succ_S$ is a family of three subsets of policy pairs $L = (L_i)_{i=1,2,3}$ (where we say "a policy pair $\{x, y\}$ has label i" if $\{x, y\} \in L_i$) such that:

L1: each policy pair has at most one label; and

L2: the three policy pairs in each 3-cycle of $\succ_S$ have distinct labels.[5]

Figures 5.1 and 5.2 illustrate examples of labeling, where "$a \to b$" indicates the relation $a \succ_S b$.

Note that a labeling L determines a unique triple $\succ = (\succ_i)_{i=1,2,3}$ of (not necessarily transitive) preferences, via the equation $D(\succ, \succ_S) = L$. Thus each label i may be seen as specifying policy pairs on which "party i" (with a possibly intransitive preference relation) disagrees with society. From this view, condition L1 ensures that at least two parties agree with society on each policy pair. Condition L2, then, says that each party disagrees with society on exactly one policy pair in each 3-cycle of the social preferences.

[5]A labeling can be formulated as a *vertex 3-coloring* of the graph $G(\succ_S) = (V, E)$ defined as follows: the vertex set V is the set of all policy pairs; the edge set E is defined so that two policy pairs are connected if and only if they belong to the same 3-cycle of $\succ_S$. Up to the colors assigned to policy pairs that do not belong to any 3-cycle, a labeling of $\succ_S$ is naturally represented by a 3-coloring of $G(\succ_S)$. There is a large literature on the problem of 3-colorability especially for planar graphs.

Theorem 5.1. *Suppose that the social preference relation $\succ_S$ is 3-party representable. Then, for any 3-party system $\succ_P$ that represents $\succ_S$, the family $D(\succ_P, \succ_S)$ of disagreement sets is a labeling of $\succ_S$.*

Proof. Since $\succ_P$ represents $\succ_S$, for each policy pair $\{x, y\}$, at most one party can disagree with the society on $\{x, y\}$. Thus L1 holds. For each party i, since $\succ_i$ is transitive, $\succ_i$ disagree with $\succ_S$ on at least one policy pair contained in each 3-cycle of $\succ_S$. Thus for each 3-cycle C, the map that assigns to each policy pair $\{x, y\}$ in C the party that disagrees with $\succ_S$ on $\{x, y\}$ must be one-to-one. Thus L2 holds. $\square$

Remark 5.2. The converse of Proposition 5.1 does not hold: for some social preference relation $\succ_S$ and some labeling L of $\succ_S$, the triple $\succ = (\succ_i)_{i=1,2,3}$ defined by $D(\succ, \succ_S) = L$ is not a 3-party system satisfying the transitivity assumption. Consider the labeling illustrated in Figure 5.2. Then for the triple $\succ$ defined as above, (a, c, e, a) is a cycle of $\succ_1$ (and (a, e, c, a) is a cycle of $\succ_2$). This example indicates that while a labeling is designed so that each label eliminates cycles that exists in the social preferences, it may give rise to a new preference cycle. $\square$

Remark 5.3. Let $\overline{n}$ be the maximum number of policies such that *any* social preference relation is 3-party representable. We do not know the exact value of $\overline{n}$. Shepardson and Tovey (2009) shows $\overline{n} \leq 7$ by explicitly showing a digraph over 8 vertices such that any tournament (i.e., a graph expression for a complete asymmetric relation) having this as a subdigraph is not 3-party realizable. Another tournament that is not 3-party realizable is called the "Paley tournament" on 19 vertices, denoted T_{19}. Recall the necessary condition for 3-party representability provided by Alon et al. (2006) which we mentioned in Introduction. It can be checked (as Graham and Spencer (1971) claim) that T_{19} does not satisfy the condition. $\square$

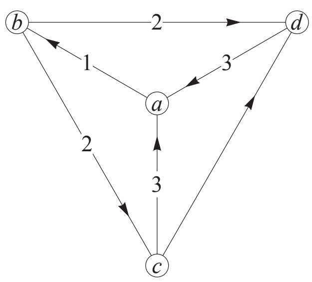

Figure 5.1: Oriented labeling

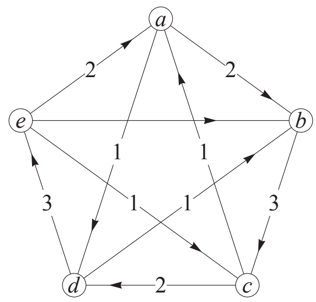

Figure 5.2: Non-oriented labeling

5.4 Oriented labeling

Remark 5.2 says that while every 3-party system representing a social preference relation can be constructed as the outcome of a labeling, some labeling fails to produce a 3-party system satisfying the transitivity condition on the parties' preferences. This occurs because when the social ordering on a policy pair is reversed according to the labeling, a new cycle may arise. In this section, we show that such effects will offset each other if we impose an additional condition on the labeling.

A labeling L of the social preference relation $\succ_S$ is called *oriented* if:

O1: whenever both (a, b, c, a) and (a, b, d, a) are 3-cycles of $\succ_S$, policy pairs $\{b, c\}$ and $\{b, d\}$ (and hence pairs $\{c, a\}$ and $\{d, a\}$) have the same label; and

O2: a policy pair has a label only if it belongs to some 3-cycle of $\succ_S$.

The labeling in Figure 5.1 is oriented, but the labeling in Figure 5.2 is not oriented.

As before, we interpret a labeling L as determining, through the equation $D(\succ_P, \succ_S) = L$, a 3-party system $\succ_P$ with possibly intransitive preferences. Recall that in the definition of a labeling, L1 requires that each party disagrees with society on exactly one policy pair in each 3-cycle. Condition O1 says that, moreover, such assignments of anti-social parties have the same order for any two adjacent 3-cycles. Condition O2 just says that the legislature unanimously agree with society on any policy pair that does not belong to any 3-cycle of the social preference relation.

Theorem 5.2. *Suppose that there exists an oriented labeling L of the social preference relation $\succ_S$, and let $\succ = (\succ_i)_{i=1,2,3}$ be the triple of binary relations determined by $D(\succ, \succ_S) = L$. Then $\succ_i$ is transitive for each i. Thus, if a social preference relation $\succ_S$ has an oriented labeling, then it is 3-party representable.*

Proof. By L1, for any policy pair $\{x, y\}$, there are at least two i's such that $\succ_i$ and $\succ_S$ agree on $\{x, y\}$.

Thus it remains to show that each party's preference relation is transitive. It suffices to check that $\succ_1$ is transitive. Suppose on the contrary that $\succ_1$ has a 3-cycle in three policies $\{x, y, z\}$.

The three policies $\{x, y, z\}$ cannot constitute a cycle of $\succ_S$: indeed, if (x, y, z, x) or (x, z, y, x) is a 3-cycle of $\succ_S$, by L2, $\succ_1$ and $\succ_S$ disagree on exactly one policy pair in $\{x, y, z\}$ so that $\succ_1$ is transitive on $\{x, y, z\}$.

Thus suppose without loss of generality that

$$x \succ_S y \succ_S z \text{ and } x \succ_S z.$$

There are two possibilities (Figures 5.3 and 5.4):

(a) pair $\{x, z\}$ is labeled 1, but pairs $\{x, y\}$ and $\{y, z\}$ are not; or

(b) pairs $\{x, y\}$ and $\{y, z\}$ are labeled 1, but pair $\{x, z\}$ is not.

Case (a). Because pair $\{x, z\}$ is labeled 1, by O2, there is a policy $u \notin \{x, y, z\}$ such that $C_u = (x, z, u, x)$ is a 3-cycle of $\succ_S$.

Suppose $y \succ_S u$. Then (x, y, u, x) is also a 3-cycle of $\succ_S$. Since pair $\{x, z\}$ is labeled 1 in 3-cycle $C_u = (x, z, u, x)$, by O1, pair $\{x, y\}$ is labeled 1, a contradiction.

Now suppose $u \succ_S y$. Then (y, z, u, y) is a 3-cycle of $\succ_S$. Since pair $\{x, z\}$ is labeled 1 in 3-cycle $C_u = (x, z, u, x)$, by O1, pair $\{y, z\}$ is labeled 1, a contradiction.

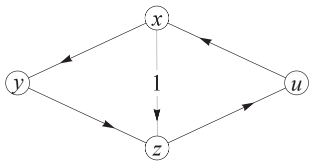

Figure 5.3: Case (a)

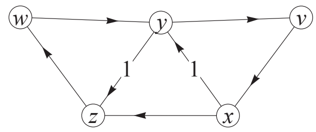

Figure 5.4: Case (b)

Thus, case (a) is impossible.

Case (b). Since pairs $\{x,y\}$ and $\{y,z\}$ are labeled 1, by O2, there are two policies $v,w \notin \{x,y,z\}$ such that $C_v = (x,y,v,x)$ and $C_w = (y,z,w,y)$ are 3-cycles of $\succ_S$. In particular, $y \succ_S v$, $w \succ_S y$, and hence $v \neq w$.

Suppose $v \succ_S w$. Then (w,y,v,w) is a 3-cycle of $\succ_S$. Since pair $\{x,y\}$ is labeled 1 in 3-cycle $C_v = (x,y,v,x)$, by O1, pair $\{w,y\}$ is labeled 1. This is impossible, because it implies that 3-cycle $C_w = (y,z,w,y)$ contains two pairs labeled 1, which violates L2.

Now suppose $w \succ_S v$.

First consider the case where $w \succ_S x$. In this case, (x,z,w,x) is a 3-cycle of $\succ_S$. Since pair $\{y,z\}$ is labeled 1 in 3-cycle $C_w = (y,z,w,y)$, by O1, pair $\{x,z\}$ is labeled 1, which contradicts (b).

Now suppose $x \succ_S w$. Then (x,w,v,x) is a 3-cycle of $\succ_S$. Then since $C_v = (x,y,v,x)$ is a 3-cycle of $\succ_S$, by O1, pair $\{w,x\}$ is labeled 1. Then, since $x \succ_S z \succ_S w$, $x \succ_S w$, and only pair $\{x,w\}$ is labeled 1 in $\{x,z,w\}$ (because $\{x,z\}$ is not labeled 1 by the assumption of case (b) and $\{z,w\}$ belongs to C_w in which $\{y,z\}$ is labeled 1), this falls into case (a) for the triple $\{w,x,z\}$.

Thus, case (b) is also impossible. Hence, $\succ_1$ has no 3-cycle, and is therefore transitive. $\square$

Remark 5.4. The existence of an oriented labeling is not necessary for 3-party representability. Consider the social preference relation $\succ_S$ illustrated in Figure 5.5. It can be checked that no labeling of $\succ_S$ is oriented. Figure 5.5 illustrates a non-oriented

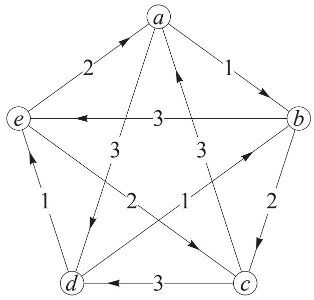

Figure 5.5: Social preference relation having no oriented labeling

labeling. Yet, this labeling defines a 3-party system representing $\succ_S$. □

Remark 5.5. Proposition 5.2 shows that an oriented labeling, if it exists, provides a method to construct a set of three voters that induces a given preference. Erdös and Moser (1964) present a different construction that needs more voters, but always succeeds. □

5.5 A sufficient condition for 3-party representability

We have seen that the existence of an oriented labeling is sufficient for 3-party representability. In this section we provide a sufficient condition for the existence of an oriented labeling. The condition restricts more explicitly the structure of the social preference relation.

A *closed path of 3-cycles* of the social preference relation $\succ_S$ is a cyclic sequence of distinct 3-cycles of $\succ_S$, $C = (C_1, C_2, \ldots, C_k, C_1)$ with $k \geq 2$, such that:

C1: for each j, C_j and C_{j+1} share a policy pair (denoted $\{x_j, y_j\}$); and

C2: $\{x_j, y_j\} \neq \{x_h, y_h\}$ for $j \neq h$.[6]

[6]Here all subscripts are mod k.

The social preference relation illustrated in Figure 5.5 contains the closed path of 3-cycles $((a,b,c,a),(b,c,d,b),(c,d,e,c),(d,e,a,d),(e,a,b,e),(a,b,c,a))$. In contrast, in Figure 5.1, the sequence $((a,b,c,a),(a,b,d,a),(a,b,c,a))$ is not a closed path of 3-cycles: indeed, it does not satisfy C2.

Theorem 5.3. *If the social preference relation $\succ_S$ has no closed path of 3-cycles, then $\succ_S$ has an oriented labeling, and hence is 3-party representable.*

Proof. The set of 3-cycles of $\succ_S$ is partitioned so that two 3-cycles belong to the same class if they share a policy pair. Fix a class $\mathcal{C}$. We show that we can implement an oriented labeling for policy pairs belonging to 3-cycles in $\mathcal{C}$ as follows:

- Step 1. Choose any 3-cycle $C_1 \in \mathcal{C}$ and label the policy pairs of C_1 (of course, subject to L1 and L2).
- Step $t \geq 2$. For each 3-cycle $C_t \in \mathcal{C}$ which has been partially labeled at Step $t-1$, label the remaining policy pairs of C_t in such a way that O1 holds for the labels assigned so far.
- Continue until the policy pairs of all 3-cycles in $\mathcal{C}$ are labeled.

Assume that all Steps $j \leq t-1$ $(t \geq 2)$ have successfully done, and we are at Step t. Take $C_t \in \mathcal{C}$ for which Step t applies. It remains to check that C_t has two unlabeled policy pairs for which we have full degree of freedom to assign labels (so that labeling subject to O1 is possible); that is,

(a) C_t has two unlabeled policy pairs; and

(b) there is no $C'_t \neq C_t$ in $\mathcal{C}$ sharing a policy pair with C_t for which Step t applies.

We only prove (a). (The proof of (b) is similar.) Suppose on the contrary that C_t has two labeled policy pairs, $\{x,y\}$ and $\{y,z\}$. Then there are two sequences of 3-cycles, $P = (C_1, C_2, ..., C_{t-1}, C_t)$ and $P' = (C_1, C'_2, ..., C'_{t-1}, C_t)$, such that: Step j applies to C_j

and C'_j; any two consecutive 3-cycles in each sequence share a policy pair; and $\{x, y\}$ and $\{y, z\}$ belong to C_{t-1} and C'_{t-1}, respectively.

Let $\{x_j, y_j\}$ ($\{x'_j, y'_j\}$) be the policy pair shared by the j-th and $(j+1)$-th elements in sequence P (P'). Then, for sequence P, $\{x_j, y_j\} \neq \{x_h, y_h\}$ for $j < h$, because otherwise all policy pairs of C_{h+1} would be labeled at Step $j+1$. (The same is true for sequence P'.)

Now, C_{t-1} and C'_{t-1} are distinct because otherwise they would coincide with C_t. Moreover, $\{x_{t-1}, y_{t-1}\} = \{x, y\} \neq \{y, z\} = \{x'_{t-1}, y'_{t-1}\}$. Thus letting s be the maximum j such that $C_j = C'_j$ or $\{x_j, y_j\} = \{x'_j, y'_j\}$, we have $1 \leq s \leq t-2$. If $\{x_s, y_s\} = \{x'_s, y'_s\}$, then, since C_{s+1} and C'_{s+1} share $\{x_s, y_s\}$, the cyclic sequence $(C_{s+1}, \ldots, C_t, C'_{t-1}, \ldots, C'_{s+1}, C_{s+1})$ is a closed path of 3-cycles. If $\{x_s, y_s\} \neq \{x'_s, y'_s\}$, then, since $C_s = C'_s$, the cyclic sequence $(C_s, C_{s+1}, \ldots, C_t, C'_{t-1}, \ldots, C'_{s+1}, C_s)$ is a closed path of 3-cycles. Both cases contradict the assumption of the proposition. □

5.6 Conclusion

Thus, a cyclic social preference relation is realized by some three-party system, unless 3-cycles constitute a closed path. This condition reveals a region of social preference relations that are compatible with some three-party legislature but not with a legislature having a single-party majority.

We have imposed no restriction on the composition of a legislature, apart from the bound on the number of parties. But in reality a society is endowed with a set of established parties. The preferences and seat shares of these parties may be limited to certain ranges, irrespective of institutional arrangements the society makes. Under such constraints, which type of party system achieves a better representation of society is not obvious. Future research may extend the framework of this paper to include such cases.

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