Considerations on Filtrations and Ambiguity in Mathematical Finance

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In Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy (Graduate School of International Corporate Strategy (ICS) Finance Program)

Hitotsubashi University
Tokyo, Japan

2014

(Defended February 17, 2014)
Acknowledgements

I owe a special debt of gratitude to my advisor Professor Hidetoshi Nakagawa who was abundantly helpful and offered invaluable assistance, support and guidance by providing fair, accurate and of course valuable comments and corrections. His attitude of research has influenced me quite a lot. He also has been supporting my attendance at various conferences, which gave me valuable experiences.

Besides my advisor, I would like to thank the rest of my thesis committee: Professor Toshiki Honda (Hitotsubashi University) and Professor Shigeo Kusuoka (The University of Tokyo) for their encouragement, insightful comments, and hard questions.

I wish also to express my gratitude to Professor Ryozo Miura. He is the man who pushed me to go back to academia when I was 50, and that was the trigger of everything I have experienced since then. Actually the result of the first topic in the thesis is based on my master thesis that was written under his guidance. Even after I moved to Nakagawa seminar, he kindly continued to support my research in several aspects.

Additionally, I would like to thank Professor Marek Rutkowski (The University of Sydney) who kindly read one of the earliest drafts of Chapter 2 and gave some important hints to go further in the next step.

The thesis was written in part during my visit at EMLyon Business School where I could concentrate on writing it under a quite comfortable environment. I wish to say a big merci beaucoup to Professor Olivier Le Courtois for the invitation.

I appreciate the financial support from Japan Society for the Promotion of Science (Grant-in-Aid for Scientific Research (A) No. 20241038) that funded a part of the research discussed in the thesis and from Foundation of Global Life Learning Center that encouraged me to get the Ph.D. degree.

Let me allow to list here the various persons who I have not mentioned above...
but who gave me many useful comments, suggestions and encouragements throughout my research.

Professor Fumio Hayashi, Professor Akitoshi Ito, Professor Nobuhiro Nakamura, Professor Kazuhiko Ohashi, Professor Tatsuyoshi Okimoto, Professor Shingo Oue and Professor Daisuke Yokouchi from the faculty of ICS, Hitotsubashi University, Professor Takahiko Fujita (Chuo University), the former member of the faculty, and the faculty secretaries Ms. Eri Inoue, Ms. Yukiko Ishikawa, Ms. Emiko Nishinami and Ms. Kyoko Oikawa gave me their seamless support throughout my Ph.D. program. Professor Koichiro Takaoka (Hitotsubashi University) provided me several supports as well as valuable technical comments.

Professor Alex Novikov (University of Technology, Sydney) and Ms. Julie Jerbic supported my first debut outside Japan at Quantitative Methods in Finance in Sydney where I met Professor Monique Jeanblanc (Evry Val d’Essonne University) for the first time. Professor Xin Guo and Dr. Zhao Ruan (University of California, Berkeley) aided my presentation about follower processes at Workshop on Probability and Statistics in Finance in Berkeley. Professor Darrell Duffie (Stanford University) encouraged me to go forward as an academic researcher even at my age when I visited Stanford. Professor Van Son Lai (University of Laval) commented kindly to my first presentation about categorical risk measure theory at French Finance Association Conference in Lyon. Professor Stephane Crepey, Professor Monique Jeanblanc, Professor Thomas Lim, and Professor Shiqi Song (Evry Val d’Essonne University) invited me to make a presentation at Bachelier seminar in Paris where I had a quite fertile conversation about filtrations with Professor Song. Professor François Le Grand, Professor François Quittard-Pinon, Professor Lorenz Schneider, Dr. Abdou Kelani and Ms. Muriel Thiémé (EMLyon Business School) provided their sincere support while I stayed in Lyon and made presentations at Séminaire CEFRA (Center for Financial Risk Analysis) there.

JAFEE (Japanese Association of Financial Econometrics and Engineering) was an
incubator to me in which I have grown up and made presentations with receiving a lot of suggestions and encouragements from its participants including Professor Takuji Arai (Keio University), Professor Yuki Itoh (Yokohama National University), Professor Kensuke Ishitani (Meijo University), Professor Takeaki Kariya (Meiji University), Professor Yoshio Miyahara (Nagoya City University), Dr. Yuji Morimoto (The University of Tokyo), Professor Yoshifumi Muroi (Tohoku University), Professor Katsushi Nakajima (Waseda University), Professor Keita Owari (The University of Tokyo), Dr. Tomoaki Shoda (Morgan Stanley), Professor Hideyuki Takada (Toho University), Professor Tetsuya Takaishi (Hiroshima University of Economics), Professor Hideatsu Tsukahara (Seijo University), Dr. Yoshihiko Uchida (Institute for Monetary and Economic Studies, Bank of Japan), Mr. Toshihiro Yamada and Dr. Suguru Yamanaka (Mitsubishi UFJ Trust Investment Technology Institute) and Professor Kazuhiro Yasuda (Hosei University). The Japanese Society for Mathematical Economics was another academic conference where I made a presentation and received quite important comments from Professor Chiaki Hara (Kyoto University) and Professor Shigeo Kusuoka (The University of Tokyo). Especially, the comments from Professor Kusuoka always contained hints showing the right directions that my research should go toward. Professor Toshikazu Kimura (Kansai University) provided me an opportunity to make a presentation at Kyoto university. In the same old city, I had a chance to have a suggestive comment on my categorical framework from Professor Freddy Delbaen (ETH Zürich).

Professor Masaaki Fukasawa, Professor Takashi Kato, Professor Teppei Ogiwara, Professor Masamitsu Ohnishi, Professor Kosuke Oya and Professor Jun Sekine (Osaka University) provided me many opportunities to present my earlier ideas at Osaka University including two CSFI (Center for the Study of Finance and Insurance) workshops where I had quite useful comments from Professor Shin Kanaya (Aarhus University) and Professor Masaaki Kijima (Tokyo Metropolitan University). Professor Syoiti Ninomiya (Tokyo Institute of Technology) gave me an instructive comment to an early stage of my categorical risk measure theory
at a workshop held in Tokyo Institute of Technology. Professor Ryoki Fukushima and Professor Kazumasa Kuwada (Tokyo Institute of Technology) gave me an opportunity to make a presentation at *Tokyo Probability Seminar*. Professor Motonari Kurasawa, Professor Katsumasa Nishide and Professor Norio Takeoka (Yokohama National University) and Professor Kyoko Yagi (Akita Prefectural University) gave me opportunities of presentations and useful comments, especially from a micro economists’ point of view.

Professor Jiro Akahori (Ritsumeikan University) is one of the best supporters to me. Members of his team (some people call them *The Akahori family*) including Professor Yuri Imamura, Dr. Nien-Lin Liu, Dr. Hideyuki Tanaka, Professor Takahiro Tsuchiya (Ritsumeikan University) and Professor Flavia Barsotti (UniCredit Group) gave me great environments and stimuli for the research including an invitation to *CREST and Ritsumeikan-Florence Workshop*. I had a privilege to attend a series of very mathematics meetings called *How do you like saturday?*\(^1\) whose members are Dr. Kazufumi Fujimoto (Osaka University), Professor Keisuke Hara (Mynd Inc.), Mr. Yuji Hishida (Mizuho Securities), Ms. Kaori Okuma (Quick Corp.), Professor Takashi Omoto (Nomura Asset Management) and Dr. Yi Zhou (Hitotsubashi University). The series was also prepared by Professor Akahori, which has kept stimulating me.

Professor Richard Blute (University of Ottawa), an editor of *Theory and Applications of Categories* and an anonymous referee provided me a lot of instructive comments on the contents of Chapter 4.

I have been blessed with intellectual and delightful fellow students, Mr. Hiroshi Sasaki and Dr. Yi Zhou (Hitotsubashi University). Especially, Dr. Zhou provided me valuable comments through long hours’ discussion about the topic

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\(^1\)The formal name of the series is *JAFEE Derivatives Division Seminar*. But we prefer this casual name in which the lower case letter ‘s’ is not a typo. The series was held at the Tokyo campus of Ritsumeikai University every two weeks.
in Chapter 3.

My sincere thanks goes to all of them.

Finally, the thesis would never be written without the understanding and the cooperation given by my family, especially by my wife, Nobuko as well as our kids, Mamiko, Toma and Panda\(^2\). I feel a deep gratitude to them.

\(^2\)He is a Holland Lop, sitting by my side while I was writing the thesis at our living room.
Abstract

We provide three different specifications to represent filtrations and ambiguity.

The first specification is about the filtration representing information delay. We introduce a stochastic process called a follower process that is a non-decreasing sequence of random times whose values \( f_t \) do not exceed the value \( t \). A follower filtration is a filtration modulated by the follower process. We show that conditional expectations given idempotent follower filtrations have some Markov property in a binomial setting, which is useful for pricing defaultable financial instruments.

The second specification is to introduce a new concept of ambiguity, called state ambiguity by extending a state space \( \Omega \). We introduce a concept called extended state that is defined as a history of an observer’s perception of the state of the world \( \Omega \) with a filtration \( G \). A set of extended states represents the observer’s ability to perceive the world. We apply the concept to dynamic choice theory by calculating value functions that characterize preference relations between consumption plans. The resulting functions are sensitive not only to prior ambiguity but also to state ambiguity.

The third specification is done by category theory. We introduce a category \( \chi \) that represents varying risk as well as ambiguity, and give a generalized conditional expectation as a contravariant functor on the category. We reformulate dy-
namic monetary value measures as a contravariant functor on $\chi$. We show that an axiom of time consistency introduced in the classical setting is deduced as a theorem in the new formulation, which may be one of the evidences that the axioms are natural. After presenting a robust representation of concave value measures in the formulation, we provide a technique to break the time consistency condition with an observer’s prior structure that is represented as a Grothendieck co-topology. We also demonstrate a topology-as-axioms paradigm in order to give a theoretical criteria with which we can pick up appropriate sets of axioms required for monetary value measures to be good.
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Chapter 1

Introduction

1.1 Motivation and background

Whenever thinking about multi-period problems in mathematical finance, we cannot live without filtrations in order to discuss their formulation and solutions. For example, when we make pricing of contingent claims, when we try to hedge them, when we seek the solution of optimization problems of controlling portfolios, it is indispensable to represent information that varies over time by filtrations in order to formulate and compute the original problems. Given such a filtration, we compute conditional expectations over time that will determine a probability distribution per time. A risk can be computed under the distribution that might be thought as an objective probability.

On the other hand, our recent experiences including two typical crises - the subprime crisis and the Lehman shock, have been pushing the financial industry to think a situation where we cannot determine a unique probability distribution to compute the risk. In other words, we need multiple subjective probabilities with which we should compute a set of risks. Actually, this kind of situation has been investigated for a long time in decision theory, starting by Savage [Sav54], or even before him by Knight [Kni21] and De Finetti [DF37]. The situation is called uncertainty or ambiguity 1.

1In this thesis, we use the word ambiguity for denoting the situation that cannot be explained by risk. We reserve the word uncertainty for denoting the situation including both risk and ambiguity.
1.1 Motivation and background

In this thesis, we provide a few new specifications about filtrations and ambiguity and discuss their applications to finance theory. We have three topics.

The first topic we will discuss in Chapter 2 is about information delay that may come by asymmetric information. In order to represent it, we introduce a new concept called a follower process that is a stochastic process consisting of a non-decreasing sequence of random times \( f_t \) whose values do not exceed \( t \). It was originally introduced for representing information delay in structural credit risk models. The follower process is an extension of a time change process introduced by Guo, Jarrow and Zeng [GJZ09] in the sense that each component of the follower process is not required to be a stopping time. Then, we introduce a class of follower processes called idempotent, which contains natural examples including follower processes driven by renewal processes. We show that any idempotent follower process is hard to be an example of time change processes.

We define a filtration modulated by the follower process and show that it is a natural extension of the continuously delayed filtration that is the filtration modulated by the time change process. Finally, we show that conditional expectations given idempotent follower filtrations have some Markov property in a binomial setting, which is useful for pricing defaultable financial instruments.

The concept of follower processes is a means of representing asymmetricity of information. However, it does not try to comprehend the intuition power of an individual who penetrate the information. In Chapter 3, we discuss the second topic in which we introduce a concept of extended states in order to represent this kind of personal abilities of the penetration.

An extended state is a history of an observer’s perception of the state of the world \( \Omega \) with a filtration \( \mathcal{G} \). Then in some cases, \( \Omega \) is naturally embedded into the set of all extended states \( \Omega[\mathcal{G}] \).

A set of extended states represents her ability to perceive the world. Here, the ability is determined not only by her personal talent such as her penetration
1.1 Motivation and background

power which we call an internal ability but also determined by her external environment such as constraints delivered by asymmetric information. We also introduce a class of subsets of $\Omega[G]$ called dominant in order to specify the observer’s internal ability. If we use a dominant set to specify her internal ability and use a follower filtration to specify her external environment, then the resulting subset of $\Omega[G]$ may be a good guideline to build a set of scenarios systematically for stress tests required by authorized rules such as Basel III.

We then apply the concept to dynamic choice theory by calculating value functions that characterize preference relations between consumption plans. The resulting functions are sensitive not only to prior ambiguity but also to state ambiguity.

In the third topic presented in Chapter 4, we formulate filtrations and ambiguity by using category theory. Within the formulation, we develop a theory of dynamic monetary value measures that have values with opposite signs to the values of monetary risk measures.

Ok. But why and what is category theory? Here is an explanation.

Suppose that we have a filtration $\{G_t\}_{t \in [0,T]}$ and a terminal value (payoff) represented by a random variable $X$ at a fixed time horizon $T$, that is, $X$ is a $G_T$-measurable function. Then, it is common in mathematical finance that we need to valuate $X$ at time $t$ that is ahead of $T$. The resulting random variable $\phi_t(X)$ which is $G_t$-measurable, is called a conditional value of $X$ at time $t$.

If you think about a conditional value measure that has values with opposite signs to the values of a conditional risk measure defined in Chapter 11 of Föllmer and Schied [FS11], it is a perfect example of the conditional value function $\phi_t$.

Now the point of conditional functions is that their domain is constrained to random variables measurable at the horizon time $T$. Then what if we have a relative conditional functions such as $\phi_t^s$ that maps a $G_t$-measurable random variable to $G_s$-measurable random variable when $s < t \leq T$. It may be natural to require
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that $\varphi_t = \varphi^t_T$ and that $\varphi^s_t \circ \varphi^u_t = \varphi^u_u$ for $s \leq t \leq u \leq T$.

The latter requirement, which guarantees some consistency, suggests a possibility of the usage of an abstract function theory known as category theory that is a theory widely used in many branches of mathematics and physics such as geometry, logic and string theory, but has not been applied to finance theory so far.

Let us go back to the story of Chapter 4. First, given a measurable space $(\Omega, \mathcal{G})$, we introduce a category $\chi$ whose objects are ordered pairs of sub-$\sigma$-fields of $\mathcal{G}$ and subjective probability measures definable on the measurable space. The category represents varying risk as well as ambiguity. Then, we give a generalized conditional expectation as a contravariant functor on the category.

Next, we reformulate dynamic monetary value measures as a contravariant functor on the category. We show that it has a time consistency property that was introduced as an axiom in the classical setting, which may be one of the evidences that the axiom is natural. It is good! However, if we remind the result of Kupper and Schachermayer [KS09] saying that every dynamic risk measure satisfying both axioms of time consistency and law invariance is an entropic risk measure, our result may be too restrictive if we insist on an idea of introducing the axiom of law invariance.

Artzner et al. [ADEH99] developed a theory of robust representation of monetary risk measures. We do the same thing for our categorical monetary value measures when they are concave.

Inspired by the robust representation, we step in to develop a technique for breaking the time consistency. But, since it is already proved that every monetary value measure defined as a contravariant functor on $\chi$ has the time consistency property, the resulting generalized monetary value measures would be no more
functors. To overcome the situation, we define a structure of subjectivity that the observer has as a Grothendieck cotopology on subcategory $\chi$ of $\mathcal{X}$, and then define the generalized monetary value measure as a family of functions per some good structure belonging to a cobasis that generates the cotopology. We see the resulting generalized version of monetary value measures do not satisfy the time consistency condition in general.

Since the axiomatization of monetary risk measures was initiated by Artzner et al. [ADEH99], many axioms including time consistency and law invariance have been presented. Those investigations are valuable in both theoretical and practical senses. However, it may be expected to have some theoretical criteria of picking appropriate sets of axioms out of them. Thinking about the recent events such as the CDS (Credit Default Swap) hedging failure at JP Morgan Chase in 2012, the importance of selecting appropriate axioms of monetary risk measures becomes even bigger than before. In Sections 4.3 and 4.4 of Chapter 4, we demonstrate a topology-as-axioms paradigm in order to give a theoretical criteria with which we can pick up appropriate sets of axioms required for monetary value measures to be good. That is, we suggest a criteria in which a given set of axioms is considered good if and only if there exists a Grothendieck topology under which the set of all monetary value measures satisfying the set of axioms coincides with the set of sheaves over $\chi$.

In Chapter 5, we have concluding remarks and some future plans based on the thesis.

1.2 Notation

Before embarking on the actual topics, we summarize some notation we are using in this thesis.
1.2 Notation

All discussions are under a measurable space

$$(\Omega, \mathcal{G})$$  \hspace{1cm} (1.1)

where $\mathcal{G}$ is a $\sigma$-field over $\Omega$. For a $\sigma$-field $\mathcal{F} \subset \mathcal{G}$. We denote the set of all bounded $\mathbb{R}$-valued $\mathcal{F}$-measurable functions by

$L^\infty(\Omega, \mathcal{F})$  \hspace{1cm} (1.2)

which becomes a linear space.

Let $\mathbb{P}$ be a probability measure defined on the measurable space $(\Omega, \mathcal{G})$, which provides us a probability space

$$(\Omega, \mathcal{F}, \mathbb{P}|\mathcal{F}).$$  \hspace{1cm} (1.3)

We write

$L^\infty(\Omega, \mathcal{F}, \mathbb{P}|\mathcal{F})$  \hspace{1cm} (1.4)

for the quotient space of $L^\infty(\Omega, \mathcal{F})$ under the equivalence relation $\sim_{\mathbb{P}}$ defined by

$$X \sim_{\mathbb{P}} Y \text{ iff } X = Y \text{ $\mathbb{P}$-a.s.}$$  \hspace{1cm} (1.5)

Then, the space (1.4) becomes a Banach space with the usual sup norm.

Let $\mathcal{T}$ be a fixed time domain that has the least element 0, equipped with an adequate topology. For $s, t \in \mathcal{T}$, we define

$$[s, t]_\mathcal{T} := \{u \in \mathcal{T} \mid s \leq u \leq t\}.$$  \hspace{1cm} (1.6)

We similarly define $[s, t[\mathcal{T}, s, t]\mathcal{T}$ and $]s, t[\mathcal{T}$. We write $\mathcal{T}_+ \text{ for } \mathcal{T} - \{0\}$. For a function $f$ whose domain is $\mathcal{T}$, $f(t-) := \lim_{s \to t-0} f(s)$ and $f(t+) := \lim_{s \to t+0} f(s)$. Note that in the case $\mathcal{T} = \{n\delta \mid n = 0, 1, \ldots\}$, $t- = t - \delta$ for $t \in \mathcal{T}_+$ and $t+ = t + \delta$ for $t \in \mathcal{T}$. \hspace{1cm}
The measurable space \((\Omega, \mathcal{G})\) may come with a filtration \(G = \{\mathcal{G}_t\}_{t \in T}\) indexed by \(T\), making a filtered measurable space

\[
(\Omega, \mathcal{G}, G = \{\mathcal{G}_t\}_{t \in T}) \tag{1.7}
\]

or a filtered probability space

\[
(\Omega, \mathcal{G}, G = \{\mathcal{G}_t\}_{t \in T}, \mathbb{P}). \tag{1.8}
\]

The filtration \(G\) satisfies the usual condition for continuous time domains unless stated otherwise.
Chapter 2

Follower processes

2.1 Introduction

The original motivation of this chapter\(^1\) comes from the theory of credit risk models \(^2\). In the credit risk theory, it is crucial to introduce a sort of incompleteness into its model when adopting a so-called structural approach. In order to make it, Guo, Jarrow and Zeng [GJZ09] introduced a process called time change.

We have a filtered probability space (1.8). Then the time change process is defined like the following.

**Definition 2.1.1.** [Time change processes (Guo, Jarrow and Zeng [GJZ09])] A G-time change process is a G-adapted stochastic process \( f : \mathcal{T} \times \Omega \to \mathcal{T} \) satisfying the following conditions,

1. \( f_0 = 0 \) \( \mathbb{P} \)-a.s.,
2. \( f_t \leq t \) \( \mathbb{P} \)-a.s. for all \( t \in \mathcal{T} \),
3. \( t_1 \leq t_2 \implies f_{t_1} \leq f_{t_2} \) \( \mathbb{P} \)-a.s. for all \( t_1, t_2 \in \mathcal{T} \),
4. for all \( t \in \mathcal{T} \), \( f_t \) is a G-stopping time.

\(^1\)This chapter is a revised version of Adachi, Miura and Nakagawa [AMN13].
\(^2\)For those who are not familiar with credit risk theory, please refer to Bielecki and Rutkowski [BR04] for more information.
2.1 Introduction

The time change process \( f_t \) represents the amount of delayed time \( t - f_t \). It can be read in the context of the credit risk theory that if the market knows an event at time \( t \), then the event actually happened at time \( f_t \) (ahead of \( t \)) when managers learned it. So, it models the fact that the market would know the information possibly after the managers know it, that is, representing asymmetric information.

Guo, Jarrow and Zeng succeeded to make their credit risk model an incomplete one by using a filtration \( \{ \mathcal{G}_{f_t} \}_{t \in T} \) called a continuously delayed filtration. The results were somehow consistent with empirically observed data.

Note that the continuously delayed filtration is well-defined since \( f_t \) is a \( \mathcal{G} \)-stopping time.

Now let us suppose a natural example representing delayed information, which we call a renewal follower process \( \{ f_t \}_{t \in T} \).

**Example 2.1.2.** [Renewal follower processes]

1. \( X_n \sim \text{i.i.d. random variables such that } 0 < \mathbb{E}^P[X_n] < \infty \) for \( n = 1, 2, \ldots \),
2. \( S_n := \sum_{k=1}^{n} X_k \),
3. \( N_t := \sup \{ n \mid S_n \leq t \} \),
4. \( f_t := S_{N_t} \).

Intuitively, the random variable \( X_n \) specifies an interval time between \((n - 1)\)-th and \( n \)-th jumps when the follower process catch up with the current time, i.e. \( f_t = t \). Figure 2.1.1 shows a sample trajectory of a renewal follower process where the random variables \( X_n \) above obey an exponential distribution \( \text{Exp}(10) \).

One of the possible interpretation of this example in reality is the situation that the firm makes all its insider information available to the market only when it is under an audit activity by authorities (at the jumping time).
2.1 Introduction

Figure 2.1.1: Renewal follower process

However, we will see in Section 2.2.2 that it is hard for renewal follower processes to be examples of the time change processes because of the strong condition (4) in Definition 2.1.1. This is the main reason we introduced the following new concept by dropping the condition.

Definition 2.1.3. [Follower processes]

A **raw follower process** is a stochastic process $f : \mathcal{T} \times \Omega \rightarrow \mathcal{T}$ satisfying the following conditions,

1. $f_0 = 0$ $\mathbb{P}$-a.s.,
2. $f_\tau \leq \tau$ $\mathbb{P}$-a.s. for all $\tau \in \mathcal{T}^*$,
3. $\tau_1 \leq \tau_2 \Rightarrow f_{\tau_1} \leq f_{\tau_2} \text{ } \mathbb{P}$-a.s. for all $\tau_1, \tau_2 \in \mathcal{T}^*$,

where $\mathcal{T}^*$ is the set of all $\mathcal{T}$-valued random times.

A **G-follower process** is a raw follower process which is $G$-adapted.
2.1 Introduction

Apparently, renewal follower processes are raw follower processes.

The difference of Definition 2.1.3 from the first three conditions of Definition 2.1.1 is the use of $T^*$ instead of $T$. This change is necessary if we consider the case when we need to make a reasoning such as $f_s \leq t$ implies $f_{f_s} \leq f_t$. Therefore, the original definition of time change processes in Definition 2.1.1 is also better to be rewritten with $t_i$ varying in $T^*$ instead of in $T$. Therefore, the original definition of time change processes in Definition 2.1.1 is also better to be rewritten with $t_i$ varying in $T^*$ instead of in $T$. So in the rest of this chapter, we assume that any $G$-time change process is a $G$-follower process.

The remainder of this chapter consists of three sections.

In Section 2.2, we begin with giving some properties and a couple of examples of follower processes. After seeing that the set of all follower processes forms a monoid, We introduce an important class of idempotent follower processes that satisfy $f_{f_t} = f_t$ for all $t \in T$. It is obvious that the class contains all renewal follower processes. Then, we show that an idempotent follower process fails to be an example of time change processes. Next, we provide some characterizations of idempotent follower processes. In the end of Section 2.2, we show that an idempotent follower process consists of a sequence of honest times, and that any honest time can be represented as a limit of an idempotent follower process.

One of our motivations to introduce the concept of follower processes is to use it for modulating a given filtration, which is necessary for pricing defaultable securities. However, except the case when $f_t$ is a stopping time, it is not generally obvious how to define the $\sigma$-field $G_{f_t}$ and the filtration consisting of those $\sigma$-fields. In Section 2.3, we present a definition of filtrations modulated by follower processes and show that our filtration is a natural extension of the continuously delayed filtrations in the sense that they coincide each other when the underlying follower process consists of stopping times.

Once we try to apply the theory of follower processes to the credit risk theory, we would face the necessity to calculate some conditional expectations given the filtration defined in Section 2.3. Especially, we need a strong Markov property
like the following:

\[ \mathbb{E}^P[ g(Y_s) \mid \mathcal{G}_t^f ] = \mathbb{E}^P[ g(Y_s) \mid f_t, Y_{f_t} ] \]

where \( \mathcal{G}_t^f \) is the filtration modulated by the follower process \( f_t \). In Section 2.4, we prove this when \( f_t \) is idempotent in a binomial model.

### 2.2 Follower processes

#### 2.2.1 Properties and examples of follower processes

Here are some simple properties of follower processes whose proofs are left to readers.

**Proposition 2.2.1.** Let \( f \) and \( g \) be raw follower processes. Then, so are the following processes:

1. \( f_t \land g_t \),
2. \( f_t \lor g_t \),
3. for a fixed \( s \in T \), \( h_t := \begin{cases} f_t & \text{if } t \leq s, \\ f_s + g_{t-s} & \text{if } t > s \end{cases} \).

The simplest example of follower processes is the identity process \( \{t \}_{t \in T} \). Other than that and renewal follower processes, we give a couple of examples in the rest of this subsection.

**Example 2.2.2.** [Constantly delayed follower processes]

Lindset et al. introduced the two time lags for markets and managers in [LLP08]. Their lags are constant and not stochastically varying like ours.

Let \( d \) be a positive constant. A raw follower process \( f = \{f_t\}_{t \in T} \) is called a **constantly delayed follower process** with delay \( d \) if for all \( t \in T \),

\[ f_t := \max\{t - d, 0\} \].
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Example 2.2.3. [Occupation times]

This example is taken from Example 6.2 in Chapter 3 of Karatzas and Shreve [KS98]. Let $W = \{W_t\}_{t \in T}$ be a Brownian motion and $B \in B(\mathbb{R})$ be a Borel set. Then, the *occupation time* of $B$ by the Brownian path up to time $t$ is the process $f$ defined by

$$f_t := \int_0^t 1_B(W_s) \, ds.$$

Obviously, any occupation time $f$ is a raw follower process. However, the occupation time will not recover to the managers’ time (that is, $f_t \neq t$) once it had a chance to walk out of the Borel set $B$. More precisely speaking, the delay $t - f_t$ is increasing as time passes, and never shrinks. Therefore, the converse is untrue.

Similarly, for a given continuous semimartingale $X = \{X_t\}_{t \in T}$, its local time $L = \{L_t\}_{t \in T}$ is a follower process.

2.2.2 Idempotent follower processes

In this subsection, we introduce a class of follower processes whose elements are called idempotent. We show that the class contains all renewal follower processes and that every idempotent follower process is hard to be a time change process.

First, we introduce a composition operator defined on the space of follower processes.

A process $f$ is a raw follower process if there is a raw follower process $f'$ such that $f_\tau = f'_\tau$ $\mathbb{P}$-a.s. for all $\tau \in T^*$. Therefore, we can treat the space of follower processes as a quotient space safely.

Definition 2.2.4. [Space of follower processes]

1. $\mathcal{M}$ is the set of all raw follower processes.

2. For $f^1, f^2 \in \mathcal{M}$, the *composite process* $f^1 \circ f^2$ is defined by for $t \in T$ and $\omega \in \Omega$,

$$(f^1 \circ f^2)_t(\omega) = (f^1 \circ f^2)(t, \omega) := f^1(f^2(t, \omega), \omega).$$
2.2 Follower processes

3. \( M := M / \sim \), where \( \sim \) is a binary relation on \( M \) defined by for any pair of \( f^1 \) and \( f^2 \) in \( M \), \( f^1 \sim f^2 \) iff \( f^1_\tau = f^2_\tau \) \( P \)-a.s. for all \( \tau \in T^* \).

For \( f \in M \), we write \( f \in M \) by identifying \( f \) with the equivalence class \([f]_\sim \in M\) if it leads no confusion.

4. An identity process is a process \( 1^M \in M \) defined by \( 1^M_\tau(\omega) = t \) for all \( t \in T \) and \( \omega \in \Omega \).

**Theorem 2.2.5.** The structure \( \langle M, \circ, 1^M \rangle \) forms a monoid, that is, a semigroup with identity, where \( \circ \) is a well-defined operator on \( M \) induced by the operator \( \circ \) on \( M \).

**Proof.** Straightforward.

**Definition 2.2.6.** [Idempotent follower processes] A raw follower process \( f \) is called idempotent if for all \( \tau \in T^* \),

\[
f_{f_\tau} = f_\tau \text{ P-a.s..} \tag{2.1}
\]

We would see (2.1) as \( f \circ f = f \). The word idempotent comes from the fact.

You can easily verify that the identity follower process and the renewal follower process are idempotent follower processes. On the other hand, neither constantly-delayed follower processes nor occupation times are idempotent.

**Proposition 2.2.7.** A raw follower process \( f \) is idempotent iff for all \( \tau_1, \tau_2 \in T^* \), \( \tau_1 \leq \tau_2 \leq \tau_1 \Rightarrow f_{\tau_1} = f_{\tau_2} \) \( P \)-a.s..

**Proof.** If part. For \( \tau \in T^* \), we have

\[
\{f_\tau \leq \tau\} \cap \{f_\tau \leq f_\tau \leq \tau \Rightarrow f_\tau = f_{f_\tau}\} \subset \{f_\tau = f_{f_\tau}\}.
\]

By the assumption, the probability of the left hand set is 1. Therefore, \( P\{f_\tau = f_{f_\tau}\} = 1 \) as well.
2.2 Follower processes

*Only if part.* For any \( \tau_1, \tau_2 \in T^* \), define a set \( A \) by

\[
A := \{ f_{\tau_1} = f_{\tau_1} \} \cap \{ f_1 \leq \tau_2 \Rightarrow f_{\tau_1} \leq f_{\tau_2} \} \cap \{ \tau_2 \leq \tau_1 \Rightarrow f_{\tau_2} \leq f_{\tau_1} \}.
\]

Then, we have \( \mathbb{P}(A) = 1 \) since \( f \) is an idempotent raw follower process. Now, observing

\[
A \cap \{ f_1 \leq \tau_2 \leq \tau_1 \} = A \cap \{ f_1 \leq \tau_2 \} \cap \{ \tau_2 \leq \tau_1 \} \subset \{ f_{\tau_1} = f_{\tau_1} \} \cap \{ f_{\tau_1} \leq f_{\tau_2} \} \cap \{ f_1 \leq f_{\tau_1} \}
\]

we have \( A \subset \{ f_1 \leq \tau_2 \leq \tau_1 \Rightarrow f_1 = f_{\tau_1} \} \). Therefore, \( \mathbb{P}\{ f_1 \leq \tau_2 \leq \tau_1 \Rightarrow f_1 = f_{\tau_2} \} = 1 \).

Here is one of the important implications derived from Proposition 2.2.7.

**Corollary 2.2.8.** Let \( f = \{ f_t \}_{t \in T} \) be an idempotent \( G \)-follower process where each \( f_t \) is a \( G \)-stopping time. Then, for every pair \( t \) and \( s \) in \( T \) with \( t \geq s \), we have \( \{ f_t = f_s \} \in G_s \).

**Proof.** Let \( A \subset \Omega \) be the set defined by \( A := \{ f_t \leq s \Rightarrow f_t = f_s \} \cap \{ f_s \leq s \} \). Then, since \( f \) is a follower process and by Proposition 2.2.7, we get \( \mathbb{P}(A) = 1 \).

Now, under the assumption \( s \leq t \), we have

\[
A \cap \{ f_t \leq s \} = A \cap (\{ f_t \leq s \Rightarrow f_t = f_s \} \cap \{ f_s \leq s \}) \subset A \cap \{ f_t = f_s \}
\]

and

\[
A \cap \{ f_t = f_s \} = A \cap (\{ f_s \leq s \} \cap \{ f_t = f_s \}) \subset A \cap \{ f_t \leq s \}.
\]

Thus \( A \cap \{ f_t \leq s \} = A \cap \{ f_t = f_s \} \). Therefore \( \{ f_t \leq s \} \triangle \{ f_t = f_s \} \subset \Omega - A \). Hence \( \mathbb{P}(\{ f_t \leq s \} \triangle \{ f_t = f_s \}) = 0 \). Since \( \{ f_t \leq s \} \) is \( G_s \)-measurable and \( G_s \) is complete, we have \( \{ f_t = f_s \} \in G_s \). \( \square \)
Let us think that \( s \) is the current time and that \( t \) is any future time. Then by Corollary 2.2.8, we can know if the information will have increased since now by any future time \( t \), which is not realistic. So, we should conclude that requiring each random time \( f_t \) to be a stopping time is not practical in the case that \( f \) is idempotent while some of the idempotent follower processes are quite interesting both in the practical and the theoretical sense. This is our original motivation to develop a delayed theory that does not depend on stopping times.

Next, we show a characterization of idempotent follower processes.

**Definition 2.2.9.** 1. For a random set \( F \subset \mathcal{T} \times \Omega \), define a process \( f^F : \mathcal{T} \times \Omega \rightarrow \mathcal{T} \) by

\[
f^F_t(\omega) := \sup \{ s \leq t \mid (s, \omega) \in F \},
\]

where we use the convention \( \sup \emptyset = 0 \).

2. For a raw follower process \( f \), define a random set \( F^f \) by

\[
F^f := \{ (t, \omega) \in \mathcal{T} \times \Omega \mid f_t(\omega) = t \}.
\]

Note that \( f^F_t \) is the end of the random set \( F_t := F \cap ([0, t]) \times \Omega \). Here, the end of a random set \( A \subset \mathcal{T} \times \Omega \) is the random time \( E_A \) defined by

\[
E_A(\omega) := \sup \{ t \in \mathcal{T} \mid (t, \omega) \in A \}.
\]

**Proposition 2.2.10.** 1. Let \( F \subset \mathcal{T} \times \Omega \) be a random set. Then, the process \( f^F \) is an idempotent raw follower process.

2. Let \( f \) be an idempotent raw follower process, Then, \( f^f_\tau = f_\tau \) \( \mathbb{P} \)-a.s. for all \( \tau \in \mathcal{T}^* \).

**Proof.** 1. It is clear that \( f^F \) is a raw follower process. So, let us show it is also idempotent. Let \( \omega \in \Omega, \tau \in \mathcal{T}^* \) and \( s := f^F_t(\omega) \). Then,

\[
s \leq \tau(\omega) \quad \text{and} \quad (\forall u \in \mathcal{T}) \left( u \leq \tau(\omega) \land (u, \omega) \in F \Rightarrow u \leq s \right).
\]
Now, it is enough to show that \( \{ u \leq s \mid (u, \omega) \in F \} = \{ u \leq \tau(\omega) \mid (u, \omega) \in F \} \). Since \( s \leq \tau(\omega) \), it is obvious that LHS \( \subset \) RHS. Let \( u \in \text{RHS} \). Then by (2.5), \( u \leq s \). Therefore, \( u \in \text{LHS} \).

2. For \( \omega \in \Omega, s \in \mathcal{T} \) and \( \tau \in \mathcal{T}^* \), We have

\[
\tau^F(\omega) = \sup\{ s \leq \tau(\omega) \mid (s, \omega) \in F \} = \sup\{ s \in \mathcal{T} \mid s \leq \tau(\omega) \land f_s(\omega) = s \}. \tag{2.6}
\]

Then by (2.6), we have \( \{ f_\tau \leq \tau \} \cap \{ f_{f_s} = f_\tau \} \subset \{ \tau^F \geq f_\tau \} \).

Here, the probability of the left hand set of the above equation is 1 since \( f \) is an idempotent follower process. Therefore, \( \mathbb{P}\{ \tau^F \geq f_\tau \} = 1 \).

On the other hand,

\[
\begin{align*}
\{ f_\tau < s \leq \tau \Rightarrow f_\tau = f_s \} \cap \{ s \leq \tau \land f_s = s \} \cap \{ f_\tau < s \} \\
= & \{ f_\tau < s \leq \tau \Rightarrow f_\tau = f_s \} \cap \{ f_\tau < s \leq \tau \} \cap \{ f_\tau < s \} \cap \{ f_s = s \} \\
\subset & \{ f_\tau = f_s \} \cap \{ f_\tau < s \} \cap \{ f_s = s \} \\
\subset & \{ f_s < s \} \cap \{ f_s = s \} = \emptyset.
\end{align*}
\]

Therefore,

\[
\{ f_\tau < s \leq \tau \Rightarrow f_\tau = f_s \} \subset \{ (s \leq \tau \land f_s = s) \Rightarrow s \leq f_\tau \} \\
\subset \{ \tau^F \leq f_\tau \}. \tag{2.7}
\]

Here the last inclusion holds by (2.6). The probability of the left most statement of (2.7) is 1 by Proposition 2.2.7. Therefore, \( \mathbb{P}\{ \tau^F \leq f_\tau \} = 1 \).

We have the following characterization theorem for idempotent raw follower processes.
2.2 Follower processes

Theorem 2.2.11. Let \( f : \mathcal{T} \times \Omega \to \mathcal{T} \) be a process. Then, \( f \) is an idempotent raw follower process iff there exists a random set \( F \subset \mathcal{T} \times \Omega \) such that \( f_\tau^F = f_\tau \) for all \( \tau \in \mathcal{T}^* \).

Proof. Immediate from Proposition 2.2.10. \( \square \)

The following is an example of idempotent follower process.

Example 2.2.12. [Starting times for excursions]

Let \( W = \{W_t\}_{t \in \mathcal{T}} \) be a standard \( \mathcal{G} \)-Brownian motion, and define a random set \( Z \) by

\[
Z := \{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid W_t(\omega) = 0\}.
\]

Then, the idempotent follower process \( f^Z \) picks the starting times for the excursions out of 0 of \( B \).

Proposition 2.2.13. Let \( F \) be a \( \mathcal{G} \)-progressive set. Then, the process \( f^F \) is an idempotent \( \mathcal{G} \)-follower process.

Proof. It is enough to show that \( f^F \) is \( \mathcal{G} \)-adapted. Since \( F \) is \( \mathcal{G} \)-progressive, \( F_t = F \cap ([0,t] \times \Omega) \) is \( \mathcal{B}[0,t] \otimes \mathcal{G}_t \)-measurable. Then, since \( f_t^F \) is the end of \( F_t \), it is \( \mathcal{G}_t \)-measurable. \( \square \)

Theorem 2.2.14. Let \( f : \mathcal{T} \times \Omega \to \mathcal{T} \) be a càdlàg process. Then, \( f \) is an idempotent \( \mathcal{G} \)-follower process iff there exists a \( \mathcal{G} \)-optional set \( F \subset \mathcal{T} \times \Omega \) such that \( f_\tau^F = f_\tau \) for all \( \tau \in \mathcal{T}^* \).

Proof. If part. By Proposition 2.2.13 and the remark after Definition A.1.1.

Only if part. All we need to show is that the random set \( F^f \) defined by (2.3) is \( \mathcal{G} \)-optional when \( f \) is \( \mathcal{G} \)-adapted.

For \( n \in \mathbb{N} \), define processes \( p^n \) by \( p^n := \mathbb{1}_{\{(t,\omega) \mid f_t(\omega) \leq t < f_t(\omega) + \frac{1}{n}\}} \). Then, \( p^n \) is obviously \( \mathcal{G} \)-adapted and càdlàg. Therefore, \( (p^n)^{-1}(1) = \{(t, \omega) \mid f_t(\omega) \leq t < f_t(\omega) + \frac{1}{n}\} \) is a \( \mathcal{G} \)-optional set. Thus, so is \( F^f = \bigcap_{n \in \mathbb{N}} (p^n)^{-1}(1) \). \( \square \)

It is easily checked that the idempotent follower process in Example 2.2.12 is an idempotent \( \mathcal{G} \)-follower process.
2.2.3 Honest times

In Corollary 2.2.8, we were discouraged to make a follower process consist of stopping times when it is idempotent.

In this subsection, we revisit the issue by adopting a wider class of random times than the class of stopping times.

A random time $\tau$ is called $G$-honest with respect to a $G$-adapted process $\{\tau_t\}_{t \in T_+}$ on $T$ if $\tau = \tau_t$ on $\{\tau \leq t\}$ for every $t \in T_+$, i.e. $\tau 1_{\{\tau \leq t\}} = \tau_t 1_{\{\tau_t \leq t\}}$.

A random time $\tau$ is called $G$-honest if there exists a $G$-adapted process $\{\tau_t\}_{t \in T_+}$ such that $\tau$ is $G$-honest with respect to $\{\tau_t\}_{t \in T_+}$.

It is well known that every $G$-stopping time is $G$-honest (See e.g. page 373 of Protter [Pro04] or page 384 of Nikeghbali [Nik06]).

Another characterization of honest times by optional processes.

**Theorem 2.2.15.** [ [Pro04] Theorem VI.16] A random time $\tau$ is $G$-honest if and only if there exists a $G$-optional set $A$ such that $\tau = E_A$, where $E_A$ is the end of $A$.

The following is a very nice characterization of honest times developed by Yor [Yor78].

**Theorem 2.2.16.** [ [Yor78]] A random time $\tau$ is $G$-honest if and only if for every $u \in [0, s]_{T}$, there exists $A \in \mathcal{G}_s$ such that $\{\tau \leq u\} = A \cap \{\tau \leq s\}$.

Our first question in this subsection is for a given follower process $f = \{f_t\}_{t \in T}$, if there exists an honest time $\tau$ with respect to $f$.

Here is a necessary and sufficient condition of the existence of such $\tau$.

**Proposition 2.2.17.** Let $f = \{f_t\}_{t \in T}$ be a $G$-follower process. Then, a random time $\tau : \Omega \to T \cup \{\infty\}$ is $G$-honest with respect to $f$ if and only if $f_\infty(\omega) := \lim_{t \to \infty} f_t(\omega) = \tau(\omega) = f_{\tau(\omega)}(\omega)$ for every $\omega \in \Omega$.

**Proof.** Note that the random time $\tau$ is $G$-honest with respect to $f$ iff for every $t \in T_+$ and $\omega \in \Omega$,

$$\tau(\omega) \leq t \Rightarrow f_t(\omega) = \tau(\omega).$$

(2.8)
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Only if part: Since $f_t$ is monotonic, $\lim_{t \to \infty} f_t(\omega) = \sup_{t \in T} f_t(\omega)$. Therefore, the result comes immediately by (2.8).

If part: Since $\sup_{t \in T} f_t(\omega) = \tau(\omega)$, $f_t(\omega) \leq \tau(\omega)$ for every $t \in T_+$. So, it is sufficient to show $f_t(\omega) \geq \tau(\omega)$, assuming $\tau(\omega) \leq t$. But, by the monotonicity of $f_t$ and the assumption $\tau(\omega) = f_{\tau(\omega)}(\omega)$, we have $\tau(\omega) = f_{\tau(\omega)}(\omega) \leq f_t(\omega)$. \qed

As an implication of Proposition 2.2.17, we missed the possibility of making whole follower process be characterized by one honest time if the follower process is unbounded. However, we have the following theorem of asserting each $f_t$ becomes an honest time for some follower processes including renewal follower processes.

**Theorem 2.2.18.** If $f = \{f_t\}_{t \in T}$ is an idempotent $G$-follower process, then for every $t \in T$, $f_t$ is a $G$-honest time.

**Proof.** Define a random field $\{\tau^F_t\}_{t \in T}$ by $\tau^F_t := f_t \wedge s$.

Then, it is obvious that $\tau^F_t$ is $G$-measurable. So, all we need to show is $\tau^F_t = f_t$ on $\{f_t \leq s\}$.

If $s \geq t$, we have $\tau^F_t = f_t$ on $\Omega$. Hence, we concentrate on the case $s < t$. Now for any $\omega \in \{f_t \leq s\}$, $f_t(\omega) \leq s < t$. Then, since $f$ is idempotent, we get $f_t(\omega) = f_{f_t(\omega)}(\omega) \leq s(\omega) \leq f_t(\omega)$. Therefore, $f_t(\omega) = s(\omega) = \tau^F_t(\omega)$. \qed

Here is another characterization of honest times by using idempotent follower processes.

**Theorem 2.2.19.** A random time $\tau : \Omega \to \bar{T}$ is $G$-honest if and only if there exists an idempotent $G$-follower process $f$ such that for every $t \in T_+$, $\tau = f_t$ on $\{\tau \leq t\}$, i.e. $\tau = f_\infty$.

**Proof.** If part. Immediate by the definition of honest times.

Only if part. By Theorem 2.2.15, there exists a $G$-optional set $F$ such that $\tau = E_F$ since $\tau$ is $G$-honest.

Let $f := f^F$. Then, by Theorem 2.2.14, $f$ is an idempotent $G$-follower process.
2.3 Follower filtrations

On the other hand, for $\omega \in \{\tau \leq t\}$, we have

$$f_t(\omega) = \sup\{s \leq t \mid (s, \omega) \in F\}$$

$$= \sup\{s \in \mathcal{T} \mid (s, \omega) \in F\} \quad \text{since} \quad s \leq \tau(\omega) \leq t$$

$$= E_F(\omega).$$

Therefore, $f_t = E_F = \tau$. \hfill $\Box$

2.3 Follower filtrations

As stated in Section 2.1, one of our motivations to introduce the concept of follower processes is to use it for modulating a given filtration. Here is a definition to make it.

**Definition 2.3.1.** [Follower filtrations] Let $f = \{f_t\}_{t \in \mathcal{T}}$ be a $G$-follower process. The **follower filtration** modulated by the $G$-follower process is the filtration $G^f = \{G^f_t\}_{t \in \mathcal{T}}$ defined by for $t \in \mathcal{T}$,

$$G^f_t := \bigvee_{s \in [0, t]} G_{fs}. \quad (2.9)$$

In Definition 2.3.1, $G_{fs}$ is the $\sigma$-field defined in Definition A.1.2.

**Theorem 2.3.2.** Let $f = \{f_t\}_{t \in \mathcal{T}}$ be a $G$-follower process. Then the follower filtration $G^f$ is a subfiltration of $G$.

**Proof.** It is obvious that $G^f$ is a filtration. So all we need to show is that $G^f_t \subset G_t$ for any $t \in \mathcal{T}$. But for any $s \leq t$, since $f_s \leq f_t \leq t$, we have $G_{fs} \subset G_t$ by Theorem A.1.4. Therefore, $G^f_t = \bigvee_{s \in [0, t]} G_{fs} \subset G_t$. \hfill $\Box$

The following theorem shows that our follower filtration is a natural extension of the continuously delayed filtration of Guo, Jarrow and Zeng [GJZ09].
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**Theorem 2.3.3.** Let \( f = \{ f_t \}_{t \in \mathcal{T}} \) be a \( \mathcal{G} \)-follower process where each \( f_t \) is a \( \mathcal{G} \)-stopping time. Then, \( \mathcal{G}^f_t = \mathcal{G}_{f_t} \).

**Proof.** Let \( s, t \in \mathcal{T} \) with \( s \leq t \). Then, \( f_s \leq f_t \).

First, we want to show \( \mathcal{G}_{f_s} \subset \mathcal{G}_{f_t} \). Let \( A \in \mathcal{G}_{f_s} \). Then, by Theorem A.1.5, for any \( u \in \mathcal{T} \), we have \( A \cap \{ f_s \leq u \} \in \mathcal{G}_u \).

On the other hand, since \( f_s \leq f_t \), we have \( A \cap \{ f_t \leq u \} = \left( A \cap \{ f_s \leq u \} \right) \cap \{ f_t \leq u \} \).

The first term of the right hand side belongs to \( \mathcal{G}_u \) by the assumption, while the second term is also in \( \mathcal{G}_u \) since \( f_t \) is a \( \mathcal{G} \)-stopping time. So again by Theorem A.1.5, we get \( A \in \mathcal{G}_{f_t} \).

Then, we have \( \mathcal{G}^f_t = \bigvee_{s \in [0,t]} \mathcal{G}_{f_s} = \mathcal{G}_{f_t} \).

Since a constant time is considered as a stopping time, we have the following corollary.

**Corollary 2.3.4.** Assume that a \( \mathcal{G} \)-follower process \( f \) is deterministic, i.e. there exists a deterministic function \( g : \mathcal{T} \rightarrow \mathcal{T} \) such that for all \( t \in \mathcal{T} \) and \( \omega \in \Omega \), \( f_t(\omega) = g(t) \). Then, we have for all \( t \in \mathcal{T} \), \( \mathcal{G}^f_t = \mathcal{G}_{g(t)} \).

Next, we investigate the shape of follower filtrations when the underlying follower processes are idempotent.

**Lemma 2.3.5.** Let \( f \) be an idempotent \( \mathcal{G} \)-follower process which is càdlàg. Then for every pair of \( s, t \in \mathcal{T} \) with \( s < t \), \( f_s \) is \( \mathcal{G}_{f_t} \)-measurable.

**Proof.** Let \( s \in \mathcal{T} \) and \( B \in \mathcal{B}(\mathcal{T}) \) be fixed. For any \( n \in \mathbb{N} \), define processes \( p^n \) and \( q^n : \mathcal{T} \times \Omega \rightarrow \mathbb{R} \) by

\[
p^n := \mathbb{1}_{\left\{ (u,\omega) \in \mathcal{T} \times \Omega | f_s(\omega) \in B, f_s(\omega) \leq u < f_s(\omega) + \frac{1}{n} \right\}},
\]

\[
q^n := \mathbb{1}_{\left\{ (u,\omega) \in \mathcal{T} \times \Omega | f_s(\omega) \in B, u \geq s + \frac{1}{n} \right\}}.
\]
Then, since $f_s$ is càdlàg, $G_s$-adapted and $G_{f_t}$-adapted by Proposition A.1.3, both $p^n$ and $q^n$ are $G$-adapted and càdlàg. Therefore, $P^n, Q^n \in G_{f_t}$ where

$$P^n := (p^n_{f_t})^{-1}(1) = \{ f_s \in B, f_s \leq f_t < f_s + \frac{1}{n} \},$$

$$Q^n := (q^n_{f_t})^{-1}(1) = \{ f_s \in B, f_t \geq s + \frac{1}{n} \}.$$

Then, we have

$$P := \bigcap_{n \in \mathbb{N}} P^n = \{ f_s \in B, f_s = f_t \} \in G_{f_t},$$

$$Q := \bigcup_{n \in \mathbb{N}} Q^n = \{ f_s \in B, f_t > s \} \in G_{f_t}.$$

Therefore

$$P \cup Q = \{ f_s \in B \} \cap (\{ f_s = f_t \} \cup \{ f_t > s \}) \in G_{f_t}.$$ 

On the other hand, under the assumption $s \leq t$, Proposition 2.2.7 implies that the two sets $\{ f_t \leq s \}$ and $\{ f_s = f_t \}$ are identical by ignoring a null-measured difference. Hence

$$\{ f_s \in B \} \cap (\{ f_t \leq s \} \cup \{ f_t > s \}) = \{ f_s \in B \} \in G_{f_t}.$$

Therefore, $f_s$ is $G_{f_t}$-measurable.

\[ \square \]

**Theorem 2.3.6.** Let $f$ be an idempotent $G$-follower process which is càdlàg. Then, for every $t \in \mathcal{T}$, we have $G^f_t = G_{f_t}$.

**Proof.** Immediate by Lemma 2.3.5 and Theorem A.1.4.

\[ \square \]

### 2.4 Follower processes in a binomial model

When we apply the theory of follower processes to the credit risk theory, we need to calculate some conditional expectations given a follower filtration in order to valuate defaultable financial instruments. In doing so, it would be quite welcome
2.4 Follower processes in a binomial model

if the follower filtration has a sort of strong Markov property such as

\[ \mathbb{E}^P[g(Y_s) | \mathcal{F}_t^f] = \mathbb{E}^P[g(Y_s) | f_t, Y_{f_t}], \] (2.10)

However, it seems a difficult task to prove (2.10) for an arbitrary time domain.

In this section we show this when \( f_t \) is idempotent in a binomial model, and
leave the continuous time domain case to future work.

2.4.1 The setup

In this subsection, we define a binomial model.

We fix the time domain \( \mathcal{T} := \{ n\delta \mid n = 0, 1, 2, \ldots, N \} \), where \( \delta \) is a given positive number. We denote its horizon \( N\delta \) by \( T \).

We define \( \Omega := \{ \S_t, \T \}^{T^+} \) and \( \omega(0) := \perp \) for \( \omega \in \Omega \), where \( \S_t, \T \) and \( \perp \) are distinct constants. For \( t \in \mathcal{T} \), we define a binary relation \( \sim_t \) on \( \Omega \) by \( \omega \sim_t \omega' \) iff \( \omega(s) = \omega'(s) \) for all \( s \in [0, t]_T \). Then, define a \( \sigma \)-field \( \mathcal{G}_t \) by \( \sigma(\Omega/ \sim_t) \). We also define \( \mathcal{G} := \mathcal{G}_T \).

We sometimes see the set \( \Omega \) as a topological space equipped with the discrete topology. In other words, any subset of \( \Omega \) is an open set.

We define a probability measure \( P \) on \( \Omega \) by \( P(A) := \sum_{\omega \in A} p^{\#\omega}(1 - p)^{N - \#\omega} \) for \( A \in \mathcal{G} \), where \( p \in [0, 1] \) is a given number and \( \#\omega \) is the cardinality of \( \omega^{-1}(\S_t) \).

Throughout this section, all discussions are under the filtered probability space \( (\Omega, \mathcal{G}, P = (\mathcal{G}_t)_{t \in \mathcal{T}}, P) \). We also fix a state space \( (E, \mathcal{E}) \) satisfying \( \{ x \} \in \mathcal{E} \) for all \( x \in E \). Note that both \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) and \( (\mathcal{T}, 2^\mathcal{T}) \) satisfy this condition.

Now it is easy to show that a function \( X : \Omega \to E \) is \( \mathcal{G}_t \)-measurable iff \( \omega \sim_t \omega' \) implies \( X(\omega) = X(\omega') \) for any \( \omega, \omega' \in \Omega \). Consequently, we can conclude that a process \( Z : \mathcal{T} \times \Omega \to E \) is \( \mathcal{G} \)-adapted iff \( \omega \sim_t \omega' \) implies \( Z(t, \omega) = Z(t, \omega') \) for all \( t \in \mathcal{T} \) and \( \omega, \omega' \in \Omega \).

**Definition 2.4.1.** [The universal process]

1. \( \Omega^* := \cup_{t \in \mathcal{T}} \{ \S_t, \T \}^{[0,t]_T}, \) where \( \{ \S_t, \T \}^{\perp} := \{ \perp \} \).
2.4 Follower processes in a binomial model

2. For \( \omega \in \Omega \) and \( t \in \mathcal{T} \), a function \( \omega|t \in \{\Omega, \mathcal{F}\}^{[0,t]} \) is defined by \( \omega|t := \omega|_{[0,t]} \) whose domain is expanded to \([0,t]\) by defining \((\omega|t)(0) := \bot\),

3. The universal process is a process \( \pi : \mathcal{T} \times \Omega \to \Omega^* \) defined by \( \pi(t, \omega) := \omega|t \).

The following theorem says that the universal process has a so-called universal property.

**Theorem 2.4.2.** Let \( Z : \mathcal{T} \times \Omega \to E \) be any \( G \)-adapted process.

1. There exists a unique function \( g : \Omega^* \to E \) such that \( Z = g \circ \pi \),

2. For any \( t \in \mathcal{T} \), \( \sigma(Z_t) \subset \sigma(\pi_t) \).

**Proof.** Left to readers. \( \square \)

2.4.2 Follower filtrations in a binomial model

In the rest of this section, we assume that \( f : \mathcal{T} \times \Omega \to \mathcal{T} \) is an arbitrary but fixed idempotent \( G \)-follower process.

**Proposition 2.4.3.** \( \mathcal{O}_G = \sigma\{ \{t\} \times [\omega]_{\sim} \mid t \in \mathcal{T}, \omega \in \Omega \} \), where \( \mathcal{O}_G \) is the optional \( \sigma \)-field defined in Definition A.1.1.

**Proof.** Since \( \Omega \) is equipped with the discrete topology, any function whose domain is \( \Omega \) is continuous. Therefore,

\[
\mathcal{O}_G := \sigma\{ Z \mid \text{Z is a G-adapted càdlàg process.} \}
= \sigma\{ Z \mid \text{Z is a G-adapted \ process.} \}.
\]

Then by Theorem 2.4.2 (2), we have \( \mathcal{O}_G = \sigma(\pi) \) since \( \pi \) itself is \( G \)-adapted.

Now remind that any element of \( \Omega^* \) can be represented as \( \omega|t \) for \( \omega \in \Omega \) and \( t \in \mathcal{T}_+ \). Then, we have the desired equation since \( \pi^{-1}(\omega|t) = \{t\} \times [\omega]_{\sim} \) for any \( \omega \in \Omega \) and \( t \in \mathcal{T}_+ \).

**Corollary 2.4.4.** A process \( Z : \mathcal{T} \times \Omega \to E \) is \( G \)-optional iff it is \( G \)-adapted.
2.4 Follower processes in a binomial model

**Proposition 2.4.5.** \( G^f_t = \sigma(\pi^f_{t}) \).

**Proof.** By Theorem 2.3.6, Corollary 2.4.4 and Theorem 2.4.2 (2). \( \square \)

Now we investigate the shape of the set \( \pi^{-1}_{f_t}(x) \) for \( x \in \Omega^* \) in order to characterize \( G^f_t \).

**Definition 2.4.6.** For a random time \( \tau \), a *neighborhood* of \( \omega \in \Omega \) at \( \tau \) is the set \( N_\tau(\omega) := [\omega]_{\sim_\tau(\omega)} \).

**Lemma 2.4.7.** For \( \omega, \omega_0 \in \Omega \), \( \omega \in N_{f_t}(\omega_0) \) implies \( f_t(\omega) \geq f_t(\omega_0) \).

**Proof.** Since \( f \) is G-adapted and \( \omega \sim_f \omega_0 \),

\[
f_{f_t(\omega_0)}(\omega) = f_{f_t(\omega_0)}(\omega_0) = f_t(\omega_0).
\]

The right most equality holds because \( f \) is idempotent. On the other hand, we have \( f_t(\omega_0) \leq t \). Therefore, \( f_{f_t(\omega_0)}(\omega) \leq f_t(\omega) \). \( \square \)

**Definition 2.4.8.** Let \( \tau \) be a random time, and \( \omega_0 \in \Omega \).

1. \( K_\tau(\omega_0) : = \{ \omega \in N_\tau(\omega_0) \mid \tau(\omega) > \tau(\omega_0) \} \),
2. \( K_\tau(\omega_0) : = \{ \omega \in K_\tau(\omega_0) \mid (\forall \omega' \in K_\tau(\omega_0))(N_\tau(\omega) \subset N_\tau(\omega') \Rightarrow N_\tau(\omega) = N_\tau(\omega')) \} \).

**Proposition 2.4.9.** Let \( t \in \mathcal{T}, \omega_0 \in \Omega \) and \( x_0 : = \pi_{f_t}(\omega_0) \). Then,

\[
\pi^{-1}_{f_t}(x_0) = N_{f_t}(\omega_0) \cup \{N_{f_t}(\omega) \mid \omega \in K_{f_t}(\omega_0)\}. \quad (2.11)
\]

**Proof.** Let \( \omega \in \pi^{-1}_{f_t}(x_0) \). Then, \( \pi_{f_t}(\omega) = \pi_{f_t}(\omega_0) \). Thus, \( \omega|_{f_t(\omega)} = \omega|_{f_t(\omega_0)} \). Therefore, \( f_t(\omega) = f_t(\omega_0) \) and \( \omega \sim_{f_t(\omega_0)} \omega_0 \), which implies \( \omega \in N_{f_t}(\omega_0) \).

Now, we show that \( \omega' \in K_{f_t}(\omega_0) \) implies \( \omega \notin N_{f_t}(\omega') \). Since \( \omega' \in K_{f_t}(\omega_0) \), we have \( \omega' \in N_{f_t}(\omega_0) \) and \( f_t(\omega') > f_t(\omega_0) \). Suppose \( \omega \in N_{f_t}(\omega') \). Then by Lemma 2.4.7, \( f_t(\omega) \geq f_t(\omega') > f_t(\omega_0) \), which contradicts to \( f_t(\omega) = f_t(\omega_0) \). Therefore, we conclude \( \omega \notin N_{f_t}(\omega') \) and \( \text{LHS} \subset \text{RHS} \).
Next, we show the opposite inclusion. Let $\omega \in N_{f_i}(\omega_0) - \cup \{ N_{f_i}(\omega) \mid \omega \in K_{f_i}(\omega_0) \}$. We want to show $\omega \in \pi_{f_i}^{-1}(x_0)$.

Since $\omega \in N_{f_i}(\omega_0)$, we have $f_i(\omega) \geq f_i(\omega_0)$ by Lemma 2.4.7. Suppose $f_i(\omega) > f_i(\omega_0)$. Then, $\omega \in K_{f_i}(\omega_0)$. We can pick $\omega' \in K_{f_i}(\omega_0)$ such that $N_{f_i}(\omega') > N_{f_i}(\omega)$. Therefore, $\omega \in N_{f_i}(\omega) \subset N_{f_i}(\omega')$. But this contradicts to the way of the selection of $\omega$. Hence, we have $f_i(\omega) = f_i(\omega_0)$.

On the other hand, we have $\omega|_{f_i}(\omega_0) = \omega_0|_{f_i}(\omega_0)$ since $\omega \in N_{f_i}(\omega_0)$. Therefore,

$$\pi_{f_i}(\omega) = \omega|_{f_i}(\omega) = \omega|_{f_i}(\omega_0) = \omega_0|_{f_i}(\omega_0) = \pi_{f_i}(\omega_0) = x_0.$$ 

\[ \square \]

**Corollary 2.4.10.** $G_{f_i}^\prime = \sigma \{ N_{f_i}(\omega) \mid \omega \in \Omega \}$.

### 2.4.3 Conditional expectations given a follower filtration

We keep assuming that $f$ is an idempotent $G$-follower process throughout this subsection.

**Theorem 2.4.11.** Let $Y$ be a random variable and $X$ be a $G_{f_i}$-measurable random variable. Then, $E[Y \mid G_{f_i}^\prime] = X$ iff $E[\mathbb{1}_{N_{f_i}(\omega_0)}Y] = E[\mathbb{1}_{N_{f_i}(\omega_0)}X]$ for all $\omega_0 \in \Omega$.

**Proof.** The only-if part is trivial. So we assume the right hand side. Let

$$\mathcal{F} := \{ A \in G_{f_i}^\prime \mid \int_A Yd\mathbb{P} = \int_A Xd\mathbb{P} \}.$$ 

Then, all we need to show is $\mathcal{F} = G_{f_i}^\prime$.

By the assumption, for any $\omega_0 \in \Omega$, we have $N_{f_i}(\omega_0) \in \mathcal{F}$. Now seeing (2.11) and noticing that the following relations are satisfied for any $\omega_1, \omega_2 \in K_{f_i}(\omega_0)$,

1. $N_{f_i}(\omega_1) \subset N_{f_i}(\omega_0)$,
2. $N_{f_i}(\omega_1) = N_{f_i}(\omega_2)$ or $N_{f_i}(\omega_1) \cap N_{f_i}(\omega_2) = \emptyset$, 

we conclude $\mathcal{F} = G_{f_i}^\prime$. 


we have the following equation where all unions are disjoint-sum:

\[ N_{fi}(\omega_0) = \pi_{fi}^{-1}(\pi_{fi}(\omega_0)) \cup \left( \bigcup \{ N_{f_i}(\omega) \mid \omega \in K_{fi}(\omega_0) \} \right). \]

Therefore, by the assumption, we have

\[ \mathcal{H} := \{ \pi_{fi}^{-1}(\pi_{fi}(\omega_0)) \mid \omega_0 \in \Omega \} \subset \mathcal{F}. \]

Again, the elements of \( \mathcal{H} \) are disjoint each other, and obviously \( \cup \mathcal{H} = \Omega \). So, any element of \( G_{fi}^t = \sigma(\pi_{fi}) \) can be represented as a disjoint sum of the elements of \( \mathcal{H} \), which concludes \( G_{fi}^t \subset \mathcal{F} \).

Now we define a process \( Y \) appeared in (2.10). As a proxy of Brownian motion, we define a process \( M \) by \( M_t(\omega) := \sum_{s \in [0,t) \cap T} X_s(\omega) \) for \( t \in T \), where \( \{ X_t \}_{t \in T} \) is a Bernoulli process defined by

\[ X_t(\omega) = \begin{cases} \sqrt{\delta} & \text{if } \omega(t) = S_f \\ -\sqrt{\delta} & \text{if } \omega(t) = T_f \end{cases} \quad (2.12) \]

Then, we define the process \( Y \) by

\[ Y_t(\omega) := y_0 + \nu t + \sigma M_t(\omega) \quad (2.13) \]

where \( y_0, \nu \) and \( \sigma \geq 0 \) are constants.

**Proposition 2.4.12.** Let \( g : \mathbb{R} \to \mathbb{R} \) be a given function. Then, for any \( s \geq t \),

\[ \mathbb{E}^P [g(Y_s) \mid G^f_t] = h(s - f_t, Y_f) \quad (2.14) \]
2.4 Follower processes in a binomial model

where the function \( h : \mathcal{T} \times \mathbb{R} \rightarrow \mathbb{R} \) is defined by

\[
h(0, y) := g(y),
\]

\[
h(t+, y) := ph(t, y + \nu \delta + \sigma \sqrt{\delta}) + (1 - p)h(t, y + \nu \delta - \sigma \sqrt{\delta}).
\]

Proof. By Theorem 2.4.11, since \( h(s - f_t, Y_{ft}) \) is \( \mathcal{G}_t \)-measurable, all we need to show is for all \( \omega_0 \in \Omega \),

\[
\mathbb{E}^P[\mathbbm{1}_{N_{ft}^t(\omega_0)} g(Y_{s})] = \mathbb{E}^P[\mathbbm{1}_{N_{ft}^t(\omega_0)} h(s - f_t, Y_{ft})].
\]

Thinking about the shape of the set \( N_{ft}^t(\omega_0) \), we can prove it by showing for all \( C \in \mathbb{R} \),

\[
\mathbb{E}^P[\mathbbm{1}_{N_{ft}^t(\omega_0)} g(Y_{s} + u + C)] = \mathbb{E}^P[\mathbbm{1}_{N_{ft}^t(\omega_0)} h(u, Y_{ft} + C)]
\]

by induction on \( u \in [0, s - f_t] \).

When \( u = 0 \), it is trivial. Assume (2.15) holds at \( u \in [0, s - f_t] \). Then, we have

\[
\mathbb{E}^P[\mathbbm{1}_{N_{ft}^t(\omega_0)} g(Y_{u+} + C)]
\]

\[
= \mathbb{E}^P[\mathbb{E}^P[\mathbbm{1}_{N_{ft}^t(\omega_0)} g(Y_{u+} + C) \mid \mathcal{G}_{u+}]]
\]

\[
= \mathbb{E}^P[p \mathbbm{1}_{N_{ft}^t(\omega_0)} g(Y_{u+} + \nu \delta + \sigma \sqrt{\delta} + C) + (1 - p) \mathbbm{1}_{N_{ft}^t(\omega_0)} g(Y_{u+} + \nu \delta - \sigma \sqrt{\delta} + C)]
\]

\[
= \mathbb{E}^P[p \mathbbm{1}_{N_{ft}^t(\omega_0)} (ph(u, (Y_{ft} + C) + \nu \delta + \sigma \sqrt{\delta}) + (1 - p)h(u, (Y_{ft} + C) + \nu \delta - \sigma \sqrt{\delta}))]
\]

\[
= \mathbb{E}^P[\mathbbm{1}_{N_{ft}^t(\omega_0)} h(u+, Y_{ft} + C)].
\]

Therefore, (2.15) holds at \( u+ \) as well, which completes the proof.

Corollary 2.4.13. Let \( g \) be a given function. Then, for any \( s \geq t \),

\[
\mathbb{E}^P[g(Y_{s}) \mid \mathcal{G}_t^f] = \mathbb{E}^P[g(Y_{s}) \mid f_t, Y_{ft}).
\]

Note 2.4.14. Practically, Corollary 2.4.13 is enough to price defaultable securities under a follower-process based model since we can make it as accurate as possible
by making $\delta$ smaller.

Suppose $p = \frac{1}{2}$. Then, as $\delta \to 0$, the process $M$ converges to a standard Brownian motion in distribution by the Central Limit Theorem, and the process $Y$ will satisfy the equation $Y_t = y_0 + vt + \sigma B_t$. The function $h$ defined in Proposition 2.4.12 will be specified with an appropriate partial differential equation, and (2.16) may hold at this continuous case.
Chapter 3

Extended states

3.1 Introduction

In dynamic choice theory starting from Kreps and Porteus [KP78] through Strza-lecki [Str13], we usually think a set of preference relations $\succeq_{t,\omega}$ indexed by a time and a state $(t,\omega) \in T \times \Omega$. However, in case $\Omega$ is an infinite set, the measure of a singleton set $\{\omega\}$ is (usually) 0. Then, what is the meaning of thinking of a preference relation whose domain has measure 0?\footnote{There is a very good concise textbook for choice theory written by Gilboa [Gil09].}

On the other hand, at time $t$, can an observer pick an exact state $\omega$ of the world where she lives? Isn’t it more natural to assume that she perceives a (possibly non-singleton) set of states for representing her view of the current world?

In order to answer these issues, we replace a single $\omega$ by a narrowing-down process, called an extended state.

An extended state is a history of an observer’s perception of the state of the world where she has lived in. Then, a set of extended states represents her ability to perceive the world. Here, the ability is determined not only by her personal talent such as her penetration but also determined by her external environment such as constraints delivered by asymmetric information. The important point is that the ability varies per person. So, we can use the set of extended states to treat some ambiguity just like we do with subjective probabilities. We will see the

\footnote{This chapter is a revised version of Adachi [Ada13b].}
3.2 Extended states

detail of the concept of extended states in Section 3.2.

In Section 3.3, we will apply the concept of extended states to calculate value functions that characterize preference relations between consumption plans. The resulting value functions will be sensitive not only to usual prior ambiguity but also to state ambiguity. We will also see that the value functions are more conservative than those defined in classical settings.

3.2 Extended states

All the discussions in this chapter are on a filtered measurable space (1.7).

3.2.1 Extended states

First, let us see the following simple discrete example that we frequently come back on.

Example 3.2.1. 1. \( \Omega = \{ \omega_0, \omega_1, \omega_2, \omega_3 \} \),

2. \( \mathcal{T} = \{0, 1, 2\} \),

3. \( \mathcal{G}_0 = \{ \emptyset, \Omega \} \),

4. \( \mathcal{G}_1 = \{ \emptyset, \{\omega_0, \omega_1\}, \{\omega_2, \omega_3\}, \Omega \} \),

5. \( \mathcal{G} = \mathcal{G}_2 = 2^\Omega \).

We can identify this structure with the binary tree in Figure 3.2.1 where the time increases from left to right and each node in the tree corresponds to a measurable set in \( \mathcal{G}_t \).

The structure in Figure 3.2.1 may come with a probability measure.

Suppose that we have two sets \( A \) and \( B \) in \( \mathcal{G} \) with \( A \subset B \). Then, the set \( A \) represents a finer information or a better perception about a situation than \( B \) does. On the other hand, the fineness of our perception at time \( t \) about the situation is limited to the sets ranging in \( \mathcal{G}_t \).
Now, thinking about a process representing an improvement of our perception as time goes by, it is natural to define the process as a decreasing sequence of sets whose member at time $t$ is in $G_t$. In Figure 3.2.2, paths $e_0$ through $e_9$ are those narrowing-down processes called extended states when the true state is $\omega_0$.

There are ten extended states that share $\omega_0$ as a possible true state. The top of them, $e_0$ in Figure 3.2.2, is the most efficient narrowing-down process. On the other hand, in the bottom of them, $e_9$, is the worst process with no update all the time.

Naturally, we are able to introduce a partial order among these extended states, according to the ‘superior-to’ relation between them. Figure 3.2.3 shows the partial order among extended states specified in Figure 3.2.2, which forms a complete lattice.

Here is a formal definition of extended states for general $(\Omega, G)$. 

**Definition 3.2.2.** [Extended states]

1. $\Omega[G] := \{e : T \rightarrow G \mid (\forall t \in T)e(t) \in G_t - \{\emptyset\} \text{ and } (\forall s, t \in T)[s \leq t \Rightarrow e(s) \supset e(t)]\}$. 

   An element of $\Omega[G]$ is called an extended state.

2. Binary relations $\leq$ and $\leq_t$ on $\Omega[G]$ are defined by for $e, d \in \Omega[G], e \leq d \iff (\forall t \in T)e(t) \supset d(t)$ and $e \leq_t d \iff (\forall s \in [0, t]T)e(s) \supset d(s)$. 

3.2 Extended states

Figure 3.2.2: Narrowing-down processes

Figure 3.2.3: Partial order among extended states
3. Let \( \bot : \mathcal{T} \to \mathcal{G} \) be a function satisfying \( (\forall t \in \mathcal{T}) \bot(t) = \Omega \).

Then \( \bot \) is the least element of the partially ordered set (poset) \( (\Omega[\mathcal{G}], \leq) \).

Figure 3.2.4 shows the structure \( \Omega[\mathcal{G}] \) corresponding to \((\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t \in \mathcal{T}})\) defined in Figure 3.2.1. Note that in general the structure is \textit{not} expected to come with probability measures.

The concept of extended states was inspired from a concept of \textit{pure experiences} advocated by a Japanese philosopher Kitarō Nishida in his eminent book “An Inquiry into the Good” [Nis11]. A pure experience is an elementary and immediate experience without any thought or speculation on it. It does not contain any cognitive perception of oppositions such as those of subject and object, and body and mind. The state of true experience itself without the addition of the least thought or discrimination. We just perceive it by intuition. In other words, for an extended state \( e, e(t) \) can be seen as a pure experience that the observer perceives at time \( t \). Selecting a state \( \omega \in e(t) \) out of the pure experience is a \textit{recognition} or an \textit{interpretation} of the current state.

Another way for interpreting the concept of extended states is to treat an extended state as a history of \textit{superpositions} in quantum mechanics. Then, \( \omega \in e(t) \) means an \textit{observation}.

\textbf{Definition 3.2.3.} 1. For \( \omega \in \Omega \) and \( t \in \mathcal{T} \), a subset \( \overline{\omega}(t) \subset \Omega \) is defined by

\[
\overline{\omega}(t) := \bigcap \{ A \in \mathcal{G}_t \mid \omega \in A \}. \tag{3.1}
\]

2. For \( e \in \Omega[\mathcal{G}] \) and \( \omega \in \Omega \), we write \( \omega \in e \) if \( (\forall t \in \mathcal{T}) \omega \in e(t) \).

\textbf{Definition 3.2.4.} A filtered measurable space \((\Omega, \mathcal{G}, \mathcal{G} = \{\mathcal{G}_t\}_{t \in \mathcal{T}})\) is called \textit{exhaustive} if \( \overline{\omega}(t) \) is in \( \mathcal{G}_t \) for any pair of \( \omega \in \Omega \) and \( t \in \mathcal{T} \).

Note that if \( \Omega \) is a finite set, \((\Omega, \mathcal{G}, \mathcal{G})\) is always exhaustive for any filtration \( \mathcal{G} \).
Figure 3.2.4: $\Omega[G]$
Remark 3.2.5. [Embedding of $\Omega$ into $\Omega[G]$] If a filtered measurable space $(\Omega, \mathcal{G}, G)$ is exhaustive, we have

$$\overline{\omega} = \bigvee \{e \in \Omega[G] \mid \omega \in e\}. \quad (3.2)$$

Moreover, if $\{\omega\} \in \mathcal{G}$ for every $\omega \in \Omega$, then the mapping

$$\Omega \xrightarrow{\sim} \Omega[G]$$
$$\omega \mapsto \overline{\omega}$$

is an embedding. This is why we call an element of $\Omega[G]$ an extended state.

3.2.2 Sets of extended states

Each observer has a subset of $\Omega[G]$ corresponding to her narrowing-down ability. Some ability may be determined by external reasons such as the size of the accessible information that came from the asymmetricity of information, whereas some may be determined by internal reasons such as her penetration.

Definition 3.2.6. [Neighborhoods and regular subsets]

Let $S$ be a subset of $\Omega[G]$.

1. A neighborhood $\mathcal{N}_S(\omega)$ of $\omega \in \Omega$ is a subset of $S$ defined by

$$\mathcal{N}_S(\omega) := \{e \in S \mid \omega \in e\}. \quad (3.3)$$

2. A subset $S$ is called regular if for every $\omega \in \Omega$, $\mathcal{N}_S(\omega)$ is non-empty, and has a sup in $S$. We write the sup by $\omega_S$.

Note that $\Omega[G]$ itself is a regular subset of $\Omega[G]$ if $(\Omega, \mathcal{G}, G)$ is exhaustive.

Remark 3.2.7. If a subset $S \in \Omega[G]$ is regular and $\{\omega\} \in \mathcal{G}$ for every $\omega \in \Omega$, then
3.2 Extended states

the mapping

\[ \Omega \longrightarrow S \]

\[ \psi \psi \]

\[ \omega \longrightarrow \omega_S \]

is an embedding.

Definition 3.2.8 defines a type of the subsets that may not contain some dumb extended states in the original set.

**Definition 3.2.8.** [Dominant subsets] Let \( S \) be a subset of \( \Omega[G] \).

1. \( \Omega_S := \{ \omega \in \Omega \mid (\exists e \in S)\omega \in e \} \).

2. A set \( D \subset S \) is called a **dominant** subset of \( S \) if it satisfies the following two conditions:

   (a) \( \Omega_D = \Omega_S \),
   
   (b) \( (\forall e \in D)(\forall d \in S)e \leq d \) implies \( d \in D \).

If \( D \) is a dominant subset of \( S \), an observer who has an ability represented by \( D \) is considered to be relatively smarter than an observer who has an ability represented by \( S \).

**Example 3.2.9.** For \( \varepsilon > 0 \), define \( \Omega^\varepsilon[G] \) by

\[ N^\varepsilon_G(\omega) := \{ e \in N_{\Omega[G]}(\omega) \mid (\forall t \in T)t \geq \varepsilon \Rightarrow \[ e(t) \notin G_{t-\varepsilon} \text{ or } (\forall A \in G_t)[\omega \in A \subset e(t) \text{ implies } A = e(t)] \} \}, \]

\[ \Omega^\varepsilon[G] := \bigcup_{\omega \in \Omega} N^\varepsilon_G(\omega). \]

Then, \( \Omega^\varepsilon[G] \) is a dominant subset of \( \Omega[G] \).

The tree shown in Figure 3.2.2 is considered as \( N_{\Omega[G]}(\omega_0) \). Then, \( N^2_G(\omega_0) \) is a tree in Figure 3.2.5, which is created just by removing the worst narrowing-down process \( e_9 \) from the tree in Figure 3.2.2.
Next we will think about a situation where an observer has a difficulty to access full information. In other words, the information she can access is limited to a subfiltration \( F = \{F_t\}_{t \in \mathcal{T}} \) of \( G \), where \( F_t \subset G_t \) for all \( t \in \mathcal{T} \). Then, it is easy to check that \( \Omega[F] \subset \Omega[G] \).

One of the examples of the situation comes with a filtration specifying information delay.

**Example 3.2.10.** Let \( f = \{f_t\}_{t \in \mathcal{T}} \) be a \( G \)-follower process introduced in Definition 2.1.3 and let \( G' = \{G'_t\}_{t \in \mathcal{T}} \) be a follower filtration modulated by \( f \) introduced in Definition 2.3.1. We can see the follower process \( f \) as a constraint enforced to the observer. Then the subset \( \Omega[G'] \subset \Omega[G] \) is a set of extended states representing her external ability.

By combining the results of Example 3.2.9 and Example 3.2.10, one of the specifications of subsets of \( \Omega[G] \) is of the form

\[
S := \Omega^f[G']
\] (3.6)
with a $G$-follower process $f$ and $\varepsilon > 0$ that can represent both the constraints coming from asymmetric information and the observer’s penetrating ability.

We may utilize this type of specifications when we build a set of scenarios systematically for stress tests required by authorized rules such as Basel III.

Here is a natural subfiltration of $G$, generated by a set of extended states.

**Definition 3.2.11.** Let $S \subset \Omega[G]$. A filtration $G^S = \{G^S_t\}_{t \in T}$ is defined by

\[ G^S_t := \sigma\{e(s) \mid e \in S, s \leq t\}. \]  

(3.7)

The resulting filtration, however, loses some characteristics of the original set of extended states since we cannot recover the set from the filtration though we have a following relation.

**Proposition 3.2.12.** For $S \subset \Omega[G]$, $S \subset \Omega[G^S]$.

**Proof.** Immediate. \qed

### 3.2.3 Worlds

As we stated in Section 3.1, our original motivation of introducing extended states was to replace $\Omega$ in $T \times \Omega$ with $\Omega[G]$. Definition 3.2.13 introduces this new “$T \times \Omega$”.

**Definition 3.2.13.** [Worlds] Let $S$ be a subset of $\Omega[G]$.

1. For $t \in T$ and $e \in \Omega[G]$, $e^t := e|_{[0,t]}$.

2. The set of *worlds* denoted by $W(S)$ is defined by

\[ W(S) := \{(t,e^t) \mid t \in T, e \in S\}. \]  

(3.8)

3. For $w = (t,e^t) \in W(S)$, $\overline{w} := e(t)$. 


3.3 Recursive utility functions

4. A binary relation \( \leq \) on \( \mathcal{W}(S) \) is defined by for \( (s, e^s), (t, d^t) \in \mathcal{W}, (s, e^s) \leq (t, d^t) \) iff \( s \leq t \) and \( e^s = d^t \).

**Proposition 3.2.14.** Let \( S \) be a subset of \( \Omega[G] \).

1. \( (\mathcal{W}(S), \leq) \) is a poset.

2. If \( \mathcal{G}_0 = \{ \emptyset, \Omega \} \), then \( (0, \bot^0) \in \mathcal{W}(S) \) is the least element.

**Proof.** Straightforward. \( \square \)

If \( S \subset \Omega[G] \) is a regular set, we have a natural map

\[
\begin{align*}
\mathcal{T} \times \Omega & \longrightarrow \mathcal{W}(S) \\
\cup & \\
(t, \omega) & \longrightarrow (t, (\omega|_{S})^t)
\end{align*}
\]

which plays an important role in Section 3.3.

### 3.3 Recursive utility functions

In this section, we assume the time domain is discrete and has its terminal (horizon) time \( T \), that is, \( T = \{0, 1, 2, \ldots, T\} \).

Let \( X \) be a Polish space with the Borel \( \sigma \)-field \( \mathcal{B}(X) \). A **consumption plan** is a bounded \( G \)-adapted process \( h : \mathcal{T} \times \Omega \rightarrow X \). Let \( \mathcal{H} := \mathcal{H}[G] \) be a set of all consumption plans.

#### 3.3.1 Recursive utility functions in classical settings

Now we proceed to review recursive utility functions in classical settings.

**Definition 3.3.1.** [Recursive utility functions in classical settings]

\( V : (\mathcal{T} \times \Omega) \rightarrow (\mathcal{H} \rightarrow \mathbb{R}) \) is a \( G \)-adapted process defined by

\[
V(t, \omega)(h) = \begin{cases} 
  u(h(t, \omega)) + \beta f(t, \omega)(V(t + 1, -)(h)) & \text{if } t < T, \\
  u(h(T, \omega)) & \text{if } t = T,
\end{cases}
\]

(3.9)
3.3 **Recursive utility functions**

where

1. \( u : X \to \mathbb{R} \) is a vNM type utility function,
2. \( \beta \in ]0, 1[ \),
3. \( J : \mathcal{T} \times \Omega \to (\Omega \to \mathbb{R}) \to \mathbb{R} \).

We can extend the first case of (3.9) like

\[
V(t, \omega)(h) = W(h(t, \omega), J(t, \omega)(V(t + 1, -)(h)))
\]

(3.10)

with a nonlinear aggregator \( W \). But, we keep using (3.9) just because we want to make things simpler in the subsequent discussion of the thesis. However, we may need some nonlinearity that is captured by the curvature of \( W \) in the future research.

We provide two examples of \( J \) defined in Definition 3.3.1. They are for the EU model and the MEU model, respectively

**Definition 3.3.2.** [Typical \( J \)'s]

1. For a probability measure \( \mu \) on \( (\Omega, \mathcal{G}) \), \( t \in \mathcal{T} \) and \( \omega \in \Omega \), define a probability measure \( \mu(t, \omega) \) on \( (\Omega, \mathcal{G}) \) by for \( A \in \mathcal{G} \),

\[
\mu(t, \omega)(A) := \mu(A \mid \mathcal{G}_t)(\omega).
\]

(3.11)

2. **EU model**

\[
J(t, \omega)(\xi) := \int_{\Omega} \xi d\mu_{t, \omega}(t, \omega),
\]

(3.12)

where \( \mu_{t, \omega} \) is a prior defined on \( (\Omega, \mathcal{G}) \).

3. **MEU model**

\[
J(t, \omega)(\xi) := \inf_{\mu \in \mathcal{P}(t, \omega)} \int_{\Omega} \xi d\mu(t, \omega),
\]

(3.13)

having prior ambiguity, where \( \mathcal{P}(t, \omega) \) is a set of probability measures on \( \Omega \).
3.3 Recursive utility functions

3.3.2 Recursive utility functions with state ambiguity

We want to replace $T \times \Omega$ appeared in Definition 3.3.1 by $W(S)$ in order to allow the value function $V$ to treat state ambiguity as well as prior ambiguity.

In the following discussion, let $S \subset \Omega[G]$ be a fixed set of extended states. Before defining sets of priors, we need some auxiliary sets that relate to possible next steps from a given world $w \in W(S)$.

**Definition 3.3.3.** Let $w = (t, e_t) \in W(S)$.

1. For $\omega \in \Omega$, a subset $N(w, \omega) \subset W(S)$ is defined by
   \[
   N(w, \omega) := \{(t + 1, d^{t+1}) \mid d \in N_S(\omega), d^t = e^t\}. \tag{3.14}
   \]

2. A subset $D(w) \subset \Omega$ is defined by
   \[
   D(w) := \{\omega \in \overline{w} \mid N(w, \omega) \neq \emptyset\}. \tag{3.15}
   \]

3. A subset $S \subset \Omega[G]$ is called **proper** if $D(w) \in \mathcal{G}_t$ for every $w = (t, e_t) \in W(S)$.

**Definition 3.3.4.** [Priors $\mathcal{P}$]

1. Let $w = (t, e_t) \in W(S)$ and $\mu$ be a probability measure on $(\Omega, \mathcal{G})$. $\mu$ is said **conditionable** with $w$ if the conditional probability measure $\mu(- \mid \overline{w})$ is well-defined on $\mathcal{G}_t$. \footnote{We want to avoid situations like the Borel-Kolmogorov paradox.}

2. A set-valued function $\Delta$ on $\mathcal{G}$ is defined by for $A \in \mathcal{G}$,
   \[
   \Delta(A) := \{\mu \mid \text{a probability measure on } (\Omega, \mathcal{G}) \text{ with } \mu(A) = 1\}. \tag{3.16}
   \]

3. The set of **extended worlds** denoted by $\text{EW}(S)$ is defined by
   \[
   \text{EW}(S) := \bigoplus_{w \in W(S)} \Delta(\overline{w}). \tag{3.17}
   \]
4. A subset $\mathcal{P} \subset \mathbf{EW}(S)$ is called to satisfy the \textbf{rectangularity condition} if the following three conditions hold:

(a) for any $w \in \mathcal{W}(S)$, there exists $\mu \in \Delta(w)$ such that $(w, \mu) \in \mathcal{P}$,

(b) if $(v, \mu) \in \mathcal{P}$ and $w \in \mathcal{W}(S)$ with $v \leq w$ and $\mu$ is conditionable with $w$, then $(w, \mu(- \mid w)) \in \mathcal{P}$,

(c) let $t \in \mathcal{T}$, $I$ be an index set either $\{0, 1, 2, \ldots, N-1\}$ or $\mathbb{N}$, $e, d_n \in S$ for all $n \in I$ such that $\{d_n(t)\}_{n \in I}$ are mutually disjoint and $\bigcup_{n \in I} d_n(t) = e(t)$. If $((t, e'), \mu), ((t, d_n), v_n) \in \mathcal{P}$ for all $n \in I$, then $((t, e'), \sum_{n \in I} \mu(d_n(t))v_n) \in \mathcal{P}$.

\textbf{Proposition 3.3.5.} Suppose that $\mathcal{P} \subset \mathbf{EW}(S)$ satisfies the rectangularity condition. Then, if $e(t) = d(t)$, we have $\mathcal{P}(t, e') = \mathcal{P}(t, d')$, where $\mathcal{P}(w) := \{\mu \mid (w, \mu) \in \mathcal{P}\}$.

\textbf{Proof.} Let $N := 1, d_0 := d$ and $v_0 := v$ in the condition (c) of Definition 3.3.4 (3). Then, since $e(t) = d(t)$, the assumptions of the condition (c) are satisfied. Therefore, $((t, e'), \mu), ((t, d'), v) \in \mathcal{P}$ imply $((t, e'), v) \in \mathcal{P}$. Then, by the condition (a) of Definition 3.3.4 (3), $\mathcal{P}(t, d') \subset \mathcal{P}(t, e')$. Similarly we have $\mathcal{P}(t, e') \subset \mathcal{P}(t, d')$. \hfill $\square$

\textbf{Definition 3.3.6.} When $\mathcal{P} \subset \mathbf{EW}(S)$ satisfies the rectangularity condition, $\overline{\mathcal{P}} : \bigoplus_{t \in \mathcal{T}} \mathcal{G}_t \rightarrow \text{Set}$ is a function defined by $\overline{\mathcal{P}}(t, \overline{w}) := \mathcal{P}(w)$.

Note that Definition 3.3.6 is well-defined by Proposition 3.3.5. On the other hand, if we have a function $\overline{\mathcal{P}} : \bigoplus_{t \in \mathcal{T}} \mathcal{G}_t \rightarrow \text{Set}$, we can recover $\mathcal{P} \subset \mathbf{EW}(S)$ by

$$\mathcal{P} := \{((t, e'), \mu) \mid \mu \in \overline{\mathcal{P}}(t, e(t))\}. \quad (3.18)$$

\textbf{Example 3.3.7.} Define $\mathcal{P}$ over the structure defined in Example 3.2.1.

1. For $(r_0, r_1, r_2) \in [0, 1]^3$, a probability measure on $(\Omega, \mathcal{G})$

   $\mu_{r_0,r_1,r_2}$ is defined by the tree in Figure 3.3.1.
2. For $r_L, r_H \in \mathbb{R}$ such that $0 < r_L \leq r_H < 1$, 

$$
\mathcal{P} := \{(w, \mu_{r_0r_1r_2}(- | \overline{w})) | w \in \mathbb{W}(S), (r_0, r_1, r_2) \in [r_L, r_H]^3\}. \tag{3.19}
$$

Then, it is obvious that $\mathcal{P}$ satisfies the rectangularity condition.

Now we are at the position to define our own recursive utility functions whose domains are $\mathbb{W}(S)$ instead of $\mathcal{T} \times \Omega$ as a straightforward extension of (3.9).

**Definition 3.3.8.** [State-ambiguity-sensitive recursive utility functions] $V : \mathbb{W}(S) \rightarrow (\mathcal{H} \rightarrow \mathbb{R})$ is a function defined by for $w = (t, e^t) \in \mathbb{W}(S)$ and $h \in \mathcal{H},$

$$
V(w)(h) = \begin{cases} 
I(\overline{w})(h(t, -)) + \beta J(\overline{w})(h) & \text{if } t < T, \\
I(\overline{w})(h(T, -)) & \text{if } t = T 
\end{cases} \tag{3.20}
$$

where

1. $I : \mathcal{G} \rightarrow ((\Omega \rightarrow X) \rightarrow \mathbb{R}),$

2. $\beta \in ]0, 1[,$

3. $J : \mathbb{W}(S) \rightarrow (\mathcal{H} \rightarrow \mathbb{R}).$

First, we give a typical form of the function $J$ and see what is going on with it through an example.
Definition 3.3.9. [Typical J] Assume that \( S \subset \Omega[G] \) is a proper subset. Then, typical form of function \( J \) for \( V \) defined in Definition 3.3.8 is

\[
J(w)(h) := \inf_{\eta \in \mathcal{P}(w)} \int_{\mathcal{D}(w)} \left( \inf_{v \in \mathcal{N}(w, \omega)} V(v)(h) \right) d\eta(\omega). \tag{3.21}
\]

In general, the values of the new \( J \) defined by (3.21) are smaller (or more conservative) than those of the old \( J \) defined by (3.13) since the new \( J \) has an extra inf to pick the minimum value in each neighborhood \( \mathcal{N}(w, \omega) \).

Example 3.3.10. In this example, we demonstrate how to calculate the value function with the function \( J \) in Definition 3.3.9 on top of the structure in Example 3.2.1.

Let \( S := \Omega^2[G] \) be a a dominant set whose subset \( \mathcal{N}_2^2(\omega_0) \) is shown in Figure 3.2.5, with priors defined in Example 3.3.7. We name the extended states \( e_0 \) through \( e_2 \) as shown in Figure 3.3.2. For \( t = 1, 2 \) and \( i = 0, 1, 2 \), let

\[
x_{t,i} := I(e_i(t))(h(t, -)). \tag{3.22}
\]
3.3 Recursive utility functions

Here is a backward calculation of $V(1, e_0^1)(h)$.

\[
V(2, e_0^1)(h) = I(e_0(2))(h(2, -)) = x_{2,0},
\]
\[
V(2, e_1^1)(h) = I(e_1(2))(h(2, -)) = x_{2,1},
\]
\[
V(2, e_2^1)(h) = I(e_2(2))(h(2, -)) = x_{2,2},
\]
\[
N((1, e_0^1), \omega) = \begin{cases}
\{(2, e_0^1), (2, e_2^1)\} & \text{if } \omega = \omega_0 \\
\{(2, e_1^1), (2, e_2^1)\} & \text{if } \omega = \omega_1 \\
\emptyset & \text{otherwise}
\end{cases}
\]
\[
D(1, e_0^1) = \{\omega_0, \omega_1\},
\]
\[
V(1, e_0^1)(h) = I(e_0(1))(h(1, -)) + \beta \inf_{\eta \in P(1 e_0^1)} \int_{D(1, e_0^1)} \left( \inf_{v \in N((1, e_0^1), \omega)} V(v)(h) \right) d\eta(\omega)
\]
\[
= x_{1,0} + \beta \inf_{\eta \in P(1 e_0^1)} \left( r_1(x_{2,0} \wedge x_{2,2}) + (1 - r_1)(x_{2,1} \wedge x_{2,2}) \right)
\]
\[
= x_{1,0} + \beta \begin{cases}
  r_H(x_{2,0} \wedge x_{2,2}) + (1 - r_H)(x_{2,1} \wedge x_{2,2}) & \text{if } x_{2,0} \leq x_{2,1} \\
  r_L(x_{2,0} \wedge x_{2,2}) + (1 - r_L)(x_{2,1} \wedge x_{2,2}) & \text{if } x_{2,0} \geq x_{2,1}
\end{cases} \tag{3.23}
\]

where $x \wedge y$ stands for $\inf\{x, y\}$.

Note that (3.23) becomes $x_{1,0} + \beta x_{2,2}$ when $x_{2,2} \leq x_{2,0} \wedge x_{2,1}$.

Supposing that we live in a world $(t, e') \in \mathcal{W}(S)$, we next try to find a concrete implementation of the function $I$ in order to calculate values $x_{t,i}$ in (3.22) that we reproduce below.

\[
x_{t,i} := I(e(t))(h(t, -)) \tag{3.24}
\]

As we noted in Section 3.2.1, we may think $e(t)$ in (3.24) as a pure experience that the observer perceives at time $t$. The pure experience is, by definition, a state before adding any discrimination. Here is a quote from Nishida [Nis11]:

“Pure experience is the intuition of facts just as they are and it is devoid of meaning.”
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\[
\begin{array}{c}
e(t) \quad \text{pure experience} \\
\downarrow \\
\omega \in e(t) \quad \text{meaning} \\
\downarrow \\
u(h(t, \omega)) \quad \text{judgment} \\
\end{array}
\]

Figure 3.3.3: An interpretation of extended states

So as shown in Figure 3.3.3, adding a thought on the pure experience \(e(t)\) in order to get its meaning is corresponding to an extraction of \(\omega\) from \(e(t)\). We make a judgment upon the meaning using a vNM type utility function appeared in Definition 3.3.1, whose value is \(u(h(t, \omega))\). But of course it depends on the selection of \(\omega \in e(t)\).

According to this line, one possible implementation is

\[
I(e(t))(h(t, -)) := \inf_{\omega \in e(t)} u(h(t, \omega)) \tag{3.25}
\]

for which we will see a better form of definition in Definition 3.3.11.

**Definition 3.3.11.** [A typical \(I\)] A typical implementation of the function \(I : \mathcal{G} \to ((\Omega \to X) \to \mathbb{R})\) used in Definition 3.3.8 is

\[
I(B)(\xi) := \inf_{\omega \in B} u(\xi(\omega)) \tag{3.26}
\]

where \(u : X \to \mathbb{R}\) is a vNM type utility function.

**Example 3.3.12.** This example performs the calculation of \(V(1, e^1_0)(h)\) as we did in Example 3.3.10 but with the function \(I\) introduced in Definition 3.3.11. For \(t = 1, 2\) and \(i = 0, 1, 2\), let

\[
u_{ij} := u(h(t, \omega_i)) \tag{3.27}
\]
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$$u_{0,0} = u(h(\omega_0))$$

$$u_{1,1} = u(h(\omega_1))$$

$$u_{2,0} = u(h(\omega_0))$$

$$u_{2,1} = u(h(\omega_1))$$

$$u_{2,2} = u(h(\omega_2))$$

$$u_{2,3} = u(h(\omega_3))$$

Figure 3.3.4: \( u \circ h : T \times \Omega \rightarrow \mathbb{R} \)

as shown in Figure 3.3.4.

$$x_{2,0} = I(e_0(2))(h(2, \cdot)) = \inf_{\omega \in \{\omega_0\}} u(h(2, \omega)) = u_{2,0},$$

$$x_{2,1} = I(e_1(2))(h(2, \cdot)) = \inf_{\omega \in \{\omega_1\}} u(h(2, \omega)) = u_{2,1},$$

$$x_{2,2} = I(e_2(2))(h(2, \cdot)) = \inf_{\omega \in \{\omega_0, \omega_1\}} u(h(2, \omega)) = u_{2,0} \land u_{2,1},$$

$$x_{1,0} = I(e_0(1))(h(1, \cdot)) = u_{1,0} \land u_{1,1} = u_{1,0},$$

$$V(1, e_0^1)(h) = x_{1,0} + \beta x_{2,2} = u_{1,0} + \beta (u_{2,0} \land u_{2,1}).$$ (3.29)

The last equation in (3.28) holds since \( h \) is \( G \)-adapted.

Getting back to the very intention of Nishida when he introduced the concept of pure experiences, the reality of the pure experience will break down once we add a thought on it. Here are some quotes from Nishida [Nis11]:

“What is immediate reality before we have added the fabrications of thinking?”

“The present of pure experience is not the present in thought, for once one thinks about the present, it is no longer present.”

“Reality is a succession of events that flow without stopping.”

So, it would be far better if we could interpret the function \( I \) without extracting a meaning \( \omega \) from the pure experience \( e(t) \), but by treating the intuition \( e(t) \) as
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a whole. Unfortunately, even if the consumption process $h$ is adapted with the filtration $G^S$ in Definition 3.2.11, the value $h(t, \omega)$ is not uniquely determined for $\omega \in \varepsilon(t)$ in general. Therefore, it might be required to introduce a new mechanism like a function having a domain $\Omega[G]$. Anyway, there needs much more work to go before reaching the goal.
Chapter 4

A categorical framework for filtrations and ambiguity

4.1 Introduction

In Chapter 3, we introduced ambiguity by extending the set of states $\Omega$ with a given filtration. In this chapter\(^1\), we will give another point of view to see filtrations and ambiguity. The formulation will be done with the help of category theory which is an area of study in mathematics that examines in an abstract way the properties of maps (called morphisms or arrows) satisfying some basic conditions.

Category theory has been applied in many fields including geometry, logic, computer science and string theory. Even for measure theory, there are some attempts to apply category theory such as Jackson [Jac06] or Breitsprecher [Bre77]. However, in finance theory, as far as we know, there has been nothing. So, this is a challenging topic.

In Section 4.2, we introduce a category called $\chi$ that can represent both risk and ambiguity. It is a structure having a time dimension as well as an ambiguity (spatial) dimension. Then we use $\chi$ as a domain of some useful functors (“functions” between categories). One of the functors is an generalized conditional

\(^1\)This chapter is based on Adachi [Ada14], Adachi [Ada13a] and Adachi [Ada12].
4.2 General settings

The following is a reproduction of (A.4) in Section A.2, which shows a family of functions representing a classical dynamic monetary value measure.

\[ \varphi = \{ \varphi_t : L(G_T) \to L(G_t) \}_{t \in [0,T]} \]  

\[ (4.1) \]
Now assume that $t < s < T$. Then, the family will be written like the following diagram.

\[
\begin{array}{ccc}
L(G_t) & \xrightarrow{\phi_t} & L(G_s) & \xleftarrow{\phi_s} & L(G_T)
\end{array}
\]

Let us try to extend this to like

\[
\begin{array}{ccc}
L(G_t) & \xrightarrow{\phi_t^l} & L(G_s) & \xleftarrow{\phi_s^l} & L(G_T)
\end{array}
\]

satisfying

\[\phi_T^l = \phi_t^l \circ \phi_s^l. \tag{4.2}\]

Then, this will allow us to investigate relative value functions, instead of restricting their origin to the horizon $T$.

If you compare the above diagram with the diagram in Definition A.3.1, you will see that it is natural to formulate monetary value measures in a categorical setting, which we will develop in this chapter.

Let us remind that $(\Omega, \mathcal{G})$ is the measurable space of functions valued with states of nature on which we define subjective probabilities.

### 4.2.1 Categories $\chi_F$, $\chi_P$ and $\chi$

In this section, we give a category called $\chi$ which will be a base category throughout this chapter, and define a generalized conditional expectation functor on it.

**Definition 4.2.1.** [Categories $\chi_F$ and $\chi_P$]

1. Let $\chi_F := \chi_F(\mathcal{G})$ be the partially ordered set of all sub-$\sigma$-field of $\mathcal{G}$ with a set-inclusion order. Then by Example A.3.3 we can think $\chi_F$ as a category.

2. Let $\chi_P := \chi_P(\mathcal{G})$ be the preordered set whose underlying set is the set of all probability measures defined on $(\Omega, \mathcal{G})$, equipped with a preorder $\leq_{\chi_P}$ on
4.2 General settings

\( \chi_{\mathcal{F}} \) defined by for \( \mu, \nu \in \chi_{\mathcal{F}} \),

\[
\mu \leq_{\chi_{\mathcal{F}}} \nu \; \text{iff} \; \mu \gg \nu
\]  

(4.3)

where \( \mu \gg \nu \) means that \( \nu \) is absolutely continuous to \( \mu \). Then again by Example A.3.3 we can think \( \chi_{\mathcal{F}} \) as a category.

**Definition 4.2.2.** [Category \( \chi \)] Let \( \chi := \chi(\mathcal{G}) \) be the product category\(^2\) \( \chi_{\mathcal{F}} \times \chi_{\mathcal{P}} \). In other words, it consists of all pairs of the form \( (\mathcal{F}, \mu) \), where \( \mathcal{F} \in \chi_{\mathcal{F}} \) and \( \mu \in \chi_{\mathcal{P}} \). For an object \( \mathcal{U} \in \chi \), we denote its \( \sigma \)-field and probability measure by \( \mathcal{F}_\mathcal{U} \) and \( \mathbb{P}_\mathcal{U} \), respectively. That is, \( \mathcal{U} = (\mathcal{F}_\mathcal{U}, \mathbb{P}_\mathcal{U}) \). Then, there is a unique arrow from \( \mathcal{V} \) to \( \mathcal{U} \) in \( \chi \) if and only if

\[
\mathcal{F}_\mathcal{V} \subset \mathcal{F}_\mathcal{U} \text{ and } \mathbb{P}_\mathcal{V} \gg \mathbb{P}_\mathcal{U}.
\]

(4.4)

We have projection functors \( \pi_{\mathcal{F}} \) and \( \pi_{\mathcal{P}} \) such as

\[
\begin{array}{ccc}
\chi_{\mathcal{F}} & \xrightarrow{\pi_{\mathcal{F}}} & \chi_{\mathcal{F}} \\
\downarrow \psi & & \downarrow \psi \\
\mathcal{F}_\mathcal{U} & \xleftarrow{\pi_{\mathcal{F}}} & \mathcal{U} & \xrightarrow{\psi} & \mathbb{P}_\mathcal{U}
\end{array}
\]

For objects \( \mathcal{V} \) and \( \mathcal{U} \) of \( \chi \) satisfying (4.4), we denote the unique arrow between them by \( *_{\mathcal{U}}^{\mathcal{V}} \).

We may be able to think the category \( \chi \) having two dimensions; one is a temporal dimension or risk dimension that is represented in a horizontal direction in Diagram 4.2.1, and the other is a spacial dimension or ambiguity dimension representing in a vertical direction.

Note that for objects \( \mathcal{U}, \mathcal{V} \in \chi, \mathcal{U} \) is isomorphic to \( \mathcal{V} \) (we write this by \( \mathcal{U} \simeq \mathcal{V} \)) if and only if \( \mathcal{F}_\mathcal{V} = \mathcal{F}_\mathcal{U} \) and \( \mathbb{P}_\mathcal{V} \approx \mathbb{P}_\mathcal{U} \) (equivalent).

---

\(^2\)See Example A.3.2 for a definition of product categories.
4.2 General settings

4.2.2 Functor $L$

**Definition 4.2.3.** For an object $U$ in $\chi$ and $X \in L^\infty(\Omega, F_U)$, define a subset $[X]_U \subset L^\infty(\Omega, F_U)$ by

$$[X]_U := \{ Y \in L^\infty(\Omega, F_U) \mid Y \sim_{P_U} X \}. \quad (4.5)$$

**Proposition 4.2.4.** Suppose that there are arrows $W \rightarrow V \rightarrow U$ in $\chi$.

1. $L^\infty(\Omega, F_W, P_W|_{F_W}) = \{ [X]_W \mid X \in L^\infty(\Omega, F_W) \}$.

2. For $X \in L^\infty(\Omega, F_W)$, $[X]_W \subset [X]_V$.

3. For $X, Y \in L^\infty(\Omega, F_W)$, $[X]_W = [Y]_W$ implies $[X]_V = [Y]_V$.

4. For $X \in L^\infty(\Omega, F_W)$ and $Z \in [X]_V$, $[X]_U = [Z]_U$.

**Proof:**
1. For $X, Y \in L^\infty(\Omega, F_W)$, $\{ X = Y \} \in F_W$. Therefore, $P_W\{ X = Y \} = P_W|_{F_W}\{ X = Y \}$. Thus, $X \sim_{P_W|_{F_W}} Y = X \sim_{P_W} Y$.

2. Since $F_W \subset F_V$, $L^\infty(\Omega, F_W) \subset L^\infty(\Omega, F_V)$. Also since $P_V \ll P_W$, $Y \sim_{P_W}$ $X$ implies $Y \sim_{P_V} X$.

3. By 2, we have $[X]_V \supset [X]_W = [Y]_W \subset [Y]_V$. Then, since $[X]_W$ is nonempty, two equivalence classes $[X]_V$ and $[Y]_V$ coincide.
4. Since $|Z|_V = |X|_V$, it is an immediate consequence by 3.

Proposition 4.2.4 makes the following definition be well-defined.

**Definition 4.2.5. [Functor $L$]**

A functor$^3$ $L : \chi \to \textbf{Set}$ is defined by for $V \to U$ in $\chi$,

\[
\begin{array}{ccl}
\mathcal{V} & \xrightarrow{L} & L_V := \mathcal{L}^\infty(\Omega, \mathcal{F}_V, \mathbb{P}_V|_{\mathcal{F}_V}) \\
\mathcal{U} & \xrightarrow{L} & L_U := \mathcal{L}^\infty(\Omega, \mathcal{F}_U, \mathbb{P}_U|_{\mathcal{F}_U})
\end{array}
\]

\[\mathcal{P}_U \ni [X]_{L_V} \subseteq [X]_{L_U}\]

**4.2.3 Generalized conditional expectations**

Now we are ready to develop one of the key functors in this chapter, a generalized conditional expectation that will be well-defined by the following proposition.

**Proposition 4.2.6.** For $V \hookrightarrow V \to U$ in $\chi$ and $X \in L_U$,

1. $\mathbb{E}^{P_U}[X|\mathcal{F}_V] \frac{dP_U}{dP_V}|_{\mathcal{F}_V} = \mathbb{E}^{P_V}[X \frac{dP_U}{dP_V}|_{\mathcal{F}_V}]$ $\mathbb{P}_V$-a.s.,

2. $\frac{dP_U}{dP_V}|_{\mathcal{F}_U} \times \frac{dP_U}{dP_V}|_{\mathcal{F}_U} = \frac{dP_U}{dP_V}|_{\mathcal{F}_U} \mathbb{P}_U$-a.s.

where $\frac{dP_U}{dP_V}$ is a Radon-Nikodym derivative of $\mathbb{P}_U$ with respect to $\mathbb{P}_V$.

**Proof.**

1. When $Q \ll P$ and $\mathcal{F} \subset \mathcal{G}$, we have

\[
\frac{dQ}{dP}|_{\mathcal{F}} = \mathbb{E}^{P}[\frac{dQ}{dP}|_{\mathcal{F}}] \quad \mathbb{P}$-a.s. \hfill (4.6)

and

\[
\mathbb{E}^{Q}[X|\mathcal{F}] = \frac{\mathbb{E}^{P}[X \frac{dQ}{dP}|_{\mathcal{F}}]}{\mathbb{E}^{P}[\frac{dQ}{dP}|_{\mathcal{F}}]} \quad \mathbb{Q}$-a.s. \hfill (4.7)

$^3$See Definition A.3.6 for a definition of functors.
by Proposition A.11 and Proposition A.12 in Föllmer and Schied [FS11]. Then, by (4.6) and since $X$ is $\mathcal{F}_U$-measurable, we have with $\mathbb{P}_V$-a.s.,

$$
\mathbb{E}^{P_V}[X \frac{d\mathbb{P}_U}{d\mathbb{P}_V} |_{\mathcal{F}_U} |_{\mathcal{F}_V}] = \mathbb{E}^{P_V}[X \mathbb{E}^{P_V}[\frac{d\mathbb{P}_U}{d\mathbb{P}_V} |_{\mathcal{F}_U} |_{\mathcal{F}_V}]] = \mathbb{E}^{P_V}[X \mathbb{E}^{P_V}[\frac{d\mathbb{P}_U}{d\mathbb{P}_V} |_{\mathcal{F}_U} |_{\mathcal{F}_V}]] = \mathbb{E}^{P_V}[X \frac{d\mathbb{P}_U}{d\mathbb{P}_V} |_{\mathcal{F}_V}].
$$

Therefore, again by (4.6) and (4.7), we get the desired equation.

2. By (4.6), (4.7) and again by (4.6), we have with $\mathbb{P}_U$-a.s.,

$$
\frac{d\mathbb{P}_U}{d\mathbb{P}_V} |_{\mathcal{F}_U} = \mathbb{E}^{P_V}[\frac{d\mathbb{P}_U}{d\mathbb{P}_V} |_{\mathcal{F}_U}] = \frac{\mathbb{E}^{P_W}[\frac{d\mathbb{P}_U}{d\mathbb{P}_V} |_{\mathcal{F}_U}]}{\mathbb{E}^{P_W}[\frac{d\mathbb{P}_V}{d\mathbb{P}_W} |_{\mathcal{F}_U}]} = \frac{\mathbb{E}^{P_W}[\frac{d\mathbb{P}_U}{d\mathbb{P}_V} |_{\mathcal{F}_U}]}{\mathbb{E}^{P_W}[\frac{d\mathbb{P}_V}{d\mathbb{P}_W} |_{\mathcal{F}_U}]} = \frac{\frac{d\mathbb{P}_U}{d\mathbb{P}_V} |_{\mathcal{F}_U}}{\frac{d\mathbb{P}_V}{d\mathbb{P}_W} |_{\mathcal{F}_U}}.
$$

\[ \square \]

**Definition 4.2.7.** [Generalized conditional expectation] A **generalized conditional expectation** is a contravariant functor $^\varepsilon : \chi^\text{op} \to \text{Set}$ defined by for $V \to U$ in $\chi$,

$$
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{\varepsilon} & \mathcal{E}(\mathcal{V}) := L_{\mathcal{V}} \\
\downarrow & & \uparrow \varepsilon^\mathcal{V}_{\mathcal{U}} \\
\mathcal{U} & \xrightarrow{\varepsilon} & \mathcal{E}(\mathcal{U}) := L_{\mathcal{U}}
\end{array}
$$

where

$$
\varepsilon^\mathcal{V}_{\mathcal{U}}(X) := \mathbb{E}^{P_U}[X |_{\mathcal{F}_V}] \frac{d\mathbb{P}_U}{d\mathbb{P}_V} |_{\mathcal{F}_V} \quad (4.8)
$$

for $X \in L_{\mathcal{U}}$.

Note that $\varepsilon^\mathcal{V}_{\mathcal{U}}$ in Definition 4.2.7 is well-defined by Proposition 4.2.6. See also Diagram 4.2.2.

---

\[ ^4 \text{See Definition A.3.8 for a definition of contravariant functors.} \]
4.24 The Yoneda lemma and stochastic processes

Now, we mention an interpretation of one of the most important theorems in category theory called the Yoneda lemma in our setting\(^5\).

**Theorem 4.2.8.** [The Yoneda lemma] For any monetary value measure \( \varphi : \chi^{op} \rightarrow \text{Set} \) and an object \( U \) in \( \chi \), there exists a bijective correspondence \( y_{\varphi,U} \) specified by the following

---

\(^5\)See Lemma A.3.14 for the original Yoneda Lemma.
4.3 Monetary value measures

In this subsection, we formulate monetary value measures with the category $\chi$ defined in Section 4.2, and show that the resulting monetary value measures satisfy the time consistency condition that was provided as an axiom in the classical setting. Next, we investigate a robust representation of concave monetary value measures in our framework, which was originally developed by Artzner et al. in [ADEH99] in the classical setting. Finally, we present a possible technique to break the time consistency condition by extending the definition of monetary value measures.

Let us see the representable functor $\chi(-, U)$ as a generalized time domain with the time horizon $U$. Then a natural transformation from $\chi(-, U)$ to $\varphi$ can be seen as a stochastic process that is (in a sense) adapted to $\varphi$, and its corresponding $\mathcal{F}_U$-measurable random variable represents a terminal value (payoff) at the horizon.

The Yoneda lemma says that we have a bijective correspondence between those stochastic processes and random variables.

\begin{equation}
\begin{aligned}
y_{\varphi, U}: \text{Nat}(\chi(-, U), \varphi) &\xrightarrow{\cong} L_U \\
\alpha &\longmapsto \alpha_u(*_u^{\varphi}) \\
\tilde{X} &\longmapsto X
\end{aligned}
\end{equation}

where $\tilde{X}$ is a natural transformation defined by for any $V \to U$ in $\chi$, $\tilde{X}_V(*_u^V) := \varphi_u^V(X)$.

Moreover, the correspondence is natural in both $\varphi$ and $U$. 

**Diagram:**

\[ y_{\varphi, U} : \text{Nat}(\chi(-, U), \varphi) \xrightarrow{\cong} L_U \]

\[ \alpha \longmapsto \alpha_u(*_u^{\varphi}) \]

\[ \tilde{X} \longmapsto X \]

where $\tilde{X}$ is a natural transformation defined by for any $V \to U$ in $\chi$, $\tilde{X}_V(*_u^V) := \varphi_u^V(X)$. 

Moreover, the correspondence is natural in both $\varphi$ and $U$. 

**4.3 Monetary value measures**

In this subsection, we formulate monetary value measures with the category $\chi$ defined in Section 4.2, and show that the resulting monetary value measures satisfy the time consistency condition that was provided as an axiom in the classical setting. Next, we investigate a robust representation of concave monetary value measures in our framework, which was originally developed by Artzner et al. in [ADEH99] in the classical setting. Finally, we present a possible technique to break the time consistency condition by extending the definition of monetary value measures.
4.3 Monetary value measures

4.3.1 A categorical framework for monetary value measures

**Definition 4.3.1.** [Monetary value measures] A monetary value measure is a contravariant functor\(^6\)

\[ \varphi : \chi^{\text{op}} \rightarrow \text{Set} \tag{4.9} \]

satisfying the following two conditions:

1. for \( U \in \chi \), \( \varphi(U) := L_U \),
2. for \( V \rightarrow U \) in \( \chi \), the map \( \varphi_U^V := \varphi(V \rightarrow U) : L_U \rightarrow L_V \) satisfies

   (a) Cash invariance: \( (\forall X \in L_U)(\forall Z \in L_V) \varphi_U^V(X + L_U^V(Z)) = \varphi_U^V(X) + Z \mathbb{P}_V\)-a.s.,

   (b) Monotonicity: \( (\forall X \in L_U)(\forall Y \in L_U) X \leq Y \Rightarrow \varphi_U^V(X) \leq \varphi_U^V(Y) \mathbb{P}_V\)-a.s.,

   (c) Normalization: \( \varphi_U^V(0_{L_U}) = 0_{L_V} \mathbb{P}_V\)-a.s. if \( \mathbb{P}_V = \mathbb{P}_U \).

At this point, we do not require the monetary value measures to satisfy some familiar conditions such as concavity, positive homogeneity or law invariance. Instead of doing so, we want to see what kind of properties are deduced from this minimal setting.

The most crucial key points of Definition 4.3.1 is that \( \varphi \) does not move only toward time direction but also moves over several absolutely continuous probability measures internally. This means we have a possibility to develop risk measures including ambiguity within this formulation.

Another key points of Definition 4.3.1 is that \( \varphi \) is a contravariant functor. So, for any triple \( \mathcal{W} \rightarrow \mathcal{V} \rightarrow \mathcal{U} \) in \( \chi \), we have, as seeing in Diagram 4.3.1,

\[ \varphi_{\mathcal{W}}^{\mathcal{U}} = 1_{\mathcal{L}_\mathcal{U}} \text{ and } \varphi_{\mathcal{V}}^{\mathcal{W}} \circ \varphi_{\mathcal{U}}^{\mathcal{V}} = \varphi_{\mathcal{U}}^{\mathcal{V}}. \tag{4.10} \]

\(^6\)Actually, this is a presheaf (See Definition A.3.13). We will investigate monetary value measures as presheaves in Section 4.4.
Example 4.3.2. [Entropic value measure] For a positive real number $\lambda$, an entropic value measure is a contravariant functor $\varphi : \chi^{op} \to \textbf{Set}$ defined by

$$\varphi(U) := L_U \quad \text{and} \quad \varphi^V_U(X) := \lambda^{-1} \log E^V_U \left( e^{\lambda X} \right)$$

for $V \to U$ in $\chi$ and $X \in L_U$. Then, it is easy to see that the contravariant functor $\varphi$ is well-defined and is a monetary value measure.

Now in case $F_V = F_U$, we have

$$\varphi^V_U(X) = \lambda^{-1} \log E^V_U \left( e^{\lambda X} \right) = \lambda^{-1} \log \left( e^{\lambda X} \frac{dP_U}{dP_V} |_{F_V} \right) = X + \lambda^{-1} \log \left( \frac{dP_U}{dP_V} |_{F_V} \right).$$

Especially, we have $\varphi^V_U(0_{L_U}) = \lambda^{-1} \log \left( \frac{dP_U}{dP_V} |_{F_V} \right)$, which is not $0_{L_V}$ unless $P_V = P_U$ on $F_V$.

This is the reason we require the assumption $P_V = P_U$ in the normalization condition in Definition 4.3.1.

Here are some properties of monetary value measures.

**Proposition 4.3.3.** Let $\varphi : \chi^{op} \to \textbf{Set}$ be a monetary value measure, and $W \to V \to U$ be arrows in $\chi$.

1. If $P_V = P_U$, we have $\varphi^V_U \circ L^V_U = 1_{L_V}$.

2. Idempotence: If $P_V = P_U$, we have $\varphi^V_U \circ L^V_U \circ \varphi^V_U = \varphi^V_U$. 

Diagram 4.3.1
3. Local property: \((\forall X \in L_u)(\forall Y \in L_u)(\forall A \in V)\) \(q^V_{U}(\mathbb{1}_A X + \mathbb{1}_A Y) = \mathbb{1}_A q^V_{U}(X) + \mathbb{1}_A q^V_{U}(Y)\).

4. Dynamic programming principle: If \(P_V = P_U\), we have \(q^V_{U} = q^V_{U} \circ L^V_{U} \circ q^V_{U}\).

5. Time consistency: \((\forall X \in L_u)(\forall Y \in L_u)\) \(q^V_{U}(X) \leq q^V_{U}(Y) \Rightarrow q^V_{U}(X) \leq q^V_{U}(Y)\).

Proof:
1. For \(X \in L_V\), we have by cash invariance and normalization, \(q^V_{U}(L^V_{U}(X)) = q^V_{U}(0) + L^V_{U}(X) = q^V_{U}(0) + X = X\).

   3. Immediate by (1).

   3. First, we show that for any \(A \in V\),

   \[
   \mathbb{1}_A q^V_{U}(X) = \mathbb{1}_A q^V_{U}(\mathbb{1}_A X). \tag{4.12}
   \]

   Since \(X \in L^\infty(\Omega, U, \mathbb{P})\), we have \(|X| \leq \|X\|\). Therefore,

   \[
   \mathbb{1}_A X - \mathbb{1}_A \|X\| \leq \mathbb{1}_A X + \mathbb{1}_A \|X\| \leq \mathbb{1}_A X + \mathbb{1}_A \|X\|_\infty.
   \]

   Then, by cash invariance and monotonicity,

   \[
   q^V_{U}(\mathbb{1}_A X) - \mathbb{1}_A \|X\| = q^V_{U}(\mathbb{1}_A X - \mathbb{1}_A \|X\|) \\
   \leq q^V_{U}(X) \\
   \leq q^V_{U}(\mathbb{1}_A X + \mathbb{1}_A \|X\|) = q^V_{U}(\mathbb{1}_A X) + \mathbb{1}_A \|X\|_\infty.
   \]

   Then,

   \[
   \mathbb{1}_A q^V_{U}(\mathbb{1}_A X) = \mathbb{1}_A (q^V_{U}(\mathbb{1}_A X) - \mathbb{1}_A \|X\|) \\
   \leq \mathbb{1}_A q^V_{U}(X) \\
   \leq \mathbb{1}_A (q^V_{U}(\mathbb{1}_A X) + \mathbb{1}_A \|X\|_\infty) = \mathbb{1}_A q^V_{U}(\mathbb{1}_A X).
   \]

   Therefore, we get (4.12).
Next by using (4.12) twice, we have
\[
\phi^V_U(1_A X + 1_A Y) = 1_A \phi^V_U(1_A X + 1_A Y) + 1_A \phi^V_U(1_A X + 1_A Y)
\]
\[
= 1_A \phi^V_U(1_A (1_A X + 1_A Y)) + 1_A \phi^V_U(1_A (1_A X + 1_A Y))
\]
\[
= 1_A \phi^V_U(1_A X) + 1_A \phi^V_U(1_A Y)
\]
\[
= 1_A \phi^V_U(X) + 1_A \phi^V_U(Y).
\]

4. By (2) and (4.10), we have
\[
\phi^W_U = \phi^W_V \circ \phi^V_U = (\phi^W_V \circ L^V_U \circ \phi^V_U)
\]
\[
= (\phi^W_V \circ \phi^V_U) \circ (L^V_U \circ \phi^V_U) = \phi^W_U \circ L^V_U \circ \phi^V_U.
\]

5. Assume \(\phi^V_U(X) \leq \phi^V_U(Y)\). Then, by monotonicity and (4.10),
\[
\phi^W_U(X) = \phi^W_U(\phi^V_U(X)) \leq \phi^W_U(\phi^V_U(Y)) = \phi^W_U(Y).
\]

In Proposition 4.3.3, two properties, dynamic programming principle and time consistency are usually introduced as axioms ([IDS06]). But, we derive them naturally here from the fact that the monetary value measure is a contravariant functor as a proposition. This may be seen as an evidence that the two axioms are natural.

### 4.3.2 Robust representation of concave value measures

**Definition 4.3.4.** [Concave value measure] A monetary value measure \(\phi\) is said to be a **concave value measure** if for any arrow \(V \to U\) in \(\chi\), \(X, Y \in L_U\) and \(\lambda \in [0, 1]\),
\[
\phi^V_U(\lambda X + (1 - \lambda)Y) \geq \lambda \phi^V_U(X) + (1 - \lambda) \phi^V_U(Y).
\]  \hspace{1cm} (4.13)
Now we proceed to examine a robust representation of concave value measures that was originally studied in Artzner et al. [ADEH99] in our framework.

**Definition 4.3.5.** [Acceptance sets] Let $\varphi : \chi^{\sigma} \to \text{Set}$ be a monetary value measure and $\mathcal{V} \to \mathcal{U}$ be in $\chi$.

1. $\mathcal{A}_\mathcal{U} := \{ X \in L_\mathcal{U} \mid X \geq 0_{L_\mathcal{U}} \mathbb{P}_\mathcal{U}\text{-a.s.} \}$.

2. An **acceptance set** $\mathcal{A}_\mathcal{U}^\mathcal{V}(\varphi) \subset L_\mathcal{U}$ is the set making the following diagram a pullback.

   \[
   \begin{array}{ccc}
   \mathcal{A}_\mathcal{U}^\mathcal{V}(\varphi) & \longrightarrow & \mathcal{A}_\mathcal{V} \\
   \downarrow & & \downarrow \\
   L_\mathcal{U} & \overset{\varphi_\mathcal{U}^\mathcal{V}}{\longrightarrow} & L_\mathcal{V}
   \end{array}
   \]

3. For $X \in L_\mathcal{U}$, $\mathbb{E}_\mathcal{U}^\mathcal{V}[X] \in L_\mathcal{V}$ is defined by $\mathbb{E}_\mathcal{U}^\mathcal{V}[X] := \mathbb{E}_\mathcal{V}[X \mid \mathcal{F}_\mathcal{V}]$.

4. **Reward function** : $\alpha_\mathcal{U}^\mathcal{V} := \mathbb{P}_\mathcal{V}\text{-ess inf} \mathbb{E}_\mathcal{U}^\mathcal{V}[\mathcal{A}_\mathcal{U}^\mathcal{V}(\varphi)]$.

It is well-known that the pullback of a monic arrow\(^7\) (that is, a natural embedding function) is considered as an inverse image. Therefore, it is straightforward to show that this representation of acceptance sets coincides with that of Artzner et al. [ADEH99].

The following two propositions are quite standard ones in the sense that they only discuss monetary value measures for fixed two $\sigma$-fields, which means they basically concern with one-period conditional value measure theory [FS11].

**Proposition 4.3.6.** Let $\mathcal{V} \to \mathcal{U}$ be an arrow in $\chi$.

1. $\mathcal{A}_\mathcal{U}^\mathcal{V}(\varphi) = \{ X \in L_\mathcal{U} \mid \varphi_\mathcal{U}^\mathcal{V}(X) \geq 0_{L_\mathcal{V}} \mathbb{P}_\mathcal{V}\text{-a.s.} \}$.

2. For $X \in \mathcal{A}_\mathcal{U}^\mathcal{V}(\varphi)$ and $Y \in L_\mathcal{U}$, $Y \geq X \mathbb{P}_\mathcal{U}\text{-a.s.}$ implies $Y \in \mathcal{A}_\mathcal{U}^\mathcal{V}(\varphi)$.

3. $\text{ess inf}_{L_\mathcal{U}}(\mathcal{A}_\mathcal{U}^\mathcal{V}(\varphi) \cap L_\mathcal{U}^\mathcal{V}(L_\mathcal{V})) = 0_{L_\mathcal{U}}$.

**Proof.**

1. It is immediate since $\mathcal{A}_\mathcal{U}^\mathcal{V} = (\varphi_\mathcal{U}^\mathcal{V})^{-1}(\mathcal{A}_\mathcal{V})$.

---

\(^7\)See Definition A.3.4 for a definition of monics.
2. By 1 and the monotonicity of \( \varphi^V_U \).

3. By 1, \( \mathcal{A}^V_U(\varphi) \cap L_U^V(Y) = \{ L_U^V(Y) \mid Y \in L_V, \ \varphi^V_U(L_U^V(Y)) \geq 0 \ \mathbb{P}_V^- \text{-a.s.} \} = \{ L_U^V(Y) \mid Y \in L_V, \ Y \geq 0 \ \mathbb{P}_V^- \text{-a.s.} \} \). The result is a direct consequence of this equation.

Proposition 4.3.7. \( \varphi^V_U(X) = \text{ess sup}_{L_V} \{ Y \in L_V \mid X - Y \in \mathcal{A}^V_U(\varphi) \} \).

Proof. For \( Y \in L_V, X - Y \in \mathcal{A}^V_U(\varphi) \) iff \( \varphi^V_U(X - Y) \geq 0 \mathbb{P}_V^- \text{-a.s.} \). But, \( \varphi^V_U(X - Y) = \varphi^V_U(X) - Y \) by cash invariance. Hence, \( X - Y \in \mathcal{A}^V_U(\varphi) \) iff \( \varphi^V_U(X) \geq Y \mathbb{P}_V^- \text{-a.s.} \). Therefore all we need to show is

\[
\varphi^V_U(X) = \text{ess sup}_{L_V} \{ Y \in L_V \mid \varphi^V_U(X) \geq Y \mathbb{P}_V^- \text{-a.s.} \}.
\]

But this is straightforward.

Next, we will see a dynamic property of acceptance sets.

Definition 4.3.8. For \( S, T \subset L_U \), the subset \( S + T \subset L_U \) is defined by

\[
S + T := \{ X + Y \mid X \in S, Y \in T \}.
\]

Proposition 4.3.9. For \( W \to V \to U \) in \( \chi \), we have \( \mathcal{A}^V_U(\varphi) = L_U^V(\mathcal{A}^V_U(\varphi)) + \mathcal{A}^V_U(\varphi) \).

Proof. For \( X \in \mathcal{A}^V_U(\varphi) \), Define \( Y := \varphi^V_U(X) \) and \( Z := X - L_U^V(Y) \). Then obviously, \( Y \in L_V \) and \( Z \in L_U \). Moreover, \( \varphi^V_Y(Y) = \varphi^V_Y(\varphi^V_U(X)) = \varphi^V_U(X) \geq 0 \) and \( \varphi^V_U(Z) = \varphi^V_U(X - L_U^V(Y)) = \varphi^V_U(X) - Y = 0 \). Therefore, \( Y \in \mathcal{A}^V_Y(\varphi) \) and \( Z \in \mathcal{A}^V_U(\varphi) \).

Conversely, let \( X = L_U^V(Y) + Z \) where \( Y \in \mathcal{A}^V_Y(\varphi) \subset L_V \) and \( Z \in \mathcal{A}^V_U(\varphi) \). Then, obviously \( X \in L_U \). Moreover, \( \varphi^V_U(X) = \varphi^V_Y(\varphi^V_U(L_U^V(Y) + Z)) = \varphi^V_Y(Y + \varphi^V_U(Z)) \geq \varphi^V_Y(Y) \geq 0 \). Hence, \( X \in \mathcal{A}^V_U(\varphi) \).
4.3 Monetary value measures

Note that the proof of Proposition 4.3.9 usually requires an extra time-consistency axiom, but we do not need it since it is already contained as a consequence of the fact that monetary value measures are functors.

**Definition 4.3.10.** [Subsets of $\chi$] Let $\mathcal{V} \to \mathcal{U}$ be an arrow in $\chi$. Then, we define the following sets of objects in $\chi$.

1. $Q_{\mathcal{V}} := \{ W \in \chi \mid W \simeq \mathcal{V} \}$,
2. $P_{\mathcal{V}}^{\mathcal{V}} := \{ W \in Q_{\mathcal{V}} \mid \mathcal{P}_{W}|_{\mathcal{F}_{W}} = \mathcal{P}_{\mathcal{U}}|_{\mathcal{F}_{\mathcal{V}}} \}$,
3. $P_{\mathcal{U}}^{\mathcal{V}} := \{ W \in P_{\mathcal{U}}^{\mathcal{V}} \mid \mathbb{E}_{W}^{\mathcal{V}}[a_{W}^{\mathcal{V}}] > -\infty \},$

where $W \simeq \mathcal{V}$ means that the object $W$ is isomorphic to the object $\mathcal{V}$ in the category $\chi$.

Obviously, $P_{\mathcal{U}}^{\mathcal{V}} \subset P_{\mathcal{U}}^{\mathcal{V}} \subset Q_{\mathcal{V}} \subset \chi$.

The following theorem is almost same as Theorem 11.2 in Föllmer and Schied [FS11].

**Theorem 4.3.11.** [Robust representation of monetary value measures] For a concave value measure $\varphi : \chi^{op} \to \textbf{Set}$, the following properties are equivalent where $\mathcal{V} \to \mathcal{U}$ is an arrow in $\chi$, $\{X_{n}\}$ is any sequence in $L_{\mathcal{U}}$ and $X \in L_{\mathcal{U}}$.

1. Let $Q$ be any set satisfying $P_{\mathcal{U}}^{\mathcal{V}} \subset Q \subset Q_{\mathcal{V}}$.

\[
\varphi_{\mathcal{V}}^{\mathcal{U}}(X) = \underset{W \in Q}{\text{ess inf}} \varphi_{W_{0}}^{V}(L_{W_{0}}^{W}(\mathbb{E}_{W}^{V}[X] - a_{W}^{V})) \quad (4.14)
\]

where $W_{0} := (\mathcal{F}_{V}, \mathcal{P}_{\mathcal{U}})$.

2. $\varphi_{\mathcal{U}}^{\mathcal{V}}$ has the reverse Fatou property, i.e.,

\[
\limsup_{n \to \infty} \varphi_{\mathcal{V}}^{\mathcal{U}}(X_{n}) \leq \varphi_{\mathcal{U}}^{\mathcal{V}}(X) \quad \mathcal{P}_{U}\text{-a.s.} \quad (4.15)
\]

if $\{X_{n}\} \subset L_{\mathcal{U}}$ is bounded and converges $\mathcal{P}_{U}$-a.s. to $X$. 
4.3 Monetary value measures

3. \( \phi^V_U \) is continuous from above, i.e.,

\[
X_n \downarrow X \mathbb{P}_U\text{-a.s.} \Rightarrow \phi^V_U(X_n) \nearrow \phi^V_U(X) \mathbb{P}_V\text{-a.s.}. \tag{4.16}
\]

**Proof.** The three statements 1, 2 and 3 are corresponding to the statements (a), (b) and (c) of Theorem 11.2 in F"ollmer and Schied [FS11]. The difference is that Theorem 11.2 treats (conditional) monetary risk measures \( \rho \) while our theorem treats monetary value measures \( \phi \). So, we need to translate the result of Theorem 11.2 to our framework by reverting the signs in equations. Upon the reverting, we should note that \( \phi^V_U = \phi^V_{W_0} \circ \phi^W_{U_0} \) and that we can apply Theorem 11.2 directly to the function \( \phi^W_{U_0} \) since \( W_0 \) and \( U \) have same probability measure \( \mathbb{P}_U \). Then, the monotonicity of the function \( \phi^V_{W_0} \) will become the final piece to complete the proof.

Let us have a closer look at the representation (4.14) in a categorical way. If we define a function \( \eta^W_U(X) \) by

\[
\eta^W_U(X) := L^W_{W_0} (E^W_U[X] - \alpha^W_U), \tag{4.17}
\]

then, (4.14) can be written like

\[
\phi^V_U = \text{ess inf}_{W \in \mathcal{Q}} (\phi^V_{W_0} \circ \eta^W_U). \tag{4.18}
\]

Now, Suppose \( \mathcal{Q} \) as a discrete category\(^8\), and let \( L^L_V \) be a partially ordered set of all monotonic functions from \( L_U \) to \( L_V \), which considered as a category. Then, by defining a functor \( R^V_U : \mathcal{Q} \rightarrow L^L_V \) by for \( W_1, W_2 \in \mathcal{Q} \),

\[
W_1 \xrightarrow{R^V_U} R^V_U(W_1) := \phi^V_{W_0} \circ \eta^W_U
\]

\[
W_2 \xrightarrow{R^V_U} R^V_U(W_2) := \phi^V_{W_0} \circ \eta^W_U
\]

\(^8\)See Example A.3.3 for a definition of discrete categories.
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we can represent $\phi^V_{U_t}$ as a limit\(^9\) of the functor $R^V_{U_t}$. That is,

$$
\phi^V_{U_t} = \lim_{\leftarrow} R^V_{U_t},
$$

(4.19)

which is a categorically robust representation of $\phi^V_{U_t}$ in a sense.

4.3.3 Breaking the time consistency

In [KS09], Kupper and Schachermayer proved that the axiom of time consistency together with the axiom of *law invariance* admits only one type of value measures, *entropic value measures*. On the other hand, in Proposition 4.3.3, we showed that our monetary value measures always satisfy the time consistency condition. Therefore, if we want to require our monetary value measures to hold the law invariance condition, our theory falls into just a theory of entropic value measures, which is not so great in two points.

The first point is that entropic value measures are not widely used by practitioners in the real world. The second point is that we see in many occasions in (again) the real world the situations where the time consistency condition, a sort of a linearity nature does not hold.

So, we want to relax the condition of time consistency to accept other types of dynamic monetary value measures by extending the definition of our monetary value measures.

In this subsection, we treat the issue how to break the time consistency paradigm.

A hint is in the form of (4.14) which is a representation of concave value measures.

$$
\phi^V_{U_t}(X) = \essinf_{W \in Q} \phi^V_{W_0}(L^W_{W_0}(E^V_{U_t}[X] - r^W_{U_t}))
$$

The equation succeeds to introduce a non-linearity by using “ess inf” over the set

---

\(^9\)See Definition A.3.16 for a definition of limits.
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\[ \begin{align*}
W & \xrightarrow{\mathcal{F}_W, \mathbb{P}_W} V \xrightarrow{\mathcal{F}_V, \mathbb{P}_V} U \\
\mathcal{F}_W, \mathbb{P}_V & \xrightarrow{\mathcal{F}_V, \mathbb{P}_V} \mathcal{F}_U, \mathbb{P}_V \\
\mathcal{F}_W, \mathbb{P}_U & \xrightarrow{\mathcal{F}_V, \mathbb{P}_U} \mathcal{F}_U, \mathbb{P}_U
\end{align*} \]

Diagram 4.3.2

\( \mathcal{Q} \). We can think \( \mathcal{Q} \) as a set of priors since all elements of

\[ \mathcal{Q} \subset \mathcal{Q}_V = \{ W \in \chi \mid W \simeq V \} = \{ W \in \chi \mid \mathcal{F}_W = \mathcal{F}_V, \mathbb{P}_W \approx \mathbb{P}_V \} \]

shares one \( \sigma \)-field \( \mathcal{F}_V \) in their first fields.

So, in the rest of this subsection, we focus on defining an adequate sets of priors that vary over objects of the category \( \chi \).

First, we define a subcategory\(^{10}\) of \( \chi \).

**Definition 4.3.12. [Category \( {}^{e}\chi \)]**

1. A category \( {}^{e}\chi_\mathbb{P} \) is a subcategory of \( \chi_\mathbb{P} \) whose objects are same as \( \chi_\mathbb{P} \) and arrows are all equivalence of probability measures.

2. A category \( {}^{e}\chi \) is a product category

\[ {}^{e}\chi := \chi_\mathcal{F} \times {}^{e}\chi_\mathbb{P}. \]  \hspace{1cm} (4.20)

All vertical arrows in Diagram 4.3.2 denote equivalence of probability mea-

---

\(^{10}\text{See Definition A.3.5 for a definition of subcategories.} \)
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\[ P_W \approx P_V \approx P_U. \]  \hfill (4.21)

Now we introduce an important material for defining sets of priors, which is borrowed from Theory of Grothendieck cotopology. It is called a cobasis and the following definition is its specialized version for \( \epsilon \chi \).

**Definition 4.3.13.** [Cobases] A cobasis on \( \epsilon \chi \) is a function \( \mathcal{P} \) which assigns to each object \( W \) in \( \epsilon \chi \), a collection \( \mathcal{P}(W) \) consisting of families of non-empty sets of arrows with domain \( W \) such that

1. If \( f : W \rightarrow U \) is an isomorphism\(^{12}\), then \( \{f\} \in \mathcal{P}(W) \).

2. **Stability:** If \( \{f_i : W \rightarrow V_i \mid i \in I\} \in \mathcal{P}(W) \) and \( W \rightarrow U \) in \( \epsilon \chi \), then the family of arrows

   \[ \{g_i : U \rightarrow (\sigma(F_U \cup F_{V_i}), P_U) \mid i \in I\} \in \mathcal{P}(U). \]  \hfill (4.22)

Members of the family are known as pushouts\(^{13}\).

3. **Transitivity:** If \( \{f_i : W \rightarrow V_i \mid i \in I\} \in \mathcal{P}(W) \) and for each \( i \in I \), \( \{g_{ij} : V_i \rightarrow U_{ij} \mid j \in I_i\} \in \mathcal{P}(V_i) \), then

   \[ \{g_{ij} \circ f_i \mid i \in I, j \in I_i\} \in \mathcal{P}(W). \]  \hfill (4.23)

**Definition 4.3.14.** [Coeval Sets] Let \( \mathcal{P} \) be a cobasis on \( \epsilon \chi \) and \( W \) be an object of \( \epsilon \chi \).

1. A set \( R \in \mathcal{P}(W) \) is called coeval if

   \[ (\exists F \subset G)(\forall f \in R) F_{\text{cod}(f)} = F. \]  \hfill (4.24)

   We write \( \text{cod}(R) \) for this \( F \).

---

\(^{11}\)See Definition A.3.29 for its original definition.

\(^{12}\)See Definition A.3.4 for a definition of isomorphisms.

\(^{13}\)See the note after Definition A.3.17 for a definition of pushouts.
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\[ \mathcal{U} \xrightarrow{g_i} (\sigma(\mathcal{F}_U \cup \mathcal{F}_V), \mathcal{P}_U) \]
\[ \mathcal{W} \xrightarrow{f_i} \mathcal{V}_i \]

Diagram 4.3.3: Stability condition

\[ \begin{array}{c}
\mathcal{W} \\
\xrightarrow{f_1} \mathcal{V}_1 \\
\xrightarrow{f_2} \mathcal{V}_2 \\
\xrightarrow{f_3} \mathcal{V}_3 \\
\end{array} \xrightarrow{g_{11}} \mathcal{U}_{11} \xrightarrow{g_{12}} \mathcal{U}_{12} \xrightarrow{g_{21}} \mathcal{U}_{21} \xrightarrow{g_{31}} \mathcal{U}_{31} \xrightarrow{g_{32}} \mathcal{U}_{32} \xrightarrow{g_{33}} \mathcal{U}_{33} \]

Diagram 4.3.4: Transitivity condition

2. Let \( \mathcal{P}^c \) be a function which assigns to each object \( \mathcal{W} \) in \( \mathcal{E}_\chi \),
\[
\mathcal{P}^c(\mathcal{W}) := \{ R \in \mathcal{P}(\mathcal{W}) \mid R \text{ is coeval} \}. \tag{4.25}
\]

**Proposition 4.3.15.** Let \( R \in \mathcal{P}^c(\mathcal{W}) \). Then for any pair \( f, g \in R \), \( L_{\text{cod}(f)} = L_{\text{cod}(g)} \).

We write \( L_R \) for \( L_{\text{cod}(f)} \).

**Proof.** By the definition of coeval sets, \( \mathcal{F}_{\text{cod}(f)} = \mathcal{F}_{\text{cod}(g)} \). On the other hand, since the structure in Diagram 4.3.5 is in the category \( \mathcal{E}_\chi \), \( \mathcal{P}_{\text{cod}(f)} \approx \mathcal{P}_{\text{cod}(g)} \). Therefore, \( L_{\text{cod}(f)} = L_{\text{cod}(g)} \).

**Definition 4.3.16.** [Extended Monetary Value Measures] Let \( \varphi : \mathcal{E}_\chi^{op} \rightarrow \text{Set} \) be a monetary value measure, and
\[
R = \{ f_i : \mathcal{W} \rightarrow \mathcal{V}_i \mid i \in I \} \in \mathcal{P}^c(\mathcal{W}). \tag{4.26}
\]
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An extended monetary value measure is a function \( \varphi(R) : L_R \to L_W \) defined by

\[
\varphi(R) := \text{ess inf}_{i \in I} \varphi^W_{V_i}. \tag{4.27}
\]

Note that an extended monetary value measure does not satisfy the time consistency condition, in general, which was the goal of this subsection.

Now Proposition A.3.30 shows us that we can generate a Grothendieck cotopology by the cobasis on \( ^e \chi \). Here is the version of the proposition for our case.

**Proposition 4.3.17.** Let \( \mathcal{P} \) be a cobasis on \( ^e \chi \). Define a function \( \mathcal{J} \) which assigns to each object \( W \) in \( ^e \chi \), a collection \( \mathcal{J}(W) \) of cosieves\(^{14}\) on \( ^e \chi \) in such a way that

\[
S \in \mathcal{J}(W) \iff (\exists R \in \mathcal{P}(W)) R \subset S.
\]

\(^{14}\)See Definition A.3.27 for a definition of cosieves.
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Then, \( \mathcal{J} \) is a Grothendieck cotopology\(^{15}\) on \( \mathcal{C} \).

Proposition 4.3.17 allows us to say that an individual (observer) has her own cotopology that is generated by her prior system.

### 4.4 Monetary value measures as sheaves

In general, a contravariant functor \( \varphi : \mathcal{C}^{\text{op}} \to \text{Set} \) is called a presheaf for a category \( \mathcal{C} \). We write \( \hat{\mathcal{C}} \) for the category of all presheaves\(^{16}\) on \( \mathcal{C} \). By definition, a monetary value measure is a presheaf for \( \chi \). The name presheaf suggests that it is related to another concept sheaves, which is a quite important concept in some classical branches in mathematics such as algebraic topology\(^{17}\). So, what makes a presheaf be a sheaf?

For a given set, a topology defined on it provides a criteria to identify good (= continuous) functions within functions on the set. In a similar way, there is a concept called a Grothendieck topology defined on a given category that gives a criteria to identify good presheaves (= sheaves) among presheaves on the category. In both cases, a (Grothendieck) topology can be seen as a vehicle to identify good functions (presheaves) among general functions (presheaves).

On the other hand, if we have a set of functions that we want to make good (= continuous), we can find the weakest topology that makes the functions continuous. In a similar way, if we have a set of presheaves that we want to make good, it is known that we can pick a Grothendieck topology with which the presheaves become sheaves. See Table 4.4.1 for the analogy.

Since a monetary value measure is a presheaf, if we have a set of good monetary value measures (= the monetary value measures that satisfy a given set of axioms), we may find a Grothendieck topology with which the monetary value measures become sheaves. We will see a concrete shape of the Grothendieck topology in Section 4.4.1.

\(^{15}\)See Definition A.3.28 for a definition of Grothendieck cotopology.

\(^{16}\)\( \hat{\mathcal{C}} \) is defined as a functor category. See Definition A.3.13.

\(^{17}\)See MacLane and Moerdijk [MM92].
Now suppose we have a weak topology that makes given functions continuous. This, however, does not imply the fact that any continuous function w.r.t. the topology is contained in the originally given functions. Similarly, Suppose that we have a Grothendieck topology that makes all monetary value measures satisfying a given set of axioms sheaves. It, however, does not mean that any sheaf w.r.t. the Grothendieck topology satisfies the given set of axioms. We will investigate this situation in Section 4.4.2.

### 4.4.1 A Grothendieck topology as axioms

The following two subsections are devoted to standard or straightforward discussions in the context of sheaf theory about which we have a general introduction in Section A.3.4. However, we think it is worth to specialize those stuff here in the context of risk measure theory.

In this subsection, we see a concrete shape of the Grothendieck topology with which all monetary value measures satisfying a given set of axioms become sheaves.

First, we review two concepts of Grothendieck typologies\(^{18}\) and sheaves.

**Definition 4.4.1.** [Grothendieck topology] A **Grothendieck topology** \(J\) on \(\chi\) consists in giving, for each object \(U \in \chi\), a family \(J(U)\) of subfunctors of the representable functor \(\chi(-, U)\), satisfying the following axioms:

\(^{18}\)See Proposition A.3.23.
1. for every $U \in \chi$, $\chi(-, U) \in J(U)$.

2. for any $V \rightarrow U$ in $\chi$ and $R \in J(U)$, the presheaf $R^V$ defined as a pullback in Diagram 4.4.1 belongs to $J(V)$.

3. let $R \rightarrow \chi(-, U)$ and $Q \in J(U)$. If we have $R^V \in J(V)$ for the pullback defined in Diagram 4.4.1 whenever $V \rightarrow U$ is in $Q(V)$, then $R \in J(U)$.

\[
\begin{array}{ccc}
R^V & \rightarrow & \chi(-, V) \\
\downarrow & & \downarrow \chi(-, U) \\
R & \rightarrow & \chi(-, U)
\end{array}
\]

Diagram 4.4.1

Since Diagram 4.4.1 is a pullback in Set and the cardinality of the set $\chi(V, U)$ is at most 1, we have for every $W \rightarrow V \in \chi$,

\[
R^V(W) = \begin{cases} 
\{ \ast_W \} & \text{if } R(W) = \{ \ast_U \}, \\
\emptyset & \text{if } R(W) = \emptyset.
\end{cases}
\] (4.28)

Here is a well-known property of Grothendieck topologies.

**Theorem 4.4.2.** Let $\{ J_a \mid a \in A \}$ be a collection of Grothendieck topologies on $\chi$. Then a system $J$ defined by $J(U) := \bigcap_{a \in A} J_a(U)$ for every object $U \in \chi$ is a Grothendieck topology. We write this $J$ by $\bigcap_{a \in A} J_a$.

Now, we can introduce the concept of sheaves.

**Definition 4.4.3.** [Sheaves] A presheaf $\varphi$ on $\chi$ is called a sheaf for a Grothendieck topology $J$ when, given $U \in \chi$ and $R \in J(U)$, every natural transformation $X : R \rightarrow \varphi$ extends uniquely to $\chi(-, U)$.

In the rest of this subsection, we will try to find a Grothendieck topology for which a given class of monetary value measures specified by a given set of (extra) axioms are sheaves.
The following proposition assures the existence of a Grothendieck topology making a given monetary value measure a sheaf.

**Proposition 4.4.4.** Let \( \varphi \in \hat{\chi} \) be a monetary value measure and \( U \in \chi \). Define a set of subfunctors \( J_{\varphi}(U) \) by

\[
J_{\varphi}(U) := \left\{ R \rightarrow \chi(-,U) \mid (\forall \mathcal{V} \rightarrow U \text{ in } \chi) \forall \chi \varphi \chi Y \right\}
\]

(4.29)

where \( R^\mathcal{V} \) is a presheaf making (4.4.1) a pullback. Then, \( J_{\varphi} \) is the largest Grothendieck topology for which \( \varphi \) is a sheaf.

**Proof.** Refer Example 3.2.14c in Borceux [Bor94].

By combining Proposition 4.4.4 and Theorem 4.4.2, we have the following corollary.

**Corollary 4.4.5.** Let \( \mathcal{M} \subset \hat{\chi} \) be the collection of all monetary value measures satisfying a given set of axioms. Then, there exists a Grothendieck topology for which all monetary value measures in \( \mathcal{M} \) are sheaves, where the topology is largest among topologies representing the axioms. We write \( J_{\mathcal{M}} \) for the topology.

**Proof.** Let \( J_{\mathcal{M}} := \cap_{\varphi \in \mathcal{M}} J_{\varphi} \). Then, it is the largest Grothendieck topology for which every monetary value measure in \( \mathcal{M} \) is a sheaf.

### 4.4.2 Complete sets of axioms

Let \( \mathcal{A} \) be a fixed set of axioms for monetary value measures. Then, for a given arbitrary monetary value measure \( \varphi \), can we make a good alternative for it? In other words, can we find a monetary value measure that satisfies \( \mathcal{A} \) and is the best approximation of the original \( \varphi \)? This is the theme of this subsection.

For a Grothendieck topology \( J \) on \( \chi \), define \( \text{Sh}(\chi, J) \subset \hat{\chi} \) to be a full subcategory whose objects are all sheaves for \( J \). Then, it is well-known that there exists a
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left adjoint\(^{19}\) \(\pi_J\) in the following diagram\(^{20}\).

\[
\begin{align*}
\text{Sh}(\chi, J) & \xrightarrow{\pi_J} \hat{\chi} \\
\Uparrow & \Uparrow
\end{align*}
\]

\[
\pi_J(\varphi) \leftarrow \varphi
\]

(4.30)

The functor \(\pi_J\) is well-known with the name sheafification functor, which comes with the following limit cone:

\[
\begin{array}{c}
\cdots \\
\xrightarrow{\text{Nat}(R, \varphi)} \\
\xrightarrow{S^R_\alpha} \\
\xrightarrow{S^0_\alpha} \\
\xrightarrow{\text{Nat}(Q, \varphi)} \\
\xrightarrow{\text{Nat}(\alpha, \varphi)} \\
\xrightarrow{\text{colim}_{R \in \hat{J}(U)} \text{Nat}(R, \varphi)} \\
\pi_J(\varphi)(U) := \text{colim}_{R \in J(U)} \text{Nat}(R, \varphi)
\end{array}
\]

(4.31)

for \(\alpha : Q \to R\) in \(\hat{\chi}\). It also satisfies the following theorem.

**Theorem 4.4.6.**

1. \(\pi_J(\varphi)\) is a sheaf for \(J\).

2. If \(\varphi\) is a sheaf for \(J\), then \(\pi_J(\varphi) \simeq \varphi\).

Theorem 4.4.6 suggests that for an arbitrary monetary value measure, the sheafification functor provides one of its closest monetary value measures that may satisfy the given set of axioms.

Let \(J_A\) be the largest Grothendieck topology on \(\chi\) such that any monetary value measure \(\varphi : \chi^{op} \to \text{Set}\) satisfying \(\mathcal{A}\) becomes a sheaf for it\(^{21}\). Then, in general, a sheaf \(\pi_{J_A} (\varphi)\) does not always satisfy \(\mathcal{A}\). If there is no such case, that is, all sheaves in the form of \(\pi_{J_A} (\varphi)\) satisfies \(\mathcal{A}\), then we call the set of axioms \(\mathcal{A}\) complete. In other words, the set of axioms \(\mathcal{A}\) is complete if it has enough members to characterize itself through a corresponding Grothendieck topology.

Reminding that the current ways of selecting axioms of risk measures in practice

\(^{19}\)See Definition A.3.20 for a definition of left adjoints.

\(^{20}\)The category \(\text{Sh}(\chi, J)\) is a reflective subcategory of \(\hat{\chi}\) and \(\pi_J\) is a reflection that is a left adjoint for the inclusion functor in (4.30), preserving finite limits (See Theorem A.3.26 and Theorem 3.3.12 in Borceux [Bor94]).

\(^{21}\)For the existence of a largest such Grothendieck topology, see Example 3.2.14.d in [Bor94].
are kind of ad hoc, it would not be so nonsense to have a new regulation for banks that requires their using monetary value measures satisfy some complete set of axioms since at least it guarantees some logical consistency.

Here is a formal definition of completeness of a set of axioms.

**Definition 4.4.7.** Let \( \mathcal{A} \) be a set of axioms for monetary value measures.

1. \( \mathcal{M}[\mathcal{A}] \) is the full and faithful subcategory of \( \hat{\chi} \) whose objects are all monetary value measures satisfying \( \mathcal{A} \).

2. \( I_\mathcal{A} := J_{\mathcal{O}, \mathcal{M}[\mathcal{A}]} \), that is, the largest Grothendieck topology for which every monetary value measure satisfying \( \mathcal{A} \) is a sheaf.

3. \( \mathcal{M}_0 := \mathcal{M}[\emptyset] \), that is, the category of all monetary value measures.

4. \( \mathcal{A} \) is called **complete** if there exists a functor \( \eta_\mathcal{A} : \mathcal{M}_0 \to \mathcal{M}[\mathcal{A}] \) such that the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{M}_0 & \xrightarrow{\eta_\mathcal{A}} & \hat{\chi} \\
\downarrow & & \downarrow \pi_{I_\mathcal{A}} \\
\mathcal{M}[\mathcal{A}] & \xrightarrow{\pi_{I_\mathcal{A}}} & \text{Sh}(\chi, I_\mathcal{A})
\end{array}
\]  

(4.32)

Note that the existence of the inclusion functor in the bottom of (4.32) is guaranteed by the definition of \( I_\mathcal{A} \). And the functor \( \eta_\mathcal{A} \) in (4.32) is actually a restriction of \( \pi_{I_\mathcal{A}} \) to \( \mathcal{M}_0 \). So, we have the following main result.

**Theorem 4.4.8.** Let \( \mathcal{A} \) be a complete set of axioms. Then, for a monetary value measure \( \varphi \), \( \pi_{I_\mathcal{A}}(\varphi) \) is the monetary value measure that is the best approximation satisfying axioms \( \mathcal{A} \).

**Proof.** Since \( \mathcal{A} \) is complete, for every \( \varphi \in \mathcal{M}_0 \) we have \( \pi_{I_\mathcal{A}}(\varphi) = \eta_\mathcal{A}(\varphi) \in \mathcal{M}[\mathcal{A}] \). Therefore, \( \pi_{I_\mathcal{A}}(\varphi) \) is a monetary value measure satisfying \( \mathcal{A} \). \( \square \)

Theorem 4.4.8 is especially important for practitioners since it is sometimes difficult to check whether a monetary value measure at hand is adequate and **safe**
4.4 Monetary value measures as sheaves

to use, in other words, whether it satisfies the given set of axioms. But, Theorem 4.4.8 tells us that they can get a closest safe monetary value measure by remedying the original monetary value measure through the functor $\pi_{J_A}$, in case $A$ is complete.

We expect that some of the well-known sets of axioms such as those for concave monetary value measures are complete. As a discussion toward this line, we will see in the next subsection an example for a quite small $\Omega$ with which the axiom set of concave monetary value measures is not complete if we restrict the category $\chi$ to the category that is not allowed to vary its probability measures, i.e. no ambiguity version $\chi_F$. However, we have no significant result so far for the current version of $\chi$ that accepts ambiguity.

4.4.3 Completeness condition on $\Omega = \{1, 2, 3\}$

In this subsection, we investigate if the set of axioms of concave monetary value measures is complete in the case $\Omega = \{1, 2, 3\}$ with a $\sigma$-field $F := 2^\Omega$ in the category $\chi_F$ that has no ambiguity.

Possible Monetary value measures on $\Omega = \{1, 2, 3\}$

First, we enumerate all possible sub-$\sigma$-fields of $\Omega$, that is, the shape of the category $\chi_F = \chi_F(\Omega)$ which is like following:

\[
\begin{array}{ccc}
U_0 & & U_2 \\
\downarrow & & \downarrow \\
U_1 & & U_3 \\
\uparrow & & \uparrow \\
U_\infty & & \end{array}
\] (4.33)
where

\[
\begin{align*}
U_\infty & := \mathcal{F} := 2^\Omega, \\
U_1 & := \{\emptyset, \{1\}, \{2,3\},\Omega\}, \\
U_2 & := \{\emptyset, \{2\}, \{1,3\},\Omega\}, \\
U_3 & := \{\emptyset, \{3\}, \{1,2\},\Omega\}, \\
U_0 & := \{\emptyset, \Omega\}. \quad (4.34)
\end{align*}
\]

The Banach spaces derived by the elements of \( \chi^F \) are:

\[
\begin{align*}
L_\infty & := L := L(U_\infty) = \{(a,b,c) \mid a,b,c \in \mathbb{R}\}, \\
L_1 & := L(U_1) = \{(a,b) \mid a,b \in \mathbb{R}\}, \\
L_2 & := L(U_2) = \{(a,b,a) \mid a,b \in \mathbb{R}\}, \\
L_3 & := L(U_3) = \{(a,a,c) \mid a,c \in \mathbb{R}\}, \\
L_0 & := L(U_0) = \{(a,a,a) \mid a \in \mathbb{R}\}. \quad (4.35)
\end{align*}
\]

Then, a monetary value measure \( \varphi : \chi_{\mathcal{F}}^{\text{op}} \to \text{Set} \) on \( \chi_{\mathcal{F}} \) is determined by the following six functions:

\[
\begin{align*}
\varphi_\infty^1 & \quad \varphi_\infty^2 & \quad \varphi_\infty^3 \\
\varphi_1^1 & \quad \varphi_1^2 & \quad \varphi_1^3 \\
\varphi_2^1 & \quad \varphi_2^2 & \quad \varphi_2^3 \\
\varphi_3^1 & \quad \varphi_3^2 & \quad \varphi_3^3 \\
L_0 & \quad L_1 & \quad L_2 & \quad L_3
\end{align*}
\]  

(4.36)

We will investigate its concrete shape one by one by considering axioms it satisfies.
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For $q^1_\infty : L_\infty \to L_1$, we have by the cash invariance axiom,

$$q^1_\infty(a, b, c) = q^1_\infty((0, b - c, 0) + (a, c, c))$$

$$= q^1_\infty((0, b - c, 0)) + (a, c, c)$$

$$= (f_{12}(b - c), f_{11}(b - c), f_{11}(b - c)) + (a, c, c)$$

$$= (f_{12}(b - c) + a, f_{11}(b - c) + c, f_{11}(b - c) + c)$$

where $f_{11}, f_{12} : \mathbb{R} \to \mathbb{R}$ are defined by $(f_{12}(x), f_{11}(x), f_{11}(x)) = q^1_\infty(0, x, 0)$. Similarly, if we define nine functions $f_{11}, f_{12}, f_{21}, f_{22}, f_{31}, f_{32}, g_1, g_2, g_3 : \mathbb{R} \to \mathbb{R}$ by

$$\begin{align*}
(f_{12}(x), f_{11}(x), f_{11}(x)) &= q^1_\infty(0, x, 0), \\
(f_{21}(x), f_{22}(x), f_{21}(x)) &= q^2_\infty(0, 0, x), \\
(f_{31}(x), f_{31}(x), f_{32}(x)) &= q^3_\infty(x, 0, 0), \\
(g_1(x), g_1(x), g_1(x)) &= q^0_1(x, 0, 0), \\
(g_2(x), g_2(x), g_2(x)) &= q^0_2(0, x, 0), \\
(g_3(x), g_3(x), g_3(x)) &= q^0_3(0, 0, x).
\end{align*} \tag{4.37}$$

We can represent the original six functions in (4.36) by the nine functions defined in (4.37).

$$\begin{align*}
q^1_\infty(a, b, c) &= (f_{12}(b - c) + a, f_{11}(b - c) + c, f_{11}(b - c) + c), \\
q^2_\infty(a, b, c) &= (f_{21}(c - a) + a, f_{22}(c - a) + b, f_{21}(c - a) + a), \\
q^3_\infty(a, b, c) &= (f_{31}(a - b) + b, f_{31}(a - b) + b, f_{32}(a - b) + c), \\
q^0_1(a, b, c) &= (g_1(a - b) + b, g_1(a - b) + b, g_1(a - b) + b), \\
q^0_2(a, b, a) &= (g_2(b - a) + a, g_2(b - a) + a, g_2(b - a) + a), \\
q^0_3(a, a, c) &= (g_3(c - a) + a, g_3(c - a) + a, g_3(c - a) + a). \tag{4.38}
\end{align*}$$
Next by the normalization axiom, we have

\[ f_{11}(0) = f_{12}(0) = f_{21}(0) = f_{22}(0) = f_{31}(0) = f_{32}(0) = g_{1}(0) = g_{2}(0) = g_{3}(0) = 0. \]  
(4.39)

Now suppose that we can partially differentiate the function \( \varphi^{1}_{\infty}(a, b, c) \) in all three arguments. Then, we have

\[
\begin{align*}
\frac{\partial}{\partial a} \varphi^{1}_{\infty}(a, b, c) &= (1, 0, 0), \\
\frac{\partial}{\partial b} \varphi^{1}_{\infty}(a, b, c) &= (f_{12}'(b - c), f_{11}'(b - c), f_{11}'(b - c)), \\
\frac{\partial}{\partial c} \varphi^{1}_{\infty}(a, b, c) &= (-f_{12}'(b - c), 1 - f_{11}'(b - c), 1 - f_{11}'(b - c)).
\end{align*}
\]

Therefore, by the monotonicity, we have \( f_{12}'(x) = 0 \) and \( 0 \leq f_{11}'(x) \leq 1 \). Then by (4.39), we have for all \( x \in \mathbb{R}, f_{12}(x) = 0 \). Hence, for all \( x \in \mathbb{R}, \)

\[ f_{12}(x) = f_{22}(x) = f_{32}(x) = 0. \]
(4.40)

With this knowledge, let us redefine the three functions \( f_{1}, f_{2}, f_{3} : \mathbb{R} \to \mathbb{R} \) by

\[
\begin{align*}
(0, f_{1}(x), f_{1}(x)) &= \varphi^{1}_{\infty}(0, x, 0), \\
(f_{2}(x), 0, f_{2}(x)) &= \varphi^{2}_{\infty}(0, 0, x), \\
(f_{3}(x), f_{3}(x), 0) &= \varphi^{3}_{\infty}(x, 0, 0).
\end{align*}
\]
(4.41)
4.4 Monetary value measures as sheaves

Then, we have a new representation of the original six functions in (4.36):

\[
\begin{align*}
\phi_1^\infty(a, b, c) &= (a, f_1(b - c) + c, f_1(b - c) + c), \\
\phi_2^\infty(a, b, c) &= (f_2(c - a) + a, b, f_2(c - a) + a), \\
\phi_3^\infty(a, b, c) &= (f_3(a - b) + b, f_3(a - b) + b, c), \\
\phi_1^0(a, b, b) &= (g_1(a - b) + b, g_1(a - b) + b, g_1(a - b) + b), \\
\phi_2^0(a, b, a) &= (g_2(b - a) + a, g_2(b - a) + a, g_2(b - a) + a), \\
\phi_3^0(a, a, c) &= (g_3(c - a) + a, g_3(c - a) + a, g_3(c - a) + a).
\end{align*}
\] (4.42)

Thinking about the composition rule \( \phi_0^\infty = \phi_1^0 \circ \phi_2^\infty = \phi_0^2 \circ \phi_3^\infty \), we have

\[
\begin{align*}
g_1(a - f_1(b - c) - c) + f_1(b - c) + c \\
= g_2(b - f_2(c - a) - a) + f_2(c - a) + a \\
= g_3(c - f_3(a - b) - b) + f_3(a - b) + b.
\end{align*}
\] (4.43)

**Proposition 4.4.9.** \( \phi \) is concave iff all of \( f_1, f_2, f_3, g_1, g_2 \) and \( g_3 \) are concave.

**Grothendieck topologies on \( \chi_F \)**

Any Grothendieck topology on \( \chi_F \) we are discussing in the following has at least one sheaf for it. Therefore, we can assume any sieve \( I \) on \( \mathcal{U} \) satisfies \( \bigvee I = \mathcal{U} \).

**Proposition 4.4.10.** Let \( J \) be a Grothendieck topology on \( \chi_F \). Then,

\[
J(\mathcal{U}_k) = \{ \downarrow \mathcal{U}_k \}
\] (4.44)

for \( k = 0, 1, 2 \) or 3.

So, we only discuss about \( J(\mathcal{U}_\infty) \) below.

For \( k = 0, 1, 2, 3, \infty \), define sieves \( I_k \) on \( \mathcal{U}_k \) by \( I_k := \downarrow \mathcal{U}_k \). Followings are all possible sieves on \( \mathcal{U}_\infty \).

\[
I_{12} := I_1 \cup I_2, \quad I_{13} := I_1 \cup I_3, \quad I_{23} := I_2 \cup I_3, \quad \text{and} \quad I_{123} := \]
4.4 Monetary value measures as sheaves

$I_1 \cup I_2 \cup I_3$.

Now, we define two Grothendieck Topologies $J_0$ and $J_1$.

- $J_0$ is defined by $J_0(U_k) = \{ I_k \}$ for $k = 0, 1, 2, 3$ or $\infty$.
- $J_1$ is defined by $J_1(U_k) = \{ I_k \}$ for $k = 0, 1, 2$ or $3$ and $J_1(U_\infty) = \{ I_\infty, I_{123} \}$.

We can easily show that any Grothendieck topology on $\chi_F$ that has at least one sheaf on $\chi_F$ other than $J_0$ contains $J_1$. In other words, $J_1$ is the smallest Grothendieck topology on $\chi_F$ next to $J_0$.

The following diagram shows the unique extension from $I_{123}$ to $I_\infty$.

\[
\begin{array}{ccc}
(a, b, c) & \xrightarrow{\phi_1^\infty} & (a, c', c') \\
\downarrow \phi_2^\infty & & \downarrow \phi_3^\infty \\
(a', b, a') & & (b', b', c) \\
\downarrow \phi_1^g & & \downarrow \phi_2^g \\
(Z, Z, Z) & & \\
\end{array}
\]

(4.45)

So, we have a necessary and sufficient condition for a monetary value measure to be a $J_1$-sheaf.

**Proposition 4.4.11.** $\phi$ becomes a sheaf for $J_1$ iff for all $a, a', b, b', c, c' \in \mathbb{R}$,

\[
g_1(a - c') + c' = g_2(b - a') + a' = g_3(c - b') + b' \\
\Rightarrow (c' = f_1(b - c) + c) \land (a' = f_2(c - a) + a) \land (b' = f_3(a - b) + b). \quad (4.46)
\]

**Entropic value measures on $\chi_F$**

Let $\mathbb{P}$ be a probability measure on $\Omega$ defined by $\mathbb{P} = (p_1, p_2, p_3)$ and $\phi$ be an entropic value measure defined by

\[
\phi_U^\mathbb{P}(X) := \frac{1}{\lambda} \log \mathbb{E}^\mathbb{P}[e^{\lambda X} | \mathcal{V}] . \quad (4.47)
\]
4.4 Monetary value measures as sheaves

Then the function $\varphi_\infty^1$ in (4.36) is

$$
\varphi_\infty^1(a,b,c) = \frac{1}{\lambda} \log \mathbb{E}^P[(e^{\lambda a}, e^{\lambda b}, e^{\lambda c}) \mid \mathcal{U}_1]
$$

$$
= (a, \frac{1}{\lambda} \log \frac{p_2 e^{\lambda b} + p_3 e^{\lambda c}}{p_2 + p_3}, \frac{1}{\lambda} \log \frac{p_2 e^{\lambda b} + p_3 e^{\lambda c}}{p_2 + p_3}). \quad (4.48)
$$

Therefore, the corresponding six functions defined in (4.37) and (4.41) are

- $f_1(x) = \frac{1}{\lambda} \log \frac{p_2 e^{\lambda x} + p_3}{p_2 + p_3},$
- $f_2(x) = \frac{1}{\lambda} \log \frac{p_3 e^{\lambda x} + p_1}{p_3 + p_1},$
- $f_3(x) = \frac{1}{\lambda} \log \frac{p_1 e^{\lambda x} + p_2}{p_1 + p_2},$
- $g_1(x) = \frac{1}{\lambda} \log (p_1 e^{\lambda x} + p_2 + p_3),$
- $g_2(x) = \frac{1}{\lambda} \log (p_1 + p_2 e^{\lambda x} + p_3),$
- $g_3(x) = \frac{1}{\lambda} \log (p_1 + p_2 + p_3 e^{\lambda x}).$

So, the question is if the entropic value measure is a $J_1$-sheaf. By Proposition 4.4.11, its necessary and sufficient condition becomes like the following:

$$
p_1 e^{\lambda a} + (1 - p_1) e^{\lambda c'} = p_2 e^{\lambda b} + (1 - p_2) e^{\lambda a'} = p_3 e^{\lambda c} + (1 - p_3) e^{\lambda b'} =: Z
$$

$$
\Rightarrow Z = p_1 e^{\lambda a} + p_2 e^{\lambda b} + p_3 e^{\lambda c}.
$$

However, this does not hold in general.

**Theorem 4.4.12.** $\varphi$ is not a $J_1$-sheaf.

**Corollary 4.4.13.** Any set of axioms over $\Omega = \{1, 2, 3\}$ that accepts concave monetary value measures is not complete.
Chapter 5

Concluding remarks

Throughout this thesis, we tried to provide a framework for representing uncertainty. There are two types of uncertainty. One is the uncertainty changing per time, depending on the information an observer can see and grasp. The other one is the uncertainty changing per space, depending on the subjective probability the observer has, which we call prior ambiguity.

Chapters 2 and 3 were devoted to the first type of uncertainty, while Chapter 4 was a trial for unifying both types of uncertainty in a single framework. Let us look back the results we had in the thesis, one by one.

In Chapter 2, we introduced a stochastic process called a follower process consisting of a non-decreasing sequence of random times $f_i$ whose values do not exceed $t$. It was introduced for representing information delay that comes from reasons uncontrollable by the observer such as asymmetricity of the information. We showed that the follower process is an extension of a time change process introduced by Guo, Jarrow and Zeng in the sense that each component of the follower process is not required to be a stopping time. We introduced a class of follower processes called idempotent and showed that the class contains natural examples including follower processes driven by renewal processes. Then, we proved that any idempotent follower process is hard to be an example of time change processes.
We defined a filtration modulated by the follower process and showed that it is a natural extension of the continuously delayed filtration that is the filtration modulated by the time change process. We showed that conditional expectations given idempotent follower filtrations have some Markov property in a binomial setting, which is useful for pricing defaultable financial instruments.

In order to represent internal reasons of the observer such as her penetrating ability as well as the external reasons that we treated by follower processes, in Chapter 3 we introduced a concept of extended states as penetration histories of the observer. We showed that each set of extended states is a proper vehicle for representing the observer’s narrowing-down ability determined by her internal reasons as well as constraints coming from her external environment.

We gave a candidate of specifications of subsets of extended states that were made by combinations of follower filtrations and dominant subsets. We may be able to utilize this type of specifications when we build a set of scenarios systematically for stress tests required by authorized rules such as Basel III.

We applied the concept of extended states to dynamic choice theory by formulating a recursive value function that is sensitive not only to prior ambiguity but also to state ambiguity. The resulting function is more conservative than a classical value function in the sense that the value of our function is not greater than that of the classical one.

Considerations made in Chapters 2 and 3 were on uncertainty coming from information change along time. In Chapter 4, upon the time dimension, we added another dimension - a space dimension - by introducing a category $\chi$ that represents varying risk as well as (prior) ambiguity. We gave a generalized conditional expectation as a contravariant functor on $\chi$, which works not only in risk direction but also in ambiguity direction.

We specified a concept of monetary value measures as a presheaf for $\chi$. The resulting monetary value measures satisfy naturally so-called time consistency.
condition as well as dynamic programming principle.

After reformulating the robust representation theorem of concave monetary value measures, we provided a means for breaking the time consistency condition by introducing a system of priors as a Grothendieck cotopology.

Finally, we discussed a possibility of applying the topology-as-axioms paradigm for getting the best approximation of the monetary value measure that satisfies given axioms from a monetary value measure at hand, which works in case the axioms are complete.

Before completing the chapter, we raise a few plans for future work starting from the thesis.

Behind the concept of extended states, we have an idea of pure experiences borrowed from Nishida philosophy. In the light of the philosophy, our interpretation of extended states using the function $I$ in Definition 3.3.11 may spoil the core power of the concept. Finding a direct approach to interpret an extended state as a whole is a crucial goal to be expected.

There seems an obvious similarity between the rectangularity condition required for the prior system defined in Definition 3.3.4 and the conditions in Definition A.3.28 of Grothendieck cotopology. We should explore their relationship in order to provide a better legitimacy to the definition of the prior systems.

We have not included the state ambiguity into the category $\chi$. It may be a good direction to seek a unified version of them.

We will investigate the possibility to represent each individual axiom of monetary value measures as a specific Grothendieck topology which may give us an insight about different aspects of the axioms of monetary value measures, as well as investigating the completeness condition against the important sets of axioms such as those of concave monetary value measures. We may be able to propose a new set of axioms that is complete as a foundation of safe monetary value measure theory.
Appendices
Appendix A

Appendix

A.1 General theory of stochastic processes

A process \( X = \{X_t\}_{t \in \mathcal{T}} \) is called \( \mathcal{G} \)-progressive if for every \( t \in \mathcal{T} \), \( X|_{[0,t]} \times \Omega \) is \( \mathcal{B}[0,t] \otimes \mathcal{G}_t \)-measurable. A random set is called \( \mathcal{G} \)-progressive if its indicator function is \( \mathcal{G} \)-progressive.

Every right continuous \( \mathcal{G} \)-adapted process is \( \mathcal{G} \)-progressive (See Rogers and Williams [RW00] Lemma VI.3.3 ).

Definition A.1.1. [Optional processes] The optional \( \sigma \)-field with respect to \( \mathcal{G} \) is the \( \sigma \)-field \( \mathcal{O}^G \) defined on \( \mathcal{T} \times \Omega \) such that

\[
\mathcal{O}^G := \sigma \{ X \mid X = \{X_t\}_{t \in \mathcal{T}} \text{ is a } \mathcal{G} \text{-adapted càdlàg process. } \}. \tag{A.1}
\]

An element of \( \mathcal{O}^G \) is called a \( \mathcal{G} \)-optional set. A process \( X = \{X_t\}_{t \in \mathcal{T}} \) is called \( \mathcal{G} \)-optional if the map \((t, \omega) \mapsto X_t(\omega)\) is \( \mathcal{O}^G \)-measurable.

Every \( \mathcal{G} \)-optional process is a \( \mathcal{G} \)-progressive process, and every \( \mathcal{G} \)-optional set is a \( \mathcal{G} \)-progressive set.

The following is one of the standard \( \sigma \)-fields generated by arbitrary random times. See Definition XX.25 in Dellacherie, Maisonneuve and Meyer [DMM92].
Definition A.1.2. Let \( \tau \) be a random time. The \( \sigma \)-field \( G_\tau \) is defined by
\[
G_\tau := \sigma\{ Z_\tau \mid Z = \{ Z_t \}_{t \in T} \text{ is a } G\text{-optional process.} \}.
\]

The \( \sigma \)-field \( G_\tau \) consists of events which depend on what happens up to and including time \( \tau \).

Proposition A.1.3. Every random time \( \tau \) is \( G_\tau \)-measurable.

Proof. Let \( Z \) be a process defined by \( Z(t, \omega) = t \) for all \( t \in T \) and \( \omega \in \Omega \). Then \( Z \) is obviously optional and \( Z_\tau = \tau \).

Theorem A.1.4. [\cite{DMM92} Théorème XX.27] Let \( \tau_1 \) and \( \tau_2 \) be two random times such that \( \tau_1 \leq \tau_2 \). If \( \tau_1 \) is \( G_{\tau_2} \)-measurable, we have \( G_{\tau_1} \subset G_{\tau_2} \).

Theorem A.1.5. [\cite{RW00} Lemma VI.17.5] If \( \tau \) is a \( G \)-stopping time, then
\[
G_\tau = \{ A \in G_\infty \mid (\forall u \in T)A \cap \{ \tau \leq u \} \in G_u \}.
\]

Especially, if there exists a constant \( t \in T \) such that for any \( \omega \in \Omega \), \( \tau(\omega) = t \), then \( G_\tau = G_t \).

A.2 Dynamic risk measure theory

First, we review the case of one period monetary risk measures.

Definition A.2.1. A one period monetary risk measure is a function
\[
\rho : L^\infty(\Omega, G, \mathbb{P}) \to \mathbb{R}
\]
satisfying the following axioms

1. Cash invariance: \((\forall X)(\forall a \in \mathbb{R}) \rho(X + a) = \rho(X) - a, \)
2. Monotonicity: \((\forall X)(\forall Y) X \leq Y \Rightarrow \rho(X) \geq \rho(Y), \)
3. Normalization: $\rho(0) = 0$.

Here are examples of one period risk measures.

**Example A.2.2.** [One period monetary risk measures]

1. **Value at Risk**

   \[ V@R_\alpha(X) := \inf \{ m \in \mathbb{R} \mid \mathbb{P}(X + m < 0) \leq \alpha \} \]

2. **Average Value at Risk**

   \[ AVaR_\alpha(X) := \frac{1}{\alpha} \int_{0}^{\alpha} V@R_u(X) du \]

Now, we will define the notion of dynamic monetary risk measures. However, we actually adopt the way of using a *monetary value measure* $\varphi$ instead of using a *monetary risk measure* $\rho$ below by conforming the manner in recent literature such as Artzner et al. [ADE+07] and Kusuoka and Morimoto [KM07], where we have a relation $\varphi(X) = -\rho(X)$ for any possible scenario (i.e. a random variable) $X$. So from now on, we think a *monetary value measure* $\varphi$ instead of a monetary risk measure $\rho$, defined by $\varphi(X) := -\rho(X)$.

In the rest of this subsection, we assume that the time domain $\mathcal{T}$ has a time horizon $T$, that is, $\mathcal{T} = [0, T]$.

**Definition A.2.3.** For a $\sigma$-field $\mathcal{F} \subset \mathcal{G}$, we write $L(\mathcal{F})$ for $L^\infty(\Omega, \mathcal{F}, \mathbb{P}|_{\mathcal{F}})$. Let $\mathcal{G} = \{ \mathcal{G}_t \}_{t \in \mathcal{T}}$ be a *filtration*, that is, a $\mathcal{T}$-indexed increasing sequence of $\sigma$-fields. A *dynamic monetary value measure* is a collection of functions

\[ \varphi = \{ \varphi_t : L(\mathcal{G}_T) \to L(\mathcal{G}_t) \}_{t \in \mathcal{T}} \quad \text{(A.4)} \]

satisfying

1. **Cash invariance:** $(\forall X \in L(\mathcal{G}_T))(\forall Z \in L(\mathcal{G}_t)) \varphi_t(X + Z) = \varphi_t(X) + Z,$
2. Monotonicity: \( (\forall X \in L(G_T)) (\forall X \in L(G_T)) X \leq Y \Rightarrow \varphi_t(X) \leq \varphi_t(Y), \)

3. Normalization: \( \varphi_t(0) = 0. \)

Note that the directions of some inequalities in Definition A.2.1 are different from those of Definition A.2.3 because we now think monetary value measures instead of monetary risk measures.

Since dynamic monetary value measures treat multi-period situations, we may require some extra axioms to regulate them toward the time dimension. Here are two possible such axioms.

**Axiom A.2.4.** [Dynamic programming principle] For \( 0 \leq s \leq t \leq T, \)

\[
(\forall X \in L(G_T)) \varphi_s(X) = \varphi_s(\varphi_t(X)).
\] (A.5)

**Axiom A.2.5.** [Time consistency] For \( 0 \leq s \leq t \leq T, \)

\[
(\forall X, \forall Y \in L(G_T)) \varphi_t(X) \leq \varphi_t(Y) \Rightarrow \varphi_s(X) \leq \varphi_s(Y).
\] (A.6)

### A.3 Category theory

The description about category theory presented in this section is very limited. For those who are interested in more detail about category theory, please consult MacLane [Mac97], Maclane and Moerdijk [MM92] and Borceux [Bor94].

#### A.3.1 Examples of categories

**Definition A.3.1.** [Categories] A category is a structure \( \mathcal{C} \) consisting of a collection \( \mathcal{O}_C \) of objects and a collection \( \mathcal{M}_C \) of arrows or morphisms that satisfy the following conditions.

1. There are two functions

\[
\mathcal{M}_C \xrightarrow{\text{dom}} \mathcal{O}_C.
\] (A.7)
Let \( f \in \mathcal{M}_C \) and \( A, B \in \mathcal{O}_C \). When \( \text{dom}(f) = A \) and \( \text{cod}(f) = B \), we write

\[ f : A \to B. \tag{A.8} \]

A **hom-set** of objects \( A \) and \( B \) is a set \( \text{Hom}_C(A, B) \) defined by

\[ \text{Hom}_C(A, B) := \{ f \in \mathcal{M}_C \mid f : A \to B \}. \tag{A.9} \]

We sometimes write \( C(A, B) \) for \( \text{Hom}_C(A, B) \).

2. For arrows \( f : A \to B \) and \( g : B \to C \), there is an arrow \( g \circ f : A \to C \), called the **composition** of \( g \) and \( f \). The composition operator \( \circ \) satisfies the associative law. That is, for \( h : C \to D \),

\[ (h \circ g) \circ f = h \circ (g \circ f). \tag{A.10} \]

3. Every object \( A \) is associated with an arrow called an **identity arrow** \( 1_A : A \to A \) satisfying

\[ f \circ 1_A = f \quad \text{and} \quad 1_A \circ g = g \tag{A.11} \]

where \( \text{dom}(f) = A \) and \( \text{cod}(g) = A \).

\[ \xymatrix{ A \ar[r]^f & B \ar[r]^g & C \ar[lu]_{1_A} \ar[lu]_{1_B} \ar[lu]_{1_C} } \]

Diagram A.3.1: Objects and arrows

A **finite category** is a category \( \mathcal{C} \) such that both \( \mathcal{O}_C \) and \( \mathcal{M}_C \) are finite sets.

**Example A.3.2.** [Examples of categories]

1. **Set** : the category of small sets\(^1\)

\(^1\)If we consider a collection of all sets and assume that the collection is a set, then it would lead to
A.3 Category theory

- \( \mathcal{O}_{\text{Set}} := \) collection of all small sets,
- \( \mathcal{M}_{\text{Set}} := \) collection of all functions between small sets.

2. \textbf{Mble} : the category of measurable spaces
   - \( \mathcal{O}_{\text{Mble}} := \) collection of all measurable spaces,
   - \( \mathcal{M}_{\text{Mble}} := \) collection of all measurable functions between measurable spaces.

3. \textbf{Top} : the category of topological spaces
   - \( \mathcal{O}_{\text{Top}} := \) collection of all topological spaces,
   - \( \mathcal{M}_{\text{Top}} := \) collection of all continuous functions between topological spaces.

4. \textit{Opposite category} \( \mathcal{C}^{op} \)

   Let \( \mathcal{C} \) be a given category. Then we define its \textit{opposite category} \( \mathcal{C}^{op} \) by the following way.
   - \( \mathcal{O}_{\mathcal{C}^{op}} := \mathcal{O}_{\mathcal{C}} \),
   - for \( A, B \in \mathcal{O}_{\mathcal{C}} \), \( \text{Hom}_{\mathcal{C}^{op}}(A, B) := \text{Hom}_{\mathcal{C}}(B, A) \).

5. \textit{Product category} \( \mathcal{C} \times \mathcal{D} \)

   Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories. Then we define their \textit{product category} \( \mathcal{C} \times \mathcal{D} \) by the following way.
   - \( \mathcal{O}_{\mathcal{C} \times \mathcal{D}} := \mathcal{O}_{\mathcal{C}} \times \mathcal{O}_{\mathcal{D}} \),
   - for \( C, C' \in \mathcal{O}_{\mathcal{C}} \) and \( D, D' \in \mathcal{O}_{\mathcal{D}} \),
     \[ \text{Hom}_{\mathcal{C} \times \mathcal{D}}((C, D), (C', D')) := \text{Hom}_{\mathcal{C}}(C, C') \times \text{Hom}_{\mathcal{D}}(D, D'). \]

---

some of the famous paradoxical sets such as \( \{ x \mid x \notin x \} \). In order to avoid that, all sets considered are required to be members of an adequately predefined set \( U \) called a \textit{universe}. A modifier \textit{small} occurred in the definition means the (underlying set of) object belongs to \( U \). In this thesis, all categories are supposed to consist of small objects if they are sets with some structures such as measurability or topology. Therefore, \( \mathcal{O}_{\mathcal{C}} \) and \( \text{Hom}_{\mathcal{C}}(A, B) \) are always sets.
Example A.3.3. [Preordered sets as categories]

A preordered set (sometimes we call it a proset) \((S, \leq)\), where the binary relation \(\leq\) on \(S\) is reflexive and transitive, can be considered as a category defined in the following way.

- \(\mathcal{O}_S := S\),
- for \(a, b \in S\), \(\text{Hom}_S(a, b) := \begin{cases} \{a, b\} & \text{if } a \leq b, \\ \emptyset & \text{otherwise.} \end{cases}\)

We see the correspondence between axioms of prosets and categories below.

1. Reflexivity \((a \leq a)\) is represented by identity arrows.
2. Transitivity \((a \leq b \text{ and } b \leq c \implies a \leq c)\) is represented by composition arrows.

\[
\begin{aligned}
\text{Diagram A.3.2: A proset as a category}
\end{aligned}
\]

Especially when the proset is discrete, meaning a proset such that \(a \leq b\) implies \(a = b\), the resulting category is called discrete. A discrete category has no arrows except identity arrows.

Definition A.3.4. [Monics, epis and isomorphisms] Let \(\mathcal{C}\) be a category.

1. An arrow \(m : A \to B\) is monic in \(\mathcal{C}\) when for any two parallel arrows \(f_1, f_2 : D \to A\) the equality \(m \circ f_1 = m \circ f_2\) implies \(f_1 = f_2\).
2. An arrow \(e : A \to B\) is epi in \(\mathcal{C}\) when for any two parallel arrows \(g_1, g_2 : B \to C\) the equality \(g_1 \circ e = g_2 \circ e\) implies \(g_1 = g_2\).
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3. An arrow \( i : A \rightarrow B \) is an isomorphism in \( C \) if there exists an arrow \( j : B \rightarrow A \) such that \( j \circ i = 1_A \) and \( i \circ j = 1_B \).

In \( \textbf{Set} \), an arrow is monic iff it is 1-1, and the arrow is epi iff it is onto.

**Definition A.3.5.** [Subcategories] A category \( S \) is a subcategory of a category \( C \) if \( O_S \subset O_C \) and for every pair of objects \( A \) and \( B \) in \( S \), \( \text{Hom}_S(A, B) \subset \text{Hom}_C(A, B) \).

The subcategory \( S \) is said full if \( \text{Hom}_S(A, B) = \text{Hom}_C(A, B) \) holds for every pair of objects \( A \) and \( B \) in \( S \).

A.3.2 Functors and natural transformations

There are maps between categories, called functors.

**Definition A.3.6.** [Functors] Let \( C \) and \( D \) be two categories. A functor \( F : C \rightarrow D \) consists of two functions, \( F_O : O_C \rightarrow O_D \) and \( F_M : M_C \rightarrow M_D \) satisfying

1. \( f : A \rightarrow B \) implies \( F(f) : F(A) \rightarrow F(B) \),
2. \( F(g \circ f) = F(g) \circ F(f) \),
3. \( F(1_A) = 1_{F(A)} \).

\[
\begin{align*}
\begin{array}{ccc}
\text{Diagram A.3.3: Functor } F
\end{array}
\end{align*}
\]

For a functor application, we sometimes write \( FC \) and \( Ff \) for \( F(C) \) and \( F(f) \) by dropping parentheses if there is no risk of confusion.
Definition A.3.7. [Full and faithful functors] Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor.

1. \( F \) is called \textit{full} when to every pair \( C, C' \) of objects of \( \mathcal{C} \) and to every arrow \( g : FC \to FC' \) of \( \mathcal{D} \), there is an arrow \( f : C \to C' \) of \( \mathcal{C} \) with \( g = Ff \).

2. \( F \) is called \textit{faithful} when to every pair \( C, C' \) of objects of \( \mathcal{C} \) and to every pair \( f_1, f_2 : C \to C' \) of parallel arrows of \( \mathcal{C} \), the equality \( Ff_1 = Ff_2 : C \to C' \) implies \( f_1 = f_2 \).

Definition A.3.8. [Contravariant functors] A functor \( F : \mathcal{C}^{\text{op}} \to \mathcal{D} \) is called a \textit{contravariant functor}, if two conditions 1 and 2 in Definition A.3.6 are replaced by

1. \( f : A \to B \) implies \( Ff : FB \to FA \),

2. \( F(g \circ f) = Ff \circ Fg \).

Here are important examples of contravariant functors.

Example A.3.9. [Representable functors] For a category \( \mathcal{C} \), a \textit{representable functor} is a contravariant functor \( \mathcal{C}(-, C) : \mathcal{C}^{\text{op}} \to \text{Set} \) defined by Diagram A.3.4.

\[
\begin{aligned}
\mathcal{C}^{\text{op}} & \xrightarrow{\mathcal{C}(-, C)} \text{Set} \\
A \downarrow f & \quad \mathcal{C}(A, C) \ni g \circ f \\
B & \quad \mathcal{C}(B, C) \ni \bar{g}
\end{aligned}
\]

Diagram A.3.4: Representable functor

Now we have maps between functors, called natural transformations.

Definition A.3.10. [Natural transformations] Let \( \mathcal{C} \xrightarrow{F} \mathcal{D} \) be two functors. A \textit{natural transformation} \( \alpha : F \Rightarrow G \) consists of a family of arrows \( \{\alpha_C | C \in \mathcal{C}\} \)
making the following diagram commute:

\[
\begin{array}{ccc}
C_1 & \xrightarrow{\alpha_{C_1}} & GC_1 \\
\downarrow f & & \downarrow Pf \\
C_2 & \xrightarrow{\alpha_{C_2}} & GC_2
\end{array}
\]

When this holds, we also say that \( \alpha_C : FC \rightarrow GC \) is \textit{natural} in \( C \).

We write \( \text{Nat}(F, G) \) for the set of all natural transformations from \( F \) to \( G \).

Let \( F, G \) and \( H \) be three functors from \( C \) to \( D \), and \( \alpha : F \rightarrow G \) and \( \beta : G \rightarrow H \) be natural transformations. Then, the \textit{composition} of \( \alpha \) and \( \beta \), denoted by \( \beta \circ \alpha \) is defined by for every object \( C \) in \( C \),

\[
(\beta \circ \alpha)_C := \beta_C \circ \alpha_C.
\]  

\[\text{(A.12)}\]

**Example A.3.11.** [Natural transformations between representable functors] Let \( f : S \rightarrow R \) and \( g : C \rightarrow D \) be arrows in \( C \). Two natural transformations

\[
C(-, f) : C(-, S) \rightarrow C(-, R)
\]  

and

\[
C(g, -) : C(D, -) \rightarrow C(C, -)
\]  

are defined by Diagram A.3.5.

**Definition A.3.12.** [Functor categories] Let \( C \) and \( D \) be categories. A \textit{functor category} \( \mathcal{D}^C \) is the category such that

- \( \mathcal{O}_{\mathcal{D}^C} := \) collection of all functors from \( C \) to \( D \),

- \( \text{Hom}_{\mathcal{D}^C}(F, G) := \) collection of all natural transformations from \( F \) to \( G \).

**Definition A.3.13.** [Presheaves] For a category \( C \), a contravariant functor \( F : C^{\text{op}} \rightarrow \text{Set} \) is called a \textit{presheaf}. The functor category \( \text{Set}^{C^{\text{op}}} \) of all these presheaves is written as \( \hat{C} \).
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Diagram A.3.5: Natural transformations between representable functors

Now we have a very important tool known as **Yoneda lemma**.

**Lemma A.3.14.** [Yoneda] If $K : C \to \text{Set}$ is a functor and $R$ an object in $C$, there is a bijection

$$y_R : \text{Nat}(C(-, R), K) \cong KR$$  \hspace{1cm} (A.15)

which sends each natural transformation $\alpha : C(-, R) \to K$ to $\alpha R 1_R$.

The proof of Lemma A.3.14 is almost straightforward by the fact that for any arrow $f : C \to R$ in $C$

$$\alpha_C f = Kf(\alpha R 1_R)$$  \hspace{1cm} (A.16)

which comes from the following diagram.

The next lemma is the obvious dual variant of Lemma A.3.14.

**Lemma A.3.15.** [Dual Yoneda] If $K : C^{\text{op}} \to \text{Set}$ is a contravariant functor and $R$ an
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object in $C$, there is a bijection

$$\tilde{g}_R : \text{Nat}(C(-, R), K) \cong KR$$  \hspace{1cm} (A.17)

which sends each natural transformation $\alpha : C(-, R) \to K$ to $\alpha_R 1_R$.

A.3.3 Limits and colimits

**Definition A.3.16.** [Limits] A limit for a functor $F : \mathcal{J} \to C$ is a pair $\langle R, v \rangle$ where

$R$ is an object of $C$ and $v$ is a family of arrows $v = \{v_J\}$ in $C$ indexed by objects $J$ of $\mathcal{J}$ satisfying the following two conditions.

1. For any arrow $h : J \to K$ in $\mathcal{J}$, $Fh \circ v_J = v_K$.

2. For any pair $\langle S, u \rangle$ where $S$ is an object of $C$ and $u$ is a family of arrows $u = \{u_J\}$, if for any arrow $h : J \to K$ in $\mathcal{J}$, $Fh \circ u_J = u_K$, then there exists a unique arrow $t : S \to R$ such that $u_J = v_J \circ t$ for all object $J$ of $\mathcal{J}$.

$$J \quad FJ \quad R \quad S$$

$$\downarrow h \quad \downarrow Fh \quad \downarrow v_J \quad \downarrow u_J \quad \quad \exists! \ t \quad \downarrow v_K \quad \downarrow u_K$$

When this holds, we write

$$R = \lim_{\leftarrow} F.$$  \hspace{1cm} (A.18)

The category $\mathcal{J}$ in Definition A.3.16 is an index category. We can define many special limits by specifying $\mathcal{J}$ concretely.

For example, when $\mathcal{J}$ is a discrete category such that $\mathcal{O}_\mathcal{J} = \{1, 2\}$ which has no arrows except identities. Then the functor $F : \mathcal{J} \to C$ is just specifying two
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objects of \( \mathcal{C} \) and the corresponding limit is their **product**.

\[
\begin{align*}
1 & \quad \xymatrix{ & C_1 \ar[dl]_{p_1} \ar[dr]^{q_1} & \ar@{=} C_1 \times C_2 \ar@{=}[ld]_{p_2} \ar@{=}[rd]^{q_2} \\
2 & \quad C_2 & \end{align*}
\]

Arrows \( p_1 \) and \( p_2 \) are called **projections**.

Similarly, we have a limit called a **pullback** if \( \mathcal{J} \) is a category like the following:

\[
\begin{align*}
1 & \quad \xymatrix{ & 0 \ar[dl]_{p_1} \ar[dr]^{q_1} & \ar@{=} C_1 \times C_0 \ar@{=} [ld]_{p_2} \ar@{=} [rd]^{q_2} \\
2 & \quad C_2 & \end{align*}
\]

Here is a pullback diagram.

\[
\begin{align*}
1 & \quad \xymatrix{ & C_1 \ar[dl]_{p_1} \ar[dr]^{q_1} & \ar@{=} C_1 \times C_0 \ar@{=} [ld]_{p_2} \ar@{=} [rd]^{q_2} \\
0 & \quad C_0 & \end{align*}
\]

Usually, we write this in the form

\[
\begin{align*}
\quad \xymatrix{ & C_1 \times C_0 \ar[dl]_{q_1} \ar[dr]^{q_2} & \ar@{=} C_2 \ar@{=} [ld]_{p_2} \ar@{=} [rd]^{p_1} \\
\quad C_1 & \ar@{=} [d] \ar@{=} d & \ar@{=} [d] \ar@{=} C_0 & \end{align*}
\]

and call it a **pullback square**.

Next, we deal with the dual concept of limit.

**Definition A.3.17.** [Colimits] A **colimit** for a functor \( F : \mathcal{J} \to \mathcal{C} \) is a pair \( \langle R, v \rangle \) where \( R \) is an object of \( \mathcal{C} \) and \( v \) is a family of arrows \( v = \{ v_J \} \) in \( \mathcal{C} \) indexed by objects \( J \) of \( \mathcal{J} \) satisfying the following two conditions.
1. For any arrow \( h : J \to K \) in \( J \), \( v_J = v_K \circ Fh \).

2. For any pair \( \langle S, u \rangle \) where \( S \) is an object of \( C \) and \( u \) is a family of arrows \( u = \{ u_J \} \), if for any arrow \( h : J \to K \) in \( J \), \( u_J = u_K \circ Fh \), then there exists a unique arrow \( t : R \to S \) such that \( u_J = t \circ v_J \) for all object \( J \) of \( J \).

When this holds, we write

\[
R = \lim_{\rightarrow} F.
\] (A.19)

The category \( J \) in Definition A.3.17 is an index category. We can define many special limits by specifying \( J \) concretely.

For example, when \( J \) is a discrete category such that \( \mathcal{O}_J = \{1, 2\} \) which has no arrows except identities. Then the functor \( F : J \to C \) is just specifying two objects of \( C \) and the corresponding colimit is their coproduct.

Arrows \( p_1 \) and \( p_2 \) are called projections.

Similarly, we have a colimit called a pushout if \( J \) is a category like the following:

Here is a pushout diagram.
Note that in $\mathcal{C}$ a pushout of $\mathcal{U} \leftarrow \mathcal{W} \rightarrow \mathcal{V}$ is

$$\mathcal{U} \bigsqcup \mathcal{W} \mathcal{V} = (\sigma(\mathcal{F}_U \cup \mathcal{F}_V), \mathcal{P}_U). \quad (A.20)$$

**Theorem A.3.18.** Let $\mathcal{C}$ be a category. Any presheaf $K \in \hat{\mathcal{C}}$ can be represented as a colimit of representable functors $\mathcal{C}(-, R)$.

**Proof.** Define an index category $\mathcal{J}$ by

- $\mathcal{O}_J := \{(C, x) \mid C \in \mathcal{O}_C, x \in KC\}$,
- $\mathcal{J}((C, x), (C', x')) := \{f \in \mathcal{C}(C, C') \mid x' = Kf x\}$,

and define a functor $M : \mathcal{J} \rightarrow \hat{\mathcal{C}}$ by for an arrow $f : (C, x) \rightarrow (C', x')$ in $\mathcal{J}$,

$$M(C, x) := \mathcal{C}(-, C), \quad Mf := \mathcal{C}(-, f). \quad (A.21)$$

Then, by using Lemma A.3.15, we can show that $\langle K, \{g^{-1}_C : K \rightarrow \mathcal{C}(-, C)\}_{C \in \mathcal{O}_C} \rangle$ is a colimit of $M$.

$\square$

**Definition A.3.19.** [Yoneda embedding] Let $\mathcal{C}$ be a category. We write

$$y : \mathcal{C} \rightarrow \hat{\mathcal{C}} \quad (A.22)$$

for the **Yoneda embedding**: $y(C) := \mathcal{C}(-, C)$ that is defined in Examples A.3.9 and A.3.11.

Theorem A.3.18 says that the image of $\mathcal{C}$ embedded by the Yoneda embedding $y$ is **dense** in $\hat{\mathcal{C}}$. 

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Definition A.3.20. [Adjunctions] Let \( C \) and \( D \) be categories. An adjunction from \( C \) to \( D \) is a triple \( \langle F, G, \varphi \rangle : C \to D \), where \( F \) and \( G \) are functors

\[
\begin{array}{ccc}
C & \xrightarrow{F} & D \\
\Downarrow & & \Downarrow \\
G & \xleftarrow{\varphi} & \end{array}
\]

while \( \varphi \) is a function which assigns to each pair of objects \( C \) in \( C \) and \( D \) in \( D \) a bijection of sets

\[
\varphi_{C,D} : D(FC, D) \cong C(C, GD)
\]  

which is natural in \( C \) and \( D \).

Given such an adjunction, the functor \( F \) is said to be a left adjoint for \( G \), while \( G \) is called a right adjoint for \( F \).

A.3.4 Grothendieck topologies and sheaves

Let \( C \) be a category.

Definition A.3.21. [Sieves] A sieve on an object \( C \) in \( C \) is a set \( S \) of arrows in \( C \) with codomain \( C \) such that we have \( f \circ g \) in \( S \) whenever \( B \xleftarrow{f} C \) in \( S \) and \( A \xrightarrow{g} B \) in \( C \).

Note that we can define a sieve \( S \) as a subfunctor \( S \xrightarrow{y} yC = C(\_, C) \) as shown in Diagram A.3.6.

\[
\begin{array}{ccc}
A & \xrightarrow{SA} & C(A, C) \\
\downarrow & & \uparrow \\
B & \xleftarrow{SB} & C(B, C)
\end{array}
\]

Diagram A.3.6: \( S \) as a subfunctor of \( C(\_, C) \)

Definition A.3.22. [Grothendieck topologies] A Grothendieck topology is a function \( J \) which assigns to each object \( C \) of \( C \) a collection \( JC \) of sieves on \( C \) in such a way that
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1. \( \{ f \mid \text{cod}(f) = C \} \in JC \).

2. Stability axiom. For \( D \xrightarrow{h} C, S \in JC \) implies \( h^*S \in JD \) where

\[
h^*S := \{ f \mid \text{cod}(f) = D, h \circ f \in S \}. \tag{A.24}
\]

3. Transitivity axiom. Let \( T \) be a sieve on \( C \in C \) and \( S \in JC \). If \( f^*T \in JC_f \) for all \( C_f \xrightarrow{f} C \) in \( S \), then \( T \in JC \).

Note that \( h^*S \) in (A.24) gives a pullback square:

\[
\begin{array}{ccc}
 f & \in & h^*S \\
\downarrow & & \downarrow \text{yh} \\
 h \circ f & \in & yC \\
\end{array}
\]

that is,

\[
h^*S = S \times yC yD. \tag{A.25}
\]

Then, we have a characterization of Grothendieck topologies by treating sieves as subfunctors.

**Proposition A.3.23.** Let \( J \) be a function assigning to each object \( C \) of \( C \) a collection \( JC \) of subfunctors of \( yC \). Then, \( J \) is a Grothendieck topology iff it satisfies

1. \( yC \in JC \).

2. Stability axiom. For \( D \xrightarrow{h} C, S \in JC \) implies \( S \times yC yD \in JD \)

3. Transitivity axiom. Let \( T \) be a subfunctor of \( yC \) and \( S \in JC \). If \( T \times yC yD \in JD \) in

\[
\begin{array}{ccc}
 T \times yC yD & \xrightarrow{yD} & yD \\
\downarrow & & \downarrow yf \\
 T & \xrightarrow{yC} & yC \\
\end{array}
\]

for all object \( D \) in \( C \) and \( D \xrightarrow{f} C \) in \( S \), then \( T \in JC \).
Definition A.3.24. [Sites] A site is a pair \((C, J)\) consisting of a category \(C\) and a Grothendieck topology \(J\) on \(C\). If \(S \in JC\), we say that \(S\) is a covering sieve, or that \(S\) covers \(C\).

Definition A.3.25. [Sheaves] Let \((C, J)\) be a site, and \(P\) be a presheaf on \(C\). Then \(P\) is a \(J\)-sheaf if all \(J\)-covering sieves \(S\) of objects \(C\) of \(C\), any natural transformation \(\alpha : S \to P\) has a unique extension to \(yC\), as in the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\alpha} & P \\
\downarrow & & \downarrow \\
yC & \nearrow & \\
\end{array}
\]

\(\text{Sh}(C, J)\) denotes the full subcategory of \(\hat{C}\) consisting of \(J\)-sheaves.

Theorem A.3.26. [Sheafification functor] The inclusion functor \(\text{Sh}(C, J) \to \hat{C}\) has a left adjoint\(\pi : \hat{C} \to \text{Sh}(C, J)\),\n(A.26)
called the sheafification functor. Moreover, this functor \(\pi\) commutes with finite limits.

Here a finite limit is a limit of \(J \to C\) with the category \(J\) finite.

A.3.5 Grothendieck cotopologies

Let \(\mathcal{C}\) be a category with pushouts.

Definition A.3.27. [Cosieves] A cosieve on an object \(C\) in \(\mathcal{C}\) is a set \(S\) of arrows in \(\mathcal{C}\) with domain \(C\) such that we have \(g \circ f\) in \(S\) whenever \(C \xrightarrow{f} A\) in \(S\) and \(A \xrightarrow{\delta} B\) in \(\mathcal{C}\).

Note that we can define a cosieve \(S\) as a subfunctor \(S : \to \mathcal{C}(C, -)\) as shown in Diagram A.3.7.

Definition A.3.28. [Grothendieck cotopologies] A Grothendieck cotopology is a function \(J\) which assigns to each object \(C\) of \(\mathcal{C}\) a collection \(JC\) of cosieves on \(C\) in such a way that
Diagram A.3.7: $S$ as a subfunctor of $C(C, -)$

1. $\{f \mid \text{dom}(f) = C\} \in JC$.

2. **Stability axiom.** For $C \xrightarrow{h} D, S \in JC$ implies $h_*S \in JD$ where
   \[
   h_*S := \{f \mid \text{dom}(f) = D, f \circ h \in S\}. \tag{A.27}
   \]

3. **Transitivity axiom.** Let $T$ be a cosieve on $C \in C$ and $S \in JC$. If $f_*T \in JC_f$ for all $C \xrightarrow{f} C_f$ in $S$, then $T \in JC$.

**Definition A.3.29.** [Cobases] A **cobasis** (for a Grothendieck cotopology) on $C$ is a function $K$ which assigns to each object $C$ in $C$ a collection $KC$ consisting of families of arrows with domain $C$ such that

1. if $C \xrightarrow{f} C'$ is an isomorphism, then $\{f\} \in KC$,

2. **Stability axiom.** if $\{ C \xrightarrow{f_i} C_i \mid i \in I \} \in KC$ and $C \xrightarrow{h} D$, then $\{ \pi^2_i \mid i \in I \} \in KD$ where $\pi^2_i$ is an arrow in the following pushout diagram.

3. **Transitivity axiom.** if $\{ C \xrightarrow{f_i} C_i \mid i \in I \} \in KC$ and for each $i \in I$,
   \[
   \{ C_i \xrightarrow{g_{ij}} D_{ij} \mid j \in I_i \} \in KC_{i j}, \text{ then } \{g_{ij} \circ f_i \mid i \in I, j \in I_i\} \in KC.
   \]

**Proposition A.3.30.** Let $K$ be a cobasis on $C$. Define a function $J$ which assigns to each object $C$ in $C$ a collection $JC$ of cosieves on $C$ in such a way that $S \in JC$ iff there exists
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$R \in KC$ such that $R \subset S$. Then, $J$ is a Grothendieck cotopology on $C$.

We call $J$ a cotopology generated by $K$.

Proof. We prove that $J$ satisfies the conditions in Definition A.3.28 one by one.

1. Since $\{ f \mid \text{dom}(f) = C \}$ is a sieve and $\{ f \mid \text{dom}(f) = C \} \supset \{ 1_C \} \in KC$, it is clear.

2. Let $S \in JC$. Then, there exists $R \in KC$ such that $R \subset S$. On the other hand, let $R := \{ C \xrightarrow{f_i} C_i \mid i \in I \}$. Then by the stability axiom, $T := \{ \pi_i^2 \mid i \in I \} \in KD$. Since $f_i \in R \subset S$ and $S$ is a cosieve, we have $\pi_i^1 \circ f_i \in S$. Therefore, $\pi_i^2 \circ h = \pi_i^1 \circ f_i \in S$. Thus, $\pi_i^2 \in h_*S$, concluding $T \subset h_*S$. Hence, by the definition of $J$, $h_*S \in KD$.

3. Let $T$ be a cosieve on $C$ on $C$ and $S \in JC$ such that for all $f \in S$, $f_*T \in JC_f$ where $C_f := \text{cod}(f)$. Thus, $C \xrightarrow{f} C_f$. Note that by the definition of $J$, there exist $R \in KC$ and $R_f \in KC_f$ such that $R \subset S$ and $R_f \subset f_*T$. Then for all $g \in R_f$, $g \circ f \in T$ since $f_*T = \{ g \mid \text{dom}(g) = C_f, g \circ f \in T \}$. On the other hand, by the transitivity axiom, $\{ g \circ f \mid f \in R, g \in R_f \} \in KC$ which is a subset of $T$. Therefore, by the definition of $J$, $T \in JC$.

\qed
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