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DECOMPOSITION OF THE POPULATION DYNAMIC THEIL'S ENTROPY AND ITS APPLICATION TO FOUR EUROPEAN COUNTRIES*

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Abstract

In this paper we propose a modification of the Dynamic Theil's Entropy that considers the inequality in the whole population. We decompose it into three addends and we show how to compute them within a Markov model of income evolution. In this way the income inequality can be measured in the whole population and not only among a given number of classes in which the economic agents are classified. The model is implemented with statistics from Eurostat data applied on France, Germany, Greece and Italy. The results reveal different inequality behaviors characterizing the considered European countries.

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I. *Introduction*

The problem of the income inequality measurement is an actual and relevant object of

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investigation. Recent contributions include the study of top income shares, quintile share ratio and the statistical techniques of inference, [e.g. Brzezinski (2013); Langel and Tillé (2011)]. Moreover the influences of political regimes, financial reforms, state sector and tax effects on income inequality have been extensively studied, [e.g. Kemp-Benedict (2011); Agnello et al. (2012); Lee (2013); D'Amico et al. (2013)].

Another recent line of research was interested in measuring the income inequality through dynamic indices instead of the classical static indices, [Theil (1967)]. The first contribution in this direction was the paper by D'Amico and Di Biase (2010) where the Dynamic Theil's Entropy (DTE) has been proposed as a valuable tool for measuring and forecasting income inequality dynamically. This generalization was made possible by considering a population that evolves over time among a finite number of classes according to a semi-Markov process and by considering the income of each economic agent as a reward process. Therefore it is possible to justify changes in indices when the population configuration varies over time. The mathematical apparatus coincides with Markov and semi-Markov systems widely developed and investigated by Vassiliou and Papadopoulou (1992) and subsequent contributions.

The model was implemented in D'Amico et al. (2011) to simulate an artificial economic system with immigration, however critical points that occur when handling with real-world applications were never faced. A further advancement was presented in D'Amico et al. (2012) were the authors proposed an application methodology that makes possible the application of the model when only the averages and medians evolution of the incomes in a country are available. The methodology was considered for a Markov Chain model of income evolution.

A different approach to the measurement of the dynamic inequality based on lognormal stochastic volatility model appeared in Nishimo et al. (2012).

Several types of decompositions have been proposed for the static Theil's Entropy, [e.g. Shorrocks (1984); Duro and Esteban (1998); GoerlichGisbert (2001)] and the extension of the decomposability of the Theil's Entropy appeared in Kakamu and Fukushige (2009). Motivated by these articles in this note we extend the DTE and we propose a decomposition into three addenda which permit the evaluation of the inequality on the whole considered population and not only among the classes of agents as done in D'Amico and Di Biase (2010). Indeed, it is important to evaluate the inequality on the total population and therefore we propose here a modification of the DTE that permits the execution of more realistic applications thorough a more reliable procedure. Definitely, in this paper, it was advanced a more general dynamic inequality index that permits the relaxing of the homogeneity assumption between agents belonging to the same income class in favour of the more realistic heterogeneity hypothesis.

The model is applied on Eurostat data for four European countries and reveals to be able to reproduce different inequality behaviors characterizing the considered European countries.

The paper is organized as follows: Section II describes the Dynamic inequality indices, the decomposition and the computation methodology. Section III presents an application to real data for France, Germany, Greece and Italy. Finally, some conclusions can be found in Section IV.

II. *Dynamic Inequality Indices*

One of the most popular income inequality measure is the Theil's Entropy (TE) defined in

Theil (1967) by

$$TE=\sum_{i=1}^{N} a_i(\log Na_i), \tag{1}$$

where a_i represents the share of the total income of agent i and N is the number of the agents in the economic system. This index was generalized into a dynamic version by D'Amico and Di Biase (2010). The generalization allows the shares of income to be random processes rather than constant deterministic numbers. Obviously, if the shares of income are known constant in time, then the classic Theil's index and the dynamic version coincide.

Suppose that each agent produces at time $t \in \mathbb{N}$ a quantity $y_i(t)$ of income. We classify each agent by allocating it depending on its own income, at each time, in one of K mutually exclusive classes of income $E=\{C_1, C_2, ..., C_K\}$ following any rule. An operative criterion of allocation has been adopted in Quah (1996) and D'Amico et al. (2012).

We suppose that once the economic agents are allocated into the classes they can leave the initial class and enter a new income class according to a discrete time Markov Chain with transition probability matrix $\mathbf{P}$. The element p_{ij} denotes the probability that an agent, now allocated in C_i, will enter the next allocation C_j.

We would like to remark that the Markovian hypothesis could be relaxed in favor of the weaker semi-Markovian hypothesis that was originally advanced by D'Amico and Di Biase (2010). Anyway here we consider a Markov chain model of income dynamic for easiness of exposition and for avoiding the computational complexity of the semi-Markov framework.

We assume that each agent allocated in the class $C_j \in E$ produces an income equal to y_{C_j}.

Let $t=0$ be the initial observation time, $n_{C_i}(0)$ be the number of agents in class C_i at time zero and $\underline{n}(0)=\{n_{C_1}(0), n_{C_2}(0), ..., n_{C_k}(0)\}$ be the population structure at time zero. Moreover let $a_{C_j}(\underline{n}(0))$ be the initial share of income due to class C_j.

Given the population configuration $\underline{n}(0)$ and the vector of average incomes $\underline{y}(0)=(y_{C_1}(0), y_{C_2}(0), ..., y_{C_K}(0))$, the DTE, see D'Amico and Di Biase (2010), is the stochastic process:

$$DTE(t;K):=\sum_{i=1}^{K} a_{C_i}(\underline{n}(t))(\log Ka_{C_i}(\underline{n}(t))), \tag{2}$$

where the process $\underline{a}(\underline{n}(t))=(a_{C_1}(\underline{n}(t)), a_{C_2}(\underline{n}(t)), ..., a_{C_K}(\underline{n}(t)))$, describes the time evolution of the shares of income among the classes of population:

$$a_{C_j}(\underline{n}(t))=\frac{n_{C_j}(t)y_{C_j}}{<\underline{n}(t), \underline{y}(t)>} \tag{3}$$

where

$$\underline{n}(t)=\{n_{C_1}(t), n_{C_2}(t), ..., n_{C_k}(t)\} \tag{4}$$

is the multivariate stochastic process describing the evolution of the population in time and $<\cdot, \cdot>$ is the usual scalar product.

The range of values of $DTE(t;K)$ is between 0 and $\log K$. At a fixed time t, the index is 0 when the income is equidistributed among the classes, whereas it reaches the value of $\log K$ when one class holds all the income.

One important point is that the DTE index (2) is a measure of the income inequality among the K classes that have to be rendered uniform with respect to the population allocation. Nevertheless it does not represent the inequality in the whole population and this constitutes a serious limitation of the model.

Indeed, when we measure the inequality with the DTE at a fixed time t implicitly we replace the current population structure (4) that produces the income $\underline{y}(t)=(y_{C_1}(t),\ y_{C_2}(t),\ ...,\ y_{C_K}(t))$, with an uniformized population $\underline{n}(t)=\{1, 1, ..., 1\}$ that produces the income $\underline{y}(t)=(n_{C_1}(t)y_{C_1}(t),\ n_{C_2}(t)y_{C_2}(t),\ ...,\ n_{C_K}(t)y_{C_K}(t))$.

Therefore the DTE suffers of two inconveniences. First it is due to the uniformization of the population. Second problem concerns the lack of a measure of inequality within the K classes of the economy.

In order to overcome these inconveniences and to be able to measure the inequality in the population we propose here a modification of the DTE and through a decomposition into three addend we show how to compute it.

As well known, the TE (1) can be represented as follows:

$$TE=\frac{1}{N}\sum_{i=1}^{N}\frac{y_i}{\bar{y}}\left(\log\frac{y_i}{\bar{y}}\right), \tag{5}$$

where $\bar{y}$ is the average income in the population and y_i is the income of the i-th agent.

If we allocate all agents in K classes we can represent the TE by using the decomposability property by

$$TE=\sum_{g=1}^{K}a_{C_g}TE(y_{C_g}\ ;\ n_{C_g})+\sum_{g=1}^{K}a_{C_g}\left(\log\frac{y_{C_g}}{\bar{y}}\right), \tag{6}$$

where a_{C_g} is the share of production of class C_g, $TE(y_{C_g}\ ;\ n_{C_g})$ is the Theil's entropy of the class C_g and y_{C_g} is the average income of the class C_g.

The entropy (6) can be represented as follows:

$$\begin{aligned}TE&=\sum_{g=1}^{K}a_{C_g}TE(y_{C_g}\ ;\ n_{C_g})+\sum_{g=1}^{K}a_{C_g}\left(\log\frac{Ky_{C_g}n_{C_g}N}{K\bar{y}\,n_{C_g}N}\right)\\&=\sum_{g=1}^{K}a_{C_g}\,TE(y_{C_g}\ ;\ n_{C_g})+\sum_{g=1}^{K}a_{C_g}\log Ka_{C_g}+\sum_{g=1}^{K}a_{C_g}\log\frac{N}{K\,n_{C_g}}.\end{aligned} \tag{7}$$

Now if the share of income a_{C_g} are assumed to be random processes as in (3), then (7) becomes the dynamic Theil's entropy on the whole population (PDTE):

$$\begin{aligned}PDTE(t;\,N)&=\sum_{g=1}^{K}a_{C_g}(\underline{n}(t))TE(y_{C_g}\ ;\ n_{C_g}(t))\\&+\sum_{i=1}^{K}a_{C_g}(\underline{n}(t))\log Ka_{C_g}(\underline{n}(t))+\sum_{g=1}^{K}a_{C_g}(\underline{n}(t))\log\frac{N}{K\,n_{C_g}(t)}.\end{aligned} \tag{8}$$

Notice that the second addendum of (8) coincides with the DTE (2) that, as remarked before, measures the income inequality among the classes after the uniformization of the population. If $K=N$ then the first and third addenda become zero and the differences between the DTE and PDTE disappear.

The entire process can be summarized by computing the first order moment addendum by addendum.

The expectation of the second addendum was computed in D'Amico et al. (2012) and is given by:

$$E\left[\sum_{g=1}^{K} a_{C_g}(\underline{n}(t))\log K a_{C_g}(\underline{n}(t))\right]$$
$$=\sum_{g=1}^{K}\sum_{\underline{n}'\in p.c.} P\left[\underline{n}(t)=\underline{n}'|\underline{n}(0)=\underline{n}\right] a_{C_g}(\underline{n}'(t))(\log K a_{C_g}(\underline{n}'(t))) \tag{9}$$
$$=\sum_{g=1}^{K}\sum_{\underline{n}'\in p.c.} \frac{N!}{\prod_{h=1}^{K} n'_{C_h}!}\prod_{h=1}^{K}(P_h(t))^{n'_{C_h}}\left(\frac{\log K a_{C_g}(\underline{n}'(t))}{a_{C_g}^{-1}(\underline{n}'(t))}\right),$$

where p.c. is the set of all possible population configurations,

$$P_i(t)=\sum_{h=1}^{K}\frac{n_{C_h}(0)}{N}p_{hi}^{(t)}$$

and $p_{hi}^{(t)}$ are the t-step transition probabilities of the Markov Chain.

The expectation of the third addendum is given by:

$$E\left[\sum_{g=1}^{K} a_{C_g}(\underline{n}(t))\log\frac{N}{Kn_{C_g}(t)}\right]$$
$$=\sum_{g=1}^{K}\sum_{\underline{n}'\in p.c.} \frac{N!}{\prod_{h=1}^{K} n'_{C_h}!}\prod_{h=1}^{K}(P_h(t))^{n'_{C_h}} a_{C_g}(\underline{n}'(t))\log\frac{N}{K\,n'_{C_h}(t)}. \tag{10}$$

Here we give an interpretation of this addendum. The quantity $\frac{N}{K}$ is equal to the average number $\bar{n}$ of agents per class. Therefore the third addendum can be rewritten for each $t\in\mathbb{N}$ as follows:

$$\sum_{g=1}^{K} a_{C_g}(\underline{n}(t))\log\frac{\bar{n}}{n_{C_g}(t)}$$
$$=-\sum_{g=1}^{K} a_{C_g}(\underline{n}(t))\log\frac{n_{C_g}(t)}{\bar{n}}=E_{\Im_t}\left[\log\frac{n_{C_g}(t)}{\bar{n}}\right], \tag{11}$$

where $\Im_t=(a_{C_1}(\underline{n}(t)), a_{C_2}(\underline{n}(t))..., a_{C_k}(\underline{n}(t)))$ is the share of production probability measure at time $t\in\mathbb{N}$. Thus formula (11) is the opposite of the mean logarithmic deviation of the population distribution (4) with respect to the uniform distribution

$$(\bar{n}, \bar{n}, ..., \bar{n}), \tag{12}$$

computed using the share of production probability measure.

In this way (11) is an inequality measure that summarizes the divergence of the population distribution (4) about the uniform population distribution (12).

It should be remembered that this third addendum is always less or equal than zero as we will demonstrate in a next proposition and, then, it represents a correction term to be applied to the DTE when computing the inequality among the classes C_1, C_2, ..., C_K that compensates the increase in the inequality caused by the uniformization of the population required for the computation of the DTE.

Proposition II.1. For each $t \in \mathbb{N}$, $-\sum_{g=1}^{K} a_{C_g}(\underline{n}(t)) \log \frac{n_{C_g}(t)}{\bar{n}} \leq 0$.

Proof.

$$-\sum_{g=1}^{K} a_{C_g}(\underline{n}(t)) \log \frac{n_{C_g}(t)}{\bar{n}} = -\sum_{g=1}^{K} \frac{n_{C_g}(t) y_{C_g}}{<\underline{n}(t), \underline{y}(t)>} \log \frac{n_{C_g}(t)/N}{\bar{n}/N}$$
$$= -\sum_{g=1}^{K} \frac{N y_{C_g}}{<\underline{n}(t), \underline{y}(t)>} \frac{n_{C_g}(t)}{N} \log \frac{n_{C_g}(t)/N}{\bar{n}/N}. \tag{13}$$

Let denote by $y_{\min} = \min_{g \in \{1, 2, \ldots K\}} y_{C_g}$ then

$$-\sum_{g=1}^{K} a_{C_g}(\underline{n}(t)) \log \frac{n_{C_g}(t)}{\bar{n}} \leq \frac{-N y_{\min}}{<\underline{n}(t), \underline{y}(t)>} \sum_{g=1}^{K} \frac{n_{C_g}(t)}{N} \log \frac{n_{C_g}(t)/N}{\bar{n}/N}$$
$$= -\left(\frac{-N y_{\min}}{<\underline{n}(t), \underline{y}(t)>}\right) \cdot KL\left(\left(\frac{n_{C_1}(t)}{N}, \ldots, \frac{n_{C_K}(t)}{N}\right); \left(\frac{\bar{n}}{N}, \ldots, \frac{\bar{n}}{N}\right)\right), \tag{14}$$

where the symbol $\mathcal{KL}$ represents the Kullback-Leiber distance between the distributions $\left(\frac{n_{C_1}(t)}{N}, \ldots, \frac{n_{C_K}(t)}{N}\right)$ and $\left(\frac{\bar{n}}{N}, \ldots, \frac{\bar{n}}{N}\right)$. The proof is complete once we note that for any pair of probability distributions the Kullback-Leibler distance is always nonnegative, see Kullback and Leibler (1951).

The moment of the first addendum is evaluated first by replacing the term $TE(y_{C_g}; n_{C_g}(t))$ with the value $TE(y_{C_g}; \bar{n}_g(t))$. The latter is the value of the Theil's Entropy function that measures the inequality within group g computed by replacing the random variable $n_{C_g}(t)$ with the expected population configuration $\bar{n}_g(t)$ that represents its average. Secondly it is possible to proceed to compute the expectation as follows:

$$E\left[\sum_{g=1}^{K} a_{C_g}(\underline{n}(t)) TE(y_{C_g}, n_{C_g}(t))\right]$$
$$= \sum_{g=1}^{K} \sum_{\underline{n}' \in p.c.} \frac{N!}{\prod_{h=1}^{K} n'_{C_h}!} \prod_{h=1}^{K} (P_h(t))^{n'_{C_h}} a_{C_g}(\underline{n}'(t)) \, TE(y_{C_g}, \bar{n}_{C_g}(t)). \tag{15}$$

The DTE index is able to capture the randomness in the inequality evolution and, consequently, through its first moment, provides a function that is an effective tool for forecasting income inequality for a given horizon time. With the new PDTE representation of the dynamic income inequality, that is adopted in this paper, the forecast of the inequality refers to the complete population and not only to the inequality among the classes, and therefore incorporates also the effect due to the heterogeneity of the agents within the same income class.

III. *Application to real data for France, Germany, Greece and Italy*

We used the Eurostat data concerning population, means and medians of the equivalised net income for France, Germany, Greece and Italy as reported in D'Amico et al. (2012). Data refer to years from 2005 to 2008. The choice of the countries and the time interval is motivated by the fact that they represent the sole case for which there are no missing data.

In order to implement the model we followed the scheme of the procedure advanced in D'Amico et al. (2012). Nevertheless, the computation of the PDTE requires the evaluation of the entropies $TE(y_{C_g}; n_{C_g})$ for all classes $g \in \{C_1, C_2, ..., C_K\}$. Therefore, the step 3 of that procedure has been modified by considering not simply the average income of the classes but the complete income distribution within each class. To do this we executed a partition of the income brackets that identifies the single class first into 100 subintervals, then in more and more finer partitions. The results on the PDTE are here reported only for the subdivision with 400 subintervals because a finer partition does not affect significantly the results.

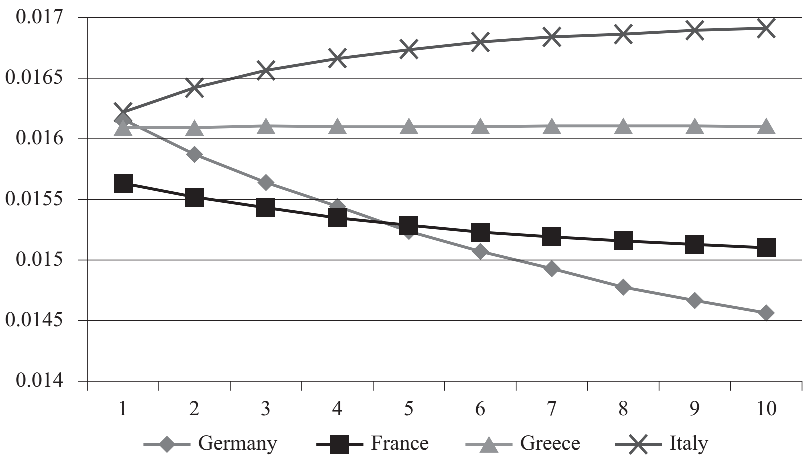
The procedure allows the recovering of the transition matrices for all the four countries that are necessary for the computation of the expectation of the PDTE. These matrices are the same reported in Table 3 in D'Amico et al. (2012) where it is possible also to find a discussion about their properties.

Once we dispose of the transition probability matrix for each country, we can then compute the expected evolution of the PDTE.

We report the expected values of (15), (9) and (10) in Figure 1, Figure 2 and Figure 3, respectively. It should be noted that the curves in Figure 2 represent the values of the DTE.

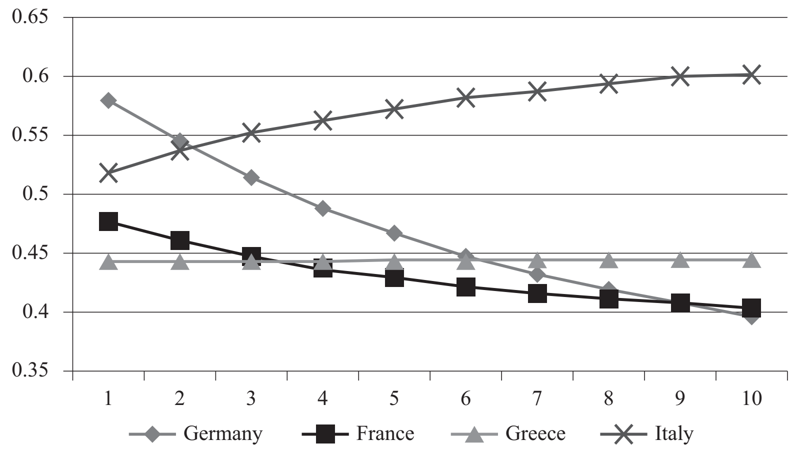
The values in Figure 1 are of one order of magnitude lower than those of Figure 2 and Figure 3. This means that the inequality within each class is less important than the others inequalities. This argument applies for all the considered countries. Moreover the inequality

FIG. 1. FIRST ADDENDUM OF THE DECOMPOSITION OF THE PDTE



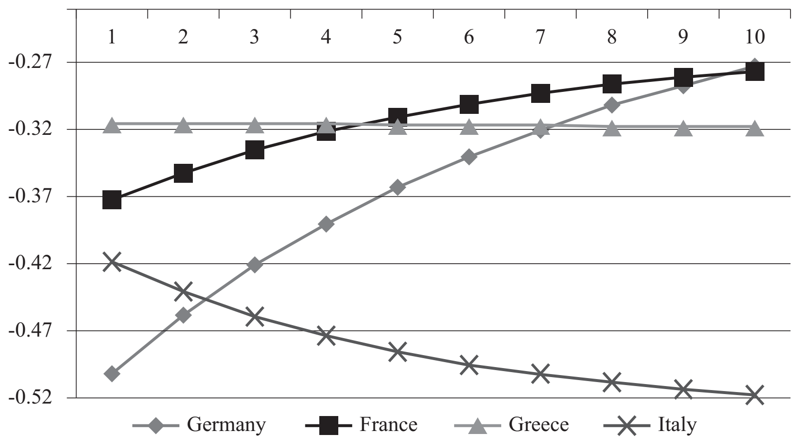
Note: x axis is t (time in years), y axis is the expectation of the PDTE.

FIG. 2. SECOND ADDENDUM OF THE DECOMPOSITION OF THE PDTE: THE DTE



Note: x axis is t (time in years), y axis is the expectation of the PDTE.

FIG. 3. THIRD ADDENDUM OF THE DECOMPOSITION OF THE PDTE



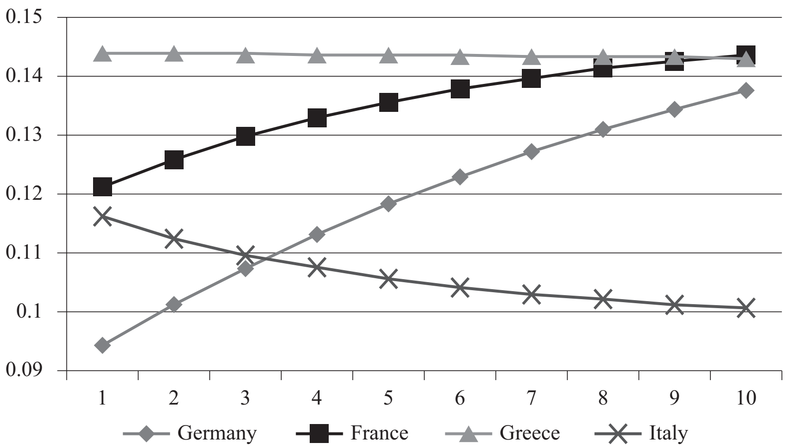
Note: x axis is t (time in years), y axis is the expectation of the PDTE.

within the classes is expected to decrease in France and Germany, to stay stable in Greece and to increase in Italy.

Figure 2 shows the values of the DTE which measures the inequality among the classes C_1, C_2, ..., C_K after the uniformization of the population. Here again Greece inequality is stable, France and Germany share similar monotonic decreasing behavior and Italy is the unique country with an increasing DTE.

Figure 3 shows the opposite of the mean logarithmic deviation of the real population structure with respect to the uniform distribution $(\bar{n}, \bar{n}, ..., \bar{n})$.

Fig. 4. Population Dynamic Theil's Entropies



Note: x axis is t (time in years), y axis is the expectation of the PDTE.

According to Proposition II.1, all the values are negative. The increasing path of France reveals that the population will move toward a more uniform distribution. The same argument applies for Germany. Also in this case Italy exhibits an opposite behavior with respect to those of France and Germany.

In Figure 4 we report the PDTE which is obtained by summing the three curves showed

As we can see, in France there is a tendency toward increasing inequality. Also for Germany the graphical forecast shows an increasing inequality. The difference is that, in the early periods, the inequality in Germany is lower than in France, but it increases more rapidly.

These results are opposite to those showed in Figure 2 and obtained using only the DTE. This fact suggests that the measurement of inequality in the whole population cannot be correctly approximated by using only the inequality among classes through an uniformization of the population, as performed by the DTE in a dynamic framework.

The information revealed in Figure 4 is, in our opinion, useful for planning an economic policy aimed at the reduction of income inequality.

IV. *Conclusion and future research*

This note extends the Dynamic Theil's Entropy and proposes a decomposition of the new index into three addenda which permit the evaluation of the inequality on the whole considered population and not only among a given number of classes of agents as done in D'Amico and Di Biase (2010).

Indeed the curves measuring the inequality are quite different in the two situations. As a consequence the forecast of the inequality in the whole population cannot be correctly executed by using only the corresponding forecast among classes as performed by the DTE in a dynamic framework.

The results showed different types of temporal evolutions of the index. These differences suggest the necessity to implement an European policy of economic integration, given the very different behaviors of inequality in the studied countries. For these reasons we think that the results are relevant for the adoption of an economic and social policy of inequality maintenance.

Possible avenues for future developments of our model include:

a) the application to the measurement of income polarization by extending different well-behaved polarization indexes (see e.g. Esteban and Ray (1994), Duro (2005) and Esteban et al. (2007)) to a dynamic framework;

b) measuring the variability of each income y_{C_g} by considering the income as a random variable and then the vector $\underline{y}(t)$ as a random process. This major complexity could be addressed by using Markov (or semi-Markov) reward processes as done for the computation of the Dynamic Herfindahl-Hirschman index by D'Amico et al. (2014).

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