Evolution of Copulas

: Continuous, Discrete, and its application to Quantitative Risk Management

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Submitted in partial fulfillment of the degree of Doctor of Philosophy in Economics

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March 2015

Contents

1. Introduction

Dependence relations among random variables are one of the most important subjects for probability and statistics research. Analysis of dependence structures is critical from both theoretical and applied viewpoints.

For example, the damage caused by typhoons in Taiwan and Japan is randomly related; some typhoons move from Taiwan to Japan, but others do not. An insurance company whose business includes these two countries must estimate the effect of this dependence. Damage is caused not only by typhoons but also by financial crises, the euro crisis, the East Japan earthquake, large hurricanes in the United States, and floods in Thailand. These catastrophic phenomena emerge suddenly, and the related losses are enormous; this concept is gaining increasing attention.

Recently, members of the financial sector, like banks and insurance companies, and their regulators have recognized that it is critical to manage these risks in a sophisticated way, that is, quantitatively. Quantitatively measured risks play a central role in this management framework. These entities face many kinds of risks, and the relations between them are very complicated. Thus, it is crucial to reflect the dependence relations to measure the risks quantitatively: the more dependence there is among risks, the less aggregated the risks are. See Appendix A and Yoshizawa, Y. [26], [27].

Linear correlation is often recognized as a satisfactory measure of dependence in risk management. However, it cannot capture the non-linear dependence relations that exist among many risk factors. It is a canonical measure only in the world of normal distributions, and is used more generally in spherical and elliptical distributions. See Embrechits, P. *et al*. [6], [10].

What expresses the dependence relations among risk factors? If we capture a multivariate joint distribution of all the risk factors, we can recognize their dependence structure probabilistically or statistically.

For this reason, there has been much interest in a copula function, or simply a copula. A copula links multivariate joint distribution and univariate marginal distributions. Copulas are often employed to investigate the dependence structure among random variables. The name copulas is derived from "couples" in Latin. The fundamental theorem of copulas, "Sklar's theorem," first appeared in 1959. See Schweizer, B., & Sklar, A. [4], and Sklar, A. [23]. This elegant theorem claims that all multivariate distributions have copulas, and that copulas are used in conjunction with univariate distributions to construct multivariate distribution. As for the study of copulas, Free, E.W. *et al.* [11] and Tsukahara, H. [25] provide useful introductive reviews. Nelsen, R.B. [3] wrote the excellent standard textbook, providing a systematic development of the theory of copulas, particularly bivariate copulas. The book by Joe, H. [1] is also helpful and covers multivariate copulas. Moreover, McNeil, A. J. *et al*. [2] introduce copulas theoretically and practically, examining quantitative risk management for financial sectors.

Because of their flexibility, copulas have been extensively studied and applied in a wide range of areas concerning dependence relations, including risk management, insurance and/or financial mathematics, and seismological analysis,. See Breymann, W.

et al. [8], Genest, G. *et al.* [12], Goda, K. [13], [14], Goorbergh, R.W.J. *et al.* [15], and Lebrun, R. [21].

Dependence relations, which transform over time, are dynamic rather than static in nature. We analyze the time variance of a dependence structure, taking foreign exchange rates as an example. We collect the daily foreign exchange rates for the U.S. dollar and the euro against the Japanese yen for 10 years, and use copulas to analyze these distributions annually. From this analysis, we can see that copula functions are transforming every year. See Appendix B.

However, copulas are useful mainly for static matters; their definitions themselves do not contain time variables. Mikosch, T. [22] suggests, "Copulas do not fit into the existing framework of stochastic processes and time series analysis; they are essentially static models and are not useful for modeling dependence through time". However, a few exceptions exist: copulas and the Markov process, as in Darsow, W.F. *et al.* [9], and dynamic copulas, as in Patton, A.J. [24]. Copulas and Markov processes can be used to analyze the dependence relations between Markov processes at different times. Dynamic copulas involve the development of dynamic time series models for financial return data using conditional copulas. Furthermore, Kallsen, J., and Tankov, P. [20] propose the Lévy copula for the Lévy process, and a survey by Bielecki, T.R. *et al.* [5] introduces some copulas for stochastic processes.

It is well known that rank correlations, one of the prevailing measures of dependence, are derived only by copulas. That is to say, copulas determine rank correlations. Thus, it is natural to analyze only copulas in the study of transformations of dependence structures through time. As a first step, we start to investigate how copulas transform, and if they evolve in accordance with the heat equation, which is one of the basic partial differential equations used to describe dynamic movements.

In this thesis, we introduce evolving copulas, which transform through time autonomously as governed by the heat equation. Our aims are to prove that their solutions are time dependent, and to analyze the transitions of their rank correlations. Moreover, we construct discrete type of the time-dependent evolution of copulas to apply empirical data analysis, investigate their properties, and prove that they converge to their original continuous type. Finally, we apply empirical data to discrete evolution copulas in order to verify their practicality

In this Chapter, we marshal the basic concepts and properties of copulas for the preparation of the subsequent chapters. We illustrate the definition of bivariate copulas; provide some examples; describe Sklar's theorem, which plays a central role in copula theories; and provide the definition and properties of rank correlations like Kendall's tau and Spearman's rho. Finally, we explain empirical copulas, which are used in the basic theory of copulas in discrete processes, in Chapter 3. For background issues in this chapter, we refer mainly to McNeil, A. J. *et al*. [2] and Nelsen, R.B. [3].

In Chapter 2, we propose that time-dependent evolving copulas transform autonomously through time. First, we prove the existence of solutions for the evolution of copulas that evolve in accordance with the heat equation; moreover, we prove that they converge to the product copula as time $t \to \infty$. Next, we prove that their rank correlations converge to zero exponentially as time $t \to \infty$. Finally, we extend the evolution of copulas backward and with coefficients. In this chapter, we refer to Ishimura, N. & Yoshizawa, Y. [16], [17], and Yoshizawa, Y. & Ishimura, N. [28].

In general, it is difficult to solve partial differential equations analytically; furthermore, numerical analysis using discrete data is suitable for calculation by computer. In Chapter 3, we study the evolution of copulas in discrete processes that satisfy the discrete version of the heat equation (we call these copulas "discrete evolution of copulas" and this type of evolution the "discrete type"). We define these copulas, and prove that they converge to the product copulas and that their rank correlations converge to zero exponentially. Moreover, we prove that these discrete evolution of copulas converge to their original evolution of copulas (we call these copulas "continuous evolution of copulas" and this type of evolution the "continuous type"). Thus, we can treat discrete evolution of copulas as an approximation of the continuous type. Next, we extend them backward and with coefficients for discrete evolution of copulas. Finally, we apply empirical data to discrete evolution of copulas to verify their practicality. In this chapter, we refer to Ishimura, N. & Yoshizawa, Y. [18], [19], and Yoshizawa, Y. & Ishimura, N. [29].

Copulas

Let us begin by recalling the definition of a copula and the fundamental theorem created by Sklar, A. [23]. Sklar, A. first developed the theorem of copulas in 1959. The structure of a copula is shown diagrammatically in Figure 1.1. In this chapter, we recall the definition of copulas, provide some examples, describe Sklar's theorem, and examine rank correlations and empirical copulas with reference to Joe, H. [1], McNail, A. J. *et al.* [2], and Nelsen, R.B. [3].

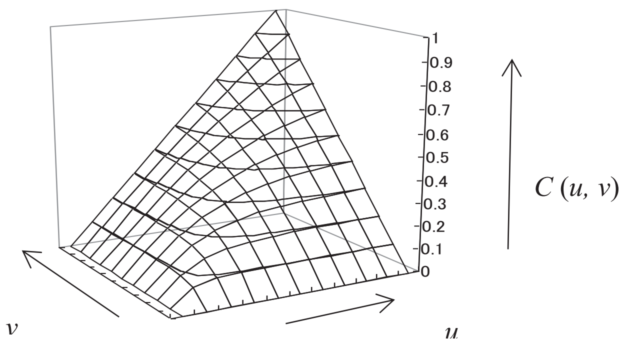


Figure 1.1. Copula

The bivariate copulas are defined by three properties described in the following (1.1), (1.2) and (1.3). The properties (1.1) and (1.2) are the boundary conditions, and the property (1.3) is called the 2-increasing condition.

Definition 1.1 (Copulas). Bivariate copula is a function $C(u,v)$ from I^2 to I defined by three properties as

$$C(u,0) = C(0,v) = 0, \tag{1.1}$$

$$C(u,1) = u \text{ and } C(1,v) = v, \tag{1.2}$$

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) - C(u_1,v_1) \geq 0, \tag{1.3}$$

for $(u_i,v_j) \in I^2 \ (i,j = 1,2), u_1 \leq u_2 \ , v_1 \leq v_2$.

The 2-increasing condition (1.3) induce that the probability density function is not less than zero. Moreover on account of this property copulas satisfy the 1-increasing condition as well as continuous properties.

Among many kinds of copulas, the following Fréchet-Hoeffding bounds of copulas, such as lower bound copula and upper bound copula, are well known. The images of these copulas are plotted in Figure 1.2. Fréchet-Hoeffding bounds of copulas are defined as

Upper bound copula; $M(u, v) = \min(u, v)$.

Lower bound copula; $W(u, v) = \max(u + v - 1, 0)$,

There are important relations among Fréchet-Hoeffding bounds copulas. $C(u, v) \leqq C(u, 1) = u$ and $C(u, v) \leqq C(1, v) = v$, thus we have $C(u, v) \leqq \min(u, v) = M(u, v)$. Using 2-increasing condition, we obtain $C(1,1) - C(u, 1) - C(u, 1) + C(u, v) \geq 0$, that is $C(u, v) \geq u + v - 1$, and $C(u, v) \geq 0$. Thus we can say $C(u, v) \geq \max(u + v - 1,0) = W(u, v)$. According to these inequities all the copulas are between lower bound copula and upper bound copula as

$$W(u, v) \leq C(u, v) \leq M(u, v).$$

Lower bound copula

$$W(u,v) = \max(u+v-1,0)$$

Upper bound copula

$$M(u,v) = \min(u,v)$$

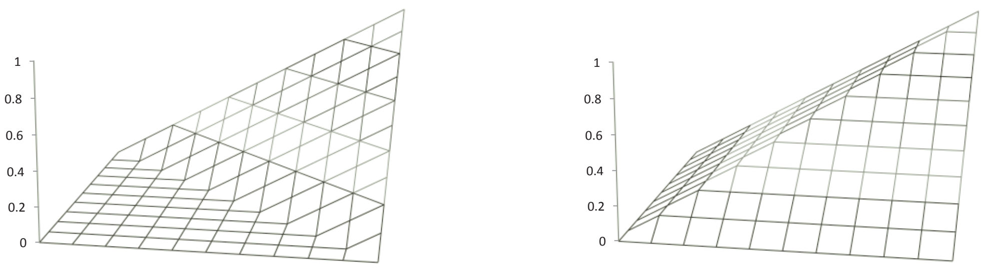


Figure 1.2. Fréchet-Hoeffding bounds of copulas

The product copula is important and often appears in this thesis. The product copula is called the independent copula, which express the independent relations between random variables. The image of the product copula is plotted in Figure 1.3.

$$\text{Product copula ;} \qquad \Pi(u,v) = uv.$$

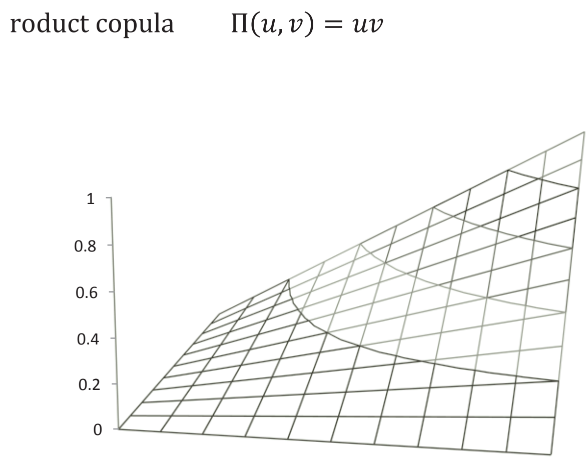


Figure 1.3. Product copula

Besides above Fréchet-Hoeffding bounds copulas and the product copula, Clayton copula C^{Cl}, Gumbel copula C^{Gu}, Frank copula C^{Gal}, Gaussian copula C_p^{Ga} and t copula$C_{v,p}^{t}$ are popular.

We introduce Clayton copula C^{Cl} , which is used in Appendix B. It is well-known that in the limit as $\theta \to 0$ Clayton copula approaches the product copula, and as $\theta \to \infty$ it approaches the Upper bound copula.

Clayton copula is defined as

$$C^{Cl}(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{\frac{-1}{\theta}}, \qquad \text{where } 0 < \theta < \infty.$$

Sklar's theorem

The following Sklar's theorem is the core theory among various copula theories. Thanks to this theorem we can construct multivariate distribution by coupling univariate marginal distributions. The Figure 1.4 is the conceptual image of the coupling using Sklar's theorem where the bivariate distribution $H\,(x, y)$ are generated by coupling the marginal distributions $F\,(x) = u$ and $G\,(y) = v$ using copula $C\,(u, v)$. As for the proof of this theorem, see Schweizer, B., & Sklar, A. [4] and Nelsen, R.B. [3].

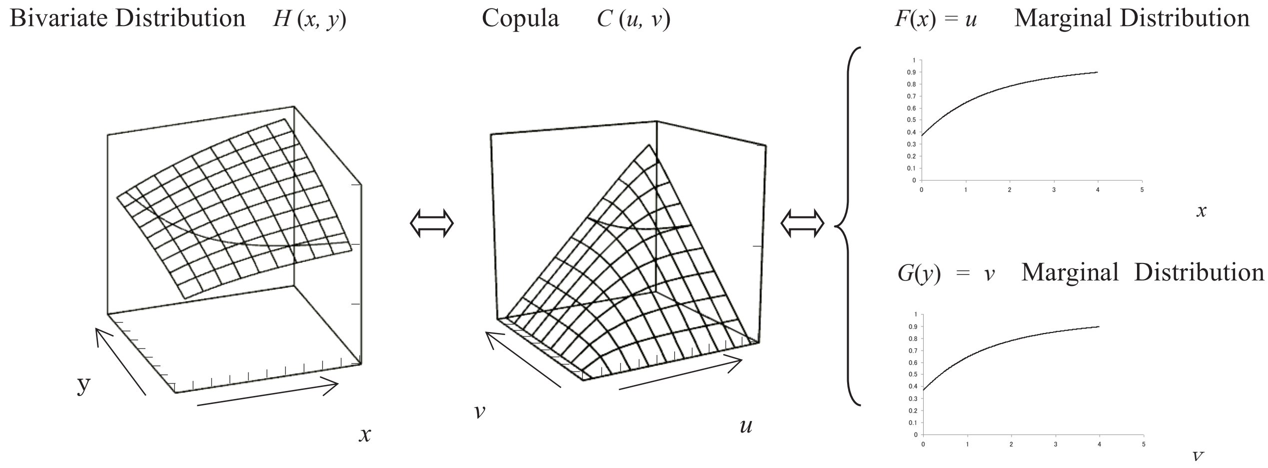


Figure 1.4. coupling of marginal distribution by Copula

Theorem 1.4 (Sklar's theorem). Let H be a bivariate joint distribution function with marginal distribution function F and G; that is

$$\lim_{y\to\infty} H(x,y) = F(x) = u \quad \text{and} \quad \lim_{x\to\infty} H(x,y) = G(y) = v. \tag{1.4}$$

Then there exists a copula, which is uniquely determined on Ran $F \times$ Ran$G$, such that

$$H(x,y) = C\left(F(x), G(y)\right) = C(u,v). \tag{1.5}$$

Conversely, if C is a copula and F and G are distribution functions, then the function H defined by (1.5) is a bivariate joint distribution function with marginal distribution functions F and G.

Rank correlations

Next we explain the concept of rank correlations. Rank correlations are a sort of dependence measures, and Kendall's tau (τ) and Spearman's rho (ρ) are typical rank correlations. Their special feature is that they do not depend on marginal distributions, but depend only on their copulas. Kendall's tau is defined in the Definition 1.5 and Spearman's rho is also defined in the Definition 1.6.

Definition 1.5 (Kendall's tau). Kendall's tau for a pair (X, Y) is the probability of concordance minus the probability of dis- concordance. Kendall's tau is defined as

$$P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0],$$

where (X_1, Y_1) and (X_2, Y_2) are independent and identically distributed random vector with joint distribution.

Moreover the popular version of Kendall's tau is derived using copula as

$$\tau_{X,Y} = 4 \iint_{I^2} C(u,v) dC(u,v) - 1, \tag{1.6}$$

where u and v are uniform random variables.

Definition 1.6 (Spearman's rho). Spearman's rho for (X_1, Y_1) and (X_2, Y_3) is proportional to the probability of concordance minus the probability of dis-cordance. Spearman's rho is defined as

$$3(P[(X_1 - X_2)(Y_1 - Y_3) > 0] - P[(X_1 - X_2)(Y_1 - Y_3) < 0]),$$

where (X_1, Y_1), (X_2, Y_2) and (X_3, Y_3) are independent and identically distributed random vectors with a common joint distribution functions.

The well-known version of Spearman's rho is derived as

$$\rho_{X,Y} = 12 \iint_{I^2} uv\, dC(u,v) - 3 = 12 \iint_{I^2} C(u,v) du dv - 3. \tag{1.7}$$

Empirical copulas

We introduce copulas for sample data, which are called empirical copulas, with reference to Nelson, R.B. [3]. Empirical copulas are used for evolution of copulas in discrete processes in Chapter 3 as the core technique and theory. The definition of empirical copulas, empirical copula frequency function and their relations are as follows.,

Definition 1.7 (Empirical copulas). Let$\{x_k, y_k\}_{k=1}^{n}$ denote a sample with size n from a continuous bivariate distribution. The empirical copula C_n is the function given by

$$C_n\left(\frac{i}{n},\frac{j}{n}\right) = \frac{\text{number of pairs } (x,y) \text{in sample with } x \le x_{(i)}, y \le y_{(j)}}{n^2}, \qquad (1.8)$$

where $x_{(i)}$ and $y_{(j)}$, $1 \le i,j \le n$, denote order statistics from the sample.

Empirical copula frequency functions. Empirical copula frequency function c_n is given by

$$c_n\left(\frac{i}{n},\frac{j}{n}\right) = \begin{cases} \frac{1}{n}, & if\ \left(x_{(i)}, y_{(j)}\right) \text{ is an element of the sample.} \\ 0, & \text{otherwise} \end{cases} \qquad (1.9)$$

Furthermore the relation between C_n and c_n is deduced by the definition as

$$C_n\left(\frac{i}{n},\frac{j}{n}\right) = \sum_{p=1}^{i}\sum_{q=1}^{j} c_n\left(\frac{p}{n},\frac{q}{n}\right), \text{ and} \qquad (1.10)$$

$$c_n\left(\frac{i}{n},\frac{j}{n}\right) = C_n\left(\frac{i}{n},\frac{j}{n}\right) - C_n\left(\frac{i-1}{n},\frac{j}{n}\right) - C_n\left(\frac{i}{n},\frac{j-1}{n}\right) + C_n\left(\frac{i-1}{n},\frac{j-1}{n}\right).$$

We present sample version of rank correlations using the above empirical copulas and empirical copulas frequency functions in the following Theorem 1.8. See Nelson, R.B. [3].

Theorem 1.8. Let C_n and c_n denote respectively the empirical copula and the empirical copula frequency function as for the sample$\{x_k, y_k\}_{k=1}^{n}$. We denote the corresponding empirical version of Kendall's tau as τ_n and that of Spearman's rho as ρ_n.

Then the empirical version of Kendall's tau τ_n is derived as

$$\frac{2n}{n-1}\sum_{i=2}^{n}\sum_{j=2}^{n}\sum_{p=1}^{i-1}\sum_{q=1}^{j-1}\left[c_n\left(\frac{i}{n},\frac{j}{n}\right)c_n\left(\frac{p}{n},\frac{q}{n}\right)-c_n\left(\frac{i}{n},\frac{q}{n}\right)c_n\left(\frac{p}{n},\frac{j}{n}\right)\right] \quad (1.11)$$

$$= \frac{2n}{n-1}\sum_{i=2}^{n}\sum_{j=2}^{n}\left[C_n\left(\frac{i}{n},\frac{j}{n}\right)C_n\left(\frac{i-1}{n},\frac{j-1}{n}\right)-C_n\left(\frac{i}{n},\frac{j-1}{n}\right)C_n\left(\frac{i-1}{n},\frac{j}{n}\right)\right],$$

and the empirical version of Spearman's rho of discrete copula ρ_n is

$$\frac{12}{n^2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[C_n\left(\frac{i}{n},\frac{j}{n}\right)-\frac{i}{n}\cdot\frac{j}{n}\right]. \quad (1.12)$$

2. Evolution of copulas

Copulas are excellent tools for investigating dependence structures among random variables. Although they vary through time (see Appendix B), copulas are used mainly for static problems. Two exceptions are the study of time-dependent copulas, such as the Markov process in Darsow, W.F. *et al.* [9], and dynamic copulas, as in Patton, A.J. [24]. The former study investigates the dependencies between Markov processes at different times. They assume that $P(s,x,t,A) = P(X_t \in A | X_s = x)$ satisfies the Chapman-Kolmogorov equation as

$P(s,x,t,A) = \int_{-\infty}^{\infty} P(r,\xi,t,A)P(s,x,r,\xi)\ d\xi$, where X_t is the stochastic process. A copula of variables X_s and X_t is denoted as $C_{st} = C_{st}(X_s, X_t)$, and product operator $*$ is defined as

$$C_1 * C_2(u,w) = \int_0^1 \partial_2 C_1(u,v)\partial_1 C_2(v,w)dv,$$

where $\partial_2 C\left(F_X(x), F_Y\big(Y(\omega)\big)\right) = \mathrm{E}(I_{X<x} | Y)(\omega)$, and $\partial_1 C(F_X(X(\omega)), F_Y(y)) = \mathrm{E}\big(I_{Y<y} | X\big)(\omega)$. Darsow, W.F. *et al.* [9] prove the relations among C_{st}, C_{sr} and C_{rt} as

$$C_{st} = C_{sr} * C_{rt}.$$

The dynamic copulas in Patton, A.J. [24] are provided as

$$C(u,v;\rho_t), \quad \text{with} \quad \rho_t = \Lambda\big(X_t(\rho_{t-1})\big),.$$

where $C(u,v;\rho_t)$ is an Archimedean copula, ρ_t is a parameter whose value belongs to some interval $J(\subset \mathrm{R})$; $X_t(\rho_{t-1})$ means some time-series model, say an ARMA(*p, q*) type process; and Λ is the transformation function designed to keep $\rho_t \in J$.

We propose that time-dependent copulas autonomously evolve as governed by the heat equation. It is well known that the heat equation is one of the basic partial differential equations which used to describe dynamic movements. Thus, it is considered natural to study these copulas.

In Section 2.1, we prove the existence of the solution for the evolution of copulas. Moreover, we prove that they converge to the product copula $\Pi(u,v) = uv$ as $t \to \infty$. In Section 2.2, we prove that rank correlations of evolution of copulas converge to zero exponentially as $t \to \infty$. In Section 2.3, we extend this evolution of copulas to backward evolution and evolution with coefficients.

2.1 Evolution of copulas

We introduce the evolution of bivariate copulas, which are a kind of time dependent copulas. The images of the evolution of bivariate copulas are graphically charted in Figure 2.1, which is an image of evolution copula varying with time. In the following Theorem 2.1 we propose evolution of bivariate copulas, and prove the existence for their solutions with reference to Ishimura, N. & Yoshizawa, Y. [16], and Yoshizawa, Y. & Ishimura, N. [28].

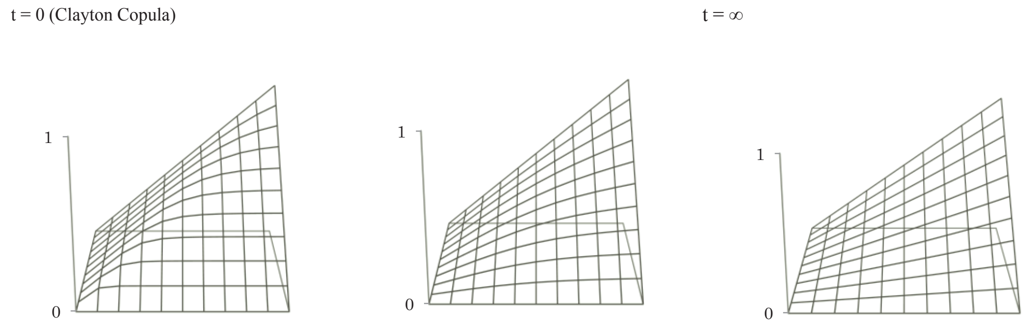


Figure 2.1 Image of evolution copulas(initial copula is Clayton copula with $\theta = 5$

Theorem 2.1 (Evolution of copulas). For any bivariate copula $C_0(u, v)$, there exists a unique family of time dependent bivariate copula$\{C(u, v, t)\}_{t \geq 0}$, which satisfies the heat equation

$$\frac{\partial C}{\partial t}(u, v, t) = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) C(u, v, t), \tag{2.1}$$

for $(u, v, t) \in I^2 \times (0. \infty)$, where $C(u, v, 0) = C_0(u, v)$ on $(u, v) \in I^2$.

The solution of the partial differential equation (2.1) is

$$C(u, v, t) = uv + 4 \sum_{m,n=1}^{\infty} e^{-\pi^2 (m^2 + n^2) t} \sin m\pi u \sin n\pi v \, K(m, n), \tag{2.2}$$

where $K(m, n) = \iint_{I^2} \sin m\pi\xi \sin n\pi\eta \; (C_0(\xi, \eta) - \xi\eta) d\xi \, d\eta$.

(Proof) We prove this theory by three steps. In first step we prove that the solution (2.2) satisfies the heat equation (2.1) as well as the boundary conditions. In second step we prove that the solution (2.2) satisfies the 2-increasing condition in the case where the initial copulas are of C^2- Class copulas. In final step we extend the result of the second step to all the continuous copulas.

First step: We prove the solution (2.2) satisfies the heat equation (2.1) with two boundary conditions, such as

$$C(u,0,t) = C(0,v,t) = 0, \qquad (2.3)$$
$$C(u,1,t) = v \text{ and } C(1,v,t) = v.$$

We define $D(u,v,t)$ as

$$D(u,v,t) := C(u,v,t) - \Pi(u,v) = C(u,v,t) - uv. \qquad (2.4)$$

Then we can rewrite the boundary conditions (2.3) as

$$D(u,0,t) = D(0,v,t) = 0, \qquad (2.5)$$
$$D(u,1,t) = D(1,v,t) = 0.$$

It is easy to verify that $D(u,v,t)$ also fulfill the heat equation with the boundary conditions (2.5). Then partial differential equation (2.1) is rewritten as

$$\frac{\partial D}{\partial t}(u,v,t) = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) D(u,v,t), \qquad \text{for } (u,v,t) \in I^2 \times (0.\infty), \qquad (2.6)$$

where $D(u,v,t) = 0$ on $(u,v,t) \in \partial I^2 \times (0.\infty)$, and $D(u,v,0) = C_0(u,v) - uv$ on $(u,v) \in I^2$.

The solution of the heat equation (2.6) are solved by use of the well-known formula involving the kernel $G(m,n)$ as

$$D(u,v,t) = 4 \sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)t} \sin m\pi u \sin n\pi v G(m,n), \qquad (2.7)$$

where $G(m,n) = \iint\limits_{I^2} \sin m\pi\xi \sin n\pi\eta \;\; D(\xi,\eta,0) d\xi \; d\xi\eta.$

By applying the relation (2.4) to the solution (2.7), we can deduce the solution (2.2) which satisfies the heat equation (2.1) and the boundary conditions (2.3).

Second step: We verify that the solution (2.2) satisfies the 2- increasing condition,

$$C(u_2,v_2,t) - C(u_2,v_1,t) - C(u_1,v_2,t) + C(u_1,v_1,t) \geq 0, \tag{2.8}$$

for $(u_i,v_j) \in I^2 \; (i,j = 1,2), u_1 \leq u_2 \,, v_1 \leq v_2, t \in (0,\infty)$, where we assume $C(u,v,0) = C_0(u,v)$ is of C^2 -Class.

First, we prove that if the initial copula $C(u,v,0) = C_0(u,v)$ is C^2- Class then $C(u,v,t)$ is also of C^2-Class for $\forall \, t > 0$.

According to (2.1), there exist $\frac{\partial^2}{\partial u^2} C(u,v,t)$ and $\frac{\partial^2}{\partial v^2} C(u,v,t)$, thus $C(u,v,t)$ is two times partial differentiable by u or v respectively. If we prove that $C(u,v,t)$ is differentiable both by u and v, then $C(u,v,t)$ is of C^2 -Class.

We define $A_{n,m}(u,v)$ as

$$A_{n,m}(u,v) := e^{-\pi^2(m^2+n^2)t} \sin m\pi u \sin n\pi v \, K(m,n).$$

Therefore we can say

$$\sum\nolimits_{n,m=1}^{\infty} \left(\frac{\partial^2}{\partial u \partial v} A_{n,m}(u,v) \right) = \frac{\partial^2}{\partial u \partial v} \sum\nolimits_{n,m=1}^{\infty} A_{n,m}(u,v), \tag{2.9}$$

Since $\sum_{n,m=1}^{\infty} \left(\frac{\partial^2}{\partial u \partial v} A_{n,m}(u,v) \right) = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{\partial^2}{\partial u \partial v} A_{n,m}(u,v) \right)$

$$= \frac{\partial^2}{\partial u \partial v} \left\{ \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} A_{n,m}(u,v) \right) \right\} = \frac{\partial^2}{\partial u \partial v} \sum_{n,m=1}^{\infty} A_{n,m}(u,v).$$

Applying this equation (2.9) to the solution (2.2), $C(u,v,t)$is two times differentiable by both u and v, which means $\frac{\partial^2 C}{\partial u \partial v}\ (u,v,t)$ exists.

Second, we verify that $\frac{\partial^2 C}{\partial u \partial v}\ (u,v,t)$ satisfies the heat equation with the Neumann boundary condition. As the initial copula $C_0(u,v) = C(u,v,0)$ is of C^2 -Class, then the 2 increasing condition is equivalent to

$$\frac{\partial^2 C_0}{\partial u \partial v}\ (u,v) \geq 0, \qquad (u,v) \in I^2. \tag{2.10}$$

We define $P(u,v,t)$ as

$$P(u,v,t) \coloneqq \frac{\partial^2 C}{\partial u \partial v}\ (u,v,t) \tag{2.11}$$

$$= 1 + 4 \sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)t} mn\pi^2\ \cos m\pi u \cos n\pi v\ \ K(m,n)$$

$$= 1 + 4 \sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)t}\ \cos m\pi u \cos n\pi v\ \ddot{K}(m,n),$$

where $\ddot{K}(m,n) = \iint_{I^2} \cos m\pi\xi \cos n\pi\eta \left(\frac{\partial^2 C_0}{\partial\xi\partial\eta}\ (\xi,\eta) - 1\right) d\xi\ d\xi\eta.$

We can confirm that $P(u,v,t)$ also satisfies the heat equation as

$$\frac{\partial P}{\partial t}(u,v,t) = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) P(u,v,t) \ \text{ for } \ (u,v,t) \in I^2 \times (0.\infty), \tag{2.12}$$

because

$$\frac{\partial P}{\partial t}(u,v,t)$$
$$= 4 \textstyle\sum_{m,n=1}^{\infty} \left(-\pi^2(m^2+n^2)\right) e^{-\pi^2(m^2+n^2)t}\ \cos m\pi u \cos n\pi v\ \ \ddot{K}(m,n)$$
$$= \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) P(u,v,t)\,.$$

Moreover $P(u,v,t)$ satisfies the Neumann boundary condition

$$\frac{\partial P}{\partial \nu}(u,v,t) = 0 \text{ on } (u,v,t) \in \partial I^2 \times (0.\infty), \tag{2.13}$$

where ν is the outer normal to ∂I^2, and ∂I^2 are the boundaries of $I^2 = [0,1] \times [0,1]$, since

$$\frac{\partial P}{\partial u}(u,v,t)\Big|_{u=0,1} = 0 \text{ and } \frac{\partial P}{\partial v}(u,v,t)\Big|_{v=0,1} = 0.$$

Hence we have verified that $P(u,v,t)$ is the solution of the heat equation with the Neumann boundary condition (2.13).

Finally, we prove $\frac{\partial^2 C}{\partial u \partial v}(u,v,t) \geq 0$, which indicates that $C(u,v,t)$ fulfill the 2-increasing condition. According to the definition (2.11) and the inequality (2.10), we obtain the initial condition as

$$P(u,v,0) = \frac{\partial^2 C_0}{\partial u \partial v}(u,v) \geq 0, \qquad \text{on } (u,v) \in I^2. \tag{2.14}$$

Furthermore the boundary value of $P(u,v,t)$ is

$$P(u,v,t) = 0, \qquad \text{on } (u,v,t) \in \partial I^2 \times (0.\infty), \tag{2 .15}$$

because

$$P(0,v,t) = \frac{\partial}{\partial v}\left(\frac{\partial C}{\partial u}(0,v,t)\right) = 0,\ P(u,0,t) = \frac{\partial}{\partial u}\left(\frac{\partial C}{\partial v}(u,0,t)\right) = 0\,,$$
$$\text{and } P(1,v,t) = \frac{\partial}{\partial v}\left(\frac{\partial}{\partial u} v\right) = 0,\ P(u,1,t) = \frac{\partial}{\partial u}\left(\frac{\partial}{\partial v} u\right) = 0.$$

Therefore, as $P(u,v,t)$ fulfills the initial condition (2.14) and boundary condition (2.15), the maximum principle implies

$$P(u,v,t) \geq 0, \qquad \text{on } (u,v,t) \in I^2 \times (0.\infty). \tag{2.16}$$

According to the inequality (2.16) and the definition (2.11) of $P(u,v,t)$, we have

$$\frac{\partial^2 C}{\partial u \partial v}(u,v,t) \geq 0, \quad \text{on } (u,v,t) \in I^2 \times (0.\infty). \tag{2.17}$$

Thus we have proved that $C(u,v,t)$ satisfies the 2-increasing condition in the case where the initial copula $C_0(u,v,t)$ is of C^2-class.

Final step: We turn our attention to the case where the initial copula $C_0(u,v) = C(u,v,0)$ is not of C^2 -Class. We approximate C_0 by a sequence of C^2-class copulas family$\{C_0^\varepsilon(u,v)\}_{\varepsilon>0}$, which converge to $C_0(u,v)$ as

$$\lim_{\varepsilon\to 0} C_0^\varepsilon(u,v) = C_0(u,v), \text{ uniformly on } (u,v) \in I^2. \tag{2.18}$$

For any $\varepsilon > 0$, it corresponds to a family of copulas $\{C^\varepsilon(u,v,t)\}_{\varepsilon>0}$ which satisfies the heat equation (2.1) with the initial value $\{C_0^\varepsilon(u,v)\}_{\varepsilon>0}$. Thanks to the proof of the second step we can say that $C^\varepsilon(u,v,t)$ verifies the 2-increasing condition as

$$\{C^\varepsilon(u_2,v_2,t) - C^\varepsilon(u_2,v_1,t)\} - \{C^\varepsilon(u_1,v_2,t) - C^\varepsilon(u_1,v_1,t)\} \geq 0, \tag{2.19}$$

for $(u_i,v_i,t) \in I^2 \times (0.\infty)$, i =1,2,$u_1 \leq u_2, v_1 \leq v_2$.

According to the solution (2.2), $C^\varepsilon(u,v,t)$ is expressed as

$$C^\varepsilon(u,v,t) = uv + 4\textstyle\sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)t} \sin m\pi u \sin n\pi v \, K^\varepsilon(m,n),$$

where $K^\varepsilon(m,n) = \iint_{I^2} \sin m\pi\xi \sin n\pi\eta \; (C_0^\varepsilon(\xi,\eta) - \xi\eta) d\xi \, d\eta$.

Therefore the difference between $C^\varepsilon(u,v,t)$ and $C(u,v,t)$ is computed as

$$|C^\varepsilon(u,v,t) - C(u,v,t)|$$

$$\leq 4 \lim_{N\to\infty} \sum_{m,n=1}^{N} |e^{-\pi^2(m^2+n^2)t} \sin m\pi u \sin n\pi v \; |K^\varepsilon(m,n) - K(m,n)||$$

$$\leq 4 \lim_{N\to\infty} \sum_{m,n=1}^{N} \iint_{I^2} |C_0^\varepsilon(\xi,\eta) - C_0(\xi,\eta)| d\xi \, d\eta.$$

By using (2.18), for any δ and N we can set ε as $|C_0^\varepsilon(u,v) - C_0(u,v)| \leq \varepsilon$, where $\varepsilon < \delta/4N^2$. Therefore we obtain

$$|C^\varepsilon(u,v,t) - C(u,v,t)| \leq 4\lim_{N\to\infty}\sum_{m,n=1}^{N}\varepsilon < \delta.$$

Thus we have verified that we can set a family of C^2 class copulas $\{C^\varepsilon(u,v,t)\}_{\varepsilon>0}$, which converges to $C(u,v,t)$ uniformly at any t as

$$\lim_{\varepsilon\to 0} C^\varepsilon(u,v,t) = C(u,v,t) \quad \text{on } (u,v,t) \in I^2 \times (0.\infty). \tag{2.20}$$

Applying (2.20) to (2.19), we have proved that $C(u,v,t)$ satisfies the 2 increasing condition, where the initial copula $C_0(u,v) = C(u,v,0)$ is not of C^2 Class. We conclude that the proof of the theorem is finally completed.

□

In addition to Theorem 2.1, evolution of copulas $C(u,v,t)$ converge to the product copula $\Pi(u,v)$as$t \to \infty$, which means their marginal distributions become independent. We prove this convergence in the following Theorem 2.2. See Ishimura, N. & Yoshizawa, Y. [16], and Yoshizawa, Y. & Ishimura, N. [28].

Theorem 2.2. Evolution of bivariate copulas converge to product copula as

$$\lim_{t\to\infty} C(u,v,t) = \Pi(u,v) \qquad \text{uniformly on } (u,v) \in I^2, \tag{2.21}$$

where $\Pi(u,v) := uv$.

(Proof) We prove that $D(u,v,t) = C(u,v,t) - \Pi(u,v)$ converges to zero uniformly. With reference to the solution (2.7), we have

$$D(u,v,t) \leq \left| 4 \sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)t} \sin m\pi u \sin n\pi v \; G(m,n) \right| \leq 4 \sum_{m,n=1}^{\infty} \left| e^{-\pi^2(m^2+n^2)t} \right|.$$

Thus we $D(u,v,t)$ converge to zero as

$$\lim_{t\to\infty} D(u,v,t) = 0 \qquad \text{uniformly} \quad \text{on } (u,v) \in I^2.$$

According to the definition (2.4), $C(u,v,t)$ converges to $\Pi(u,v)$ as

$$\lim_{t\to\infty} C(u,v,t) = \Pi(u,v) = uv \quad \text{uniformly} \quad \text{on } (u,v) \in I^2.$$

□

2.2 Rank correlations of evolution of copulas

As we explain in Chapter 1, rank correlations are simple scalar measures of dependence, moreover which depend only on their copulas, not on their marginal distributions. We introduce the special properties of rank correlations of evolution of copulas. Our conclusion is that both Kendall's tau(τ_{C_t}) and Spearman's rho (ρ_{C_t}) converge to zero exponentially as $t \to \infty$. We prove these properties in the following Theorem 2.3. See to Ishimura, N. & Yoshizawa, Y. [17], and Yoshizawa, Y. & Ishimura, N. [28].

Theorem 2.3 (Rank correlation of evolution of copulas). Rank correlations, Kendall's tau(τ_{C_t}) and Spearman' rho(ρ_{C_t}), of evolution of copulas $C(u,v,t)$ are less than $Ae^{-\alpha t}$ as

$$|\tau_{C_t}| + |\rho_{C_t}| \leq Ae^{-\alpha t}, \qquad \text{for } A, \alpha > 0. \tag{2.22}$$

Furthermore Kendall's tau (τ_{C_t}) and Spearman' rho (ρ_{C_t}) converge to zero exponentially as $t \to \infty$.

(Proof) If the following inequality

$$\left|\rho_{C_t}\right| \leq A_\rho e^{-\alpha t} \tag{2.23}$$

and

$$\left|\tau_{C_t}\right| \leq A_\tau e^{-\alpha t} \tag{2.24}$$

hold, then the inequality (2.22) is proved by setting $A = (A_\rho + A_\tau)$.

Spearman's rho$\left(\boldsymbol{\rho_{C_t}}\right)$

We prove the above inequality (2.23) regarding Spearman's rho $\left(\rho_{C_t}\right)$. Recall the formula (1.7) in Chapter 1 as

$$\rho_{C_t} = 12 \iint_{I^2} C(u,v,t)du\, dv - 3 = 12 \iint_{I^2} (C(u,v,t) - uv)du\, dv. \tag{2.25}$$

We define $\|(C-\Pi)(t)\|_2^2$ as

$$\|(C-\Pi)(t)\|_2^2 := \iint_{I^2} (C(u,v,t) - uv)^2 du\, dv. \tag{2.26}$$

According to the heat equation (2.1) and differentiating (2.26) by *t*, we obtain

$$\frac{d}{dt}\|(C-\Pi)(t)\|_2^2 = 2\iint_{I^2} (C-\Pi)\Delta(C-\Pi)du\, dv$$

$$= -2\|\nabla(C-\Pi)(t)\|_2^2 , \tag{2.27}$$

where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}, \nabla = \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$.

Using Poincaré's inequality, we obtain

$$\|\nabla(C-\Pi)(t)\|_2^2 \geq \alpha\|(C-\Pi)(t)\|_2^2, \quad \text{for } \alpha > 0, \tag{2.28}$$

where $C - \Pi = 0$ on ∂I^2.

Applying the inequality (2.28) to the differential equation (2.27), we have

$$\frac{d}{dt}\|(C-\Pi)(t)\|_2^2 \le -2\alpha\|(C-\Pi)(t)\|_2^2. \tag{2.29}$$

The differential equation (2.29) is solved as

$$\|(C-\Pi)(t)\|_2^2 \le \|(C_0-\Pi)\|_2^2 e^{-2\alpha t} \text{ for } \alpha > 0 \qquad \text{and}$$

$$\|(C-\Pi)(t)\|_2 \le \|(C_0-\Pi)\|_2 e^{-\alpha t}, \tag{2.30}$$

where $(C_0 - \Pi)$is the initial value $C(u,v,0) - uv$.

We apply this inequality (2.30) to (2.25), and then ρ_{C_t} is expressed as

$$|\rho_{C_t}| = \left|12 \iint_{I^2} (C(u,v) - \Pi)du\, dv\right|$$

$$\le 12\|(C-\Pi)(t)\|_2 \le 12\|(C_0-\Pi)\|_2 e^{-\alpha t}.$$

Replacing $12\|(C_0-\Pi)\|_2$ by A_ρ, we prove the inequality (2.23).

Kendall's tau (τ_{C_t})

We prove the above formula (2.24) for Kendall's tau (τ_{C_t}). Recall the formula (1.6) in Chapter 1 as

$$\tau_{C_t} = 4\iint_{I^2} C(u,v,t)\, dC(u,v,t) - 1$$

$$= 1 - 4\iint_{I^2} \frac{\partial C}{\partial u}(u,v,t)\frac{\partial C}{\partial v}(u,v,t)\, du\, dv. \tag{2.31}$$

We apply Spearman' rho (2.25) to Kendall' tau (2.31) as

$$\tau_{C_t} = 1 - 4\iint_{I^2} \frac{\partial C}{\partial u}(u,v)\frac{\partial C}{\partial v}(u,v)\, du\, dv$$

$$= \frac{2}{3}\rho_{C_t} - 4\iint_{I^2} \frac{\partial}{\partial u}(C(u,v) - uv)\frac{\partial}{\partial v}(C(u,v) - uv)du\, dv. \tag{2.32}$$

By multiplying $e^{\alpha t}$to the equation (2.32) and using the inequality (2.30), we obtain

$$e^{\alpha t}|\tau_{C_t}| \leq \frac{2}{3} e^{\alpha t}|\rho_{C_t}| \ + 4e^{\alpha t} \iint_{I^2} |\frac{\partial}{\partial u}(C(u,v) - \Pi)\frac{\partial}{\partial v}(C(u,v) - \Pi)| du\, dv$$

$$= 8\|(C_0 - \Pi)\|_2 + 4e^{\alpha t}\|\nabla(C - \Pi)(t)\|_2^2. \tag{2.33}$$

Next we verify that the second term of the right side of the inequality (2.33) is bounded and is less than M as

$$e^{\alpha t}\|\nabla(C - \Pi)(t)\|_2^2 \leq M < \infty. \tag{2.34}$$

As $\|\nabla(C - \Pi)(t)\|_2^2$ is continuous for t, then it is sufficient only to confirm

$$\int_t^\infty e^{\alpha s}\|\nabla(C - \Pi)(s)\|_2^2\, ds < \infty\, .$$

We differentiate $e^{\alpha t}\|(C - \Pi)(t)\|_2^2$ by t, and apply the equation (2.27) to it, and then we have

$$\frac{d}{dt}(e^{\alpha t}\|(C - \Pi)(t)\|_2^2) \ = \alpha e^{\alpha t}\|(C - \Pi)(t)\|_2^2 + e^{\alpha t}\frac{d}{dt}\|(C - \Pi)(t)\|_2^2$$

$$= \alpha e^{\alpha t}\|(C - \Pi)(t)\|_2^2 - 2e^{\alpha t}\|\nabla(C - \Pi)(t)\|_2^2. \tag{2.35}$$

Applying the inequality (2.30) to the differential equation (2.35), we obtain

$$e^{\alpha t}\|\nabla(C - \Pi)(t)\|_2^2 = \frac{\alpha}{2} e^{\alpha t}\|(C - \Pi)\|_2^2 \ - \frac{1}{2}\frac{d}{dt}(e^{\alpha t}\|(C - \Pi)(t)\|_2^2)$$

$$\leq \frac{\alpha}{2} e^{-\alpha t}\|(C_0 - \Pi)\|_2^2 - \frac{1}{2}\frac{d}{dt}(e^{\alpha t}\|(C - \Pi)(t)\|_2^2). \tag{2.36}$$

We integrate the inequality (2.36) by variable*s* s on $[\mathrm{t}, \infty]$, and apply the inequality (2.30) to it, and then we have

$$\int_t^\infty e^{\alpha s}\|\nabla(C - \Pi)(s)\|_2^2\ ds \ \leq \int_t^\infty \frac{\alpha}{2} e^{-\alpha s}\|(C_0 - \Pi)(s)\|_2^2\, ds \ + \frac{1}{2}\ e^{\alpha t}\|(C - \Pi)(t)\|_2^2$$

$$\leq \|(C_0 - \Pi)(s)\|_2^2 e^{-\alpha t}.$$

Thus we can proclaim $\int_t^\infty e^{\alpha s}\|\nabla(C - \Pi)(s)\|_2^2\, ds < \infty$, that is to say that we have verified the inequality (2.34).

Applying the inequality (2.34) to the inequality (2.33), we obtain

$$e^{\alpha t}\left|\tau_{C_t}\right| \leq 8\|(C_0 - \Pi)\|_2 + 4M.$$

Replacing $8\|(C_0 - \Pi)\|_2 + 4M$ by A_τ, the inequality (2.24) holds. Hence we have proved the inequality (2.22) by use of the inequality (2.23) and (2.24). Furthermore the inequality (2.22) indicates that Kendall's tau(τ_{C_t}) and Spearman' rho(ρ_{C_t}) converge to zero exponentially as $t \to \infty$.

□

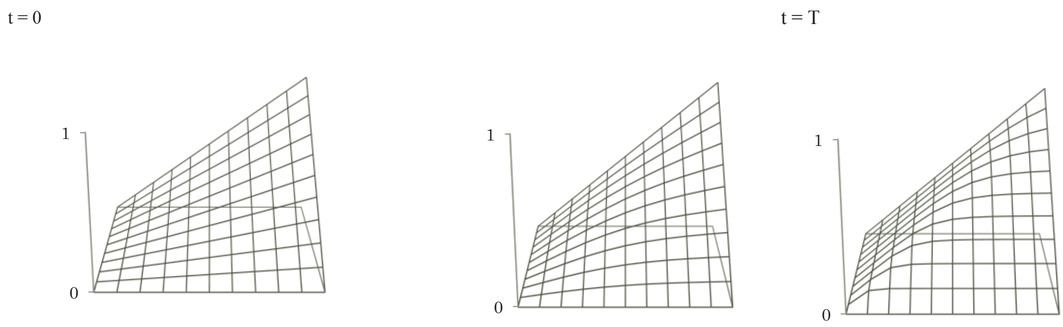
On account of Theorem 2.3, we claim that rank correlations, Kendall' s tau $\left(\tau_{C_t}\right)$ and Spearman's rho$\left(\rho_{C_t}\right)$, of evolution of copulas converge to zero exponentially as $t \to \infty$. This conclusion is consistent with the fact that as $t \to \infty$ evolution of copulas converge to the product copula $\Pi(u,v) = uv$, whose rank correlations are zero.

2.3 Extension of evolution of copulas

We extend the evolution of copulas in the Theorem 2.1 to the following backward evolution of copulas and evolution of copulas with coefficients. We introduce backward evolution of copulas is in Theorem 2.4, evolution of copulas with coefficients is from Corollary 2.5 to Corollary 2.7, and backward evolution of copulas with coefficients is in Corollary 2.8.

Backward evolution of copulas

Backward evolution of copulas has a property to strengthen the dependencies between random variables through time. Their graphical images are charted in Figure 2.2. We prove the existences for the solution of backward evolution of copulas in the following Theorem 2.4. See Yoshizawa, Y. & Ishimura, N. [28].



Fgure 2.2 Image of backward evolution copula

Theorem 2.4 (backward evolution of copulas). There exist backward evolution of copulas which satisfy

$$\frac{\partial C}{\partial t}(u,v,t) + \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) C(u,v,t) = 0, \tag{2.37}$$

for $(u,v,t) \in I^2 \times \{0 \le t \le T\}$, where $C(u,v,T) = C_T(u,v)$ on $(u,v) \in I^2$.

The solutions of (2.37) is

$$C(u,v,t) = uv + 4 \sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)(T-t)} \sin m\pi u \sin n\pi v \, K_T(m,n), \tag{2.38}$$

where $K_T(m,n) = \iint_{I^2} \sin m\pi\xi \sin n\pi\eta \; (C_T(\xi,\eta) - \xi\eta) d\xi \, d\eta$.

(Proof) Backward evolution of copulas transforms backwardly in comparison to evolution of copulas. Backward evolution of copulas at $t = 0$ is the same as evolution of copulas at maturity time T. For any midpoint of time, backward evolution of copulas at time t is the same as evolution of copulas at time $(T - t)$.

We prove the existence for the solution of the backward evolution of copulas in the similar way as evolution of copulas in Theorem 2.1. Replacing $(T - t)$ by t', the partial differential equation (2.37) is rewritten as

$$\frac{\partial C}{\partial t}(u, v, t') = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) C(u, v, t'). \tag{2.39}$$

Applying the analogy of Theorem 2.1, the solution of (2.39) is

$$C(u, v, t') = uv + 4 \sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)t'} \sin m\pi u \sin n\pi v \, K(m, n), \tag{2.40}$$

where $K(m, n) = \iint_{I^2} \sin m\pi\xi \sin n\pi\eta \;\; (C_0(\xi, \eta) - \xi\eta) d\xi \, d\eta$.

The above solution (2.40) is the same as the solution (2.2) in Theorem 2.1 except for the domain of variables. The domain of variable t' of the backward evolution of copulas is$(0, T)$, although that of variable t of evolution of copulas is $(0, \infty)$. The solution (2.2) involves the range of the solution (2.40). Replacing t' by $(T - t)$, we obtain the solution (2.38).

□

Evolution of copulas with coefficients

We extend evolution of copulas to evolution of copulas with coefficients which satisfy the heat equation with diffusion coefficient, and prove the existences for their solutions. See Yoshizawa, Y. & Ishimura, N. [28].

First, in the following Corollary 2.5 we introduce evolution of copulas with coefficient, and prove the existence for their solutions.

Corollary 2.5 For any bivariate copula $C_0(u,v)$, there is a unique family of time dependent bivariate copula$\{C(u,v,t)\}_{t\geq 0}$, which satisfies

$$\frac{\partial C}{\partial t}(u,v,t) = \nu\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)C(u,v,t), \text{for } (u,v,t) \in I^2 \times (0.\infty), \qquad (2.41)$$

where ν is the diffusion coefficient and $C(u,v,0) = C_0(u,v)$ on $(u,v) \in I^2$.

The solution of (2.41) is

$$C(u,v,t) = uv + 4\sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)\nu t} \sin m\pi u \sin n\pi v\, K(m,n), \qquad (2.42)$$

where $K(m,n) = \iint_{I^2} \sin m\pi\xi \sin n\pi\eta\ (C_0(\xi,\eta) - \xi\eta)d\xi\, d\eta$.

(Proof) We prove this theorem using variable transformation. Replacing νt by t', the equation (2.41) is rewritten as

$$\frac{\partial C}{\partial t'}(u,v,t') = \left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)C(u,v,t') \quad \text{for } (u,v,t') \in I^2 \times (0,\infty).$$

The above heat equation is the same as the equation (2.1) in Theorem 2.1. Using the analogy of (2.2), the solution is deduced to

$$C(u,v,t') = uv + 4\sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)t'} \sin m\pi u \sin n\pi v\, K(m,n),$$

where $K(m,n) = \iint_{I^2} \sin m\pi\xi \sin n\pi\eta \ (C_0(\xi,\eta) - \xi\eta) d\xi\, d\eta$.

Thus, the proof of (2.42) is completed.

□

Second, we prove that evolution of copulas with coefficient (2.41) also converge to the product copula in the following Corollary 2.6. See Yoshizawa, Y. & Ishimura, N. [28].

Corollary 2.6 The evolution of copulas with coefficients converge to the product copula $\Pi(u,v)$ as

$$\lim_{t\to\infty} C(u,v,t) = \Pi(u,v) := uv\,, \quad \text{uniformly on } (u,v) \in I^2. \tag{2.43}$$

(Proof) We prove this corollary in the same way as the Theorem 2.2. $D(u,v,t)$ is defined as $D(u,v,t) := C(u,v,t) - \Pi(u,v)$ in (2.4), and then using the equation (2.42) we obtain

$$D(u,v,t) \le \left| 4 \sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)\nu t} \sin m\pi u \sin n\pi v\, G(m,n) \right|$$

$$\le 4 \sum_{m,n=1}^{\infty} \left| e^{-\pi^2(m^2+n^2)\nu t} \right|.$$

Thus we can say that $D(u,v,t)$ converge to zero uniformly as

$$\lim_{t\to\infty} D(u,v,t) = 0 \qquad \text{uniformly} \quad \text{on } (u,v) \in I^2.$$

Recalling the definition of $D(u,v,t)$, $C(u,v,t)$ converge to the product copulas uniformly as

$$\lim_{t\to\infty} C(u,v,t) = \Pi(u,v) = uv.$$

Thus, the proof of (2.43) is completed.

□

Third, rank correlations, Kendall's(τ_{C_t}) and Spearman's rho(ρ_{C_t}), of evolution of copulas with coefficient also converge to zero exponentially as $t \to \infty$. We prove them in the following Corollary 2.7. See Yoshizawa, Y. & Ishimura, N. [28].

Corollary 2.7 The rank correlations Kendall' tau(τ_c) and Spearman' rho (ρ_c) of evolution of copulas with coefficients $C(u,v,t)$ in Corollary 2.5 is

$$\left|\tau_{C_t}\right| + \left|\rho_{C_t}\right| \le Ae^{-\alpha t}, \ \text{ for } A, \alpha > 0. \tag{2.44}$$

(Proof) By the variable transformation $\nu t = t'$, the heat equation (2.41) in the Corollary 2.5 is rewritten as

$$\frac{\partial C}{\partial t}(u,v,t') = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) C(u,v,t'),$$

and the solution (2.42) is transcribed as

$$C(u,v,t') = uv + 4\sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)t'} \sin m\pi u \sin n\pi v \, K(m,n).$$

Applying the Theorem 2.3, we can proclaim

$$\left|\tau_{C_t}\right| + \left|\rho_{C_t}\right| \le Ae^{-\alpha' t'}, \qquad \text{for } A, \alpha' t' > 0.$$

By replacing α' by α/ν, we have

$$\left|\tau_{C_t}\right| + \left|\rho_{C_t}\right| \le Ae^{-\alpha t}, \qquad \text{for } A, \alpha > 0.$$

Hence we have proved the inequality (2.44).

□

Backward evolution of copulas with coefficients

We can develop the backward evolution of copulas with coefficients and, prove the existences for the solutions in the following Corollary 2.8. See Yoshizawa, Y. & Ishimura, N. [28].

Corollary 2.8 There exist backward evolution of copulas with coefficients which satisfy

$$\frac{\partial C}{\partial t}(u,v,t)+\nu\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)C(u,v,t)=0, \tag{2.45}$$

for $(u,v,t)\in I^2\times\{0\le t\le T\}$, where $C(u,v,T)=C_T(u,v)$ on $(u,v)\in I^2$.

The resolution of (2.45) is

$$C(u,v,t)=uv+4\sum_{m,n=1}^{\infty}e^{-\pi^2(m^2+n^2)\nu(T-t)}\sin m\pi u\sin n\pi v\,K_T(m,n), \tag{2.46}$$

where $K_T(m,n)=\iint_{I^2}\sin m\pi\xi\sin n\pi\eta\ (C_T(\xi,\eta)-\xi\eta)d\xi\,d\eta$.

(Proof) We prove this corollary in the same way as the proof of Theorem 2.4 by use of the variable transformation $\nu(T-t)=t'$. The solution (2.46) is the same as (2.2) by replacing $\nu(T-t)$ to t'. The difference between backward evolution of copulas with coefficient and evolution of copulas is the domain of their variables. The domain of variables of backward evolution of copulas with coefficient is $(0,\nu T)$, although that of evolution of copulas is $(0,\infty)$. The range of the solution (2.2) involves that of the solution (2.46). By replacing t' by $\nu(T-t)$again, we obtain the solution (2.46).

□

3. Evolution of Copulas in discrete processes

In the previous chapter, we studied the bivariate evolution of copulas, which is continuous. In general, it is difficult to solve partial differential equations analytically; therefore, numerical approaches are often applied in practice, especially where analytical solutions do not exist. Moreover, numerical analysis is only suitable for computer calculation. Even if analytical solutions do exist, it is often difficult to compute their solutions numerically.

In this chapter, we study the bivariate discrete evolution of copulas, which satisfies the discrete version of the heat equation, and prove that it converges to the product copula. In Section 3.2 we prove that discrete evolution copulas converge to continuous evolution copulas. Based on this proof, we can treat discrete evolution copulas as approximate versions of continuous evolution copulas. Then, in Section 3.3, we prove that the rank correlations of discrete evolution copulas also converge to zero exponentially. In Section 3.4, we introduce the backward type discrete evolution of copulas and the discrete evolution of copulas with coefficient. Finally, in Section 3.5 we apply discrete type evolution copula to empirical data, foreign exchange rates, to confirm their practicality.

First, in the following Definition 3.1, we define the properties that bivariate discrete evolution of copulas satisfies.

Definition 3.1. Bivariate discrete evolution of copulas are defined as the functions which satisfy the following three properties.

Boundary conditions;

$$C(u_i, 0, 0) = C(0, v_i, 0) = 0, \tag{3.1}$$

$C(u_i, N, 0) = u_i$ and $C(N, v_i, 0) = v_i$.

The 2-increasing condition;

$$C\left(u_{i_2}, v_{j_2}, t\right) - C\left(u_{i_2}, v_{j_1}, t\right) - C\left(u_{i_1}, v_{j_2}, t\right) + C\left(u_{i_1}, v_{j_1}, t\right) \geq 0, \tag{3.2}$$

for $\left(u_{i_a}, v_{j_b}\right) \in I^2$ $(a, b = 1,2), u_{i_1} \leq u_{i_2}, v_{j_1} \leq v_{j_2}, t \in (0, \infty)$.

3.1 Evolution of copulas in discrete processes

We introduce bivariate discrete evolution copulas which satisfy the following discrete version of the heat equation (3.3). First, we prove that their discrete solution described by recurrence formula fulfills the three properties of copulas. Second, we define the interpolation, and we interpolate the discrete solutions to be continuous on I^2. Finally, we prove that this interpolated discrete solutions are copulas, and that converge to the product copula as $t \to \infty$. See Ishimura, N. & Yoshizawa, Y. [18], and Yoshizawa, Y. & Ishimura, N. [29].

Let $N \geq 0$ and $0 < h \leq 1$, we put

$$\triangle u = \triangle v := \frac{1}{N} = M, t := h,$$

$$\lambda := \frac{\triangle t}{(\triangle u)^2} = \frac{\triangle t}{(\triangle v)^2} = hN^2, \text{and}$$

$$u_i := i \vartriangle u = \frac{i}{N}, v_j := j \vartriangle v = \frac{j}{N}, \qquad \text{for } i, j = 0,1,\cdots,N.$$

At any $\{(u_i, v_j)\}_{i,j=0,1,2,\cdots,N}$, the value $C_{i,j}^n := C^n(u_i, v_i)$ is governed by the system of the difference equation

$$\frac{C_{i,j}^{n+1} - C_{i,j}^n}{\vartriangle t} = \frac{C_{i+1,j}^n - 2C_{i,j}^n + C_{i-1,j}^n}{(\vartriangle u)^2} + \frac{C_{i,j+1}^n - 2C_{i,j}^n + C_{i,j-1}^n}{(\vartriangle v)^2}, \tag{3.3}$$

for $i, j = 0,1,\cdots,N-1$, where $C_{i,j}^0 = C^0(u_i, v_j) := C_0(u_i, v_j)$is an initial copula, together with the boundary conditions as

$$C_{i,0}^n = C_{0,j}^n = 0, \tag{3.4}$$

$$C_{i,N}^n = u_i, C_{N,j}^n = v_j, \qquad \text{for } i, j = 0,1,\cdots,N.$$

The difference equation (3.3) approximates the heat equation (2.1), because we can describe its left hand side $\frac{\partial C}{\partial t}(u, v, t)$ as $\frac{C_{i,j}^{n+1} - C_{i,j}^n}{\Delta t}$, and its right hand side $\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) C(u, v, t)$ as $\frac{\left(C_{i+1,j}^n - C_{i,j}^n\right) - \left(C_{i,j}^n - C_{i-1,j}^n\right)}{(\Delta u)^2} + \frac{\left(C_{i,j+1}^n - C_{i,j}^n\right) - \left(C_{i,j}^n - C_{i,j-1}^n\right)}{(\Delta v)^2}$.

First, we proclaim that the solution of above difference equation (3.3) satisfies the 2- increasing condition (3.5) in the following Proposition 3.2.

Proposition 3.2. $C_{i,j}^n := C^n(u_i, v_i)$ fulfills the 2- increasing condition

$$C_{i_2,j_2}^n - C_{i_2,j_1}^n - C_{i_1,j_2}^n + C_{i_1,j_1}^n \geq 0, \tag{3.5}$$

for $\left(u_{i_a}, v_{j_b}\right) \in I^2 \ (a, b = 1,2), u_{i_1} \leq u_{i_2}, v_{j_1} \leq v_{j_2}$, where $C_{i,j}^n$ satisfies the difference equation (3.3) and the constrain

$$0 \leq \lambda \leq \frac{1}{4}. \tag{3.6}$$

(Proof) We employ mathematical induction to prove that $C_{i,j}^{n+1}$ fulfills the 2- increasing condition (3.5). By use of the relation $\lambda := \frac{\Delta t}{(\Delta u)^2} = \frac{\Delta t}{(\Delta v)^2}$, we transform the difference equation (3.3) into the recurrence formula as

$$C_{i,j}^{n} = (1-4\lambda)C_{i,j}^{n-1} + \lambda\left(C_{i+1,j}^{n-1} + C_{i-1,j}^{n-1} + C_{i,j+1}^{n-1} + C_{i,j-1}^{n-1}\right). \tag{3.7}$$

We illustrate the image of (3.7) diagrammatically in Figure 3.1.

First, we confirm that the initial conditions $C_{i,j}^{0}$ satisfy the 2- increasing condition, because they are the discrete points of the copula $C_0(u,v)$.

Second, we assume that $C_{i,j}^{n-1}$ satisfies the 2-increasing condition.

Finally, we verify that $C_{i,j}^{n}$ also fulfills the 2-increasing condition. Using the recurrence formula (3.7) we can rewrite the 2-increasing condition (3.5) as

$$\begin{aligned}
&C_{i_2,j_2}^{n} - C_{i_2,j_1}^{n} - C_{i_1,j_2}^{n} + C_{i_1,j_1}^{n} \\
&\quad = (1-4\lambda)\left(C_{i_2,j_2}^{n-1} - C_{i_2,j_1}^{n-1} - C_{i_1,j_2}^{n-1} + C_{i_1,j_1}^{n-1}\right) \\
&\quad + \lambda\left(C_{i_2+1,j_2}^{n-1} - C_{i_2+1,j_1}^{n-1} - C_{i_1+1,j_2}^{n-1} + C_{i_1+1,j_1}^{n-1}\right) + \lambda\left(C_{i_2-1,j_2}^{n-1} - C_{i_2-1,j_1}^{n-1} - C_{i_1-1,j_2}^{n-1} + C_{i_1-1,j_1}^{n-1}\right) \\
&\quad + \lambda\left(C_{i_2,j_2+1}^{n-1} - C_{i_2,j_1+1}^{n-1} - C_{i_1,j_2+1}^{n-1} + C_{i_1,j_1+1}^{n-1}\right) + \lambda\left(C_{i_2,j_2-1}^{n-1} - C_{i_2,j_1-1}^{n-1} - C_{i_1,j_2-1}^{n-1} + C_{i_1,j_1-1}^{n-1}\right),
\end{aligned}$$

Based on the constrain (3.6) as $0 < \lambda < 1/4$ and the assumption that $C_{i,j}^{n-1}$ satisfies the 2- increasing condition, we can verify

$$C_{i_2,j_2}^{n} - C_{i_2,j_1}^{n} - C_{i_1,j_2}^{n} + C_{i_1,j_1}^{n} \geq 0.$$

Hence we have proved that $C_{i,j}^{n}$ satisfy the 2-increasing condition (3.5).

□

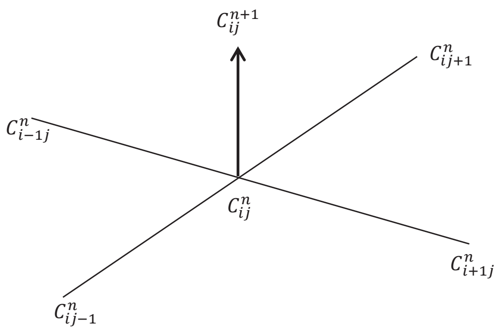


Figure 3.1. Sequence of discrete evolution copula

$C^n(u_i, v_i)$, which satisfies the difference equation (3.3) and the boundary condition (3.4), fulfills the three properties of copulas thanks to Proposition 3.2.

However copulas must be defined on I^2 and continuous, we have to extend the domain of the variable from discrete points (u_i, v_i) to continuous points (u, v) on I^2. Therefore we define the interpolation for the purpose of this extension in the following Definition 3.3.

Definition 3.3. We define the interpolation as

$$C^n(u,v) := C^n_{i,j} + \frac{C^n_{i+1,j} - C^n_{i,j}}{u_{i+1} - u_i}(u - u_i) + \frac{C^n_{i,j+1} - C^n_{i,j}}{v_{j+1} - v_j}(v - v_j) \tag{3.8}$$

$$+ \frac{C^n_{i+1,j+1} - C^n_{i+1,j} - C^n_{i,j+1} + C^n_{i,j}}{(u_{i+1} - u_i)(v_{j+1} - v_j)}(u - u_i)(v - v_j),$$

for $u_i \le u \le u_{i+1}, v_j \le v \le v_{j+1}$ and $(u, v) \in I^2$.

As to the point $(u, v) \in I^2$ other than$\{(u_i, v_i)\}_{n=0,1,2,\cdots N}$, the value $C^n(u, v)$ is provided by the interpolation (3.8). It is trivial that $C^n(u, v)$ is continuous. In the following Theorem 3.4 we proclaim and prove that $C^n(u, v)$ have the properties of copulas. See Ishimura, N. & Yoshizawa, Y. [18], and Yoshizawa, Y. & Ishimura, N. [29].

Theorem 3.4 (Evolution of copulas in discrete processes). For any initial copula C_0, there exists a sequence of copula$\{C^n(u,v)\}_{n=0,1,2,\cdots}$ on $(u,v) \in I^2$, which satisfy the system of difference equation (3.3) at every $\{(u_i, v_j)\}_{i,j=0,1,2,\cdots,N}$. We call these copulas $C^n(u,v)$ as evolution of copulas in discrete processes.

(Proof) According to Definition 3.3, $\{C^n(u,v)\}_{n=0,1,2,\cdots}$ is defined on I^2, and satisfy the system of difference equations (3.3) at every $\{(u_i, v_j)\}_{i,j=0,1,2,\cdots,N}$, where $C^n(u,v) = C^n(u_i, v_j) = C^n_{i,j}$. Then what we have to check is if $\{C^n(u,v)\}_{n=0,1,2,\cdots}$ fulfill the boundary conditions (3.4) as well as the 2- increasing condition (3.5).

First, we verify that $C^n(u,v)$ satisfy the boundary conditions. If $i = 0, u = u_i = 0$, then $C^n_{0,j} = C^n(0,v) = 0$. Because $C^n(0,v) = C^n_{0,j} + 0 + 0 + 0 = 0$ by the formula (3.8). We also obtain $C^n_{i,0} = C^n(u,0) = 0$ in the same way as $C^n_{0,j}$.

If $i = N, u = u_N = 1$ and $C^n_{N,j} = \frac{j}{N}$, then $C^n_{1,j} = C^n(1,v) = v$. Because $C^n(1,v) = C^n_{N,j} + 0 + \frac{C^n_{N,j+1} - C^n_{N,j}}{(v_{j+1} - v_j)}(v - v_j) + 0 = v_j + (v - v_j) = v$. We also obtain $C^n_{i,1} = C^n(u,1) = u$ in the same way as $C^n_{1,j}$.

Second, we verify $C^n(u,v)$ fulfills the 2-increasing condition (3.5).

Let $u_i \le u_1 \le u_2 \le u_{i+1}, v_j \le v_1 \le v_2 \le v_{j+1}$, then

$$C^n(u_2,v_2) - C^n(u_2,v_1) - C^n(u_1,v_2) + C^n(u_1,v_1) \tag{3.9}$$

$$= \frac{C^n_{i+1,j} - C^n_{i,j}}{u_{i+1} - u_i}\{(u_2 - u_i) - (u_2 - u_i) - (u_1 - u_i) + (u_1 - u_i)\}$$

$$+ \frac{C^n_{i,j+1} - C^n_{i,j}}{v_{j+1} - v_j}\{(v_2 - v_j) - (v_2 - v_j) - (v_1 - v_j) + (v_1 - v_j)\}$$

$$+ \frac{C^n_{i+1,j+1} - C^n_{i+1,j} - C^n_{i,j+1} + C^n_{i,j}}{(u_{i+1} - u_i)(v_{j+1} - v_j)}$$

$$\times \{(u_2 - u_i)(v_2 - v_j) - (u_2 - u_i)(v_1 - v_j) - (u_1 - u_i)(v_2 - v_j) + (u_1 - u_i)(v_1 - v_j)\}$$

$$= (C^n_{i+1,j+1} - C^n_{i+1,j} - C^n_{i,j+1} + C^n_{i,j}) \frac{(u_2 - u_1)(v_2 - v_1)}{(u_{i+1} - u_i)(v_{j+1} - v_j)} \geq 0.$$

Hence we have proved that $\{C^n(u,v)\}_{n=0,1,2,\cdots}$ satisfy the properties of copulas, such as two boundary conditions and the 2-increasing condition, on $(u,v) \in I^2$.

□

Next we prove that evolution of copulas in discrete processes $C^n(u,v)$ converge to the product copula $\Pi(u,v) = uv$ in the following Theorem 3.5. It is remarkable that the convergences do not depend on the fineness of mesh M, but depend only on the number of times n. See Ishimura, N. & Yoshizawa, Y. [18], and Yoshizawa, Y. & Ishimura, N. [29].

Theorem 3.5. Evolution of copulas in discrete processes $C^n(u,v)$converge to the product copula as

$$\lim_{n\to\infty} C'^n(u,v) = \Pi(u,v) = uv, \quad \text{uniformly on } I^2 . \tag{3.10}$$

(Proof) We define $D^n(u,v)$ as

$$D^n(u,v) := C^n(u,v) - uv. \tag{3.11}$$

In addition we denote $D^n_{i,j} := D^n(u_i, v_i)$. $D^n_{i,j}$is described as

$$D^n_{i,j} = C^n_{i,j} - u_i v_i = C^n_{i,j} - \frac{ij}{N^2}, \tag{3.12}$$

and the difference equation (3.3) can be rewritten by $D^n_{i,j}$ as

$$\frac{D_{i,j}^{n+1} - D_{i,j}^{n}}{\Delta\, t} = \frac{D_{i+1,j}^{n} - 2D_{i,j}^{n} + D_{i-1,j}^{n}}{(\Delta\, u)^2} + \frac{D_{i,j+1}^{n} - 2D_{i,j}^{n} + D_{i,j-1}^{n}}{(\Delta\, v)^2}\,, \tag{3.13}$$

for $i, j = 0,1, \cdots, N-1$.

We derive the null boundary conditions from the original boundary conditions (3.4) as

$$D_{i,0}^{n} = D_{0,j}^{n} = 0, \tag{3.14}$$

$$D_{i,N}^{n} = D_{N,j}^{n} = 0, \quad \text{for } \; i, j = 0,1, \cdots, N\,.$$

By use of the relation $\lambda \coloneqq \frac{\Delta t}{(\Delta u)^2} = \frac{\Delta t}{(\Delta v)^2}$ and the constrain (3.6) $0 \le \lambda \le \frac{1}{4}$, the difference equation (3.13) is rewritten as

$$D_{i,j}^{n} = (1-4\lambda) D_{i,j}^{n-1} + \lambda\left(D_{i+1,j}^{n-1} + D_{i-1,j}^{n-1} + D_{i,j+1}^{n-1} + D_{i,j-1}^{n-1}\right). \tag{3.15}$$

First, we verify that $D_{i,j}^{n}$ converges to zero on (u_i, v_i). Using the recurrent equation (3.15), we derive

$$\begin{aligned} D_{i,j}^{n} &\le (1-4\lambda)|D_{i,j}^{n-1}| + \lambda\left\{|D_{i+1,j}^{n-1}| + |D_{i-1,j}^{n-1}| + |D_{i,j+1}^{n-1}| + |D_{i,j-1}^{n-1}|\right\} \\ &\le \max_{0<i,j<N-1} \left[(1-4\lambda)|D_{i,j}^{n-1}| + \lambda\left\{|D_{i+1,j}^{n-1}| + |D_{i-1,j}^{n-1}| + |D_{i,j+1}^{n-1}| + |D_{i,j-1}^{n-1}|\right\}\right] \\ &\le \max_{0<i,j<N} \left|D_{i,j}^{n-1}\right|. \end{aligned} \tag{3.16}$$

Taking the maximum of the left hand side (3.16), we have

$$\max_{0<i,j<N} |D_{i,j}^{n}| < \max_{0<i,j<N} |D_{i,j}^{n-1}|\,.$$

We set K and θ as

$$K = \max\nolimits_{0<i,j<N} |D_{i,j}^{0}|, \text{ and}$$

$$\theta = \max\nolimits_{d=1,\cdots,n} \theta_d \quad \text{,where} \quad \theta_{n+1} = \frac{\max_{0<i,j<N} |D_{i,j}^{n}|}{\max_{0<i,j<N} |D_{i,j}^{n-1}|} < 1\,.$$

We obtain

$$\max_{(u_i, v_j) \in I^2} \left|D_{i,j}^{n}\right| < K \prod\nolimits_{1}^{n} \theta_d < K\theta^n. \tag{3.17}$$

Thus we can proclaim that $D_{i,j}^{n}$ converges to zero as $n \to \infty$.

Second, we prove $D^n(u,v)$ converge to zero on $(u,v) \in I^2$. Based on the definition (3.11) and the interpolation (3.8),

$$D^n(u,v) = C^n(u,v) - uv$$

$$= C_{i,j}^n + \frac{C_{i+1,j}^n - C_{i,j}^n}{u_{i+1} - u_i}(u - u_i) + \frac{C_{i,j+1}^n - C_{i,j}^n}{v_{j+1} - v_j}(v - v_j)$$

$$+ \frac{C_{i+1,j+1}^n - C_{i+1,j}^n - C_{i,j+1}^n + C_{i,j}^n}{(u_{i+1} - u_i)(v_{j+1} - v_j)}(u - u_i)(v - v_j) - uv.$$

$$= D_{i,j}^n\left(1 - \frac{h}{M}\right)\left(1 - \frac{k}{M}\right) + D_{i+1,j}^n\left(\frac{h}{M}\right)\left(1 - \frac{k}{M}\right) \quad (3.18)$$
$$+ D_{i,j+1}^n\left(1 - \frac{h}{M}\right)\left(\frac{k}{M}\right) + D_{i+1,j+1}^n\left(\frac{h}{M}\right)\left(\frac{k}{M}\right),$$

where $u_i = \frac{i}{N} = i\,M, v_j = \frac{j}{N} = j\,M$, $u_{i+1} - u_i = v_{j+1} - v_j = \frac{1}{N} = M, u := u_i + h, 0 \leq h \leq M, v := v_j + k, 0 \leq h \leq M.$

We take the maximum the equation (3.18) on $(u,v) \in I^2$ and apply it to the inequality (3.17), we obtain

$$\max_{(u,v)\in I^2} |D^n(u,v)| \quad (3.19)$$
$$\leq \max_{(u_i,v_j)\in I^2} \left|D_{i,j}^n\left(1 - \tfrac{h}{M}\right)\left(1 - \tfrac{k}{M}\right)\right| + \max_{(u_{i+1},v_j)\in I^2} \left|D_{i+1,j}^n\left(\tfrac{h}{M}\right)\left(1 - \tfrac{k}{M}\right)\right|$$
$$+ \max_{(u_i,v_{j+1})\in I^2} \left|D_{i,j+1}^n\left(1 - \tfrac{h}{M}\right)\left(\tfrac{k}{M}\right)\right| + \max_{(u_{i+1},v_{j+1})\in I^2} \left|D_{i+1,j+1}^n\left(\tfrac{h}{M}\right)\left(\tfrac{k}{M}\right)\right|$$
$$< K\theta^n\left\{\left(1 - \tfrac{h}{M}\right)\left(1 - \tfrac{k}{M}\right) + \left(\tfrac{h}{M}\right)\left(1 - \tfrac{k}{M}\right) + \left(1 - \tfrac{h}{M}\right)\left(\tfrac{k}{M}\right) + \left(\tfrac{h}{M}\right)\left(\tfrac{k}{M}\right)\right\} = K\theta^n.$$

We verify that $D^n(u,v)$ converge to zero as

$$\lim_{n\to\infty} D^n(u,v) = 0, \text{ uniformly on } I^2.$$

Hence the proof of the formula (3.10) is completed by use of the definition (3.11).

□

Furthermore we define the density $c_{i,j}^n$ of discrete evolution of copulas $C_{i,j}^n$, which is well defined and consistent with continuous type evolution of copulas.

Definition 3.6. The densities $c_{i,j}^n$ of discrete evolution of copula $C_{i,j}^n$ is defined as

$$c_{i,j}^n = \frac{C_{i,j}^n - C_{i,j-1}^n - C_{i-1,j}^n + C_{i-1,j-1}^n}{(\Delta u)(\Delta v)}, \text{for } 1 < i,j < N-1, \tag{3.20}$$

where $c_{0,j}^n = c_{i,0}^n = 0, c_{1,j}^n = c_{i,1}^n = 0.$

Using the above definition we can calculate densities of copulas, and their images are illustrated in Figure 3.2. Upper stands are evolution of copulas ,and lower stands are densities of evolution of copulas.

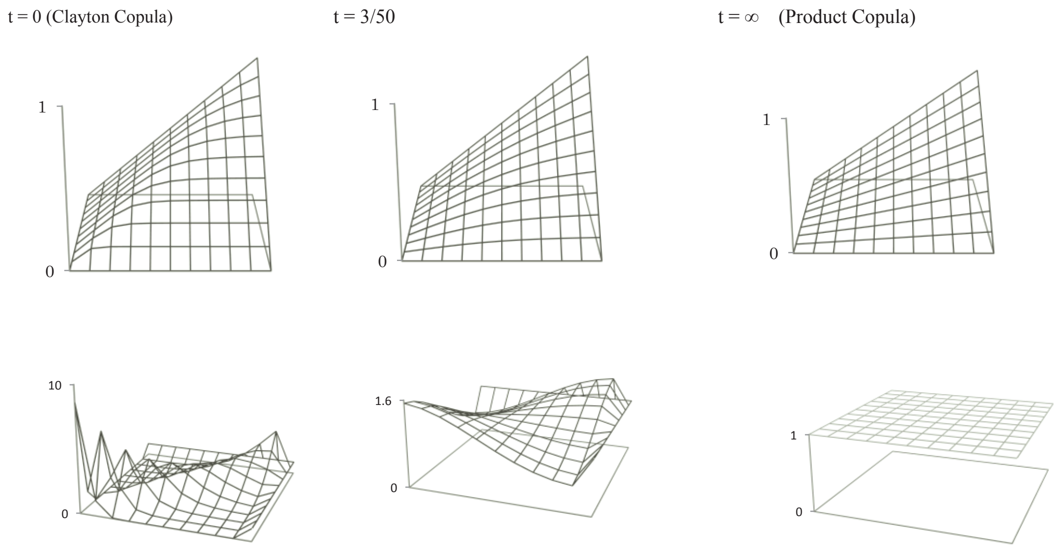


Figure 3.2. Evolution of copulas and their densities

3.2 Convergence to evolution of copulas in continuous time

We have studied both continuous evolution of copulas and discrete evolution of copulas respectively. In this section we focus on their relation, and prove that evolution of copulas in discrete processes converge to evolution of copulas in continuous time. First of all, we define the discrete points of continuous evolution of copulas in the following Definition 3.7.

Definition 3.7. Let $t_n := nh,\ h = \triangle t$. Then we define the discrete points of the corresponding continuous evolution of copulas as

$$C_{i,j}^{c,n} := C^{c,n}(u_i, v_j) = C^c(u_i, v_j, t_n) = C(u_i, v_j, t_n), \tag{3.21}$$

where $C(u, v, t)$is the continuous evolution of copulas which are already solved in Theorem 2.1, and $C(u_i, v_j, t_n)$ is the discrete point (u_i, v_j, t_n) of $C(u, v, t)$.

We also define the corresponding points of the discrete evolution of copulas as

$$C_{i,j}^{d,n} := C^{d,n}(u_i, v_j) = C^d(u_i, v_j, t_n) = C^n(u_i, v_j), \tag{3.22}$$

where $C^n(u_i, v_j)$ is derived from the boundary conditions (3.4) and the recurrent formula (3.7).

First, we prove that the discrete evolution of copulas converge to the continuous type evolution of copulas at any discrete points and times (u_i, v_j, t_n) in Proposition 3.8. Next in Proposition 3.9 we extend this domain of variables, prove this convergence holds on(u, v, t_n), where $(u, v) \in I^2$ and discrete time t_n. Finally, we prove that the discrete type converge to the continuous type on $(u, v, t) \in I^2 \times (0, \infty)$ in Theorem 3.10.

Proposition 3.8. At any points (u_i, v_j) and any times $t_n = nh$, the discrete evolution of copulas converge to the correspondent continuous evolution of copulas as

$$\lim_{N\to\infty} \max_{0\le i,j\le N, 0\le n\le\left[\frac{T}{h}\right]} \left|C_{i,j}^{c,n} - C_{i,j}^{d,n}\right| = 0, \tag{3.23}$$

for $i, j = 0,1,\cdots,N,\ n = 0,1,\cdots,\left[\frac{T}{h}\right]$, where T denote maturity time.

(Proof) We define $D_{i,j}^{c,n}$ and $D_{i,j}^{d,n}$ as

$$D_{i,j}^{c,n} := D^c(u_i, v_j, t_n) = C_{i,j}^{c,n} - u_i v_j\ , \tag{3.24}$$

$$D_{i,j}^{d,n} := D^{d,n}(u_i, v_j) \ = C_{i,j}^{d,n} - u_i v_j. \tag{3.25}$$

The discrete points of the left hand side of the heat equation (2.6) are described as

$$\frac{D_{i,j}^{c,n+1} - D_{i,j}^{c,n}}{h} = \frac{\partial D^c(u_i, v_j, t_n)}{\partial t} + \mathcal{O}(h).$$

where $\mathcal{O}(h)$ denotes Landau's symbol and $h = \Delta t$.

The discrete points of the right hand side of the heat equation (2.6) are described as

$$\frac{\left\{D_{i+1,j}^{c,n} + D_{i-1,j}^{c,n} + D_{i,j+1}^{c,n} + D_{i,j-1}^{c,n}\right\} - 4D_{i,j}^{c,n}}{M^2} = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) D^c(u_i, v_j, t_n) + \mathcal{O}(M^2),$$

where $\mathcal{O}(M^2)$ denotes Landau's symbol and $M = \Delta u = \Delta v$.

As $D^c(u_i, v_j, t_n)$ satisfies the heat equation (2.6), we obtain

$$\frac{D_{i,j}^{c,n+1} - D_{i,j}^{c,n}}{h} - \frac{\left\{D_{i+1,j}^{c,n} + D_{i-1,j}^{c,n} + D_{i,j+1}^{c,n} + D_{i,j-1}^{c,n}\right\} - 4D_{i,j}^{c,n}}{M^2} = \mathcal{O}(h + M^2).$$

We can rewrite the above equation as

$$D_{i,j}^{c,n+1} = (1 - 4\lambda) D_{i,j}^{c,n} + \lambda\left\{D_{i+1,j}^{c,n} + D_{i-1,j}^{c,n} + D_{i,j+1}^{c,n} + D_{i,j-1}^{c,n}\right\} + \mathcal{O}(h^2 + hM^2),$$

where $\lambda = \frac{\Delta t}{(\Delta u)^2} = \frac{\Delta t}{(\Delta v)^2}$ and the constrain $0 \le \lambda \le \frac{1}{4}$.

According to (3.15), the discrete evolution of copulas $D_{i,j}^{d,n}$ satisfies

$$D_{i,j}^{d,n+1} = (1-4\lambda)D_{i,j}^{d,n} + \lambda\{D_{i+1,j}^{d,n} + D_{i-1,j}^{d,n} + D_{i,j+1}^{d,n} + D_{i,j-1}^{d,n}\}.$$

We define $e_{i,j}^{n}$ as

$$e_{i,j}^{n} := D_{i,j}^{c,n} - D_{i,j}^{d,n}. \tag{3.26}$$

Then $e_{i,j}^{n}$ satisfy

$$e_{i,j}^{n+1} = (1-4\lambda)e_{i,j}^{n} + \lambda\{e_{i+1,j}^{n} + e_{i-1,j}^{n} + e_{i,j+1}^{n} + e_{i,j-1}^{n}\} + \mathcal{O}(h^2 + hM^2).$$

Both $(1-4\lambda)$ and λ are positive, thus we have

$$|e_{i,j}^{n+1}| \leq (1-4\lambda)|e_{i,j}^{n}| + \lambda\{|e_{i+1,j}^{n}| + |e_{i-1,j}^{n}| + |e_{i,j+1}^{n}| + |e_{i,j-1}^{n}|\} + \mathcal{C}(h^2 + hM^2),$$

where $\mathcal{C} > 0$ and $e_{0,j}^{n} = e_{i,0}^{n} = 0$ for any n.

Taking the maximum both side of the above inequality, we obtain

$$\begin{aligned}
&\max_{0\leq i,j\leq N-1} |e_{i,j}^{n+1}| \\
&\quad\leq (1-4\lambda) \max_{0\leq i,j\leq N-1} |e_{i,j}^{n}| + \lambda \max_{0\leq i,j\leq N-1} \{|e_{i+1,j}^{n}| + |e_{i-1,j}^{n}| + |e_{i,j+1}^{n}| + |e_{i,j-1}^{n}|\} \\
&\qquad\qquad + \mathcal{C}|h^2 + hM^2| \\
&\quad\leq (1-4\lambda) \max_{0\leq i,j\leq N} |e_{i,j}^{n}| + 4\lambda \max_{0\leq i,j\leq N} |e_{i,j}^{n}| + \mathcal{C}|h^2 + hM^2| \\
&\quad\leq \max_{0\leq i,j\leq N} |e_{i,j}^{n}| + \mathcal{C}|h^2 + hM^2|.
\end{aligned}$$

By use of $e_{i,N}^{n} = e_{N,j}^{n} = 0$, we have $\max_{0\leq i,j\leq N-1} |e_{i,j}^{n+1}| = \max_{0\leq i,j\leq N} |e_{i,j}^{n+1}|$.

Therefore we obtain

$$\max_{0\leq i,j\leq N} |e_{i,j}^{n+1}| \leq \max_{0\leq i,j\leq N} |e_{i,j}^{n}| + \mathcal{C}|h^2 + hM^2|. \tag{3.27}$$

Using the inequality (3.27) inductively, we can calculate $\max_{0\leq i,j\leq N} |e_{i,j}^{0}|$ as

$$\begin{aligned}
\max_{0\leq i,j\leq N} |e_{i,j}^{n}| &\leq \max_{0\leq i,j\leq N} |e_{i,j}^{0}| + \mathrm{n}\mathcal{C}|h^2 + hM^2| \\
&\leq T\,\mathcal{C}|h + M^2| = T\,\mathcal{C}\left|\frac{(1+\lambda)}{N^2}\right|,
\end{aligned} \tag{3.28}$$

where $nh = T$, $h = \lambda M^2$ and $M = \frac{1}{N}$.

We can rewrite (3.28) as

$$\max_{0\le i,j\le N} |\, D_{i,j}^{c,n} - D_{i,j}^{d,n}| \le T\, C \left|\frac{(1+\lambda)}{N^2}\right|.$$

Thus we verify that $D_{i,j}^{d,n}$ converges to $D_{i,j}^{c,n}$ as

$$\lim_{N\to\infty} \max_{0\le i,j\le N, 0\le n\le\left[\frac{T}{h}\right]} |D_{i,j}^{c,n} - D_{i,j}^{d,n}| = 0.$$

Applying the definitions (3.24) and (3.25) to the above equation, we can proclaim that $C_{i,j}^{d,n}$ also converge to $C_{i,j}^{c,n}$ as

$$\lim_{N\to\infty} \max_{0\le i,j\le N, 0\le n\le\left[\frac{T}{h}\right]} \left|C_{i,j}^{c,n} - C_{i,j}^{d,n}\right| = 0.$$

□

We have proved that at any discrete points (u_i, v_j) the discrete type converge to the continuous type uniformly as $N \to \infty$. Furthermore in the next Proposition 3.9 we propose that the discrete evolution of copulas converge to the continuous evolution of copulas uniformly on I^2 as $N \to \infty$. For the preparation of this proof, we define $C^{c,n}(u,v)$ as

$$C^{c,n}(u,v) \coloneqq C^c(u,v,t_n) = C(u,v,t_n),$$

where $C^{c,n}(u,v)$ is the continuous type evolution of copulas at discrete time t_n.

We also define $C^{d,n}(u,v)$ as

$$C^{d,n}(u,v) \coloneqq C^n(u,v), \qquad ,$$

where $C^{d,n}(u,v)$ is the discrete type evolution copulas $C^n(u,v)$ to which the interpolation (3.8) is applied on I^2.

Proposition 3.9. The discrete type evolution of copulas $C^{d,n}(u,v)$ converge to the continuous type $C^{c,n}(u,v)$ in nth process as

$$\lim_{N\to\infty}|C^{c,n}(u,v)-C^{d,n}(u,v)|=0, \qquad \text{uniformly on } (u,v)\in I^2, \tag{3.29}$$

for $u:=u_i+k_u, 0\le k_u\le M, u_i=\frac{i}{N}=i\,M,\quad v:=v_j+k_v, 0\le k_v\le M, v_j=\frac{j}{N}=j\,M$.

(Proof) We define $D^{c,n}(u,v)$ and $D^{d,n}(u,v)$ on I^2 as

$$D^{c,n}(u,v):=C^{c,n}(u,v)-uv,$$

$$D^{d,n}(u,v):=C^{d,n}(u,v)-uv.$$

By the above equities and the analogy of the inequity (3.19), the difference between $C^{c,n}(u,v)$ and $C^{d,n}(u,v)$ is

$$\begin{aligned}
&|\, C^{c,n}(u,v)-\ C^{d,n}(u,v)| \qquad (3.30)\\
&=|\, D^{c,n}(u,v)-\ D^{d,n}(u,v)|\\
&\le \left|D_{i,j}^{c,n}-D_{i,j}^{d,n}\right|\left(1-\frac{k_u}{M}\right)\left(1-\frac{k_v}{M}\right)+\left|D_{i+1,j}^{c,n}-D_{i+1,j}^{d,n}\right|\left(\frac{k_u}{M}\right)\left(1-\frac{k_v}{M}\right)\\
&\quad+\left|D_{i,j+1}^{c,n}-D_{i,j+1}^{d,n}\right|\left(1-\frac{k_u}{M}\right)\left(\frac{k_v}{M}\right)+\left|D_{i+1,j+1}^{c,n}-D_{i+1,j+1}^{d,n}\right|\left(\frac{k_u}{M}\right)\left(\frac{k_v}{M}\right),
\end{aligned}$$

where $0\le\frac{k_u}{M},\frac{k_v}{M}<1$.

According to (3.27) $\lim_{N\to\infty}\left|D_{i,j}^{c,n}-\ D_{i,j}^{d,n}\right|=0$ holds for any i,j , as well as $\lim_{N\to\infty}\left|C_{i,j}^{c,n}-\ C_{i,j}^{d,n}\right|=0$. In addition, $0<\left|\left(1-\frac{k_u}{M}\right)\left(1-\frac{k_v}{M}\right)\right|<1$, $0<\left|\left(\frac{k_u}{M}\right)\left(1-\frac{k_v}{M}\right)\right|<1$, $0<\left|\left(1-\frac{k_u}{M}\right)\left(\frac{k_v}{M}\right)\right|<1$, and $0<\left|\left(\frac{k_u}{M}\right)\left(\frac{k_v}{M}\right)\right|<1$. Thus we have verified that $|\, C^{c,n}(u,v)-\ C^{d,n}(u,v)|$ converge to zero as

$$\lim_{N\to\infty}|\, C^{c,n}(u,v)-\ C^{d,n}(u,v)|=0, \text{ uniformly on } I^2.$$

□

Finally, we extend this result of Proposition 3.9 to prove that the discrete evolution of copulas converge to the continuous evolution of copulas both by space and time simultaneously. We prove this convergence in the following Theorem 3.10. For the preparation of this proof, we set time t_n as

$$C(u,v,t_n) := C^{c,n}(u,v), \text{ and } C^d(u,v,t_n) := C^{d,n}(u,v),$$

where $t_{n+1} - t_n = \Delta t = h$.

We also define the interpolation of $C^d(u,v,t_n)$ as

$$C^d(u,v,t) := C^d(u,v,t_n) + \frac{C^d(u,v,t_{n+1}) - C^d(u,v,t_n)}{t_{n+1} - t_n}(t - t_n),$$

for $t = t_n + k_n, t_n \leq t \leq t_{n+1}, \Delta t = t_{n+1} - t_n = h, \ 0 \leq k_n \leq h,$

$$u := u_i + k_u, 0 \leq k_u \leq M, u_i = \frac{i}{N} = i\,M, v := v_j + k_v, 0 \leq k_v \leq M, v_j = \frac{j}{N} = j\,M.$$

The above $C^d(u,v,t)$ is rewritten as

$$C^d(u,v,t) := C^d(u,v,t_n)\frac{h - k_n}{h} + C^d(u,v,t_{n+1})\,\frac{k_n}{h}. \tag{3.31}$$

Theorem 3.10 (Convergence to continuous type). The evolution copulas in discrete processes converge to the evolution of copulas in continuous time as

$$\lim_{\substack{N\to\infty \\ h\to 0}} |C(u,v,t) - C^d(u,v,t)\,| = 0, \tag{3.32}$$

uniformly on $(u,v) \times t \in I^2 \times (0,\infty)$.

(Proof) Based on the equation (3.31) we define the interpolation of the continuous evolution of copulas between discrete time t_n and t_{n+1} as

$$C^{Ip}(u,v,t) := C(u,v,t_n)\frac{h - k_n}{h} + C(u,v,t_{n+1})\,\frac{k_n}{h}.$$

Using the equation (3.31) and the above equation, the difference between $C(u,v,t)$ and $C^d(u,v,t)$ is described as

$$|C(u,v,t) - C^d(u,v,t)| \tag{3.33}$$

$$\leq |\, C(u,v,t) - \ C^{Ip}(u,v,t)| + |\, C^{Ip}(u,v,t) - \ C^{d}(u,v,t)|$$

$$\leq |\, C(u,v,t) - C(u,v,t_n)| \left|\frac{h-k_n}{h}\right| + |\, C(u,v,t) - C(u,v,t_{n+1})| \left|\frac{k_n}{h}\right|$$
$$+|\, C(u,v,t_n) - C^{d}(u,v,t_n)| \left|\frac{h-k_n}{h}\right| + |\, C(u,v,t_{n+1}) - C^{d}(u,v,t_{n+1})| \left|\frac{k_n}{h}\right|$$

The first term $|\, C(u,v,t) - C(u,v,t_n)| \left|\frac{h-k_n}{h}\right|$ and the second term $|\, C(u,v,t) - C(u,v,t_{n+1})| \left|\frac{k_n}{h}\right|$ converge to zero as $h \to 0$, because $C(u,v,t)$ is continuous for$t$, $0 \leq (t - t_n) = k_n \leq h$, and $0 \leq \left|\frac{k_n}{h}\right|, \left|\frac{h-k_n}{h}\right| \leq 1$.

By use of Proposition 3.9 and $0 \leq \left|\frac{k_n}{h}\right|, \left|\frac{h-k_n}{h}\right| \leq 1$, the third term $|\, C(u,v,t_n) - C^{d}(u,v,t_n)| \left|\frac{h-k_n}{h}\right|$ and the fourth term$|\, C(u,v,t_{n+1}) - C^{d}(u,v,t_{n+1})| \left|\frac{k_n}{h}\right|$ converge to zero as $N \to \infty$.

Hence we have proved that the discrete evolution of copulas converge to the continuous type evolution of copulas as $\lim_{\substack{N\to\infty \\ h\to 0}} |C(u,v,t) - C^{d}(u,v,t)\,| = 0$, uniformly on $(u,v) \times t \in I^2 \times (0,\infty)$.

□

3.3 Rank correlations of evolution of copulas in discrete processes

We proved that rank correlations, the Kendall's tau and Spearman's rho, of the continuous evolution of copulas converge to zero exponentially in Theorem 2.3. Corresponding to this theorem we propose and prove that rank correlations of discrete evolution of copulas also converge to zero exponentially as n → ∞ in the following Theorem 3.11. See Yoshizawa, Y. & Ishimura, N. [29].

Theorem 3.11 For any initial copulas C_0, a sequence of copulas $\{C^n(u,v)\}_{n=1,2,\cdots}$, provided by the formula (3.3) fulfill

$$|\tau^n|, |\rho^n| \to 0 \quad \text{exponentially as } n \to \infty. \tag{3.34}$$

(Proof) Let's recall empirical copulas and their rank correlations in Chapter 1, where the discrete versions of Spearman's rho (1.12) is derived as

$$\rho^n = \frac{12}{N^2-1}\sum_{i,j=1}^{N}(C_{i,j}^n - u_i v_j), \tag{3.35}$$

and the discrete versions of Kendall's tau (1.11) is derived as

$$\tau^n = \frac{2N}{N-1}\sum_{i,j=2}^{N}\left(C_{i,j}^n C_{i-1,j-1}^n - C_{i,j-1}^n C_{i-1,j}^n\right). \tag{3.36}$$

First, we prove that nth Spearman's rho (ρ) converges to zero exponentially as $n \to \infty$. According to the formula of discrete version Spearman's rho (3.35) and the inequality (3.17), we obtain

$$\rho^n = \frac{12}{N^2-1}\sum_{i,j=1}^{N}(C_{i,j}^n - u_i v_j) \le \frac{12}{N^2-1}\sum_{i,j=2}^{N}|C_{i,j}^n - u_i v_j| \tag{3.37}$$

$$\le \frac{12N^2}{N^2-1}\max_{i,j=0,1,\cdots,N}|D_{i,j}^n| < \frac{12N^2}{N^2-1}K\theta^n.$$

Thus we have proved that Spearman's rho converges to zero as

$$\lim_{n\to\infty}|\rho^n| = 0 \quad \text{exponentially.}$$

Second, we prove that nth Kendall's tau (τ) converges to zero exponentially as $n \to \infty$. Based on the formula of discrete version Kendall's tau (3.36), the inequality (3.17) and the relation $u_i v_j u_{i-1} v_{j-1} - u_i v_{j-1} u_{i-1} v_j = 0$, we obtain

$$\tau^n = \frac{2N}{N-1}\sum_{i,j=2}^{N}(C_{i,j}^n C_{i-1,j-1}^n - C_{i,j-1}^n C_{i-1,j}^n) \tag{3.38}$$

$$= \frac{2N}{N-1}\sum_{i,j=2}^{N}\Big\{D_{i,j}^n D_{i-1,j-1}^n + D_{i,j}^n \frac{(i-1)}{N}\frac{(j-1)}{N} + D_{i-1,j-1}^n \frac{i}{N}\frac{j}{N}$$

$$-D_{i,j-1}^n D_{i-1,j}^n - D_{i,j-1}^n \frac{(i-1)}{N}\frac{j}{N} - D_{i-1,j}^n \frac{i}{N}\frac{(j-1)}{N}\Big\}$$

$$\leq \frac{2N(N-1)^2}{N-1}\max_{i,j=2,3,\cdots,N}\Big\{|D_{i,j}^n D_{i-1,j-1}^n| + |D_{i,j}^n|\frac{(i-1)}{N}\frac{(j-1)}{N} + |D_{i-1,j-1}^n|\frac{i}{N}\frac{j}{N}$$

$$+|D_{i,j-1}^n D_{i-1,j}^n| + |D_{i,j-1}^n|\frac{(i-1)}{N}\frac{j}{N} + |D_{i-1,j}^n|\frac{i}{N}\frac{(j-1)}{N}\Big\}$$

$$< 2N(N-1)\{2K^2\theta^{2n} + 4K\theta^n\} = K_1\theta^{2n} + K_2\theta^n < K_3\theta^n,$$

where $K_3 = K_1 + K_2,\ K_1 = 4N(N-1)K^2, K_2 = 8N(N-1)K.$

Thus we have proved that Kendall's tau converge to zero as

$$\lim_{n\to\infty}|\tau^n| = 0 \quad \text{exponentially.}$$

□

3.4 Extension of evolution of copulas in discrete processes

We extend continuous type evolution of copulas to backward evolution of copulas and evolution of copulas with coefficients. In order to keep the consistency with the continuous type, we extend evolution of copulas in discrete processes to backward evolution of copulas in discrete processes, and evolution of copulas with coefficients in discrete processes.

Backward evolution of copulas in discrete processes

We proposed continuous type backward evolution of copulas in Section 2.4. Therefore it is natural to think of their discrete type. We proclaim and prove that the existence of backward evolution of copulas in discrete processes in the following Theorem 3.12. We refer to Yoshizawa, Y. & Ishimura, N. [29].

Theorem 3.12. For any initial copula C_T, there exists a sequence of copula $\left\{C^{\frac{T}{h}-n}(u,v)\right\}_{n=0,1,2,\cdots}$ on $(u,v) \in I^2$, which satisfy the system of difference equations

$$\frac{C_{i,j}^{\frac{T}{h}-(n+1)} - C_{i,j}^{\frac{T}{h}-n}}{\Delta t} + \frac{C_{i+1,j}^{\frac{T}{h}-n} - 2C_{i,j}^{\frac{T}{h}-n} + C_{i-1,j}^{\frac{T}{h}-n}}{(\Delta u)^2} + \frac{C_{i,j+1}^{\frac{T}{h}-n} - 2C_{i,j}^{\frac{T}{h}-n} + C_{i,j-1}^{\frac{T}{h}-n}}{(\Delta v)^2} = 0, \quad (3.39)$$

, at every $\{(u_i, v_j)\}_{i,j=0,1,2,\cdots,N}$, for $i,j = 0,1,\cdots,N-1, \mathrm{n} = 0,1,\cdots,\left[\frac{T}{h}\right]$,

where $C^T(u,v) := C_T(u,v) = C_T$ denotes given maturity copula $C(u,v,T)$ together with the boundary condition (3.4).

We call $C^{\frac{T}{h}-n}(u,v)$ as backward evolution of copulas in discrete processes.

(Proof) First, we replace n by $(T/h - n)$ in the difference equation (3.3), then its left hand side $\frac{C_{i,j}^{n+1}-C_{i,j}^{n}}{\Delta t}$ is transformed into $-\frac{C_{i,j}^{T/h-(n+1)}-C_{i,j}^{T/h-n}}{\Delta t}$, and its right hand side $\frac{\left(C_{i+1,j}^{n}-C_{i,j}^{n}\right)-\left(C_{i,j}^{n}-C_{i-1,j}^{n}\right)}{(\Delta u)^2}+\frac{\left(C_{i,j+1}^{n}-C_{i,j}^{n}\right)-\left(C_{i,j}^{n}-C_{i,j-1}^{n}\right)}{(\Delta v)^2}$ is transformed into $\frac{C_{i+1,j}^{T/h-n}-2C_{i,j}^{T/h-n}+C_{i-1,j}^{T/h-n}}{(\Delta u)^2}+\frac{C_{i,j+1}^{T/h-n}-2C_{i,j}^{T/h-n}+C_{i,j-1}^{T/h-n}}{(\Delta v)^2}$. Therefore we have confirmed that the difference equation (3.39) holds. We also can calculate its solution backwardly in the same method as we compute the solution of the difference equation (3.3).

We can verify that $C_{i,j}^{\frac{T}{h}-n}$ satisfy the 2-increasing condition, in the same way as the proof in Proposition 3.2.

Applying the interpolation (3.8) to $C_{i,j}^{\frac{T}{h}-n}$, we obtain

$$C^n(u,v) := C_{i,j}^n + \frac{C_{i+1,j}^n - C_{i,j}^n}{u_{i+1} - u_i}(u - u_i) + \frac{C_{i,j+1}^n - C_{i,j}^n}{v_{j+1} - v_j}(v - v_j)$$

$$+ \frac{C_{i+1,j+1}^n - C_{i+1,j}^n - C_{i,j+1}^n + C_{i,j}^n}{(u_{i+1} - u_i)(v_{j+1} - v_j)}(u - u_i)(v - v_j),$$

for $u_i \le u \le u_{i+1}, v_j \le v \le v_{j+1}$ and $(u,v) \in I^2$.

We confirm that the interpolation of backward evolution of copulas in discrete processes $C^{\frac{T}{h}-n}(u,v)$ satisfy two boundary conditions and the 2-increasing condition on $(u,v) \in I^2$, in the same method as the proof in the Theorem 3.4.

Hence we have proved that $\left\{C^{\frac{T}{h}-n}(u,v)\right\}_{n=0,1,2,\cdots}$ are copulas on $(u,v) \in I^2$, which fulfill the three properties of copulas, such as two boundary conditions and the 2-increasing condition.

□

Evolution of copulas with coefficients in discrete processes

We recall that evolution of copulas with coefficient which satisfy the following heat equation with the diffusion coefficient in Corollary 2.5.

$$\frac{\partial C}{\partial t}(u,v,t) = \nu\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)C(u,v,t)$$

ν is the diffusion coefficient

Corresponding to the above partial differential equation we define the deference equation for discrete type evolution of copulas with coefficient as

$$\frac{C_{i,j}^{n+1} - C_{i,j}^{n}}{\Delta\, t} = \nu \frac{C_{i+1,j}^{n} - 2C_{i,j}^{n} + C_{i-1,j}^{n}}{(\Delta\, u)^2} + \nu \frac{C_{i,j+1}^{n} - 2C_{i,j}^{n} + C_{i,j-1}^{n}}{(\Delta\, v)^2} \tag{3.40}$$

for $1 \le i,j \le N-1$, $\nu\lambda \le 1/4$, $C_{0,j}^{n} = C_{i,0}^{n}$, $C_{i,N}^{n} = \frac{i}{N}, C_{N,j}^{n} = \frac{j}{N}$, $\{C_{i,j}^{0}\}_{i,j=0,\cdots,N}$ is copula, $\lambda := \frac{\Delta t}{(\Delta u)^2} = \frac{\Delta t}{(\Delta v)^2} = hN^2 = \frac{h}{M^2}$ and ν is the diffusion coefficient.

Corollary 3.13. For any initial copula C_0, there exists a sequence of copula $\{C^n(u,v)\}_{n=0,1,2,\cdots}$ on $(u,v) \in I^2$, which satisfy the system of difference equation (3.40) at every $\{(u_i, v_j)\}_{i,j=0,1,2,\cdots,N}$. We call these copulas $C^n(u,v)$ as evolution of copulas in discrete processes.

(Proof) We can rewrite (3.40) as the following recurrence formula as

$$C_{i,j}^{n+1} = (1 - 4\nu\lambda)C_{i,j}^{n} + \nu\lambda\left(C_{i+1,j}^{n} + C_{i-1,j}^{n} + C_{i,j+1}^{n} + C_{i,j-1}^{n}\right) \tag{3.41}$$

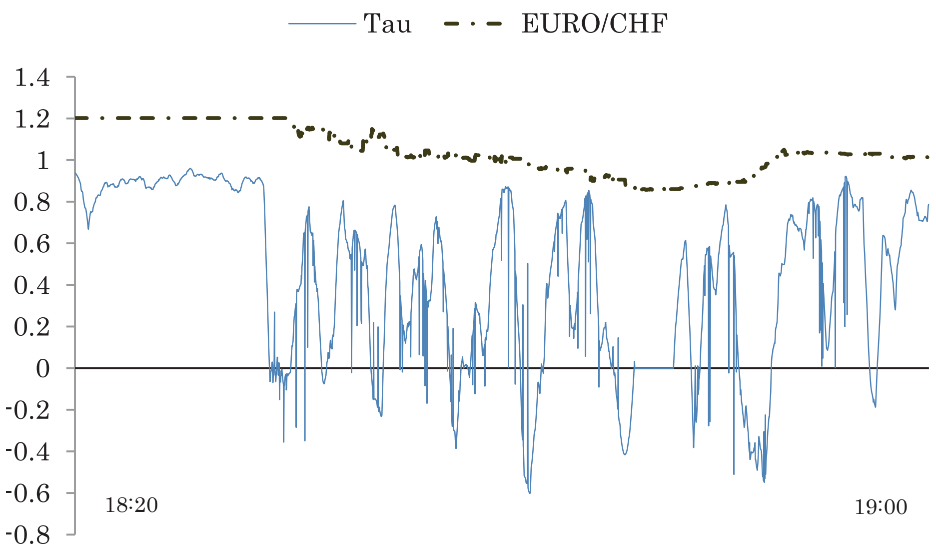
The proof is analogous Theorem 3.4, where the only difference is that λ is replaced $\nu\lambda$.

□

3.5 Application to empirical data

In this subsection, we apply the above results to real data to confirm the quality of practically using the evolution of copulas. As an example, we analyze the dependence of euro–Japanese yen foreign exchange rates with those of the Swiss franc–Japanese yen. The evolution of copulas has properties to suit events whose dependence monotonically increases or decreases. Therefore, we focus on rapidly changing events when their directivities are almost stable. We select foreign exchange rates on January 15, 2015, when the Swiss franc endured a shock breakout after the announcement that the Swiss central bank had stopped monetary policy efforts to maintain the Swiss franc against the euro at more than 1.20. Moreover, we collect data on the second time scale in order to capture their monotonic directivity.

First we construct empirical copulas of the euro–Japanese yen rates and the Swiss franc–Japanese yen rates for every second of 40 minutes using the formula (1.8). Then, we calculate their Kendall's tau correlation measure by applying the formula (1.11). Figure 3.3 charts the transition of the Kendall's tau, as well as the euro–Swiss franc foreign exchange rates.



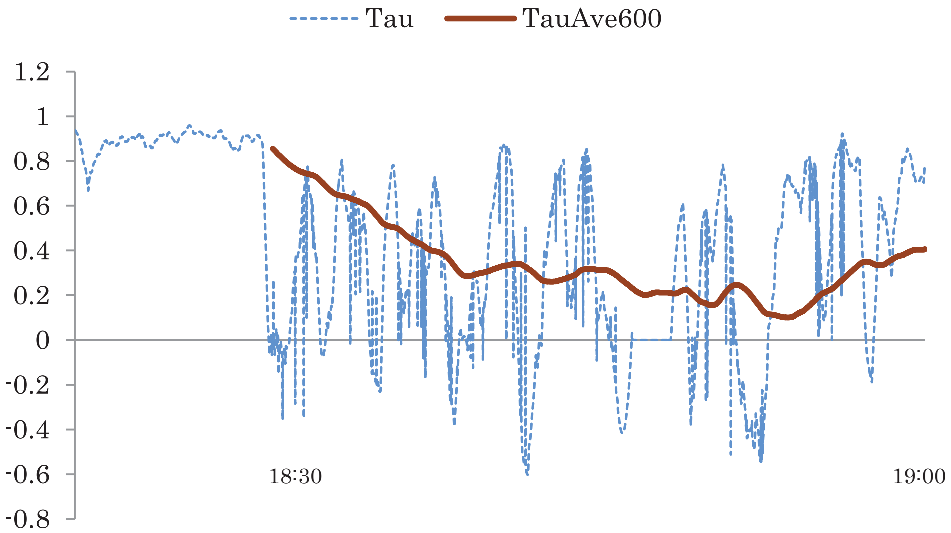
Notes: Tau: Kendall's Tau of the euro against the Japanese yen and the Swiss franc against the Japanese yen; EURO/CHF: The euro against the Swiss franc

Data source: Bloomberg, exchange rate

We collect the foreign exchange rates on the second scale interval from 18:20 to 19:00 on January 15, 2015. We calculate the euro–Japanese yen rates by multiply the Swiss franc–Japanese yen rates by the euro–Swiss franc rates. With reference to the formula (1.8), we construct empirical copulas for every second by using 60 datasets for both the euro–Japanese yen and the Swiss franc–Japanese yen, where we collect 60 datasets in 1 minute. Then, we calculate Kendall's tau by applying the formula (1.11) to the empirical copulas to every second.

Figure 3.3. Kendall's tau versus the euro against the Swiss franc

Second, we apply a smoothing technique to the transitions of Kendall's tau, since they fluctuate and include some singular data. We apply a moving average method and the results are shown in Figure 3.4.

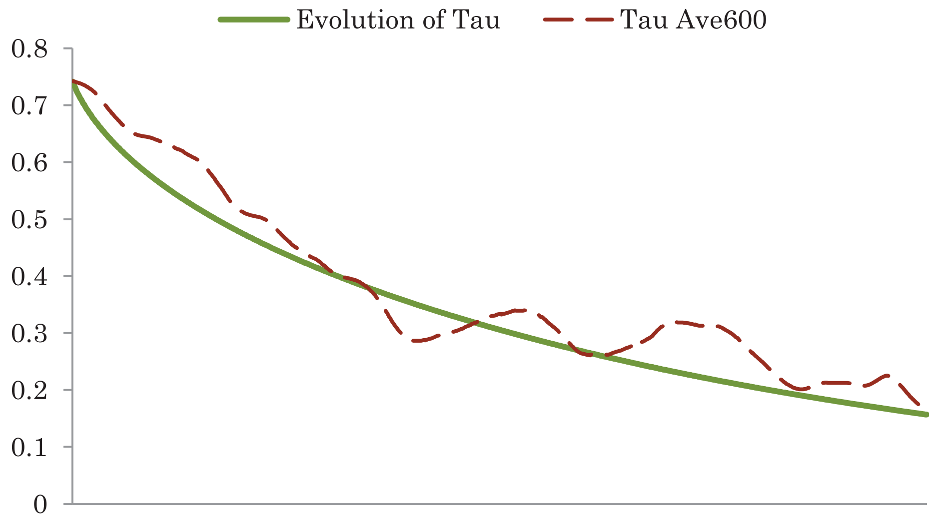


Notes: Tau: Kendall's Tau of the euro against the Japanese yen and the Swiss franc against the Japanese yen; TauAve600: Moving averages of "Tau" for 600 datasets

We apply a moving average method to 600 datasets of Kendall's tau, as shown in Figure 3.3. The average of 600 datasets for every second refers to the data average over 10 minutes. Thus, the start point of the average of 600 datasets is 18:30.

Figure 3.4. Moving averages of Kendall's tau

Finally, we compare Kendall's tau of the evolution of empirical copulas to the abovementioned moving averages of Kendall's tau of empirical copulas. We choose the start time at which Kendall's tau of the empirical copula and its moving average are almost equal. We evolve the empirical copula at the start time using the recursion equation for 20 minutes. The results are plotted in Figure 3.5, which shows that the evolution of empirical copulas approximate the smoothed transition of empirical copulas from the viewpoint of Kendall's tau.



Notes: TauAve600: Moving averages of "Tau" for 600 data; Evolution of Tau: Kendall's tau of evolution of copulas.

We extract the empirical copula at 18:31:47, at which time the Kendall's tau of the empirical copulas is almost equal to the smoothed Kendall's tau. We evolve them 1,200 times, which means for 20 minutes. For this evolution, we set parameters for the recursion equation

$C_{i,j}^{n+1} = (1-4\nu\lambda)C_{i,j}^{n} + \nu\lambda\left(C_{i+1,j}^{n} + C_{i-1,j}^{n} + C_{i,j+1}^{n} + C_{i,j-1}^{n}\right)$ (3.41), as $\Delta u = \Delta v := \frac{1}{N} = \frac{1}{60 \text{ data}}$, $\Delta t = 1$ second, $\lambda := \frac{\Delta t}{(\Delta u)^2} = \frac{\Delta t}{(\Delta v)^2} = 3{,}600$, and $\nu = 720$. Thus, we derive the equation $C_{i,j}^{n+1} = \frac{1}{5}\left(C_{i,j}^{n} + C_{i+1,j}^{n} + C_{i-1,j}^{n} + C_{i,j+1}^{n} + C_{i,j-1}^{n}\right)$, which is used to evolve the empirical copulas.

Figure 3.5. Kendall's tau of evolution of copulas

Using these flexible discrete copula models, we analyze the movement of dependence relations between the Swiss franc and the euro against the Japanese yen approximately. We can verify the practicality of some theories of the evolution of copulas, although this is restricted to events whose essential dependence does not fluctuate but transforms monotonically.

4. Conclusion

In this thesis, we studied the evolution of copulas in continuous as well as discrete processes. As a first step, we focus on the evolution of copulas as governed by the heat equation, which is a basic partial differential equation used to describe dynamic movements. We find some excellent theorems, but several problems remain to be solved. Our main discoveries are summarized in the following paragraphs.

Main Findings

First, we find that solutions exist for the evolution of copulas that transform autonomously through time in accordance with the heat equation. Moreover, we prove that they converge to the product copula and that their rank correlations converge to zero exponentially as time $t \to \infty$. These facts ensure consistency and mean that dependence decreases over time. However, for many phenomena, dependences increase from moment to moment. Thus, we propose the backward evolution of copulas, in which they evolve in reverse, in accordance with dependence-increasing events, and we prove the existence of their solutions.

Second, we study the evolution of copulas in discrete processes to satisfy the discrete version of the heat equation. We define these processes and prove that these discrete evolution copulas converge to the original continuous evolution copulas; thus, we can treat discrete evolution copulas as approximations of the continuous type. Furthermore, we prove that these discrete evolution copulas also converge to the product copula, and that their rank correlations converge exponentially to zero, regardless of their mesh size.

Third, discrete evolution copulas are a good fit for numeric analysis when constructing mathematical models on computers. In order to maintain the flexibility to create a model, we extend the plain discrete type evolution of copulas backward and/or with coefficients of the discrete as well as continuous types. Using these flexible discrete copula models, we analyze the movement of dependence relations between the Swiss franc and the euro against the Japanese yen. Hence, we can verify the practicality of several theories of the evolution of copulas.

Future Prospects

First, for conciseness, we studied only bivariate copulas in this thesis. However, most events in the real world are complicated and consist of many factors or variables. Thus, it is of use to establish some theories about the evolution of multivariate copulas, corresponding to the evolution of bivariate copulas. We ourselves have studied the evolution of multivariate copulas in discrete processes. See Ishimura, N. & Yoshizawa, Y. [18], [19].

Second, the evolution of copulas is not versatile; it is restricted to events whose essential dependences do not fluctuate but transform monotonically. The evolution of copulas fits with events where dependence decreases; conversely, the backward evolution of copulas conforms to dependence-increasing events. We expect that the evolution of copulas will be applied in many fields, such as risk management, natural science, engineering, medical science, and social science. Thus, it is our task to uncover the events, phenomena, and theories that the evolution of copulas fits and, apply them.

Third, we think that finding events with which the evolution of copulas fits is not enough. As the evolution of copulas is in the first stage, we should enter the next stage to study other time-dependent copulas that also transform autonomously as governed by the equations containing time variables. We hope that new time-dependent copulas will fit into various transformations of dependence structures, and will contribute to the analysis of all events in any field, as well as to the progress of copula theories.

Appendix A. Quantitative risk management

Progressive insurance companies in developed countries have adopted enterprise risk management (ERM), which manages all company risks; companies use it not only to avoid their own losses but also to profit by taking reasonable risks with their capital. In Europe, new solvency regulations have been introduced: the Swiss Solvency Test and SolvencyⅡ. Switzerland introduced the Swiss Solvency Test in 2008, and the European Union will introduce SolvencyⅡ in the near future. Their common factor is quantitatively measured risk, which is called economic capital (EC) in ERM and the solvency capital requirement (SCR) in solvency regulations. Economic capital plays a central role in ERM, as it is used to evaluate and control risks, and it relates not only to risk management but also to capital management. In solvency regulations, the SCR is the key criterion for judging whether an insurance company has enough capital to be solvent. The conceptual image of the capital requirement in solvency regulations is charted in Figure A.1.

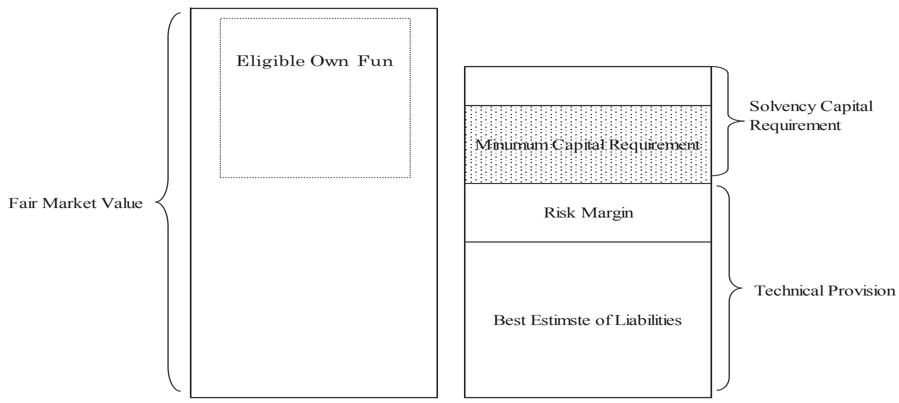


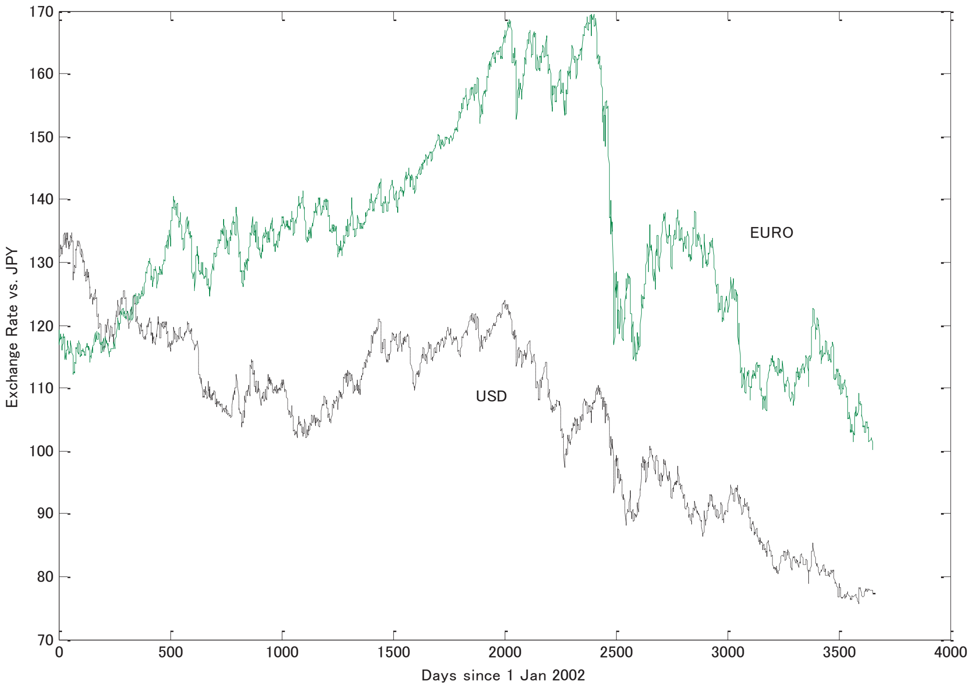
Figure A.1. Capital requirement in Solvency II; Source, Yoshizawa, Y. [27]

We cannot measure risks without Quantitative Risk Management (QRM), such as risk measures and risk aggregating approaches. Value at risk (VaR) and tail value at risk (Tail VaR) are commonly used to measure EC and SCR. In practice, many kinds of risks exist within financial entities like insurance companies, and the relations among those risks are complicated. Insurance companies' total aggregated risks vary widely based on their dependencies. When we measure risks, it is critical to consider the dependencies among risks as well as risk measures. Traditionally, the variance-covariance approach has been adapted to aggregate risks, taking account of their dependencies.

However, this approach has the defect that it does not reflect the details of risk factor distribution. It applies constant dependency factors, such as coefficients of variance and covariance, to the entire range of risk distributions, including tail risks. Therefore, copulas have recently come to attract much attention for their ability to reflect detailed dependencies. Using copulas, we can obtain a multivariate distribution of risk factors, which expresses their dependencies throughout the range. In this Appendix, we refer to Yoshizawa, Y. [26] and [27].

Appendix B. Time variance of dependence structure

We analyze the time variance of the dependence structure, taking foreign exchange rates as an example. We collect daily data for the U.S. dollar against the Japanese yen, and the euro against the yen, from January 1, 2002, to December 31, 2011; this is charted in Figure B.1.



Data source: ONDA

Figure B.1. Transition of foreign exchange rates (USD vs. JPY and EURO vs. JPY)

The chart in Figure B.2 is a scatter graph for the years 2002 to 2011, charting the foreign exchange rates of each of the U.S. dollar and the euro against the Japanese yen, and their frequencies.

Total (2002–2011)

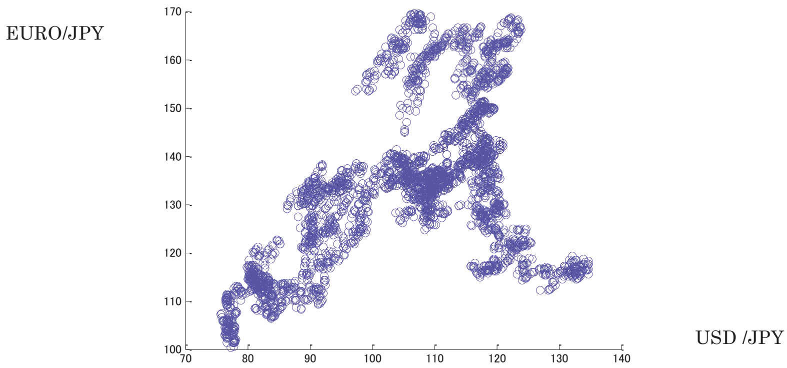


Figure B.2. Scatter graph of foreign exchange rates (2002–2011)

We use of copulas to modify the original empirical data to suit the theoretical data. We approximate the original 10 years' data, from 2002 to 2011, using copulas, and chart their distributions in Figure B.3, where we expose the original data on the left side and the data approximated using a Clayton copula on the right side. We use MATLAB R2011a to analyze these data.

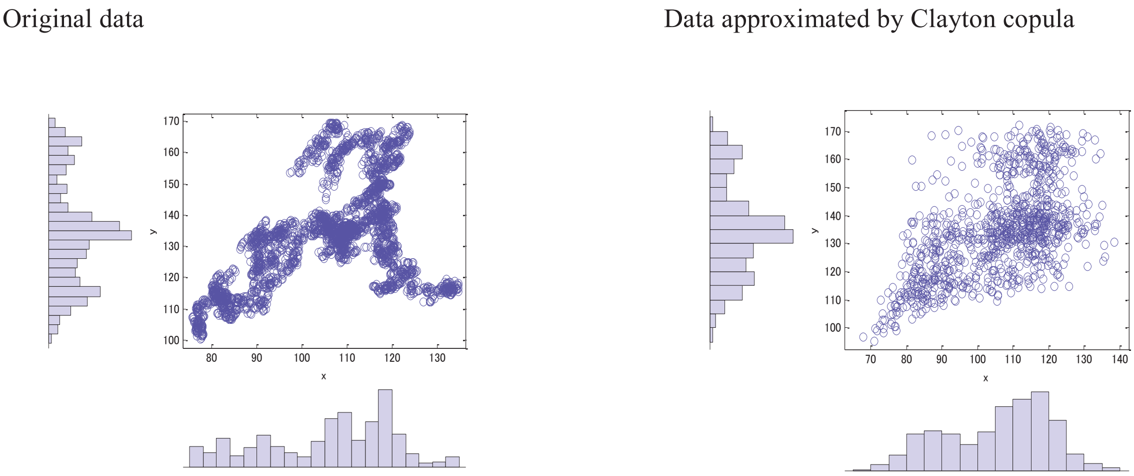


Figure B.3. Approximating dependency between foreign exchange rates using copulas (2002–2011)

We study the transition of distributions annually using the same method to analyze Figure B.3. We divide the above 10 years' data by fiscal year into 10 blocks, each of which consists of approximately 365 data points. We use Clayton copulas to approximate the original empirical data to suit the theoretical data, which are charted in Figure B.4.

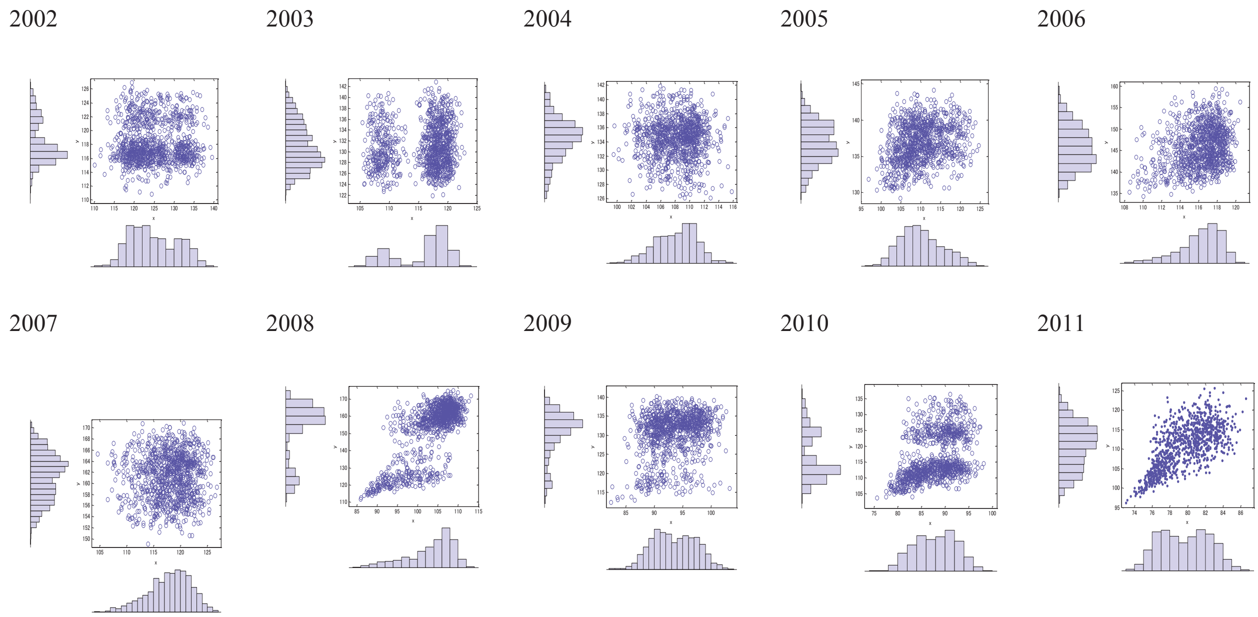


Figure B.4. Transformation of distributions approximated by Clayton copulas (2002–2011)

Moreover, we pick up the sequence of θ of Clayton copulas and draw a diagram of them in Figure B.5. The parameter θ is neither constant nor monotonous; it fluctuates. We can guess that foreign exchange rates have the complicated property of fluctuating, because they are sometimes affected by various economic, political, social, and psychological events.

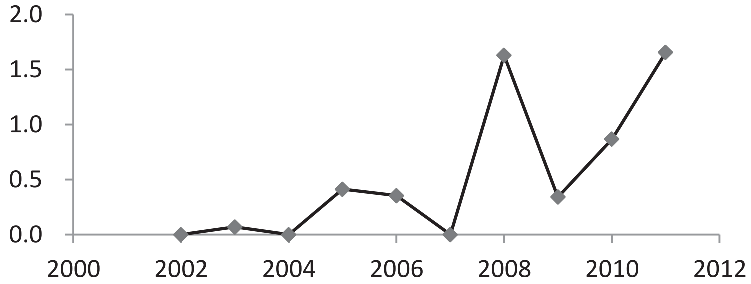


Figure B.5. Transition of parameter θ of Clayton copulas

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Acknowledgement

First of all I would like to thank a wonderful person, my professor, Dr. Naoyuki ISHIMURA, my supervisor as well as the chairman of my doctoral dissertation supervising committee, for his constant encouragement, good advice and constructive criticism. He introduced me to the fascinating subject of copulas.

Further to my supervisor, my special thanks go to Dr. Tsunemasa SHIBA, my co-supervisor, for his useful comments and advices.