Evolution of Copulas

：Continuous, Discrete, and its application to Quantitative Risk Management

By Yasukazu YOSHIZAWA

Abstract

1. Background and Purpose

Dependence relations among random variables are one of the most important subjects for probability and statistics research. Analysis of dependence structures is critical from both theoretical and applied viewpoints. Recently, members of the financial sector and their regulators have recognized that it is critical to manage these risks in a sophisticated way, that is, quantitatively. Quantitatively measured risks play a central role in this management framework. These entities face many kinds of risks, and the relations between them are very complicated. Thus, it is crucial to reflect the dependence relations to measure the risks quantitatively: the more dependence there is among risks, the less aggregated the risks are.

Linear correlation is often recognized as a satisfactory measure of dependence. However, it cannot capture the non-linear dependence relations that exist among many risk factors. What expresses the dependence relations among risk factors? If we capture a multivariate joint distribution of all the risk factors, we can recognize their dependence structure probabilistically or statistically. For this reason, there has been much interest in a copula function, or simply a

copula. A copula links multivariate joint distribution and univariate marginal distributions. Copulas are often employed to investigate the dependence structure among random variables.

Dependence relations, who transform over time, are dynamic rather than static in nature. However, copulas are useful mainly for static matters; their definitions themselves do not contain time variables. It is well known that rank correlations, one of the prevailing measures of dependence, are derived only by copulas. That is to say, copulas determine rank correlations. Thus, it is natural to analyze only copulas in the study of transformations of dependence structures through time. As a first step, we start to investigate how copulas transform, and if they evolve autonomously in accordance with the heat equation, which is one of the basic partial differential equations used to describe dynamic movements.

In this thesis, we introduce evolving copulas, which transform through time autonomously as governed by the heat equation. Moreover, we construct discrete type of the time-dependent evolution of copulas to apply empirical data analysis, investigate their properties, and prove that they converge to their original continuous type. Finally, we apply empirical data to discrete evolution of copulas in order to verify their practicality.

2. Evolution of Copulas in Continuous Time

We propose that time-dependent evolving copulas transform autonomously through time First, we prove the existence of solutions for the evolution of copulas that evolve in accordance with the heat equation in the following theorem. Second, we prove that they converge to the product copula as time $t \to \infty$, and that their rank correlations converge to zero exponentially as time $t \to \infty$. Finally, we extend the evolution of copulas backward and with coefficients.

Theorem (Evolution of copulas). For any bivariate copula $C_0(u,v)$, there exists a unique family of time dependent bivariate copula$\{C(u,v,t)\}_{t\geq 0}$, which satisfies the heat equation

$$\frac{\partial C}{\partial t}(u,v,t) = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) C(u,v,t),$$
$$\text{for } (u,v,t) \in I^2 \times (0.\infty), \text{ where } C(u,v,0) = C_0(u,v) \quad \text{on } (u,v) \in I^2.$$

The solution of the above partial differential equation is

$$C(u,v,t) = uv + 4\sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)t} \sin m\pi u \sin n\pi v \, K(m,n),$$
$$\text{where } K(m,n) = \iint_{I^2} \sin m\pi\xi \sin n\pi\eta \, (C_0(\xi,\eta) - \xi\eta) d\xi \, d\eta.$$

3. Evolution of Copulas in Discrete Processes

In general, it is difficult to solve partial differential equations analytically; furthermore, numerical analysis using discrete data is suitable for calculation by computer. We propose the evolution of copulas in discrete processes that satisfy the discrete version of the heat equation. We prove that they converge to the product copulas and that their rank correlations converge to zero exponentially. Moreover, we prove that these discrete evolution copulas converge to their original continuous type. Thus, we can treat discrete evolution copulas as an approximation of the continuous type. We extend them backward and with coefficients for discrete evolution copulas as well as continuous type.

4. Application to Quantitative Risk Management

We apply empirical data to discrete evolution copulas to verify their practicality. The evolution of copulas has properties to suit events whose dependence monotonically increases or decreases. Therefore, we focus on rapidly changing events when their directivities are almost stable. As an example, we analyze the dependence of euro–Japanese yen foreign exchange rates with those of the Swiss franc–Japanese yen on January 15, 2015, when the Swiss franc endured a shock breakout after the announcement that the Swiss central bank had stopped monetary policy efforts to maintain the Swiss franc against the euro at more than 1.20.

We collect data on the second time scale in order to capture their monotonic directivity. Next we construct empirical copulas of the euro–Japanese yen rates and the Swiss franc–Japanese yen rates, calculate their Kendall's tau, and apply a smoothing technique to the transitions of Kendall's tau. Finally, we evolve an empirical copula to construct discrete evolution copulas, and compare Kendall's tau of the discrete evolution copulas to the abovementioned moving averages of Kendall's tau of the empirical copulas. The results are that the discrete evolution copulas approximate the smoothed transition of empirical copulas from the viewpoint of Kendall's tau.