ESSAYS ON PRODUCTION EXTERNALITIES: MICROECONOMICS, TRADE AND ENVIRONMENTAL ECONOMICS

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BY

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Chapter 1

Introduction

1.1 Background

Externalities are common phenomena observed in various economic activities. A traditional example of externalities is pollution. A polluter is reluctant or short of information to consider the indirect costs of pollution, such as pressures on the ecological system, lower quality of life for those living near the polluting spot, and forgone opportunities of other economic activities like tourism. Without charges on pollution, a polluter's private costs of production would be lower than the social costs, and so he often overproduces from the social point of view.

Externalities are not necessarily bad. Research and development (R&D) often add to the general body of knowledge, and so other producers and R&D activities can benefit. Without subsidies on R&D, private firms' costs of R&D are higher than the social costs, leading to insufficient R&D.

Economists have long recognized the importance of externalities and discussed its various forms–production externalities and consumption externalities, input-generated and output-generated, economies and diseconomies, unpaid factor and creation of atmosphere–in many fields of economics including microeconomics, welfare economics, public economics, growth theory, trade theory, environmental and resource economics.

Despite the large body of literature on externalities, there are still questions not resolved or addressed yet, as well as issues newly emerging. In the dissertation, I focus on production externalities and deal with four related questions and issues in two fields of economics. In Chapter 2 and Chapter 3, I examine the production possibility frontier (PPF) from the perspective of microeconomics. In Chapter 4 and Chapter 5, I consider production externalities in the context of trade and the environment. Chapter 6 provides some concluding remarks.

1.2 Research Questions

In Chapter 2, I ask whether production efficiency requires full employment of factors of production in the presence of production externalities. Here, production efficiency indicates the situation that the economy is operating on the PPF. The question is important. First, full employment is a presumption applied in many economic models: once assuming inelasticity of factor supply, full employment follows immediately in a market-based economy. Second, if the answer is NOT necessarily, government interventions are expected. To my knowledge, surprisingly, there seems no formal answer to this basic but essential question.

In Chapter 3, I examine the properties of the PPF in the presence of strong input-generated externalities. Here, "strong" indicates the situation that full use of factors is not efficient. The problem is important. First, strong input-generated externalities such as traffic jams are often seen in real life. Second, the convexity of the PPF (or, non-convexity of the production possibility set) provides an explanation to the symmetry breaking among ex-ante identical agents, such as division of labor and comparative advantages: if the PPF is convex, each agent can enjoy higher efficiency by cooperating and specializing in narrower range of tasks.

In Chapter 4, I try to understand the endogenous link between trade, economic development, and the environment. The issue is important. In recent years, the rapid process of globalization promotes the separation of production and consumption, aggravating externality problems that affect the environment. The globalization often stimulates economic development, too, imposing more environmental pressures. It becomes an increasingly pressing issue to understand the close nexus between the three elements. However, possibly due to the complexity of the problem, existing theoretical studies on this issue are far from satisfactory.

In Chapter 5, we work on narrowing the gap between the multi-functional environment in reality and its single role in theoretical economic models. Those theoretical models usually formulate only the aspect of the environment directly relating to the issue of interest. For example, if renewable resources is the interest, the environment is then a place that grows resources. If environmental pollution is the concern, the environment then becomes a pollution sink. These extreme simplifications come at risks of misleading implications.

1.3 Main Messages

In Chapter 2, to answer whether production efficiency requires full employment in the presence of production externalities, I consider two general formulations of production externalities: input-generated and output-generated. The result is impressive. If externalities are input-generated, the answer is NOT necessarily, that is, full employment may be inefficient. In contrast, if externalities are output-generated, the answer is YES.

In Chapter 3, to highlight the effect of strong input-generated externalities on the properties of the PPF, I focus on the single-factor case and examine monotonicity, continuity, and convexity of the PPF. I show that, in the presence of strong input-generated externalities, a sufficient condition for the PPF to be (strictly) convex is the (strict) quasi-concavity of the by-product generation function. Moreover, under reasonable conditions, the PPF is either entirely strictly convex, or entirely linear.

In Chapter 4, to investigate the close link between trade, economic development, and the environment, I develop a two-sector dynamic general equilibrium model, in which economic development is described by the accumulation of private capital. In terms of model structure, my model extends Copeland and Taylor (1999) by introducing households behaving in a Ramsey fashion. I also conduct policy analysis. Among a rich set of results, the following two are especially interesting. First, in the short run, trade can be good or bad to the environment, depending on the direction of composition effect. However, in the long run, trade necessarily harms the environment thanks to the scale effect. Second, the social optimum can be achieved through a pollution tax. I show that the optimal pollution tax is a dynamic version of the Pigouvian tax.

In Chapter 5, to move one step toward modeling the multi-dimensional roles of the environment, we construct a two-sector dynamic general equilibrium model allowing environmental impacts from both sectors. This small step brings in a new agenda for considering the link between trade and the environment. Beside a rich set of results on the properties of the model, the following two provide important insights. First, countries can be categorized into two types depending on the slope of long-run supply curve: the Copeland–Taylor type with an upward-sloping supply curve and the Brander–Taylor type with a downward-sloping one. Thus, a country of Copeland–Taylor type tends to specialize in trade. Second, both countries can export their own dirty goods to each other and may lose from trade.

Chapter 2

The Efficiency of Full Employment under Production Externalities

2.1 Introduction

Full employment of factors of production is a starting point for many economic models, which comes as the equilibrium outcome of inelastic factor supplies. If production processes are independent of each other, full employment is, as well known, a necessary condition for production efficiency–operating on the production possibility frontier (PPF).[1] However, production processes often affect each other. For example, toxic wastes from a pulp mill may harm fishery resources, which is a bad news to local fishermen. So the following question arises naturally. In the presence of production externalities, does production efficiency still require full employment?

In terms of economic theory, the question is essential since production externalities are common phenomena in reality and have be extensively discussed in economics. In terms of policy, the question helps to understand whether we need regulations on investment and when. Surprisingly, in the literature there seems no formal answer to this essential question. This study attempts to provide one.

2.2 Production Externalities: Input-generated and Output-generated

In general, production externalities can come from inputs, or from outputs, or from both. Consider m goods and n factors, we can write

$$F_j\left(v_{-j}, x_{-j}, v_j, x_j\right) = 0 \;\; j = 1, ..., m, \tag{2.1}$$

[1] The underlying requirement is that all factors are marginally productive, otherwise the statement is not necessarily true. The Leontief technology is an example.

where F_j presents the production process of good j, the non-negative vector $v_j \equiv (v_{1j}, ..., v_{nj})'$ denotes the vector of inputs in good j, x_j the output of good j. Production externalities are reflected by the presence of $v_{-j} \equiv (v_1, ..., v_{j-1}, v_{j+1}, ..., v_m)$, namely the vector with all input vectors as the components except for v_j, and $x_{-j} \equiv (x_1, ..., x_{j-1}, x_{j+1}, ..., x_m)$, namely the vector with all outputs as the components except for x_j, in the production process good j. The formulation (2.1) is very general, but too general to derive clear answer to the question above.

In this note, I focus on two formulations of production externalities that are still quit general, though less general than (2.1), and widely used in economics: input-generated and output-generated production externalities (Kemp, 1955). Formally, input-generated production externalities can be expressed by

$$x_j = f_j \left(v_{-j}, v_j\right), \ \ j = 1, ..., m, \tag{2.2}$$

where f_j represents the production function of good j. The output of good j depends on the input in the production of good j, as well as the inputs in other goods. In contrast, output-generated production externalities can be expressed by

$$x_j = f_j \left(x_{-j}, v_j\right), \ \ j = 1, ..., m. \tag{2.3}$$

That is, the output of good j depends on the input in the production of good j, as well as the outputs of other goods.

Finally, the factor constraint can be written into

$$E - e \cdot v \geq 0, \tag{2.4}$$

where $E \equiv (E_1, ..., E_n)'$ is the non-negative vector of factor endowments, $v \equiv (v_1, ..., v_m)$ is the vector of input vectors, e is the unitary vector with all components as one.

2.3 Production Efficiency of Full Employment

Assume that the technology satisfies

(A1) f_j is continuous in all augments;

(A2) f_j is non-decreasing in v_{kj}, $k \in \{1, ..., n\}$, and strictly increasing if $f_j > 0$;

(A3) $f_j(\cdot, 0) = 0$;

(A4) the production possibility set (PPS) Ω is bounded.

Assumption (A2) requires the inputs to be productive when the output is positive, thereby excluding the Leontief technology and good-specific factors. Assumption (A3) formulates the idea of no free lunch, implying that the origin belongs to the PPS. Assumption (A4) ensures the existence of the PPF, which is defined as the boundary of the PPS. Formally, let $M \equiv \{1, ..., m\}$ denote the index set of goods, $M^0(x) \equiv \{j \in M; x_j = 0\}$ the index set of those goods with zero output, then the PPF can be defined as

Definition 2.1. PPF $\equiv \{x \in \Omega; \exists u \neq 0$ satisfying $u_j = 0$ if $j \in M^0(x), \forall \delta > 0, x + \delta u \notin \Omega\}$, where $u \equiv (u_1, ..., u_m)'$.

That is, staring from a point on the PPF, there exists a direction along which any movement leaves the PPS. Since we do not assume free disposal of outputs, nor exclude positive production externalities, the PPF defined above may have positively sloping intervals. These positive sloping intervals, if any, disappear once allowing for free disposal.

2.3.1 Production Efficiency under Input-generated Externalities

If production externalities are input-generated, the answer to the question is simply NOT necessarily. To see this, consider a two-good one-factor economy with the factor endowment $E = 2$ and satisfying the following technology:

$$x_1 = \begin{cases} (1 - v_2) v_1 & \text{if } v_2 \leq 1, \\ 0 & \text{if } v_2 > 1, \end{cases}$$

$$x_2 = \begin{cases} (1 - v_1) v_2 & \text{if } v_1 \leq 1, \\ 0 & \text{if } v_1 > 1. \end{cases}$$

One way to derive the PPF is to solve the maximization problem $\max_{x_2} x_1$ subject to the technology and the factor constraint. For positive x_1 and x_2, the Lagrangian can be written as

$$\mathcal{L} = (1 - v_2) v_1 - p (x_2 - (1 - v_1) v_2) + \lambda (E - v_1 - v_2).$$

The first-order condition requires that $v_1 + v_2 = 1 < E = 2$: full employment does not hold on the part of the PPF where both outputs are positive. Given the same technology, however, if the factor endowment becomes $E = 1/2$, then full employment holds on the PPF.

The following is another example:

$$x_1 = \begin{cases} (1 - v_1 - v_2) v_1 & \text{if } v_1 + v_2 \leq 1, \\ 0 & \text{if } v_1 + v_2 > 1, \end{cases}$$

$$x_2 = \begin{cases} (1 - v_1 - v_2) v_2 & \text{if } v_1 + v_2 \leq 1, \\ 0 & \text{if } v_1 + v_2 > 1, \end{cases}$$

with the factor endowment $E = 1$. Clearly, full employment is not efficient since it leads to zero outputs. The two examples above suggest that

Remark 2.2. Given (A1)–(A4), full employment of factors does not necessarily hold on the PPF in the presence of input-generated production externalities (2.2).

2.3.2 Production Efficiency under Output-generated Externalities

In contrast, if production externalities are output-generated, the answer to the question above is YES. Formally,

Theorem 2.3. *Given (A1)–(A4), full employment of factors holds on the PPF in the presence of output-generated production externalities (2.3).*

Proof. Let $x^* = (x_1^*, ..., x_m^*)'$ denote a point on the PPF and $v_j^* = \left(v_{1j}^*, ..., v_{nj}^*\right)'$ the corresponding input in good j. Toward a contradiction, assume that $\exists k \in \{1, ..., n\}$, $\sum_{j\in M} v_{kj}^* < E_k$. Then we shall show that x^* is not on the PPF. According to Definition 2.1, this is equivalent to show that for any direction $u = (u_1, ..., u_m)' \neq 0$ satisfying $u_j = 0$ if $j \in M^0(x)$, there exist $\delta > 0$ and Δ such that $x^* + \delta u$ can be produced by using $v^* + \Delta$ and that $v^* + \Delta \leq E$ holds.

For this purpose, note that the j-th component of $v^* + \Delta$, $v_j^* + \Delta_j$, satisfies $x_j^* + \delta u_j = f_j\left(x_{-j}^* + \delta u_{-j}, v_j^* + \Delta_j\right)$, where u_{-j} is obtained from u by dropping u_j. If $x_j^* = 0$, namely $j \in M^0(x^*)$, $v_j^* = 0$ and $u_j = 0$. So, we can let $\Delta_j = 0$ and δ be any number. If $x_j^* > 0$, namely $j \in M \setminus M^0(x^*)$, the following two steps help to find δ and Δ_j.

Step 1: Subtract $x_j^* = f_j\left(x_{-j}^*, v_j^*\right)$ from $x_j^* + \delta u_j = f_j\left(x_{-j}^* + \delta u_{-j}, v_j^* + \Delta_j\right)$ and obtain

$$\delta u_j = f_j\left(x_{-j}^* + \delta u_{-j}, v_j^* + \Delta_j\right) - f_j\left(x_{-j}^*, v_j^*\right) = A_j\left(\delta u_{-j}, \Delta_j\right) + B_j\left(\delta u_{-j}\right),$$

where

$$\begin{aligned} A_j\left(\delta u_{-j}, \Delta_j\right) &\equiv f_j\left(x_{-j}^* + \delta u_{-j}, v_j^* + \Delta_j\right) - f_j\left(x_{-j}^* + \delta u_{-j}, v_j^*\right), \\ B_j\left(\delta u_{-j}\right) &\equiv f_j\left(x_{-j}^* + \delta u_{-j}, v_j^*\right) - f_j\left(x_{-j}^*, v_j^*\right). \end{aligned}$$

Note that v^* and x^* are dropped in $A_j(\cdot, \cdot)$ and $B_j(\cdot)$ to save notations. It follows that

$$A_j\left(\delta u_{-j}, \Delta_j\right) = \delta u_j - B_j\left(\delta u_{-j}\right) \tag{2.5}$$

Step 2: Consider two possible cases: (i) $v_{kj}^* > 0$ or $\delta u_j - B_j\left(\delta u_{-j}\right) \geq 0$; (ii) $v_{kj}^* = 0$ and $\delta u_j - B_j\left(\delta u_{-j}\right) < 0$. Let M_1 and M_2 denote the index sets that belong to case (i) and case (ii), respectively.

In case (i), let the candidate of Δ_j take the form of $t_j e_k$, where t_j is a scalar variable and e_k is the vector with the k-th component being one and others zero. That is, the new input vector $v_j + \Delta_j$ only adjusts the input of factor k in the original v_j. In doing so, (2.5) can be rewritten into

$$A_j\left(\delta u_{-j}, t_j e_k\right) = \delta u_j - B_j\left(\delta u_{-j}\right), \ j \in M_1. \tag{2.6}$$

By (A1), $A_j\left(\delta u_{-j}, t_j e_k\right)$ is a continuous function of t_j, and $\delta u_j - B_j\left(\delta u_{-j}\right)$ is a continuous function of δ. Thus $A_j\left(\delta u_{-j}, t_j e_k\right) \to 0$ as $t_j \to 0$, and $\delta u_j - B_j\left(\delta u_{-j}\right) \to 0$ as $\delta \to 0$. Choose δ small enough such that $x_j^* + \delta u_j > 0$ holds. So, by (A2), $A_j\left(\delta u_{-j}, t_j e_k\right)$ is a strictly increasing function of t_j. As a result, t_j and $\delta u_j - B_j\left(\delta u_{-j}\right)$ have the same sign according to (2.6), and $t_j \to 0$ as $\delta \to 0$. Therefore, the non-negative constraint $v_{kj}^* + t_j \geq 0$ holds if $\delta u_j - B_j\left(\delta u_{-j}\right) \geq 0$, or if $v_{kj}^* > 0$ and δ is small enough.

In case (ii), however, the input of factor k is already zero, and thus we cannot use less factor k due to the non-negative constraint. Note that $x_j^* > 0$, by (A3), $\exists k' \in \{1, ..., n\} \neq k$ such that $v_{k'j} > 0$. Let the candidate of Δ_j take the form of $t_j e_{k'}$, then

$$A_j\left(\delta u_{-j}, t_j e_{k'}\right) = \delta u_j - B_j\left(\delta u_{-j}\right), \ j \in M_2, k' \neq k \tag{2.7}$$

It follows from the continuity that $t_j \to 0$ as $\delta \to 0$, and from $\delta u_j - B_j\left(\delta u_{-j}\right) < 0$ that $t_j < 0$. Choose δ small enough such that $v_{k'j}^* + t_j > 0$.

Finally, check if the factor constraint remains not violated. As for factor k, the required amount is $\sum_{j\in M_1}\left(v_{kj}^* + t_j\right)$ (noting that $v_{kj}^* = t_j = 0$ if $j \notin M_1$). Since $\sum_{j\in M} v_{kj}^* < E_k$ and $t_j \to 0$ as $\delta \to 0$, $\sum_{j\in M_1}\left(v_{kj}^* + t_j\right) \leq E_k$ holds as we choose $\delta > 0$ small enough. As for other factors, the required amount is the same, or smaller in case (ii). That is, $v^* + \Delta \leq E$ holds as long as $\delta > 0$ is small enough.

Therefore, if $\sum_{j\in M} v_{kj}^* < E_k$, then for any direction $u \neq 0$ satisfying $u_j = 0$, $j \in M^0(x)$, we can find $\delta > 0$ and Δ such that the new output bundle $x^* + \delta u$ can be produced by $v^* + \Delta \leq E$, which means that x^* is not on the PPF. The contradiction implies that $\forall k \in \{1, ..., n\}$, $\sum_{j\in M} v_{kj}^* = E_k$. That is, all factors are fully employed on the PPF. □

It is noteworthy that although Theorem 2.3 is derived without assuming free disposal, it holds when free disposal is available. This is because, if a point on the PPF can be achieved by dropping some outputs, the input required is the same as another point on the PPF without dropping those outputs, to which the result applies.

2.4 Discussion

2.4.1 Nonequivalence of Input-generated and Output-generated

One may misunderstand the relationship between input-generated and output-generated production externalities in the following way: the outputs are produced from inputs, so output-generated production externalities are a special case of input-generated production externalities. This statement, however, misses the point that, in the presence of output-generated production externalities, the outputs are produced not only from inputs, but also "from" other outputs, which are further produced from other inputs and, again, "from" some other outputs, and so on. It is true that in some cases we can equivalently express output-generated production externalities in the input-generated form, as in the following example:

$$\begin{aligned} x_1 &= v_1, \\ x_2 &= \begin{cases} (1 - x_1)\, v_2 = (1 - v_1)\, v_2 & \text{if } v_1 \le 1, \\ 0 & \text{if } v_1 > 1, \end{cases} \end{aligned}$$

where the scalar variables v_1 and v_2 denote the inputs in goods 1 and 2, respectively. However, in many other cases, it is impossible to rewrite one formulation into another equivalently. Consider the following two-good one-factor production externalities:

$$\begin{aligned} x_1 &= f_1 (x_2, v_1), \\ x_2 &= f_2 (x_1, v_2). \end{aligned}$$

Substitute the second equation into the first for x_2 and obtain

$$x_1 = f_1 (f_2 (x_1, v_2), v_1).$$

In general, x_1 is not necessarily a function of (v_1, v_2), depending on functional forms of f_1 and f_2. The implicit function theorem suggests that, given that both f_1 and f_2 are differentiable, x_1 cannot be written as a function of (v_1, v_2) if there exists (x_1, v_1, v_2) such that

$$1 - \frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial x_1} = 0.$$

Another specific example is as follows:

$$\begin{aligned} x_1 &= \begin{cases} (1 - x_2)\, v_1 & \text{if } x_2 \le 1, \\ 0 & \text{if } x_2 > 1, \end{cases} \\ x_2 &= \begin{cases} (1 - x_1)\, v_2 & \text{if } x_1 \le 1, \\ 0 & \text{if } x_1 > 1, \end{cases} \end{aligned}$$

where v_1 and v_2 are also scalar variables. The factor endowment is $E = 4$. Letting $(v_1, v_2) = (2, 2)$, it is easy to check that three output bundles are feasible: $(x_1, x_2) = (2, 0)$, $(2/3, 2/3)$, and $(0, 2)$. Since an input bundle can produce only a unique output bundle in the presence of input-generated production externalities as in (2.2), the example above cannot be rewritten into the input-generated form equivalently.

2.4.2 When Full Employment is Inefficient?

The analysis above focuses on the yes-no question about the production efficiency of full employment. We may ask further when that happens, since full employment can be inefficient in the presence of input-generated production externalities? To answer this question, assume in what follows that

(A1′) f_j is continuously differentiable in all augments.

Then we can apply calculus to analyze the problem precisely.

To derive the PPF, solve the maximization problem $\max_{x_{-1}} x_1$ subject to constraints. The Lagrangian can be written as

$$\mathcal{L} = x_1 - \sum_{j=1}^{m} p_j \left(x_j - f_j\left(v_{-j}, v_j\right)\right) + \sum_{i=1}^{n} \lambda_i \left(E_i - \sum_{j=1}^{m} v_{ij}\right) + \sum_{i=1}^{n}\sum_{j=1}^{m} \mu_{ij} v_{ij},$$

where p_j, λ_i, μ_{ij} are the Lagrange multipliers corresponding to the technology of producing good j, the constraint of factor i and the non-negativity of inputs. The first-order condition requires

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial x_1} = 1 - p_1 = 0,\\
&\frac{\partial \mathcal{L}}{\partial v_{lk}} = \sum_{j=2}^{m} p_j \frac{\partial f_j}{\partial v_{lk}} - \lambda_l + \mu_{lk} = 0,\ \ l = 1, \ldots, n,\ \ k = 1, \ldots, m,\\
&\lambda_l \geq 0,\ \ \lambda_l \left(E_l - \sum_{j=1}^{m} v_{ij}\right) = 0,\ \ l = 1, \ldots, n,\\
&\mu_{lk} \geq 0,\ \ \mu_{lk} v_{lk} = 0,\ \ l = 1, \ldots, n,\ \ k = 1, \ldots, m.
\end{aligned}$$

Assume that factor l' is not fully used on the PPF, namely that $E_{l'} > \sum_{j=1}^{m} v_{l'j}$. From the first-order condition, $\lambda_{l'} = 0$ and

$$\sum_{j=1}^{m} p_j \frac{\partial f_j}{\partial v_{l'k}} = -\mu_{l'k} \leq 0,\ \ k = 1, \ldots, m. \tag{2.8}$$

Define

$$G(x) \equiv \sum_{j=1}^{m} p_j x_j.$$

Note that p_j is the shadow price of good j and satisfies $p_1 = 1$, so G can be interpreted as the GDP measured by good 1. The condition (2.8) can be rewritten into

$$\frac{dG}{dv_{l'k}} = -\mu_{l'k} \leq 0,\ \ k = 1, \ldots, m.$$

Therefore,

Remark 2.4. A necessary condition for a factor not to be fully used on the PPF is that the factor's marginal effect (through any production process) on the GDP is zero or negative.

2.5 Conclusion

Focusing on two general and wildly applied formulations of production externalities: input-generated and output-generated, I show that full employment can be inefficient if production externalities are input-generated, but is a necessary condition for production efficiency if production externalities are output-generated. The result has a significant policy implication. If production externalities are input-generated, we shall be careful about whether there are too many factors and, if this is the case, the use of and investment on factors should be regulated. In contrast, if production externalities are output-generated, more factors is better and thus the policy should promote full employment.

Chapter 3

The Production Possibility Frontier under Strong Input-generated Production Externalities[1]

3.1 Introduction

The strong input-generated production externalities denotes the situation that input-generated production externalities are strong such that full employment of factors becomes inefficient. This seemingly special situation is very common in real life. For example, although we have 24 hours a day, working for 24 hours without rest is not wise. Traffic jams is another example, which is caused by too many cars on the road.

In the presence of strong input-generated production externalities, factor use along the PPF becomes endogenously determined. The purpose of this study is to carefully examine the properties of the production possibility frontier (PPF) including monotonicity, continuity, and convexity in such situation. I focus on the single-factor case to highlight the effect of strong input-generated externalities on the PPF. I show that a sufficient condition for the PPF to be (strictly) convex is the (strict) quasi-concavity of the by-product generation function. I also show that the PPF is either entirely strictly convex or entirely linear under reasonable conditions.

In the literature, the analysis of the PPF under production externalities usually focuses on output-generated externalities (e.g., Herberg and Kemp, 1969; Herberg et al., 1982; Dalal, 2006). However, as shown previously, full employment holds on the PPF in the case of output-generated production externalities, thus leaving no space for the focus of this study: how the trade-off between factor use and productivity affects the PPF. Another closely related literature is on public intermediate goods (e.g., Manning

[1]This chapter were presented at Hitotsubashi University, Summer Workshop on Economic Theory 2013.

and McMillan, 1979; Tawada and Abe, 1984). The model in this study embraces the "constant returns to scale" case in Manning and McMillan (1979) as a special case.[2] Moreover, although the model is static in the sense that time is not involved in the specification, it can be regarded as the steady-state version of a dynamic model. Thus, the properties derived from the model hold for the steady-state PPF in corresponding dynamic models.

Other results are summarized as follows. Proposition 3.6 employs calculus and is a precise version of Proposition 3.5 on the convexity of the PPF. Proposition 3.8 shows how the output of by-product changes when moving along the PPF. Corollary 3.7 and 3.9 are, respectively, the application of Proposition 3.6 and 3.8 into a special by-product generation function. Proposition 3.13 provides the sufficient condition for the set of factor use on the PPF to be convex-valued and, even stronger, single-valued. Proposition 3.14 tells when the difference in the sensitivity (to by-product) remains the same sign along the PPF. In a special case that factor use yields the same amount of by-product, Proposition 3.16 considers the change of the aggregate factor use when moving along the PPF. In a special two-factor case, Proposition 3.17 demonstrates that if the difference in factor intensities is small, the PPF tends to be convex given the quasi-concave by-product generation function.

The rest of this chapter is organized as follows. Section 3.2 gives the basic model and assumptions. Section 3.3 examines three basic properties of the PPF without using calculus. Section 3.4 obtains some calculus-based properties. Section 3.5 considers two extensions. The last section concludes.

3.2 Formulation of Strong Input-generated Externalities

There is a single factor of production (v), two final goods (x and y), and a by-product (z). The technology satisfies

$$x = G^x(z)\, v_x, \tag{3.1a}$$

$$y = G^y(z)\, v_y, \tag{3.1b}$$

$$z = R\left(v_x, v_y\right), \tag{3.1c}$$

where $v_i \geq 0$ ($i = x, y$) denotes the use of factor in good i, $R(\cdot,\cdot) \geq 0$ is the generation function of by-product, $G^i(\cdot) \geq 0$ is the productivity function representing the

[2] Tawada and Abe (1984) analyze a two-factor model of pure public intermediate goods and focus on the special case that industries have identical sensitivity to by-product. They find that the PPF is necessarily concave. Abe et al. (1986) obtain the same result while allowing non-separability of the production function and any number of factors. This study has very different focus from theirs and attempts to highlight the effect on the PPF of the difference in the sensitivity to by-product.

product-specific relationship between the productivity and the amount of by-product. It can be shown that[3]

Remark 3.1. Manning and McMillan's (1979) "constant returns to scale" (or pure public intermediate good) case is a special case of (3.1).

Following Dalal (2006), the PPF is defined by the maximum value function[4]

$$y = T(x) \equiv \max_{C(x)} G^y(z)\, v_y, \tag{3.2}$$

where $C(x)$ denotes the constraint set

$$C(x) \equiv \left\{ \left(z, v_x, v_y\right); G^x(z)\, v_x \geq x, z = R\left(v_x, v_y\right), v_x + v_y \leq E \right\}. \tag{3.3}$$

The inequality $G^x(z)\, v_x \geq x$ means that free disposal is available, E is the factor endowment. Let $S(x)$ denote the solution set

$$S(x) = \arg\max_{C(x)} G^y(z)\, v_y.$$

To exclude trivial cases, assume that the feasible maximum outputs of x and y, denoted $\bar{x}$ and $\bar{y}$, are positive. By the definition of $T(x)$, we have $T(0) = \bar{y}$ and $T(\bar{x}) = 0$.

The analysis proceeds by assuming that

(A1) R and G^i are continuous in all augments;

(A2) R is strictly increasing in all arguments;

(A3) $v_x + v_y \leq E$ is slack on the PPF.

Assumption (A2) implies that for each z, there is a bijective mapping between v_x and v_y. It also follows that

Lemma 3.2. *Given (A1) and (A2), $G^x(z)\, v_x = x$ holds on the PPF.*

Proof. Let $(x', T(x'))$ denote a point on the PPF and $\left(z', v'_x, v'_y\right)$ denote the corresponding factor use and by-product output. Assume to the contrary that $G^x(z')\, v'_x > x'$. Then it is possible to find $v''_x < v'_x$ and $v''_y > v'_y$ so that $R\left(v''_x, v''_y\right) = z'$ and $G^x(z')\, v''_x \geq x'$ hold. Hence (x', y'') is feasible, where $y'' = G^y(z')\, v''_y$. Note that $y'' > G^y(z')\, v'_y = T(x')$, which leads to a contradiction to the definition of $T(x')$. □

[3] To see this, introduce a constant L, two variables L_r and r, and two functions $f_r(\cdot)$ and $A^i(\cdot)$. Let $L_r = L - z$, $r = f_r(L_r)$, $A^i(f_r(L - z)) = G^i(z)$, then $G^i(z) = A^i(f_r(L_r)) = A^i(r)$. Moreover, let $R(v_x, v_y) = v_x + v_y$, then $L_r = L - z = L - v_x - v_y$. Now, (3.1) can be rewritten into $x = A^x(r)\, v_x$, $y = A^y(r)\, v_y$, $r = f_r(L_r)$ and $L = v_x + v_y + L_r$, which is exactly the "constant returns to scale" case in Manning and McMillan (1979).

[4] The way of defining the PPF as in (3.2) has a limitation. That is, if there is non-bijective mapping between x and y on the frontier, then (3.2) describes only the upper locus of the PPF (in terms of y). This limitation will not present a problem here since Proposition 3.3 shows that (3.2) is strictly decreasing over its domain. This means that, at most, some vertical lines are degenerated to discontinuous jump points.

Assumption (A3) is imposed so as to focus on strong input-generated production externalities. If (A3) does not hold, the shape of the PPF depends on the specific forms of R and G^i. This can be seen in the following examples.

Example 1: $x = \left(101 - v_x - v_y\right) v_x$, $y = \left(101 - v_x - v_y\right) v_y^2$.

Example 2: $x = \left(101 - v_x - v_y\right) v_x^{0.5}$, $y = \left(101 - v_x - v_y\right) v_y$.

Example 3: $x = \left(101 - v_x - v_y\right) v_x^{0.5}$, $y = \left(101 - v_x - v_y\right) v_y^2$.

Example 4: $x = \left(1.01 - v_x - v_y\right) v_x$, $y = \left(1.01 - v_x - v_y\right) v_y^2$.

Example 5: $x = \left(1.01 - v_x - v_y\right) v_x^{0.5}$, $y = \left(1.01 - v_x - v_y\right) v_y$.

Example 6: $x = \left(1.01 - v_x - v_y\right) v_x^{0.5}$, $y = \left(1.01 - v_x - v_y\right) v_y^2$.

The factor endowment is $E = 1$. Clearly, from Example 1 to Example 3, full employment holds on the PPF. The PPF derived from Example 1 is

$$y = \left(10 - \frac{x}{10}\right)^2,$$

which is convex to the origin. The PPF derived from Example 2 is

$$y = 100 - \frac{x^2}{100},$$

which is concave to the origin. The PPF derived from Example 3 is

$$y = \left(10 - \frac{x^2}{1000}\right)^2,$$

which is convex when close to x axis and concave when close to y axis.

In contrast, full employment is inefficient from Example 4 to Example 6. From the proposition in the subsequent section, the PPFs derived from these examples are convex to the origin. But if one insists on full employment, the resulting "PPFs" (precisely, transformation loci) have similar curvatures to those derived from Example 1 to Example 3.

Assumption (A3) is worthy of further explanation. First, it implicitly excludes those by-products with nonnegative effect ($dG^i(z)/dz \geq 0$ for all $z \geq 0$) such as knowledge spillover.[5] Although growth theorists might not like this, it is plausible when one is concerned with the context in which agents, such as workers, firms, regions, or countries, face some underlying constraints, such as health condition, public infrastructure, resource stock, or environmental quality.

3.3 Monotonicity, Continuity and Convexity

In what follows I establish monotonicity, continuity and convexity of $T(x)$. Define

$$\tilde{x} \equiv \inf\{x; T(x) = 0\}. \tag{3.4}$$

Thus, $\tilde{x} \in [0, \bar{x}]$. We can show that

[5]Note that (A3) does not exclude positive externalities in some ranges.

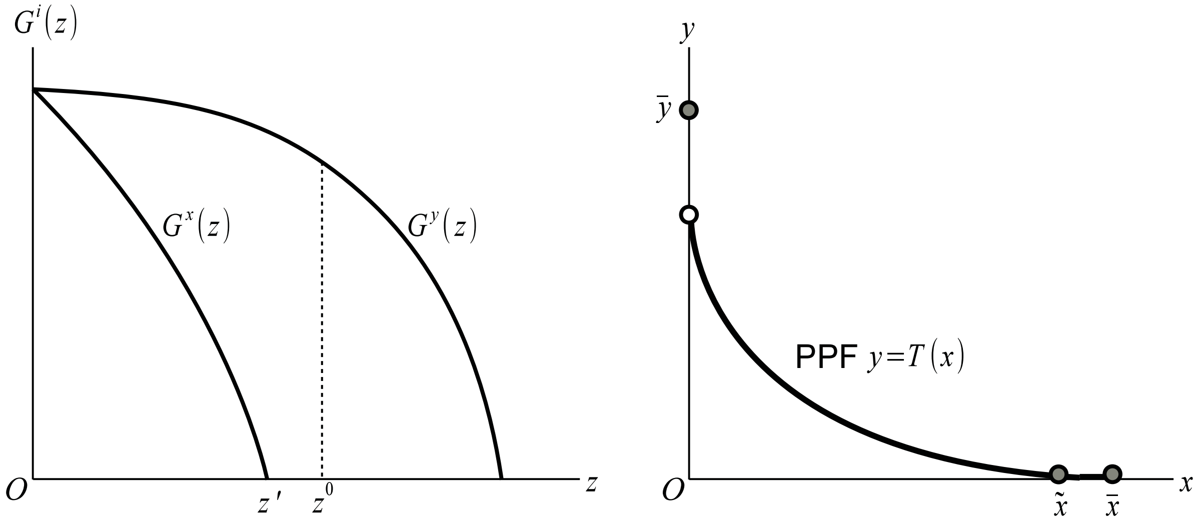

Figure 3.1: Jump discontinuity on the PPF

Proposition 3.3 (Monotonicity). *Given (A1) and (A2), $T(x)$ is strictly decreasing over $[0, \tilde{x}]$, and satisfies $T(x) = 0$ over $(\tilde{x}, \bar{x}]$.*

Proof. See Appendix 3.A.1. □

Does $T(\tilde{x}) = 0$ hold? This depends on the continuity of $T(x)$ at $x = \tilde{x}$, which is established in the following proposition. For convenience, let $Z(x)$ denote the set of corresponding by-product outputs on the PPF.

Proposition 3.4 (Continuity). *Given (A1) and (A2), $T(x)$ is continuous over $(0, \bar{x}]$. Moreover, $T(x)$ is continuous at $x = 0$ if and only if*

$$\forall \sigma > 0, \exists z^0 \in Z(0) \text{ and } z' \in \left(z^0 - \sigma, z^0 + \sigma\right) \text{ so that } G^x\left(z'\right) > 0.$$

Proof. See Appendix 3.A.2 for the proof of continuity over $(0, \bar{x}]$ and Appendix 3.A.3 for the condition for continuity at $x = 0$. □

As is well known, the PPF is discontinuous at the corner in the presence of fixed cost. This model suggests another channel for the discontinuity on the PPF. Since discontinuity, if any, arises only at $x = 0$, $T(\tilde{x}) = 0$ if $\tilde{x} > 0$. Figure 3.1 gives an example of discontinuous PPF. In the figure, y is assumed to reach the maximum $\bar{y}$ at $z = z^0$. As shown in the left diagram, $z' < z^0$ and $G^x(z) = 0$ for $z \geq z'$. As a result, as shown in the right diagram, the PPF is discontinuous at $x = 0$. Note that $G^x\left(z^0\right) = 0$ for all $z^0 \in Z(0)$ does not mean that $T(x)$ is discontinuous at $x = 0$. As long as there exists $z^0 \in Z(0)$ so that $G^x(z) > 0$ in an arbitrarily small neighborhood of z^0, $T(x)$ is continuous at $x = 0$. For example, $T(x)$ is continuous at $x = 0$ if $G^x\left(z^0\right) = 0$ and if $G^x(z) > 0$ for $z < z^0$.

The following proposition is about the convexity of the PPF.

Proposition 3.5 (Convexity). *Given (A1), (A2) and (A3), if $R(\cdot,\cdot)$ is quasi-concave, $T(x)$ is convex over $(0,\bar{x}]$. If $R(\cdot,\cdot)$ is strictly quasi-concave, $T(x)$ is strictly convex over $(0,\tilde{x})$.*

Proof. According to Lemma 3.2, the PPF can be equivalently defined by, instead of (3.2),

$$y = T(x) \equiv \max_z Y(z,x),$$

where $Y(z,x)$ is the output of good y given z and x, that is, $Y(z,x) = G^y(z)\, v_y$, $x = G^x(z)\, v_x$. There are two cases: $G^y(z) > 0$ and $G^y(z) = 0$. If $G^y(z) > 0$, we have $z = R(x/G^x(z), Y(z,x)/G^y(z))$. Given any z, changing x in $(x/G^x(z), Y(z,x)/G^y(z))$ gives a locus convex to the origin, according to (A2) and the quasi-concavity of $R(\cdot,\cdot)$. This simply means that $Y(z,x)$ is a convex function of x. If $G^y(z) = 0$, $Y(z,x) = 0$. In both cases, $Y(z,x)$ is a convex function of x. Since $T(x)$ is the upper envelope of $Y(z,x)$ by changing z and since assumption (A3) ensures that all output bundles on this upper envelope are feasible, $T(x)$ is necessarily convex over $(0,\bar{x}]$.

If $R(\cdot,\cdot)$ is strictly quasi-concave, following similar arguments we show that $Y(z,x)$ is a strictly convex function of x for any z satisfying $G^y(z) > 0$. Let Ω denote the set of z's such that $Y(z,x)$ contributes to the upper envelope. According to Proposition 3.3, $T(x) > 0$ for any $x \in (0,\tilde{x})$, which implies $G^y(z) > 0$ for any $z \in \Omega$. Since the upper envelope of $Y(z,x)$ by changing z within Ω also gives $T(x)$, it is necessarily strictly convex over $(0,\tilde{x})$. □

Proposition 3.5 provides a clear-cut result about the curvature of the PPF in the single-factor case. The intuition of the proof is straightforward. If we fix z at certain level, feasible output bundles will lie on a locus convex to the origin due to the quasi-concavity of $R(\cdot,\cdot)$. Fix z at another level, we can obtain another convex curve. Repeating this yields a family of loci convex to the origin. As shown in Figure 3.2, in which a linear $R(\cdot,\cdot)$ is assumed, five line segments are generated by changing z from z_1 to z_5. The PPF is the upper envelope of these convex loci and thus is necessarily convex.[6]

3.4 Calculus-based Properties

To derive richer results by exploiting calculus, replace assumption (A1) with

(A1′) $R(\cdot,\cdot)$ and $G^i(\cdot)$ are of class C^2.

[6]Some values of z may generate loci not contributing to the PPF, such as z_4 and z_5 in the figure.

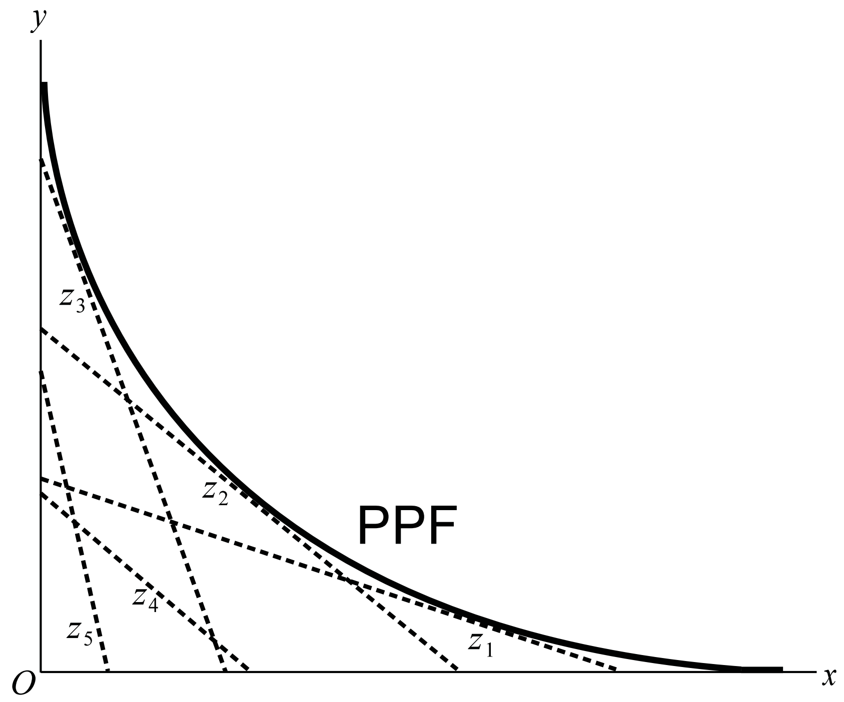

Figure 3.2: PPF as the upper envelope

It is convenient to define the *sensitivity* to by-product as the elasticity of a good's productivity with respect to the level of by-product:[7]

$$\varepsilon_i \equiv -\frac{d\ln G^i}{d\ln z}, i = x, y.$$

In the following we shall focus on the interval $(0, \tilde{x})$ for two reasons. First, $T(x) = 0$ over $[\tilde{x}, \bar{x}]$, which is of no special interest. Second, $x \in (0, \tilde{x})$ ensures the positive outputs and thus the positive optimal factor uses. This simplifies the first-order necessary conditions from Kuhn–Tucker type to a system of equations.

From Lemma 3.2 and (A3), the original problem (3.2) is equivalent to

$$T(x) \equiv \max G^y\left(R\left(v_x, v_y\right)\right) v_y,$$

subject to $G^x\left(R\left(v_x, v_y\right)\right) v_x = x$. The difference lies in that $R\left(v_x, v_y\right)$ is now substituted into $G^i(z)$ for z. The Lagrangian can be written as

$$\mathcal{L} = G^y\left(R\left(v_x, v_y\right)\right) v_y + p\left(G^x\left(R\left(v_x, v_y\right)\right) v_x - x\right),$$

where p is the Lagrange multiplier representing the shadow price of x in terms of y. For any $x \in (0, \tilde{x})$, the first-order condition requires

$$\frac{\partial \mathcal{L}}{\partial v_x} = G^{y\prime} R_x v_y + p\left(G^{x\prime} R_x v_x + G^x\right) = 0, \tag{3.5a}$$

$$\frac{\partial \mathcal{L}}{\partial v_y} = G^{y\prime} R_y v_y + G^y + p G^{x\prime} R_y v_x = 0, \tag{3.5b}$$

[7]When no confusion arises, hereafter let G^i, $G^{i\prime}$ and $G^{i\prime\prime}$ denote respectively $G^i(z)$, $dG^i(z)/dz$ and $d^2G^i(z)/dz^2$.

where R_x and R_y denotes respectively $\partial R/\partial v_x$ and $\partial R/\partial v_y$. From the first-order condition, we have

$$p = \frac{G^y R_x}{G^x R_y} > 0, \tag{3.6a}$$

$$1 = \varepsilon_x \frac{R_x v_x}{z} + \varepsilon_y \frac{R_y v_y}{z}. \tag{3.6b}$$

Define $w \equiv pG^x/R_x > 0$, then the first-order condition requires that $w = G^y/R_y = -\left(pG^{x\prime}v_x + G^{y\prime}v_y\right)$.[8] Let H denote the Hessian matrix of $\mathcal{L}$, i.e.

$$H \equiv \frac{\partial^2 \mathcal{L}}{\partial\left(p, v_x, v_y\right)^2}.$$

The second-order necessary condition requires that $|H| \geq 0$. However, if $|H| = 0$, then there is a kink on $T(x)$ and thus $T''(x)$ is not well-defined. To exploit calculus, we shall focus on the case of $|H| > 0$.

3.4.1 Local Properties

The following proposition provides a precise version of Proposition 3.5.

Proposition 3.6. *Given (A1′), (A2) and (A3), then*

$$T''(x) = \underbrace{\frac{wR_x}{G^{x2}}}_{>0} Q + \underbrace{\frac{\left(wR_xR_y\right)^2}{z^2|H|}}_{>0} D^2, x \in (0, \tilde{x}) \tag{3.7}$$

where

$$Q \equiv 2\frac{R_{xy}}{R_xR_y} - \frac{R_{xx}}{R_x^2} - \frac{R_{yy}}{R_y^2},$$

$$D \equiv \varepsilon_y\left(1 - R_yv_yq_2\right) - \varepsilon_x\left(1 - R_xv_xq_1\right),$$

$$q_1 \equiv \frac{R_{xy}}{R_xR_y} - \frac{R_{xx}}{R_x^2}, q_2 \equiv \frac{R_{xy}}{R_xR_y} - \frac{R_{yy}}{R_y^2}.$$

Proof. See Appendix 3.A.4. □

Note that Q is related to the bordered Hessian matrix of $R(\cdot,\cdot)$:

$$Q = \frac{1}{\left(R_xR_y\right)^2}\begin{vmatrix} 0 & R_x & R_y \\ R_x & R_{xx} & R_{xy} \\ R_y & R_{xy} & R_{yy} \end{vmatrix}.$$

[8]The intuition is as follows. On one hand, a unit of increase in the by-product implies $1/R_x$ ($1/R_y$) units of increase in factor use in good x (good y) if the factor use in the other good remains unchanged. This gives rise to a return of pG^x/R_x (G^y/R_y) if the productivity is the same. On the other hand, a marginal increase in the by-product affects the productivity and leads to a marginal loss of $pG^{x\prime}v_x + G^{y\prime}v_y$, which is a negative number. On the PPF, the marginal returns in both goods must be equalized, and must be equal to the absolute value of the marginal loss.

Hence, if $R(\cdot,\cdot)$ is quasi-concave, then $Q \geq 0$. Furthermore, along with (A2), if $R(\cdot,\cdot)$ is strictly quasi-concave, then we have $Q > 0$. On the other hand, if $R(\cdot,\cdot)$ is quasi-convex, then $Q \leq 0$ and the PPF is not necessarily entirely concave or convex. The curvature at each point on the PPF depends on the relative magnitude of each term in (3.7) at that point.

Moreover, if $R(\cdot,\cdot)$ takes the special form

$$R\left(v_x, v_y\right) = I\left(r_1 v_x + r_2 v_y\right), \tag{3.8}$$

where $I' > 0$ and $r_1, r_2 > 0$ are constants, then $Q = q_1 = q_2 = 0$ and $D = \left(\varepsilon_y - \varepsilon_x\right)$. Therefore,

Corollary 3.7. *Given (A1′), (A2), (A3) and by-product generation function (3.8), the PPF, $T(x)$, is strictly convex at a point $x \in (0, \tilde{x})$ if and only if $\varepsilon_x \neq \varepsilon_y$ there.*

Proof. The result directly follows that given (3.8), (3.7) becomes

$$T''(x) = \underbrace{\frac{\left(wr_1r_2\right)^2 I'^4}{z^2 |H|}}_{>0} \left(\varepsilon_y - \varepsilon_x\right)^2. \tag{3.9}$$

□

This corollary highlights how the difference in the sensitivity between two goods renders the PPF convex. It is also interesting to see how the output of by-product changes along the PPF.

Proposition 3.8. *Given assumption (A1′), (A2) and (A3), then along the PPF*

$$\frac{dz}{dx} = \underbrace{\frac{G^x G^y R_x R_y}{z|H|}}_{>0} D, x \in (0, \tilde{x}) \tag{3.10}$$

where D is defined as in Proposition 3.6.

Proof. See Appendix 3.A.5. □

The sign of dz/dx depends on that of D, and thus is ambiguous without further information on the specific forms of $R(\cdot,\cdot)$, $G^i(\cdot)$, and the value of x. If $R(\cdot,\cdot)$ is linearly homogeneous, it can be shown that

$$D = \left(\varepsilon_y - \varepsilon_x\right)\left(1 + \frac{R_{yy} v_y z}{R_x R_y v_x}\right).$$

Thus, the sign of dz/dx crucially depends on $\left(\varepsilon_y - \varepsilon_x\right)$ and the curvature of $R(\cdot,\cdot)$. Moreover, if $R\left(v_x, v_y\right)$ takes the form (3.8), we have

Corollary 3.9. *Given assumption (A1′), (A2), (A3) and by-product generation function (3.8), the sign of dz/dx on the PPF for $x \in (0, \tilde{x})$ is determined by the sign of $(\varepsilon_y - \varepsilon_x)$ there.*

Proof. If follows (3.8) and (3.10) that

$$\frac{dz}{dx} = \underbrace{\frac{G^x G^y r_1 r_2 I'^2}{z\,|H|}}_{>0} \left(\varepsilon_y - \varepsilon_x\right). \tag{3.11}$$

□

Corollary 3.7 and 3.9 are similar with Manning and McMillan's (1979) Proposition 5 and 6. Note that $R\left(v_x, v_y\right) = v_x + v_y$ in Remark 3.1 is a special case of $R\left(v_x, v_y\right) = I\left(r_1 v_x + r_2 v_y\right)$ in Corollary 3.7 and 3.9. The two corollaries are stronger results than their propositions.

3.4.2 Global Properties

So far, we have obtained several local properties of the PPF. In what follows, we shall prove two important results on the global properties of the PPF. The first is to find under what conditions the set of $\left(z, v_x, v_y\right)$ for a point on the PPF, namely the solution set $S(x)$, is convex-valued or single-valued. For this purpose, we focus on the following situation that

(A4) $\quad G^i(\cdot)$ $(i = x, y)$ is quasi-concave.

The role of (A4) is to ensure the convexity of the domain of relevant functions. That is,

Lemma 3.10. *Given (A4), $\left\{z; G^i(z) > 0\right\}$ and $\left\{(z, v_i); G^i(z)\, v_i > 0\right\}$ are open convex sets.*

Proof. The quasi-concavity $G^i(z)$ means that $\left\{z; G^i(z) > 0\right\}$ is an open connected interval, which is convex. As for $G^i(z)\, v_i$, note that $G^i(z)\, v_i > 0$ is equivalent to $G^i(z) > 0$ and $v_i > 0$ since $G^i(z) \geq 0$, i.e., $\left\{(z, v_i); G^i(z)\, v_i > 0\right\} = \left\{(z, v_i); G^i(z) > 0\right\} \cap \left\{(z, v_i); v_i > 0\right\}$, which is clearly an open convex set if $G^i(z)$ is quasi-concave. □

The following two lemmas are also useful.

Lemma 3.11. *Given (A1′), if $1/G^i(\cdot)$ is (strictly) convex, then $G^i(z)\, v_i$ is (strictly) pseudo-concave with respect to (z, v_i).*

Proof. See Appendix 3.A.6. □

Lemma 3.12. *If $G^i(z)\, v_i$ $(i = x, y)$ is (strictly) pseudo-concave with respect to (z, v_i) and if $R(\cdot, \cdot)$ is convex, then $G^i\left(R\left(v_x, v_y\right)\right) v_i$ is (strictly) pseudo-concave with respect to $\left(v_x, v_y\right)$.*

Proof. See Appendix 3.A.7. □

Using these lemmas, it can be shown that

Proposition 3.13. *Given (A1′), (A2), (A3), and (A4), the solution set* $S(x)$ *is convex-valued for any* $x \in (0, \tilde{x})$ *if* $1/G^i(\cdot)$ *(*$i = x, y$*) and* $R(\cdot, \cdot)$ *are convex. Moreover, if either* $1/G^x(\cdot)$ *or* $1/G^y(\cdot)$ *is strictly convex, then* $S(x)$ *is single-valued and can be represented by a* C^1 *vector function* $S(x) = (z(x), v_x(x), v_y(x))$.

Proof. See Appendix 3.A.8. □

Note that the condition of convex $1/G^i(\cdot)$ is not as strong as it seems to be. It is easy to check that the (strict) concavity of $G^i(\cdot)$ is sufficient for the (strict) convexity $1/G^i(\cdot)$. In Manning and McMillan (1979), applying their assumptions (11) and (12) to the constant-returns-to-scale form (9) implies that $A^i(f_r(L_r))$ is a concave function of L_r in their model. Thus, according to Remark 3.1, they actually assume concave $G^i(\cdot)$.

As shown in Corollary 3.7 and 3.9, the sign of $(\varepsilon_x - \varepsilon_y)$ determines the curvature of the PPF when the by-product generation function $R(\cdot, \cdot)$ is linearly homogeneous or takes the form (3.8). Our second result about the global property of the PPF is to answer whether the sign of $(\varepsilon_x - \varepsilon_y)$ can change along the PPF given linearly homogeneous $R(\cdot, \cdot)$. Before giving the result, define the set of the sensitivity bundles on the PPF over $(0, \tilde{x})$ by $\Psi \equiv \{(\varepsilon_x, \varepsilon_y); \varepsilon_x \in \varepsilon_x(x), \varepsilon_y \in \varepsilon_y(x), x \in (0, \tilde{x})\}$.

Proposition 3.14. *Given (A1′), (A2), (A3), and (A4), the sign of* $(\varepsilon_x - \varepsilon_y)$ *remains the same when moving along the PPF over* $(0, \tilde{x})$ *if* $1/G^i(\cdot)$ *(*$i = x, y$*) is convex and* $R(\cdot, \cdot)$ *is linearly homogeneous and quasi-convex.*

Proof. See Appendix 3.A.9. □

Figure 3.3 depicts a possible path of $(\varepsilon_x, \varepsilon_y)$ that corresponds with moving along the PPF. All points within Region I satisfy $\varepsilon_y > 1 > \varepsilon_x$, all point within Region II satisfy $\varepsilon_y < 1 < \varepsilon_x$, and the point $(1, 1)$ corresponds with $\varepsilon_x = \varepsilon_y = 1$. The first-order necessary condition and the linear homogeneity of $R(\cdot, \cdot)$ together imply that, on the PPF over $(0, \tilde{x})$, z takes the value such that $(\varepsilon_x, \varepsilon_y)$ lies within either Region I, II or one point $(1, 1)$. Proposition 3.14 says that, given the above mentioned conditions, the path of $(\varepsilon_x, \varepsilon_y)$ must either always be within Region I, or Region II, or on point $(1, 1)$. Figure 3.3 demonstrates only the case in which Ψ is located within Region I. The intuition for Proposition 3.14 is that, if Ψ is connected, the path must pass through point $(1, 1)$ when the sign of $(\varepsilon_x - \varepsilon_y)$ can change. However, given the conditions, the value of z satisfying $\varepsilon_x = \varepsilon_y = 1$ is the optimal value for all $x \in (0, \tilde{x})$. Therefore, it is impossible for some points on the PPF to satisfy $(\varepsilon_x, \varepsilon_y) = (1, 1)$, while other points satisfy $(\varepsilon_x, \varepsilon_y) \neq (1, 1)$.

From Proposition 3.14, it follows directly that

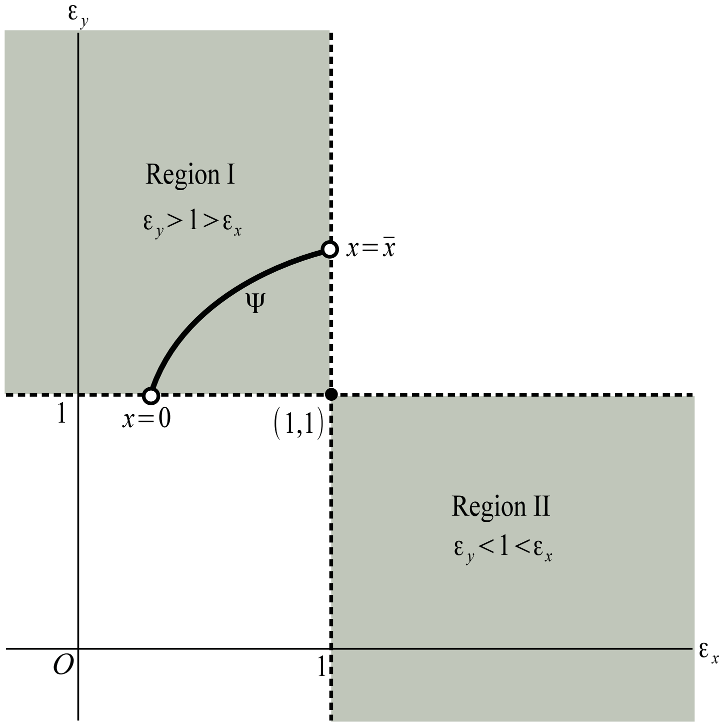

Figure 3.3: A path of $(\varepsilon_x, \varepsilon_y)$ on the PPF

Corollary 3.15. *Given (A1′), (A2), (A3), and (A4), the PPF is either entirely strictly convex or entirely linear if* $1/G^i(\cdot)$ *(*$i = x, y$*) is convex and* $R(\cdot,\cdot) = r_1 v_x + r_2 v_y$.

Manning and McMillan's (1979) Proposition 6 implies that the strictly convex and linear intervals may coexist on a PPF. Our Corollary 3.15 excludes this possibility.

3.5 Extension

This section discusses two extensions of the basic model. The first extension focuses on a special form of $R(\cdot,\cdot)$ to see what happens when assumption (A3) is relaxed. The second extension analyzes a special two-factor case of identical factor intensity between two goods.

3.5.1 Total Factor Use: A Special Case

So far, we do not consider how the total factor use $v \equiv v_x + v_y$ changes along the PPF since it depends on the specific forms of $R(\cdot,\cdot)$ and $G^i(\cdot)$. The expression of dv/dx looks even more cumbersome than that of dz/dx. However, if $R(\cdot,\cdot)$ takes the special form $R(v_x, v_y) = v_x + v_y$, then $z = v$ and consequently a simple expression for dv/dx directly follows (3.11):

$$\frac{dv}{dx} = \underbrace{\frac{G^x G^y}{v\,|H|}}_{>0}\left(\varepsilon_y - \varepsilon_x\right).$$

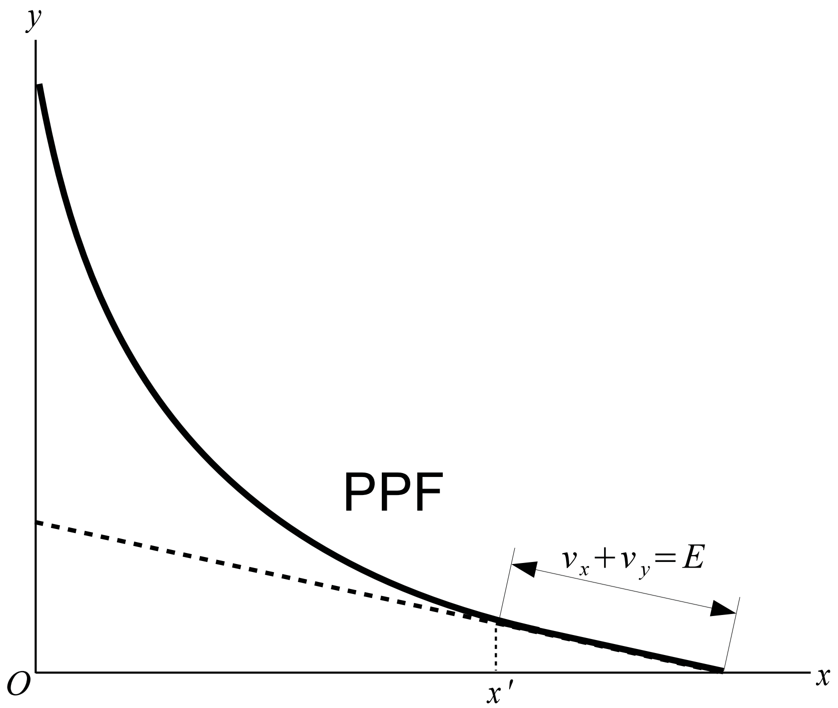

Figure 3.4: The PPF with $z = v_x + v_y$ and binding factor constraint

It can be seen clearly, in this special case, what happens when the factor constraint is binding on the PPF.

Proposition 3.16. *Given (A1′), (A2), and (A4), the part of the PPF corresponding to the binding constraint $v_x + v_y = E$ (if any) is a straight line segment and must be located on the end of the PPF if $1/G^i(\cdot)$ $(i = x, y)$ is convex and $R(v_x, v_y) = v_x + v_y$.*

Proof. According to Proposition 3.14 and Corollary 3.9, as long as $v_x + v_y < E$, the value of z either increases uniformly, decrease uniformly, or remain unchanged when moving along the PPF over $(0, \tilde{x})$. This is also true for the aggregate use of factor $v_x + v_y$ since $z = R(v_x, v_y) = v_x + v_y$. Suppose $\exists x' \in (0, \tilde{x})$ such that $z = v_x + v_y = E$. There are three cases depending on the sign of $\partial z/\partial x$, namely the sign of $\varepsilon_y - \varepsilon_x$, at x'. If $\partial z/\partial x > 0$ holds at x', then $z = v_x + v_y < E$ must hold over $(0, x')$, while $z = v_x + v_y = E$ holds over $(x', \tilde{x})$. If $\partial z/\partial x < 0$ holds at x', then $z = v_x + v_y < E$ must hold over $(x', \tilde{x})$, while $z = v_x + v_y = E$ holds over $(0, x')$. If $\partial z/\partial x = 0$ at x', $z = v_x + v_y = E$ must hold over $(0, \tilde{x})$. Since $z = E$ is a constant when the factor constraint binds, from (3.1a), the corresponding part of the PPF is necessarily linear. □

Figure 3.4 demonstrates the case in which $\partial z/\partial x > 0$ holds at x'.

3.5.2 Two Factors: A Special Case

What happens if there are more than one factors? In general, the answer is ambiguous, even given the special form of by-product generation function (3.8). This is because, compared to the difference in the sensitivity (to by-product) that renders the PPF more convex, the difference in the factor intensity between two goods works in just the opposite way driving the PPF to be more concave. Hence, the curvature at any point on the PPF depends on which force dominates there.

Here, instead of focusing on the general multi-factor case to derive the detailed condition for the PPF to be convex or concave, we examine a special two-factor model in which the factor intensity is identical between two goods. We begin by writing down the general two-good, two-factor model:

$$x = G^x(z)\, F^x(v_{1x}, v_{2x}), \tag{3.12a}$$

$$y = G^y(z)\, F^y(v_{1y}, v_{2y}), \tag{3.12b}$$

$$z = R(v_{1x}, v_{1y}, v_{2x}, v_{2y}) \tag{3.12c}$$

where v_{ji} $(j = 1,2; i = x, y)$ is the use of factor j in good i. $R(v_{1x}, v_{1y}, v_{2x}, v_{2y})$ describes the relationship between the output of by-product and factor uses. $F^i(v_{1i}, v_{2i})$ has the standard properties of a Neoclassical production function that reflect the contribution of factors.

The PPF is defined by the following maximum value function

$$y = T(x) \equiv \max_{z, v_{1y}, v_{2y}} G^y(z)\, F^y(v_{1y}, v_{2y}), \tag{3.13}$$

subject to $G^x(z)\, F^x(v_{1x}, v_{2x}) = x$ and $z = R(v_{1x}, v_{1y}, v_{2x}, v_{2y})$. Again, assume that the factor constraint is slack on the PPF. It follows from the first-order conditions that

$$\frac{F_1^x}{F_2^x} = \frac{F_1^y}{F_2^y}.$$

Two goods share the same factor intensity, i.e. F^x and F^y satisfy the following:

$$\frac{F_1^x}{F_2^x} = \frac{F_1^y}{F_2^y} \text{ if } \frac{v_{1x}}{v_{2x}} = \frac{v_{1y}}{v_{2y}}. \tag{3.14}$$

According to the first-order necessary conditions, $v_{1x}/v_{2x} = v_{1y}/v_{2y}$ on the PPF. Let c denote this ratio, then we have $v_{1x} = cv_{2x}$ and $v_{1y} = cv_{2y}$. The PPF can be expressed equivalently as

$$T(x) \equiv \max_c t(x, c),$$

where

$$t(x, c) \equiv \max_{z, v_{2y}} G^y(z)\, F^y(c, 1)\, v_{2y}, \tag{3.15}$$

subject to $G^x(z)\, F^x(c, 1)\, v_{2x} = x$ and $z = R(cv_{2x}, cv_{2y}, v_{2x}, v_{2y})$. For convenience, let $r(v_{2x}, v_{2y}) \equiv R(cv_{2x}, cv_{2y}, v_{2x}, v_{2y})$.

Clearly, problem (3.15) is the single-factor case with a constant c. From Proposition 3.5, $t(x, c)$ is convex with respect to x, given that $r(v_{2x}, v_{2y})$ is quasi-concave with respect to (v_{2x}, v_{2y}). Since the upper envelope of $t(x, c)$ by changing c constructs $T(x)$, $T(x)$ is also convex. The following proposition summarizes the argument.

Proposition 3.17. *In the two-factor case* (3.12) *and given assumptions (A1′), (A2), (A3), and (A4), the PPF is convex if two goods share the identical factor intensity and* $R(\cdot,\cdot)$ *is quasi-concave.*

3.6 Conclusion

The curvature of the PPF is crucial for many issues in economic theory. For example, if the PPFs of two economies are convex to the origin, both economies can achieve higher efficiencies by specializing and trading with each other. This provides an explanation to the source of comparative advantages. This study shows that in the presence of strong input-generated production externalities, the PPF tends to be convex. To neutralize the substitution between factors that renders the PPF more concave, the model has only a single factor. Two forces that drive the PPF to be more convex are highlighted: the quasi-concavity of the by-product generation function and, second, the difference in the sensitivity to by-product. Although the model has only two goods, most results apply when there are many goods.

3.A Appendix

3.A.1 Monotonicity

Note that the envelope theorem is not applicable because we do not assume the differentiability of $G^i(\cdot)$ and $R(\cdot,\cdot)$. We prove firstly that $T(x)$ is strictly decreasing over $[0,\tilde{x}]$. There are two possible cases: $\tilde{x}=0$ and $\tilde{x}>0$. The case of $\tilde{x}=0$ is trivial. Thus we can deal with only the case of $\tilde{x}>0$. Assume to the contrary that there exist two values of $x\in[0,\tilde{x}]$, say x' and x'', so that $x'>x''$ and $T(x')\geq T(x'')$. Note that $x'>0$ for $x'>x''\geq 0$, and that $\tilde{x}>x''$ for $\tilde{x}\geq x'>x''$. Then we have $T(x'')>0$ by the definition of $\tilde{x}$, and have $T(x')>0$ since $T(x')\geq T(x'')$. Letting $\left(v'_x,v'_y\right)$ denote the optimal factor input vector corresponding to x', then $x'=G^x(z')\,v'_x$ and $T(x')=G^y(z')\,v'_y$ where $z'=R\left(v'_x,v'_y\right)$. It follows $x'>0$ and $T(x')>0$ that $G^i(z')>0$ $(i=x,y)$. Let $\left(v''_x,v''_y\right)$ denote the factor input vector so that $x''=G^x(z')\,v''_x$ and $z'=R\left(v''_x,v''_y\right)$. Since $x'>x''$, we have $v''_x<v'_x$ and thus, by assumption (A2), $v''_y>v'_y$. This means $y''\equiv G^y(z')\,v''_y>G^y(z')\,v'_y=T(x')$. Since (x'',y'') is a feasible production bundle, we have $T(x'')\geq y''$ by the definition of $T(x)$. This implies $T(x'')>T(x')$ and leads to a contradiction.

Now we move on to the proof that $T(x)=0$ over $(\tilde{x},\bar{x}]$. There are two cases: $\tilde{x}=\bar{x}$ and $\tilde{x}<\bar{x}$. The case of $\tilde{x}=\bar{x}$ is trivial for $(\tilde{x},\bar{x}]=\emptyset$. Thus we can focus on only the case of $\tilde{x}<\bar{x}$. Assume to the contrary there exists a value of x, say x', so that $x'\in(\tilde{x},\bar{x}]$

and $T(x') > 0$. By similar procedures, we can show that $T(x) > T(x') > 0$ for any $x < x'$. This leads to a contradiction to the definition of $\tilde{x}$.

3.A.2 Continuity over $(0, \bar{x}]$

According to Berge's theorem of the maximum, if the constraint set $C(x)$ defined by (3.3) is continuous over $(0, \bar{x}]$, then $T(x)$ is also continuous. To prove this, we shall check both the upper semi-continuity and the lower semi-continuity of $C(x)$.

Upper semi-continuity Take a sequence $\{x^n\} \to x' \in (0, \bar{x}]$ so that $x^n \in (0, \bar{x}]$, and take a sequence $\left\{\left(v_x^n, v_y^n\right)\right\} \to \left(v_x^s, v_y^s\right)$ so that $\left(z^n, v_x^n, v_y^n\right) \in C(x^n)$ for all n. By the sequential characterization, if $\left(z^s, v_x^s, v_y^s\right) \in C(x')$, then $C(x)$ is upper semi-continuous.

We first consider the case that $\left\{\left(z^n, v_x^n, v_y^n\right)\right\}$ or its subsequence (for simplicity, we use the same notation here) satisfies $G^x(z^n) v_x^n > x^n$, then it is obvious that

$$\exists N \text{ so that } \forall n > N, G^x\left(z^N\right) v_x^N > x',$$

which actually implies that $\left(z^s, v_x^s, v_y^s\right) \in C(x')$.

Second, we consider the case that $\left\{\left(z^n, v_x^n, v_y^n\right)\right\}$ or its subsequence satisfying $G^x(z^n) v_x^n = x^n$. Note that real analysis requires one of the following cases to hold: (i) $\{x^n\}$ contains an increasing subsequence $\{x^{n_k}\}$; (ii) $\{x^n\}$ contains a decreasing subsequence $\{x^{n_k}\}$; (iii) $\{x^n\}$ contains both types of subsequence. In case (i), for any n_k and corresponding $(z^{n_k}, v_x^{n_k}, v_y^{n_k}) \in C(x^{n_k})$, there are further two situations: $v_y^{n_k} > 0$ and $v_y^{n_k} = 0$. We first discuss the situation of $v_y^{n_k} > 0$. Since $\{x^{n_k}\} \to x'$ and $v_y^{n_k} > 0$, there exist a number N_1 and $\left(z^{n_k}, v_x^{N_1}, v_y^{N_1}\right) \in C(x')$ satisfying $G^x(z^{n_k}) v_x^{N_1} = x'$, so that $\left\|\left(z^{n_k}, v_x^{N_1}, v_y^{N_1}\right), (z^{n_k}, v_x^{n_k}, v_y^{n_k})\right\| \leq c_1 (x' - x^{n_k})$ for all $n_k > N_1$. It is obvious that $v_x^{N_1} > v_x^{n_k}$ and $v_y^{N_1} < v_y^{n_k}$ for $x' > x^{n_k}$. Assumption (A1) ensures the existence of such constant $c_1 > 0$. The distance from $(z^{n_k}, v_x^{n_k}, v_y^{n_k})$ to $C(x')$ can be defined as follows.

$$d\left((z^{n_k}, v_x^{n_k}, v_y^{n_k}), C(x')\right) \equiv \inf_{\left(z', v_x', v_y'\right) \in C(X')} \left\|(z^{n_k}, v_x^{n_k}, v_y^{n_k}), \left(z', v_x', v_y'\right)\right\|.$$

Then we have, for any $n_k > N_1$,

$$d\left((z^{n_k} v_x^{n_k}, v_y^{n_k}), C(x')\right) \leq \left\|\left(z^{n_k}, v_x^{N_1}, v_y^{N_1}\right), (z^{n_k}, v_x^{n_k}, v_y^{n_k})\right\| \leq c_1 (x' - x^{n_k}). \tag{3.16}$$

Now we discuss the situation of $v_y^{n_k} = 0$ in case (i). It is impossible to find an element in $C(x')$, as is done previously, by replacing $v_x^{n_k}$ with a larger value without changing z^{n_k}. However, there exists a number N_2 and $\left(z^{n_k}, v_x^{N_2}, 0\right) \in C(x')$ satisfying $x' = G^x(z^{n_k}) v_x^{N_2}$, so that $\left\|\left(z^{n_k}, v_x^{N_2}, 0\right), (z^{n_k}, v_x^{n_k}, 0)\right\| \leq c_1 (x' - x^{n_k})$ for all $n_k > N_2$. Again, (A1) ensures the existence of such constant $c_2 > 0$. The distance from $\left(z^{n_k}, v_x^{n_k}, 0\right)$ to $C(x')$ satisfies, for all $n_k > N_2$,

$$d\left(\left(z^{n_k}, v_x^{n_k}, 0\right), C(x')\right) \leq \left\| \left(z^{n_k}, v_x^{N_2}, 0\right), \left(z^{n_k}, v_x^{n_k}, 0\right) \right\| \leq c_1 \left(x' - x^{n_k}\right). \tag{3.17}$$

Equations (3.16) and (3.17) together implies that $d\left(\left(z^{n_k}, v_x^{n_k}, v_y^{n_k}\right), C(x')\right) \to 0$ when $x^{n_k} \to x'$. Hence, $\left\{\left(z^{n_k}, v_x^{n_k}, v_y^{n_k}\right)\right\} \to \left(z^s, v_x^s, v_y^s\right) \in C(x')$.

The similar method can be applied to case (ii), which is rather simpler since now it is always possible, for sufficiently large n_k, to find an element in $C(x')$ by replacing $v_x^{n_k}$ by a smaller value without changing z^{n_k}. Because we have proved the upper semi-continuity in both case (i) and case (ii), case (iii) becomes trivial and requires no further discussion.

Lower semi-continuity Take a sequence of $\{x^n\} \to x' \in (0, \bar{x}]$ and a point $\left(z', v_x', v_y'\right) \in C(x')$. By the sequential characterization, if there exists a sequence $\left\{\left(z^n, v_x^n, v_y^n\right)\right\}$ so that $\left(z^n, v_x^n, v_y^n\right) \in C(x^n)$ and $\left\{\left(z^n, v_x^n, v_y^n\right)\right\} \to \left(z', v_x', v_y'\right)$, then $C(x)$ is lower semi-continuous. To show this, we construct a sequence $\left\{\left(v_x^n, v_y^n\right)\right\}$ by letting $v_x^n = a(n) + v_x'$ and $v_y^n = b(n) + v_y'$. Then we choose a number N large enough, so that there exist $a(n)$ and $b(n)$ for all $n > N$ satisfying that $x^n = G^x(z')\, v_x^n$ and $R\left(v_x^n, v_y^n\right) = z' = R\left(v_x', v_y'\right)$. Hence $\left(z', v_x^n, v_y^n\right) \in C(x^n)$ for all $n > N$.

Note that $x^n = G^x(z')\, v_x^n = G^x(z')\left(a(n) + v_x'\right) = x' + G^x(z')\, a(n)$ for all $n > N$. Thus we obtain $a(n) = (x^n - x')/G^x(z')$ since $G^x(z') > 0$ due to $x' > 0$. This implies that $a(n) \to 0$ when $x^n \to x'$. Furthermore, by assumption (A2) and $R\left(v_x^n, v_y^n\right) = z' = R\left(v_x', v_y'\right)$, $b(n) \to 0$ when $a(n) \to 0$. Therefore, $\left(z', v_x^n, v_y^n\right) \to \left(z', v_x', v_y'\right)$ when $x^n \to x'$.

3.A.3 Continuity at $x = 0$

First note that when $x = 0$, the optimal factor input in good x, $v_x^0 = 0$. Otherwise it is possible to, according to (A2), raise the output of y by reducing v_x and increasing v_y in such a way that z remains unchanged.

Sufficiency It follows from (A1) that

$$\forall \varepsilon > 0 \text{ and } \forall z^0 \in Z(0), \exists \sigma > 0 \text{ so that } \forall z \in \left(z^0 - \sigma, z^0 + \sigma\right), \left|T(0) - G^y(z)\, v_y\right| < \frac{\varepsilon}{2}, \tag{3.18}$$

where v_y satisfies $z = R(0, v_y)$. The sufficient condition implies that there exists $z^{0\prime} \in Z(0)$ satisfying that

$$\forall \sigma > 0, \exists z' \in \left(z^{0\prime} - \sigma, z^{0\prime} + \sigma\right) \text{ so that } G^x(z') > 0. \tag{3.19}$$

Together, (3.18) and (3.19) imply

$$\forall \varepsilon > 0, \exists \sigma > 0 \text{ so that } \forall z' \in \left(z^{0\prime} - \sigma, z^{0\prime} + \sigma\right), \left|T(0) - G^y(z') v'_y\right| < \frac{\varepsilon}{2},$$

where v'_y satisfies $R\left(0, v'_y\right) = z'$. Provided a small enough value of x, say $x^1 > 0$, there exist $\left(v^1_x, v^1_y\right)$ so that $G^x(z') v^1_x = x^1$ and $R\left(v^1_x, v^1_y\right) = z'$. It follows from the continuity of $R(\cdot,\cdot)$ that

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ so that } \forall x' < \delta, \left|G^y(z') v'_y - G^y(z') v^1_y\right| < \frac{\varepsilon}{2}. \tag{3.20}$$

By the definition of $T(x)$, we have $T(x') \geq G^y(z') v^1_y$. On the other hand, by Proposition 3.3, we have $T(0) \geq T(x')$. Therefore,

$$\left|T(0) - T(x')\right| \leq \left|T(0) - G^y(z') v^1_y\right|.$$

By assumption (A2), $v^1_y < v'_y$ for $v^1_x > 0$, which means $G^y(z') v'_y > G^y(z') v^1_y$ and thus

$$\left|T(0) - G^y(z') v^1_y\right| = \left|T(0) - G^y(z') v'_y\right| + \left|G^y(z') v'_y - G^y(z') v^1_y\right|.$$

Two expressions imply

$$\left|T(0) - T(x')\right| \leq \left|T(0) - G^y(z') v'_y\right| + \left|G^y(z') v'_y - G^y(z') v^1_y\right|. \tag{3.21}$$

Together with (3.19) and (3.20) we obtain

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x < \delta, \left|T(0) - T(x)\right| < \varepsilon.$$

This establishes the continuity of $T(x)$ at $x = 0$.

Necessity It is easier to prove the contrapositive: $y = T(x)$ is discontinuous at $x = 0$ if

$$\exists \sigma > 0 \text{ so that } \forall z^0 \in Z(0) \text{ and } \forall z \in \left(z^0 - \sigma, z^0 + \sigma\right), G^x(z) = 0.$$

For this purpose, define $m = \inf_z \left(T(0) - G^y(z) v^0_y\right)$ where z and v^0_y satisfy that $z \notin \left(z^0 - \sigma, z^0 + \sigma\right)$ for any $z^0 \in Z(0)$ and $R\left(0, v^0_y\right) = z$. It is evident that $m > 0$, otherwise $z \in Z(0)$ and thus $z \in \left(z^0 - \sigma, z^0 + \sigma\right)$. For any $\delta > 0$, we can pick a number $x' \in (0, \delta)$. Let $z' \in Z(x')$, then $G^x(z') > 0$ for $x' > 0$, which implies that $z' \notin \left(z^0 - \sigma, z^0 + \sigma\right)$. Let v^1_y satisfy that $R\left(0, v'_y\right) = z'$, then

$$\left|T(0) - G^y(z') v^1_y\right| \geq m.$$

On the other hand, if we let $\left(v'_x, v'_y\right)$ be the corresponding optimal factor inputs, we have $G^y(z') v^1_y > G^y\left(z^{x'}\right) v'_y = T(x')$ since $v'_y < v^1_y$ due to $v'_x > 0$. Therefore,

$$\left|T(0) - T(x')\right| > \left|T(0) - G^y(z') v^1_y\right| \geq m.$$

This implies that

$$\forall \delta > 0 \text{ and } \forall x \in (0, \delta), \exists m > 0 \text{ so that } \left|T(0) - T(x')\right| > m,$$

which establishes the discontinuity of $T(x)$ at $x = 0$.

3.A.4 Proof of Proposition 3.6

By the envelope theorem, on the PPF

$$T'(x) = \frac{\partial \mathcal{L}}{\partial x} = -p = -\frac{G^y}{G^x}.$$

The local convexity of the PPF is then characterized by

$$T''(x) = -\frac{dp}{dx}.$$

Taking the total differentiation of first-order conditions and constraints yields

$$H\left(dp, dv_x, dv_y\right)' = (1,0,0)'\,dx. \tag{3.22}$$

H is the Hessian matrix of $\mathcal{L}$

$$H = \begin{bmatrix} 0 & G^{x\prime}R_x v_x + G^x & G^{x\prime}R_y v_x \\ G^{x\prime}R_x v_x + G^x & R_x^2\left(B_1 + \frac{R_{xx}}{R_x^2}B_2 + B_3 + B_4\right) & R_x R_y\left(B_1 + \frac{R_{xy}}{R_x R_y}B_2 + B_3\right) \\ G^{x\prime}R_y v_x & R_x R_y\left(B_1 + \frac{R_{xy}}{R_x R_y}B_2 + B_3\right) & R_y^2\left(B_1 + \frac{R_{yy}}{R_y^2}B_2 + B_3 - B_4\right) \end{bmatrix}$$

where

$$\begin{aligned} B_1 &\equiv pG^{x\prime\prime}v_x + G^{y\prime\prime}v_y, \\ B_2 &\equiv pG^{x\prime}v_x + G^{y\prime}v_y = -w, \\ B_3 &\equiv \frac{pG^{x\prime}}{R_x} + \frac{G^{y\prime}}{R_y}, \\ B_4 &\equiv \frac{pG^{x\prime}}{R_x} - \frac{G^{y\prime}}{R_y}. \end{aligned}$$

Since $|H| > 0$, by Cramer's rule,

$$\frac{dp}{dx} = \frac{|H_1|}{|H|},$$

where H_i is the matrix formed from H by replacing its i-th column by $(1,0,0)'$. It follows the definition of ε_i and first-order conditions that

$$\begin{aligned} G^{x\prime}R_x v_x + G^x &= -\frac{G^{y\prime}R_y v_y}{G^y}G^x = \left(\varepsilon_y \frac{R_y v_y}{z}\right)G^x, \\ G^{x\prime}R_y v_x &= \frac{G^{x\prime}R_x v_x}{G^x}\frac{G^x R_y}{R_x} = -\left(\varepsilon_x \frac{R_x v_x}{z}\right)\frac{G^x R_y}{R_x}. \end{aligned}$$

Let $\lambda_x \equiv \varepsilon_x R_x v_x / z$, $\lambda_y \equiv \varepsilon_y R_y v_y / z$ and H_{ij} denote the entry in the i-th row and j-th column of H. Then

$$H = \begin{bmatrix} 0 & \lambda_y G^x & -\lambda_x \frac{G^x R_y}{R_x} \\ \lambda_y G^x & H_{22} & H_{23} \\ -\lambda_x \frac{G^x R_y}{R_x} & H_{23} & H_{33} \end{bmatrix}.$$

We do matrix transformation of H while keeping $|H|$ unchanged as follows.

$$
\begin{aligned}
|H| &\overset{(\text{row }1\times\frac{1}{G^xR_y},\ \text{col }1\times\frac{1}{G^xR_y})}{=} (G^xR_y)^2 \begin{vmatrix} 0 & \frac{\lambda_y}{R_y} & -\frac{\lambda_x}{R_x} \\ \frac{\lambda_y}{R_y} & H_{22} & H_{23} \\ -\frac{\lambda_x}{R_x} & H_{23} & H_{33} \end{vmatrix} \\
&\overset{(\text{row }2\times R_y,\ \text{col }2\times R_y}{=} (G^x)^2 \begin{vmatrix} 0 & \lambda_y & -\frac{\lambda_x}{R_x} \\ \lambda_y & R_y^2H_{22} & R_yH_{23} \\ -\frac{\lambda_x}{R_x} & R_yH_{23} & H_{33} \end{vmatrix} \\
&\overset{(\text{row }3\times R_x,\ \text{col }3\times R_x)}{=} \left(\frac{G^x}{R_x}\right)^2 \begin{vmatrix} 0 & \lambda_y & -\lambda_x \\ \lambda_y & R_y^2H_{22} & R_xR_yH_{23} \\ -\lambda_x & R_xR_yH_{23} & R_x^2H_{33} \end{vmatrix}.
\end{aligned}
$$

According to (3.6b), $\lambda_x + \lambda_y = 1$ on the PPF. Therefore,

$$
\begin{aligned}
|H| &\overset{(\text{row }2-\text{row }3)}{=} \left(\frac{G^x}{R_x}\right)^2 \begin{vmatrix} 0 & \lambda_y & -\lambda_x \\ 1 & R_y^2H_{22} - R_xR_yH_{23} & R_xR_yH_{23} - R_x^2H_{33} \\ -\lambda_x & R_xR_yH_{23} & R_x^2H_{33} \end{vmatrix} \\
&\overset{(\text{col }2-\text{col }3)}{=} \left(\frac{G^x}{R_x}\right)^2 \begin{vmatrix} 0 & 1 & -\lambda_x \\ 1 & R_y^2H_{22} + R_x^2H_{33} - 2R_xR_yH_{23} & R_xR_yH_{23} - R_x^2H_{33} \\ -\lambda_x & R_xR_yH_{23} - R_x^2H_{33} & R_x^2H_{33} \end{vmatrix}
\end{aligned}
$$

For the sake of notation, let $h_{22} \equiv R_yH_{22} + R_xH_{33} - 2R_xR_yH_{23}$, $h_{23} \equiv R_xR_yH_{23} - R_xH_{33}$ and $h_{33} \equiv R_xH_{33}$, then

$$
\begin{aligned}
|H| = \left(\frac{G^x}{R_x}\right)^2 \begin{vmatrix} 0 & 1 & -\lambda_x \\ 1 & h_{22} & h_{23} \\ -\lambda_x & h_{23} & h_{33} \end{vmatrix} &\overset{(\text{col }3+\text{col }2\times\lambda_x)}{=} \left(\frac{G^x}{R_x}\right)^2 \begin{vmatrix} 0 & 1 & 0 \\ 1 & h_{22} & h_{23}+\lambda_xh_{22} \\ -\lambda_x & h_{23} & h_{33}+\lambda_xh_{23} \end{vmatrix} \\
&\overset{(\text{row }3+\text{row }2\times\lambda_x)}{=} \left(\frac{G^x}{R_x}\right)^2 \begin{vmatrix} 0 & 1 & 0 \\ 1 & h_{22} & h_{23}+\lambda_xh_{22} \\ 0 & h_{23}+\lambda_xh_{22} & h_{33}+2\lambda_xh_{23}+\lambda_x^2h_{22} \end{vmatrix} \\
&= -\left(\frac{G^x}{R_x}\right)^2 \left(h_{33}+2\lambda_xh_{23}+\lambda_x^2h_{22}\right).
\end{aligned}
$$

On the other hand, recalling the steps of matrix transformation, we have

$$
|H_1| = \begin{vmatrix} H_{22} & H_{23} \\ H_{23} & H_{33} \end{vmatrix} = \frac{1}{(R_xR_y)^2} \begin{vmatrix} h_{22} & h_{23}+\lambda_xh_{22} \\ h_{23}+\lambda_xh_{22} & h_{33}+2\lambda_xh_{23}+\lambda_x^2h_{22} \end{vmatrix}.
$$

Therefore,

$$
\frac{|H_1|}{|H|} = -\frac{h_{22}}{(G^xR_y)^2} - \frac{(h_{23}+\lambda_xh_{22})^2}{(R_xR_y)^2\,|H|}
$$

The routine calculation gives that

$$
\begin{aligned}
h_{22} &= \left(R_x R_y\right)^2 wQ, \\
h_{23} + \lambda_x h_{22} &= \left[\lambda_x R_y^2 H_{22} - \lambda_y R_x^2 H_{33} + \left(\lambda_y - \lambda_x\right) R_x R_y H_{23}\right] \\
&= \frac{\left(R_x R_y\right)^2 w}{z} \left[\varepsilon_y \left(1 - R_y v_y q_2\right) - \varepsilon_x \left(1 - R_x v_x q_1\right)\right].
\end{aligned}
$$

Substituting into the expression of $|H_1| / |H|$ and recalling $T''(x) = -|H_1| / |H|$ yield (3.7).

3.A.5 Proof of Proposition 3.8

On the PPF, we have

$$
\begin{aligned}
\frac{dz}{dx} &= R_x \frac{dv_x}{dx} + R_y \frac{dv_y}{dx} \\
&= R_x \frac{|H_2|}{|H|} + R_y \frac{|H_3|}{|H|}.
\end{aligned}
$$

First calculate $|H_2|$,

$$
\begin{aligned}
|H_2| &= \begin{vmatrix} 0 & 1 & -\lambda_x \frac{G^x R_y}{R_x} \\ \lambda_y G^x & 0 & H_{23} \\ -\lambda_x \frac{G^x R_y}{R_x} & 0 & H_{33} \end{vmatrix} \overset{(\text{col } 1 \times \frac{1}{G^x R_y})}{=} G^x R_y \begin{vmatrix} 0 & 1 & H_{13} \\ \frac{\lambda_y}{R_y} & 0 & H_{23} \\ -\frac{\lambda_x}{R_x} & 0 & H_{33} \end{vmatrix} \\
&\overset{(\text{row } 2 \times R_y)}{=} G^x \begin{vmatrix} 0 & 1 & H_{13} \\ \lambda_y & 0 & R_y H_{23} \\ -\frac{\lambda_x}{R_x} & 0 & H_{33} \end{vmatrix} \overset{(\text{row } 3 \times R_x)}{=} \frac{G^x}{R_x} \begin{vmatrix} 0 & 1 & H_{13} \\ \lambda_y & 0 & R_y H_{23} \\ -\lambda_x & 0 & R_x H_{33} \end{vmatrix}.
\end{aligned}
$$

Since $\lambda_x + \lambda_y = 1$,

$$
\begin{aligned}
|H_2| &= \frac{G^x}{R_x^2} \begin{vmatrix} 0 & 1 & R_x H_{13} \\ \lambda_y & 0 & R_x R_y H_{23} \\ -\lambda_x & 0 & R_x^2 H_{33} \end{vmatrix} \overset{(\text{row } 2 - \text{row } 3)}{=} \frac{G^x}{R_x^2} \begin{vmatrix} 0 & 1 & R_x H_{13} \\ 1 & 0 & R_x R_y H_{23} - R_x^2 H_{33} \\ -\lambda_x & 0 & R_x^2 H_{33} \end{vmatrix} \\
&\overset{(\text{row } 3 + \text{row } 2 \times \lambda_x)}{=} \frac{G^x}{R_x^2} \begin{vmatrix} 0 & 1 & R_x H_{13} \\ 1 & 0 & R_x R_y H_{23} + R_x^2 H_{33} \\ 0 & 0 & \lambda_x R_x R_y H_{23} + \lambda_y R_x^2 H_{33} \end{vmatrix} = -\frac{G^x}{R_x^2} \left(\lambda_x R_x R_y H_{23} + \lambda_y R_x^2 H_{33}\right).
\end{aligned}
$$

$|H_3|$ can be calculated in a similar way,

$$|H_3| = \frac{G^x}{R_x}\begin{vmatrix} 0 & H_{12} & 1 \\ \lambda_y & R_y H_{22} & 0 \\ -\lambda_x & R_x H_{23} & 0 \end{vmatrix} \overset{(\text{col } 2\times R_y)}{=} \frac{G^x}{R_x R_y}\begin{vmatrix} 0 & R_y H_{12} & 1 \\ \lambda_y & R_y^2 H_{22} & 0 \\ -\lambda_x & R_x R_y H_{23} & 0 \end{vmatrix}$$

$$\overset{(\text{row } 2-\text{row } 3)}{=} \frac{G^x}{R_x R_y}\begin{vmatrix} 0 & R_y H_{12} & 1 \\ 1 & R_y^2 H_{22} - R_x R_y H_{23} & 0 \\ -\lambda_x & R_x R_y H_{23} & 0 \end{vmatrix}$$

$$\overset{(\text{row } 3+\text{row } 2\times\lambda_x)}{=} \frac{G^x}{R_x R_y}\begin{vmatrix} 0 & R_y H_{12} & 1 \\ 1 & R_y^2 H_{22} - R_x^2 H_{23} & 0 \\ 0 & \lambda_x R_y^2 H_{22} + \lambda_y R_x R_y H_{23} & 0 \end{vmatrix} = \frac{G^x}{R_x R_y}\left(\lambda_x R_y^2 H_{22} + \lambda_y R_x R_y H_{23}\right).$$

Therefore,

$$\begin{aligned} \frac{dz}{dx} &= \frac{G^x}{R_x |H|}\left[\lambda_x R_y^2 H_{22} - \lambda_y R_x^2 H_{33} + \left(\lambda_y - \lambda_x\right) R_x R_y H_{23}\right] \\ &= \frac{G^x R_x R_y^2 w}{z |H|}\left[\varepsilon_y \left(1 - R_y v_y q_2\right) - \varepsilon_x \left(1 - R_x v_x q_1\right)\right]. \end{aligned}$$

Substituting $w = G^y / R_y$ for w yields the result.

3.A.6 Proof of Lemma 3.11

The convexity of $1/G^i(z)$ is defined by, for any $z', z'' \in \{z : G^i(z) > 0\}$,

$$\left(z' - z''\right)\frac{d}{dz}\left(\frac{1}{G^i(z'')}\right) \leq \frac{1}{G^i(z')} - \frac{1}{G^i(z'')},$$

which can be rewritten into

$$\left(z' - z''\right) G^{i\prime}\left(z''\right) + G^i\left(z''\right)\left(\frac{G^i(z'')}{G^i(z')} - 1\right) \geq 0. \tag{3.23}$$

On the other hand, the pseudo-concavity of $G^i(z)\, v_i$ is defined by, for any $\left(z', v_i'\right), \left(z'', v_i''\right) \in \left\{(z, v_i) : G^i(z)\, v_i > 0\right\}$,

$$G^i\left(z'\right) v_i' > G^i\left(z''\right) v_i'' \Rightarrow \left[\left(z', v_i'\right) - \left(z'', v_i''\right)\right] \nabla\left(G^i\left(z''\right) v_i''\right) > 0,$$

which can be simplified into

$$G^i\left(z'\right) v_i' > G^i\left(z''\right) v_i'' \Rightarrow \left(z' - z''\right) G^{i\prime}\left(z''\right) + G^i\left(z''\right)\left(\frac{v_i'}{v_i''} - 1\right) > 0. \tag{3.24}$$

Note that

$$G^i\left(z'\right) v_i' > G^i\left(z''\right) v_i'' \Leftrightarrow G^i\left(z''\right)\left(\frac{v_i'}{v_i''} - 1\right) > G^i\left(z''\right)\left(\frac{G^i(z'')}{G^i(z')} - 1\right). \tag{3.25}$$

Combining (3.25) and (3.23) gives (3.24). Hence $G^i(z)\, v_i$ is pseudo-concave.

Similarly, the strict convexity of $1/G^i(z)$ is defined by, provided $z' \neq z''$,

$$\left(z' - z''\right) \frac{d}{dz}\left(\frac{1}{G^i(z'')}\right) < \frac{1}{G^i(z')} - \frac{1}{G^i(z'')},$$

which can be rewritten into

$$\left(z' - z''\right) G^{i\prime}\left(z''\right) + G^i\left(z''\right)\left(\frac{G^i(z'')}{G^i(z')} - 1\right) > 0. \tag{3.26}$$

On the other hand, the strict pseudo-concavity of $G^i(z)\, v_i$ is defined by, provided $\left(z', v_i'\right) \neq \left(z'', v_i''\right)$,

$$G^i\left(z'\right) v_i' \geq G^i\left(z''\right) v_i'' \Rightarrow \left[\left(z', v_i'\right) - \left(z'', v_i''\right)\right] \nabla \left(G^i\left(z''\right) v_i''\right) > 0,$$

which can be simplified into

$$G^i\left(z'\right) v_i' \geq G^i\left(z''\right) v_i'' \Rightarrow \left(z' - z''\right) G^{i\prime}\left(z''\right) + G^i\left(z''\right)\left(\frac{v_i'}{v_i''} - 1\right) > 0. \tag{3.27}$$

There are two cases. If $z' \neq z''$, then (3.27) holds according to (3.26). If $z' = z''$, it follows $G^i\left(z'\right) v_i' \geq G^i\left(z''\right) v_i''$ that $v_i' \geq v_i''$. Moreover, $z' = z''$ and $\left(z', v_i'\right) \neq \left(z'', v_i''\right)$ implies $v_i' \neq v_i''$. Hence we have $v_i' > v_i''$, which again means that (3.27) holds. This completes the proof.

3.A.7 Proof of Lemma 3.12

For concreteness, we prove the case of $i = x$. Then the case of $i = y$ can be proved similarly. The routine calculation gives that

$$\left[\left(v_x', v_y'\right) - \left(v_x'', v_y''\right)\right] \nabla \left(G^x\left(R\left(v_x'', v_y''\right)\right) v_x''\right) = A v_x'' + G^{x\prime}\left(z''\right) v_x'' B,$$

where

$$\begin{aligned}
z' &\equiv R\left(v_x', v_y'\right), z'' \equiv R\left(v_x'', v_y''\right), \\
A &\equiv \left(z' - z''\right) G^{x\prime}\left(z''\right) + G^x\left(z''\right)\left(\frac{v_x'}{v_x''} - 1\right), \\
B &\equiv \left(v_x' - v_x''\right) R_x\left(v_x'', v_y''\right) + \left(v_y' - v_y''\right) R_y\left(v_x'', v_y''\right) - \left(z' - z''\right).
\end{aligned}$$

Note that $B \leq 0$ for $R(\cdot,\cdot)$ is convex, while $G^{x\prime}(z'')$ could be either negative or non-negative. The pseudo-concavity of $G^x(z)\, v_x$ means that

$$G^x\left(z'\right) v_x' > G^x\left(z''\right) v_x'' \Rightarrow A > 0.$$

If $G^{x\prime}(z'')$ is negative, then $G^{x\prime}(z'')\, v_x'' B \geq 0$. This simply implies that

$$G^x\left(R\left(v_x', v_y'\right)\right) v_x' > G^x\left(R\left(v_x'', v_y''\right)\right) v_x'' \Rightarrow A v_x'' + G^{x\prime}\left(z''\right) v_x'' B > 0,$$

which says that $G^x\left(R\left(v_x, v_y\right)\right) v_x$ is pseudo-concave with respect to $\left(v_x, v_y\right)$. If $G^{x\prime}\left(z''\right)$ is non-negative, then $G^{x\prime}\left(z''\right) v''_x B \leq 0$. The alternative definition of pseudo-concavity requires

$$G^x\left(z'\right) v'_x < G^x\left(z''\right) v''_x \Rightarrow A < 0,$$

which implies

$$G^x\left(R\left(v'_x, v'_y\right)\right) v'_x < G^x\left(R\left(v''_x, v''_y\right)\right) v''_x \Rightarrow A v''_x + G^{x\prime}\left(z''\right) v''_x B < 0.$$

This is equivalent to saying that $G^x\left(R\left(v_x, v_y\right)\right) v_x$ is pseudo-concave.

Now consider a strictly pseudo-concave $G^x(z)\, v_x$. The strict pseudo-concavity requires, provided that $(z', v'_x) \neq (z'', v''_x)$,

$$G^x\left(z'\right) v'_x \geq G^x\left(z''\right) v''_x \Rightarrow A > 0.$$

The properties of $R(\cdot,\cdot)$, characterized by (A1) and (A2), ensures

$$(z', v'_x) \neq (z'', v''_x) \Leftrightarrow \left(v'_x, v'_y\right) \neq \left(v''_x, v''_y\right).$$

Hence, provided that $\left(v'_x, v'_y\right) \neq \left(v''_x, v''_y\right)$, if $G^{x\prime}\left(z''\right)$ is negative, then

$$G^x\left(R\left(v'_x, v'_y\right)\right) v'_x \geq G^x\left(R\left(v''_x, v''_y\right)\right) v''_x \Rightarrow A v''_x + G^{x\prime}\left(z''\right) v''_x B > 0,$$

which says that $G^x\left(R\left(v_x, v_y\right)\right) v_x$ is strictly pseudo-concave with respect to $\left(v_x, v_y\right)$. If $G^{x\prime}\left(z''\right)$ is non-negative, then we can obtain the same conclusion by using the alternative definition of strict pseudo-concavity.

3.A.8 Convexity and Uniqueness of Solution Set

Define

$$C'(x) \equiv \left\{\left(z, v_x, v_y\right) : G^x(z)\, v_x - x \geq 0, z - R\left(v_x, v_y\right) \geq 0\right\}. \tag{3.28}$$

The constraint $C(x)$ of problem (3.2) can be replaced by $C'(x)$ without changing the solution set. This is because $z - R\left(v_x, v_y\right) \geq 0$ is binding in the optimal and $v_x + v_y \leq E$ is not. Since $1/G^x(z)$ is convex, by Lemma 3.11 $G^x(z)\, v_x$ is pseudo-concave (thus quasi-concave) with respect to (z, v_x), and thus with respect to $\left(z, v_x, v_y\right)$ as well. Since $R(\cdot,\cdot)$ is convex, $z - R\left(v_x, v_y\right)$ is concave. On the other hand, since $1/G^y(z)$ is convex, the objective function $G^y(z)\, v_y$ is pseudo-concave (thus quasi-concave) with respect to $\left(z, v_y\right)$, and thus with respect to $\left(z, v_x, v_y\right)$ as well. According to the results from quasi-convex programming, the solution set $S(x)$ is convex.

The maximization problem (3.2) can be rewritten into

$$T(x) \equiv \max_{\left(v_x, v_y\right) \in C''(x)} G^y\left(R\left(v_x, v_y\right)\right) v_y,$$

where

$$C''(x) \equiv \left\{ (v_x, v_y) : G^x \left(R \left(v_x, v_y \right) \right) v_x - x \geq 0 \right\}.$$

Given that $1/G^i(z)$ ($i = x, y$) and $R(\cdot,\cdot)$ are convex, and that either $1/G^x(z)$ or $1/G^y(z)$ is strictly convex, according to Lemma 3.11 and 3.12, $G^i\left(R\left(v_x, v_y\right)\right) v_i$ ($i = x, y$) is pseudo-concave and either one of them is strictly pseudo-concave. According to the results from quasi-convex programming, there is a unique optimal $\left(v_x, v_y\right)$, denoted $\left(v_x^*, v_y^*\right)$. Since $z = R\left(v_x, v_y\right)$, the corresponding level of by-product is $z^* = R\left(v_x^*, v_y^*\right)$. Thus $S(x)$=$\left\{\left(z^*, v_x^*, v_y^*\right)\right\} = \left(z(x), v_x(x), v_y(x)\right)$. It follows the implicit-function theorem that $S(x)$ is continuously differentiable.

3.A.9 Proof of Proposition 3.14

Since $R(\cdot,\cdot)$ is linearly homogeneous and quasi-convex, it is convex. According to Proposition 3.13, the solution set $S(x)$ is convex over $(0, \tilde{x})$. Thus the set of good i's sensitivity on the PPF, denoted $\varepsilon_i(x) \equiv \left\{-G^{i\prime}(z) z / G^i(z) : z \in Z(x)\right\}$, is a connected set given any $x \in (0, \tilde{x})$. On the other hand, the constraint set $C(x)$ is continuous over $(0, \tilde{x})$ (see Appendix 3.A.2). According to the theorem of the maximum, the solution set $S(x)$ is upper semi-continuous over $(0, \tilde{x})$, and thus so is $\varepsilon_i(x)$. Therefore, the set of the sensitivity bundles on the PPF over $(0, \tilde{x})$, Ψ, is also a connected set on the $\varepsilon_x - \varepsilon_y$ plane.

It follows from the first-order conditions (3.5) that

$$\varepsilon_x \theta_x + \varepsilon_y \theta_y = 1,$$

where $\theta_i \equiv R_i v_i / z$ satisfying $\theta_x + \theta_y = 1$ since $R(\cdot,\cdot)$ is linearly homogeneous. This implies that at any point on the PPF either $\varepsilon_x > 1 > \varepsilon_y$, $\varepsilon_x = \varepsilon_y = 1$, or $\varepsilon_x < 1 < \varepsilon_y$ holds. Since Ψ is a connected set, if the sign of $\left(\varepsilon_x - \varepsilon_y\right)$ changes when moving along the PPF, then $\varepsilon_x > 1 > \varepsilon_y$, $\varepsilon_x = \varepsilon_y = 1$, and $\varepsilon_x < 1 < \varepsilon_y$ must coexist on the PPF. This is impossible. To see this, let z' denote the value of z so that $\varepsilon_x = \varepsilon_y = 1$. Using this z', we can obtain the following relationship between x and y:

$$G^x\left(z'\right) z' = R\left(x, \frac{G^x\left(z'\right)}{G^y\left(z'\right)} y\right), \tag{3.29}$$

where the linearly homogeneity of $R(\cdot,\cdot)$ is used. Let $y = Y(x, z')$ denote the relationship (3.29). Clearly $(x, Y(x, z'))$ is feasible for any $x \in (0, \tilde{x})$. In the following, we shall prove that $y = Y(x, z')$ is exactly the expression for the PPF, i.e. $T(x) = Y(x, z')$.

Assume to the contrary that there exists a feasible output bundle (x'', y'') lying outward to $y = Y(x, z')$. Let z'' denote the value of z corresponding with (x'', y''), then we also have

$$G^x\left(z''\right) z'' = R\left(x'', \frac{G^x\left(z''\right)}{G^y\left(z''\right)} y''\right). \tag{3.30}$$

According to assumption (A2), $\partial Y(x,z')/\partial x < 0$. Hence, we find a point lying on $y = Y(x,z')$, say (x''', y'''), so that

$$x''' \left(\frac{G^x(z')}{G^y(z')} y''' \right)^{-1} = x'' \left(\frac{G^x(z'')}{G^y(z'')} y'' \right)^{-1} \tag{3.31}$$

Because (x'', y'') lies outward to $y = Y(x,z')$, we have $x''' < x''$. Since (x''', y''') lies on $y = Y(x,z')$,

$$G^x(z')\, z' = R\left(x''', \frac{G^x(z')}{G^y(z')} y''' \right).$$

Hence we have $G^x(z'')\, z'' > G^x(z')\, z'$ according to the linearly homogeneity of $R(\cdot,\cdot)$.

On the other hand, according to Lemma 3.12, $G^i(z)\, v_i$ is pseudo-concave if $1/G^i(\cdot)$ is convex. This simply means, noting that $G^i(z)\, z$ can be obtained by letting $z = v_i$ in $G^i(z)\, v_i$, that $G^i(z)\, z$ is pseudo-concave as well. One of the properties of pseudo-concavity is that $G^i(z)\, z$ attains a global maximum when $d\left(G^i(z)\, z\right) = 0$, i.e. $\varepsilon_i = 1$. Since $\varepsilon_x = \varepsilon_y = 1$ when $z = z'$, $G^i(z')\, z' \geq G^x(z'')\, z''$, this leads to the contradiction. Hence, there is no feasible output bundle lying outside of $y = Y(x,z')$, which means $T(x) = Y(x,z')$. Furthermore, along the PPF we have $z = z'$ and $\varepsilon_x = \varepsilon_y = 1$.

Chapter 4

Trade, Capital Accumulation, and the Environment[1]

4.1 Introduction

The trade-off between economic development and environmental preservation has always been a challenging issue. Now, this issue becomes even more pressing thanks to the ongoing rapid globalization over the last few decades. On one hand, trade liberalization promotes the spatial separation between production and consumption, thus aggravating environmental externality issues. On the other hand, trade encourages economic development, which often means more pollution, wastes, and resource extractions. For example, some researchers (e.g., Austin, 2010) link the huge expansion of soybean industries driven by soybean exports in Latin American nations to the high deforestation rates in the region. Lathuillière et al. (2014) estimate that in the 2000s, soybean production was associated with 65% of the deforestation in Brazil's state of Mato Grosso, the largest soybean producer around the Amazon basin. The direction of impacts can also be the other way around. When the environment degrades, many industries such as agriculture, fishery, and tourism get hurt. This has two further possible consequences: it discourages a country's economic development, and nullifies a country's comparative advantages.

In this study, I develop a two-sector dynamic general equilibrium model to formulate the increasingly close nexus between trade, economic development, and the environment. In the model, economic development is represented by capital accumulation, which occurs when households behaving in a Ramsey fashion invest enough. A single final good–for consumption as well as investment–is assembled using two in-

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termediate goods–one from agriculture and the other from manufacturing. Both intermediate goods are produced with private capital; agriculture productivity is affected by environmental quality, which could change depending on the difference between the natural growth of the environment and the flow of pollution arising from production. For simplicity, trade is introduced considering a small open economy (SOE). There is no abatement and technological change.

With this setup above, the model formulates the following interaction between trade, economic development, and the environment. First, capital accumulation (economic development) leads to more pollution and harms the environment. Second, environmental changes affect agriculture productivity, which in turn affects the country's comparative advantage and capital rental (and thus capital accumulation). Third, trade induces specialization (in a cleaner or dirtier sector) and affects the environment. From this direct composition effect, trade could be good or bad for the environment. Trade also raises capital rental and stimulates capital accumulation. From this indirect scale effect, trade could be harmful for the environment. Therefore, the model captures two well-known channels through which trade affects the environment.[2] More importantly, the model captures the endogenous interaction between the two channels by introducing the Ramsey style investment.

The model yields several impressive results. First, the quality of environment converges across closed countries under laissez faire, regardless of differences in environmental endowment, including the carrying capacity and recovery rate of the environment, and pollution intensities. Thus, countries with greater carrying capacity, a faster recovery rate, or cleaner technologies have more private capital in the autarky steady state. This environmental convergence highlights the role of capital accumulation as a channel through which households sacrifice the environment for further economic development.

Second, moving from autarky to trade, although short-run trade could be good or bad for the environment, long-run trade necessarily harms the environment. This result sharply contrasts those of models without capital accumulation (e.g., Copeland and Taylor, 1999; Brander and Taylor, 1997, 1998), where trade is good for the environment as long as it specializes in a cleaner or non-resource sector. The intuition comes from realizing that trade has two effects on the environment in this model. On one hand, it raises capital rental and stimulates capital accumulation in a dynamic adjustment process that could take time. Thus, this scale effect of trade has little impact on the environment in the short run. On the other hand, trade could induce changes in the production patterns and thus in capital reallocation in a static adjustment process

[2] The third channel is the technical effect, which emphasizes the benefit of trade in promoting cleaner technologies; see Grossman and Krueger (1994). The technical effect is definitely important, but it is not the focus of this study.

that can be done instantaneously. Thus, this composition effect of trade dominates, leading to environmental change right after trade liberalization: the environment improves (degrades) if the country specializes in a cleaner (dirtier) sector. In the long run, however, the scale effect dominates: capital keeps on accumulating until the environment degrades to the extent that the trade premium of capital rental is cancelled out, which is necessarily worse than the autarky steady state. In this sense, capital accumulation provides a channel through which households can further sacrifice the environment to exploit the benefit of trade.

Third, policy analysis shows that the social optimum can be achieved in a market-based economy by imposing a dynamic Pigouvian pollution tax, with a lump-sum transfer of tax revenue to households. Pollution tax works in two directions. On one hand, it raises more proportionately the cost in the dirtier sector and thus corrects the misallocation of private capital. On the other hand, it reduces capital rental and thus helps resolve the excess investment problem. Furthermore, differing from laissez faire, trade does not necessarily harm the environment in the long run under optimal policy.

Finally, although at every point in time the trade pattern (and thus the specialization pattern) depends endogenously on quality of environment (as well as the pollution tax under optimal policy), in the long run it basically depends on parameters: that is, the economy specializes in goods with a high world price.[3] If the economy specializes in agriculture, there is a unique steady state, whereas if it specializes in manufacturing, the economy enters a growth path. This is because manufacturing provides a constant returns to scale and thus provides the economy with a growth engine similar to the AK model.

In the literature, extensive theoretical studies focus on a link between any two of trade, economic development, and the environment, but, to my knowledge, very few on a link between the three.[4] Copeland and Taylor (2004) provide an exception and survey the related literature. Their framework, however, considers economic development as exogenous and examines its impacts by conducting comparative statics. In this sense, their framework cannot capture the endogenous interaction between the three elements. This study attempts to fill the gap in the literature.

An important issue related to economic development and the environment is the environmental Kuznets curve (EKC), which suggests that environmental degradation first rises and then falls with increasing income per capita (Stern et al., 1996). Although empirical evidence still remains inconclusive (e.g., Dinda, 2004; Stern, 2004), theoreti-

[3]Here, I use the term "basically" because the situation under optimal policy is a little bit more complicated.

[4]See Brock and Taylor (2005) for an insightful review of the literature on economic growth and the environment. See also, among others, Stiglitz (1970), Baxter (1992), and Brecher et al. (2005) for discussions of economic growth and trade based on the Heckscher-Ohlin model.

cally the EKC can be derived from a model with disutility from environmental degradation, technological changes, and policies that maximize the utility, as in Copeland and Taylor (2004). Focusing on production rather than consumption externalities, this study does not intend to explain the EKC, or, precisely, the right tail of the EKC.

This study adds to the extensive literature on the interaction between trade and the environment. Theoretical work on the topic covers a wide range of contexts. Focusing on production externalities, as in this study, Brander and Taylor (1997, 1998) examine the interaction between trade and open-access renewable resource extractions. Copeland and Taylor (1999) consider the dynamic interaction between trade and pollution and show how pollution motivates trade, while Benarroch and Thille (2001) extend their model to the transboundary type of pollution. Kotsogiannis and Woodland (2013) provide a very general framework characterizing Pareto-efficient and Pareto-improving policies. Furthermore, many studies focus on consumption externalities (e.g., Markusen, 1975b,a; Asako, 1979; Copeland and Taylor, 1994, 1995; Ishikawa and Kiyono, 2006). In terms of structure, our model combines Copeland and Taylor (1999) and the Ramsey growth model, and thus formulates economic development and its endogenous interaction with trade and the environment through changes in capital rental.[5] This extension yields a significant insight as mentioned above. That is, the long-run environmental impact of trade can be largely underestimated by not considering capital accumulation through endogenous investment.

This study also contributes to the literature on the optimal control of pollution, pioneered by Keeler et al. (1972). van der Ploeg and Withagen (1991) analyze the problem in a Ramsey growth model. However, their models, like many others, are one-sector ones that ignore trade. This study extends the prevailing one-sector model in this literature to a two-sector model and considers trade. Furthermore, most studies are confined to the characterization of the optimal tax with a differential equation. However, this study moves one step forward and uses the differential equation as well as transversality condition to solve the optimal tax in an integral form. This helps us provide a clearer economic meaning to the optimal tax: it can be interpreted as a dynamic version of the Pigouvian tax. Recently, van der Ploeg and Withagen (2014) and Golosov et al. (2014) also provide the integral (summation) form of expressions of social damage. But in their models, the environment cannot recover by itself, and thus their expressions do not include the discount term associated with environmental recovery.

[5] Another minor extension is that I allow agriculture to be polluting instead of keeping it purely clean. This is motivated by the fact that agriculture has become one of the most leading sources of pollution and wastes. For example, EPA (2009) reports that agricultural activities, such as crop production, grazing, and animal feeding operations, are among the leading sources of water pollution in the United States.

4.2 The Basic Model

The model has two intermediate goods–one each from agriculture and manufacturing–and a single final good. The intermediate goods are tradable, and produced using a single factor of production, private capital. Pollution that arises from the production of the intermediate goods affects the environmental quality, and thus the productivity in agriculture, as in Copeland and Taylor (1999). The final good is assembled using the two intermediate goods, and is either consumed or invested. Households behave in a Ramsey fashion. Throughout the study, the final good is assumed to be the numeraire.

4.2.1 Households

The model has a large number of identical households, who own private capital (K) and take the rental of private capital (r) as given. The representative household receives income rK and behaves in a Ramsey fashion, by choosing between consumption (C) and investment (I) so as to maximize their discounted lifetime utility. Formally, the representative household maximizes $\int_0^\infty \ln C(t) e^{-\rho t} dt$ subject to

$$\dot{K}(t) = I(t) - \delta K(t) = rK(t) - C(t) - \delta K(t), \tag{4.1}$$

where δ is the depreciation rate of private capital. To save notations, in what follows I omit the time index whenever there could be no ambiguity. The current value Hamiltonian can be written as $H = \ln C + \gamma(rK - C - \delta K)$, where γ is a costate variable of K. The first-order condition yields the Euler equation[6]

$$\frac{\dot{C}}{C} = r - \delta - \rho, \forall t\, [0, \infty). \tag{4.2}$$

The transversality condition $\lim_{t\to\infty} \gamma K e^{-\rho t} = 0$ is required to ensure optimization.

4.2.2 Firms

The final good is assembled using the Cobb–Douglas technology

$$Y = X_a^b X_m^{1-b}, \tag{4.3}$$

where Y denotes the final output, and X_a and X_m denote the agricultural and manufacturing inputs, respectively. Given the intermediate prices p_a and p_m, the Cobb–Douglas technology with perfect competition yields

$$X_a = \frac{bpY}{p_a}, \quad X_m = \frac{(1-b)\,pY}{p_m}, \quad p = \frac{p_a^b p_m^{1-b}}{B}, \tag{4.4}$$

[6]Households behave as if they do not know the impact of capital accumulation on the rental of private capital. This is because there are a large number of households competing with each other in investment. The competition leads to the Nash equilibrium in which households make investment decisions only by the current level of capital rental, even though they have such knowledge.

where $B \equiv b^b (1-b)^{1-b}$ and p is the price of the final good. The final good is the numeraire, $p = 1$ and thus

$$p_a^b p_m^{1-b} = B. \tag{4.5}$$

The intermediate goods are produced using private capital:

$$Y_i = q_i K_i, \tag{4.6}$$

where Y_i and q_i denote respectively intermediate sector i's output and productivity, and K_i is the amount of private capital employed in that sector ($i = a, m$). Following Copeland and Taylor (1999), I assume that manufacturing productivity q_m is exogenously given and agriculture productivity q_a satisfies

$$q_a = q_a(V), \quad q_a(0) = 0, \quad q_a'(V) > 0, \tag{4.7}$$

where V is the stock of environmental capital measuring the quality of environment. Under perfect competition, firms producing the intermediate goods maximize their profits by taking the environment as given; thus, $p_i q_i = r$ holds as long as sector i is active ($K_i > 0$).

Private capital is freely and instantaneously mobile across sectors. Therefore, for the intermediate goods firms, the marginal rate of transformation (MRT) is $q_a(V)/q_m$. Since the environmental capital stock V is fixed in the short run, the MRT is also fixed, implying a short-run Ricardian structure for the model. This can also be obtained by substituting (4.6) into the clearing condition of private capital, so as to obtain $K = K_a + K_m = Y_a/q_a(V) + Y_m/q_m$, which gives the linear production possibility frontier (of intermediate outputs) in the short run. Note that since both K and V can vary over time, a feasible production schedule in the short run is not necessarily feasible in the long run.

4.2.3 The Environment

The use of private capital leads to pollution:

$$Z_i = \omega_i K_i, \tag{4.8}$$

where Z_i is the flow of pollution from sector i, and $\omega_i > 0$ gives the sector-specific pollution intensity. If $\omega_a < \omega_m$ ($\omega_a > \omega_m$), manufacturing (agriculture) is dirtier in that a unit of private capital employed causes more pollution in that sector. A positive ω_a captures the fact that some environmentally sensitive industries such as agriculture also contribute to significant pollution and wastes. The total flow of pollution can be shown as

$$Z = Z_a + Z_m = \omega_a K_a + \omega_m K_m. \tag{4.9}$$

In this study, I focus on country-specific environment. Following Copeland and Taylor (1999), the stock of environmental capital evolves according to

$$\dot{V} = g\left(\bar{V} - V\right) - Z, \tag{4.10}$$

where g and $\bar{V}$ are the environment's recovery rate and carrying capacity, respectively, and $g\left(\bar{V} - V\right)$ can be seen as its natural growth.

4.2.4 Consumption Function and Dynamic System

The dynamics of the economy can be characterized by three equations–equation (4.1) private capital accumulation, equation (4.2) the Euler equation, and equation (4.10) environmental capital evolution–and a transversality condition. Given the initial stocks (K_0, V_0), the initial consumption, C_0, can be pinned down by using the transversality condition. Then I can derive the transition path from (K_0, V_0) by solving (4.1), (4.2), and (4.10) with the initial condition (K_0, V_0, C_0).

Although a closed form of C_0 is usually not available, the logarithmic form of the utility function in our model implies that $C_0 = \rho K_0$, and consequently[7]

$$C = \rho K, \quad I = (r - \rho) K. \tag{4.11}$$

This gives us the saving rate $1 - \rho/r$. Households under laissez faire have a simple rule of investment: higher the rental of private capital, the more they invest.

Substituting (4.11) into (4.1) yields

$$\frac{\dot{K}}{K} = r - \delta - \rho. \tag{4.12}$$

That is, with the simple form of consumption function, only two equations describe the dynamics, equations (4.10) and (4.12), instead of the above three (plus the transversality condition). This significantly helps us simplify the analysis of transition dynamics.

4.3 Autarky

Before considering trade liberalization, focus on the closed economy. The analysis proceeds in two steps. First, to close the dynamic system, I use K and V to express the

[7] To see this, obtain $K(t) = e^{\int_0^t (r(\sigma)-\delta)d\sigma}\left(K_0 - \int_0^t C(s)\, e^{-\int_0^s (r(\sigma)-\delta)d\sigma} ds\right)$ from private capital accumulation (4.1) and $C(t) = C_0 e^{\int_0^t (r(\sigma)-\rho-\delta)d\sigma}$ from the Euler equation (4.2). Here, time index t, σ, and s appear explicitly to prevent ambiguity. Let $t = s$ in the expression of $C(t)$ and substitute it into $K(t)$ for $C(s)$ to obtain $K(t) = e^{\int_0^t (r(\sigma)-\delta)d\sigma}\left(K_0 - C_0 \int_0^t e^{-\rho s} ds\right)$. The first-order condition $\partial H/\partial C = 0$ gives $\gamma(t) = 1/C(t)$. Substituting these results into the transversality condition $\lim_{t\to\infty} \gamma(t) K(t) e^{-\rho t} = 0$ yields $\lim_{t\to\infty}\left(K_0/C_0 - \int_0^t e^{-\rho s} ds\right) = 0$ and consequently $C_0 = \rho K_0$, which can be plugged back into the expression of $K(t)$ to obtain $K(t) = K_0 e^{\int_0^t (r(\sigma)-\rho-\delta)d\sigma}$. Comparing this with the expression of $C(t)$ gives $C(t) = \rho K(t)$.

rental r and flow of pollution Z, respectively, in (4.10) and (4.12). I then examine the steady state and transition dynamics.

4.3.1 The Dynamic System

In autarky, demand is met by domestic supply. The clearing condition requires $X_i = Y_i$ $(i = a, m)$, which along with (4.4) and (4.6) gives the allocation of private capital:

$$K_a = bK, \quad K_m = (1-b)\,K. \tag{4.13}$$

The intermediate output follows directly: $X_a = q_a(V)\,bK$ and $X_m = q_m(1-b)\,K$, implying that

$$Y = B\left(q_a(V)\right)^b q_m^{1-b} K, \quad r = \frac{Y}{K} = B\left(q_a(V)\right)^b q_m^{1-b}. \tag{4.14}$$

From (4.8) and (4.13), the flow of pollution is

$$Z = \left(b\omega_a + (1-b)\,\omega_m\right) K \equiv \Omega K. \tag{4.15}$$

Therefore, in autarky, this model is equivalent to the model in which there is no intermediate goods, but a single final good produced using private capital with productivity $B\left(q_a(V)\right)^b q_m^{1-b}$ and pollution intensity Ω.

By substituting (4.14) into (4.12) and (4.15) into (4.10), I obtain

$$\frac{\dot{K}}{K} = B\left(q_a(V)\right)^b q_m^{1-b} - \delta - \rho, \tag{4.16}$$

$$\dot{V} = g\left(\bar{V} - V\right) - \Omega K, \tag{4.17}$$

which provide a complete description of the dynamics of the economy.

4.3.2 Autarky Steady State

We can easily obtain the autarky steady state by letting $\dot{K} = 0$ and $\dot{V} = 0$ in equations (4.16) and (4.17):

$$K^A = \frac{g}{\Omega}\left(\bar{V} - V^A\right), \quad V^A = q_a^{-1}\left(\left(\frac{\rho+\delta}{Bq_m^{1-b}}\right)^{\frac{1}{b}}\right), \tag{4.18}$$

where the superscript A denotes the values in autarky steady state and $q_a^{-1}(\cdot)$ denotes the inverse function of $q_a(\cdot)$. Note that the first equation in (4.18) represents a downward-sloping line in the (K, V) plane, and the second represents a horizontal line. Assume that $\bar{V}$ satisfies

$$Bq_a^b\left(\bar{V}\right) q_m^{1-b} - \delta - \rho > 0, \tag{4.19}$$

which ensures the existence of positive (K^A, V^A).[8] From this, it is clear that (K^A, V^A) is uniquely determined. From (4.11), consumption in the autarky steady state can simply be shown as

$$C^A = \rho K^A. \tag{4.20}$$

The stability of (K^A, V^A) in the dynamic system (4.16) and (4.17) can be verified by calculating the Jacobian at (K^A, V^A):

$$J^A = \begin{bmatrix} 0 & J_{12}^A \\ -\Omega & -g \end{bmatrix}, \tag{4.21}$$

where $J_{12}^A \equiv Bbq_a'\left(V^A\right)\left(q_a\left(V^A\right)\right)^{b-1} q_m^{1-b} K^A > 0$. From (4.21), it follows that $\det J^A = \Omega J_{12}^A > 0$ and $\mathrm{tr} J^A = -g < 0$, implying that (K^A, V^A) is locally stable.[9] We can also easily verify that (K^A, V^A, C^A) is locally saddle stable in the dynamic system (4.1), (4.2), and (4.10).

Note that $q_a(\cdot)$ is strictly increasing, and so is the inverse function $q_a^{-1}(\cdot)$. From (4.18), this suggests that the environmental quality in autarky steady state, V^A, improves with δ and ρ, and worsens with q_m. The intuition is as follows: a lower δ means more durable private capital, whereas a higher q_m implies higher rental of private capital. Both render investment more profitable. On the other hand, a lower ρ means that households care more about the future, and thus tend to consume less (and invest more) at present. All these induce more private capital in the autarky steady state, which means a worse environment at the same time. From (4.18), δ, ρ, and q_m affect K^A only through V^A, and K^A is negatively related to V^A; thus, these parameters have opposite effects on K^A.

In contrast, V^A is independent of environmental endowments ($\bar{V}$ and g) and pollution intensities (ω_i). This is so because households make investment decisions based only on the rental of private capital, which is independent of $\bar{V}$, g, and ω_i. Thus, countries with better environmental endowments (higher $\bar{V}$ or g) or cleaner technologies (smaller ω_i) have higher stock of private capital and hence consumption in the autarky steady state but the same environmental capital (as long as other parameters are the same). That is, households enjoy higher consumption by exploiting their environmental advantages through the channel of investment.

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[8] If $\bar{V}$ is so small that $Bq_a^b(\bar{V})\, q_m^{1-b} - \delta - \rho \leq 0$, the $\dot{V} = 0$ line cannot intersect with the $\dot{K} = 0$ line at a point satisfying $K > 0$, and any positive stock of private capital cannot sustain. This is of no special interest and hence excluded from the discussion.

[9] The stability is quite robust. Consider the more general form $\dot{V} = E(V, Z)$ instead of (4.10); the local stability holds as long as $\partial E/\partial V < 0$ and $\partial E/\partial Z < 0$ around the steady state.

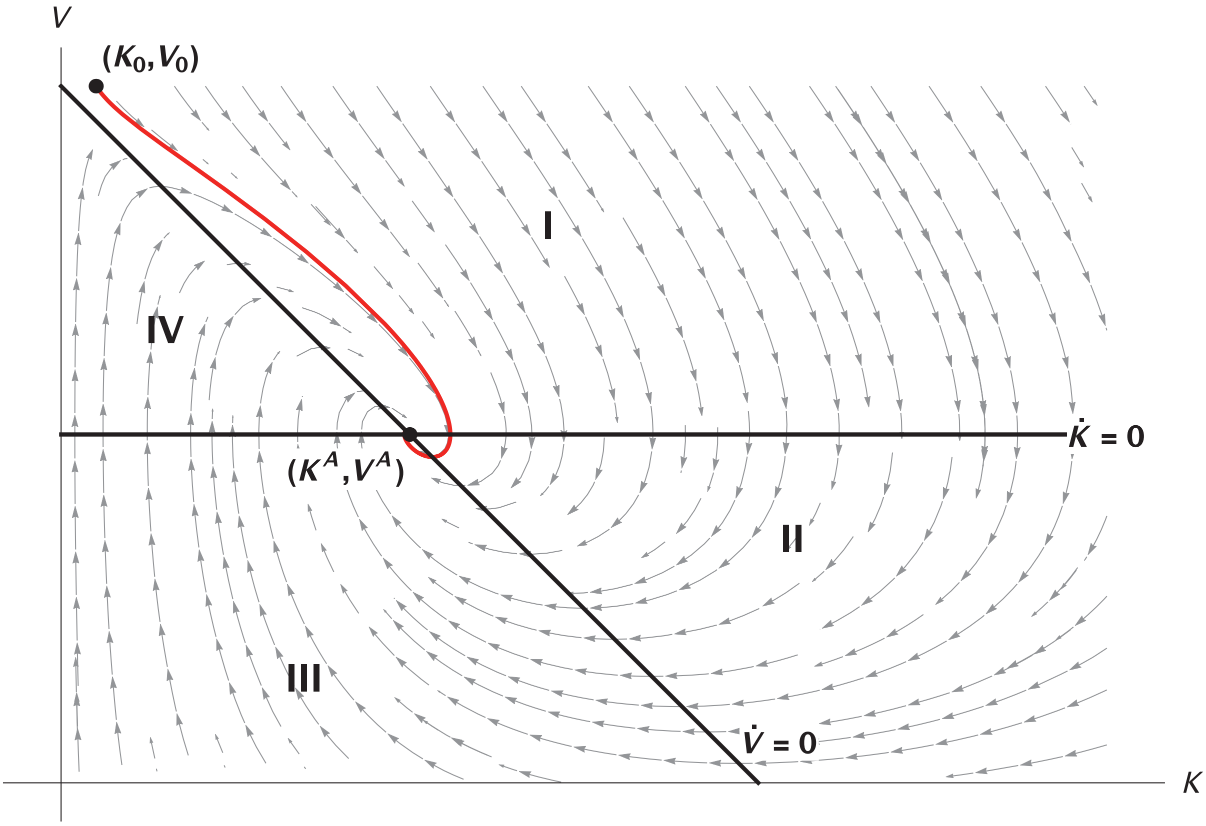


Figure 4.1: Autarky transition dynamics

4.3.3 Autarky Transition Dynamics

Figure 4.1 provides the phase diagram of the dynamic system (4.16) and (4.17). Starting from any initial value of (K, V), the transition dynamic in autarky is clearly illustrated by the streamlines in the figure. The figure shows two key lines: the horizontal $\dot{K} = 0$ line, and the downward-sloping $\dot{V} = 0$ line. Above (below) the $\dot{K} = 0$ line, private capital K increases (decreases) over time. Above (below) the $\dot{V} = 0$ line, environmental capital V decreases (increases) over time. The $\dot{K} = 0$ and $\dot{V} = 0$ lines divide the state plane into four quadrants.

Suppose that the economy starts from a point in quadrant I. In quadrant I, the stock of environmental capital K is relatively high and so the rental is high enough to attract sufficient investment for capital accumulation. On the other hand, capital accumulation raises the flow of pollution and in turn reduces environmental capital V. Thus, the trajectory of (K, V) takes the lower-right direction in quadrant I. It may cross the horizontal $\dot{K} = 0$ line or approach the steady state $\left(K^A, V^A\right)$.[10] If the former is the case, as the figure illustrates, the trajectory must enter quadrant II.

In quadrant II, the flow of pollution is still higher than the natural growth of the

[10]This depends on the sign of $\Delta \equiv g - 4b\,(\rho + \delta)\left(\bar{V} - V^A\right) q_a'\left(V^A\right) / q_a\left(V^A\right)$ and the position of the trajectory. Δ is the discriminant of the characteristic equation of (4.21). Around the steady state, (K, V) converges to $\left(K^A, V^A\right)$ along a straight line if $\Delta > 0$, and converges spirally if $\Delta < 0$.

environment, and so V continues to fall. On the other hand, the rental now falls below $\rho + \delta$, and households reduce their investment to below the maintenance level δK, resulting in a decline in K. This decline reduces the flow of pollution and consequently the speed at which V declines. Eventually, the trajectory leaves quadrant II and enters quadrant III.

In quadrant III, investment remains low and K continues to fall. However, the flow of pollution is now relatively low and so V starts to increase. Thus, the rental of private capital increases as well, and the system may enter quadrant IV or converge to the steady state. Again, the trajectory may enter quadrant IV or approach the steady state. The figure depicts the former case.

In quadrant IV, the rental becomes higher than $\rho + \delta$, again leading to capital accumulation. The flow of pollution is still relatively low, and V continues to increase over time. Eventually, the system enters quadrant I once again.

The following proposition summarizes the results in autarky:

Proposition 4.1. *A closed economy adopts the following characteristics:*

(i) The steady state is unique, locally stable, and satisfies (4.18) and (4.20).

(ii) The environment in closed economies converges to the same quality, regardless of difference in environmental endowments ($\bar{V}$ and g) and pollution intensities (ω_i).

(iii) If $\Delta < 0$, the trajectory of (K, V) follows a clockwise spiraling path.

4.4 Small Open Economy

Trade liberalization breaks down the correlation between domestic demand and supply (of intermediate goods), and changes the way capital accumulation interacts with the environment. In this section, I focus on a small open economy (SOE) to examine how the three elements–trade, capital accumulation, and the environment–interact with one another.

4.4.1 Trade Pattern and the Dynamic System in SOE

Let P denote the world relative price. Now, $P = p_m / p_a$ holds in an SOE, which along with (4.5) yields

$$p_a = BP^{b-1}, \quad p_m = BP^b. \tag{4.22}$$

To find the comparative advantage, we need to compare the MRT $q_a(V)/q_m$ with the world relative price P. If the MRT is higher (lower), the small economy has a comparative advantage in agriculture (manufacturing) and in free trade completely specializes in that good owing to the short-run Ricardian structure.

We can conveniently define V^W as

$$\frac{q_a\left(V^W\right)}{q_m} = P, \tag{4.23}$$

and so V^W can be seen as a measure of the environment for the rest of the world. The horizontal $V = V^W$ line divides the (K, V) plane into two regimes. First, the agriculture regime lies above the $V = V^W$ line, where $q_a(V)/q_m > P$ and the SOE completely specializes in agriculture. Here, the rental is determined only by agriculture productivity: $r = p_a q_a(V) = BP^{b-1} q_a(V)$. The flow of pollution is simply $Z = \omega_a K$.

Second, the manufacturing regime lies below the $V = V^W$ line is the manufacturing regime, where $q_a(V)/q_m < P$ and the economy completely specializes in manufacturing. The rental and flow of pollution are, respectively, $r = p_m q_m = BP^b q_m$ and $Z = \omega_m K$. On the $V = V^W$ line, $r = BP^{b-1} q_a\left(V^W\right) = BP^b q_m$, and so employed in whichever sector, there is no difference for private capital. To get around the indeterminacy in this knife-edge case, I simply assume that no trade would arise if $V = V^W$, meaning that $Z = \Omega K$ if $V = V^W$. By substituting these results into (4.10) and (4.12), I obtain the dynamic equations governing the motion of K and V in an SOE:

$$\frac{\dot{K}}{K} = \begin{cases} BP^{b-1} q_a(V) - \delta - \rho & \text{if } V > V^W, \\ BP^b q_m - \delta - \rho & \text{if } V = V^W, \\ BP^b q_m - \delta - \rho & \text{if } V < V^W, \end{cases} \tag{4.24}$$

$$\dot{V} = \begin{cases} g\left(\bar{V} - V\right) - \omega_a K & \text{if } V > V^W, \\ g\left(\bar{V} - V\right) - \Omega K & \text{if } V = V^W, \\ g\left(\bar{V} - V\right) - \omega_m K & \text{if } V < V^W. \end{cases} \tag{4.25}$$

4.4.2 Small Open Economies: Four Types

Equations (4.24) and (4.25) suggest the crucial role of the signs of $BP^b q_m - \delta - \rho$ and $\omega_a - \omega_m$ in determining the nature of the dynamics. Let $P^A \equiv p_m^A / p_a^A$ denote the relative price in autarky steady state. Now, by (4.18), $P^A = \left((\delta + \rho)/Bq_m\right)^{1/b}$, implying that $P^A < P$ ($P^A > P$) is equivalent to $BP^b q_m > \delta + \rho$ ($BP^b q_m < \delta + \rho$). I consider four SOE types.[11]

$$P^A < P, \quad \omega_a < \omega_m, \tag{MM type}$$

$$P^A < P, \quad \omega_a > \omega_m, \tag{MA type}$$

$$P^A > P, \quad \omega_a < \omega_m, \tag{AM type}$$

$$P^A > P, \quad \omega_a > \omega_m, \tag{AA type}$$

[11] The knife-edge events $P = P^A$ and $\omega_a = \omega_m$ are excluded since they are not of special interest.

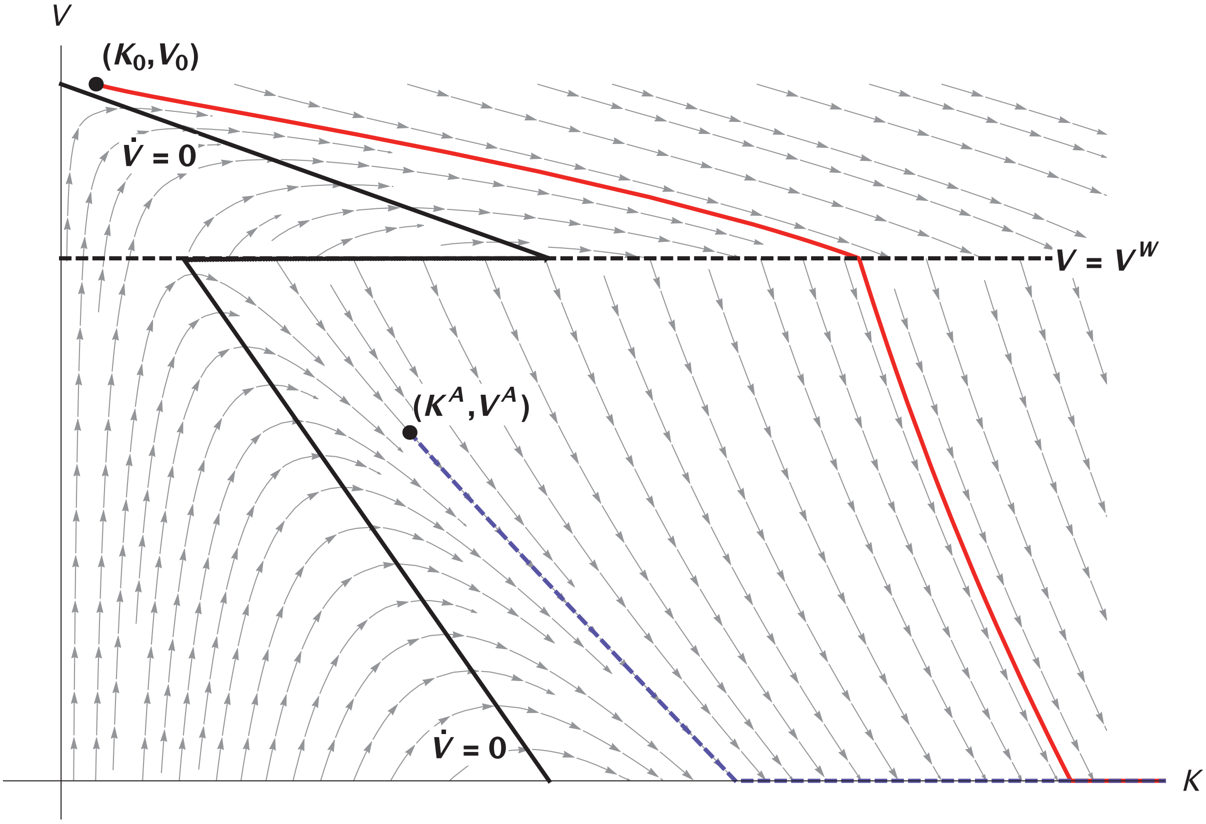


Figure 4.2: Transition dynamics in an MM type SOE

The economic implications of the types are straightforward: $P^A < P$ ($P^A > P$) implies a pre-trade comparative advantage in manufacturing (agriculture), whereas $\omega_a < \omega_m$ ($\omega_a > \omega_m$) implies that manufacturing (agriculture) is dirtier.

4.4.3 MM Type SOE

From equation (4.24) and the definition of MM type SOE, it follows that $\dot{K} > 0$ if $V \leq V^W$. Moreover, $q_a(V)/q_m > P$ and $BP^b q_m - \delta - \rho > 0$ together yield $BP^{b-1} q_a(V) > BP^b q_m > \delta + \rho$, implying that $\dot{K} > 0$ if $V > V^W$. Therefore, no steady state exists in an MM type SOE.

Figure 4.2 illustrates the dynamics in an MM type SOE. Differing from autarky, the $\dot{V} = 0$ line now has the slope $-\omega_a/g$ above the $V = V^W$ line (by specializing in agriculture) and the slope $-\omega_m/g$ below the $V = V^W$ line (by specializing in manufacturing). Since $\omega_a < \omega_m$, the lower segment is steeper. Furthermore, differing from autarky, no $\dot{K} = 0$ line exists since $\dot{K} > 0$ holds on the whole (K, V) plane.

Suppose that the SOE starts from (K_0, V_0), as illustrated in Figure 4.2. The environment is initially good, with two consequences. First, the SOE has a comparative advantage in agriculture and thus specializes in it. Second, the natural growth of the environment is too slow to surpass the flow of pollution (reflected in the figure showing (K_0, V_0) above the $\dot{V} = 0$ line), and thus the environment degrades over time.

Since the trajectory of (K, V) goes in the lower-right direction, sooner or later it crosses the $V = V^W$ line. Thus, the SOE loses its comparative advantage in agriculture and specializes in manufacturing, where the rental remains high enough to support capital accumulation (at the growth rate of $BP^b q_m - \delta - \rho$). The flow of pollution grows at the same rate. This scale effect eventually drives the stock of environmental capital to zero, that is, it leads to destruction of the environment.

The gray arrowed streamlines in the figure provide a general picture of trajectories from other positions. For example, from the autarky steady state $\left(K^A, V^A\right)$, since $V^A < V^W$, the small economy specializes in manufacturing right after trade liberalization. Since manufacturing is dirtier, trade has a negative composition effect on the environment. The trajectory of (K, V) takes the lower-right direction and eventually hits the horizontal axis. However, since consumption grows over time, there are welfare gains from trade.

4.4.4 MA Type SOE

Following steps similar to those for the MM type SOE, I can show that $\dot{K} > 0$ always holds in an MA type SOE, but with two significant differences. First, the $\dot{V} = 0$ line has a steeper segment above the $V = V^W$ line and hence the trajectory of (K, V) experiences the sliding mode if it approaches the part of the $V = V^W$ line that lies in-between the $\dot{V} = 0$ line. Around there, as illustrated in Figure 4.3, trajectories are attracted into and slide right along the $V = V^W$ line, until they reach the intersection of the $V = V^W$ line and the $\dot{V} = 0$ line. Second, $\left(K^A, V^A\right)$ lies below the $V = V^W$ line. Thus, if the economy starts from the autarky steady state, it specializes in the cleaner manufacturing sector and the environment improves right after trade liberalization.

The following proposition summarizes these discussions on the MM type and MA type SOEs:

Proposition 4.2. *In free trade, an SOE with pre-trade comparative advantage in manufacturing (MM type or MA type) adopts the following characteristics:*

(i) The economy specializes in either sector right after trade liberalization, depending on the initial condition, but will eventually specialize in manufacturing, and there is no steady state.

(ii) Environmental capital increases or decreases right after trade liberalization, depending on the initial condition, but will eventually be destroyed;

(iii) Private capital and consumption increase over time, eventually growing at the rate of $BP^b q_m - \delta - \rho$; there are welfare gains from trade.

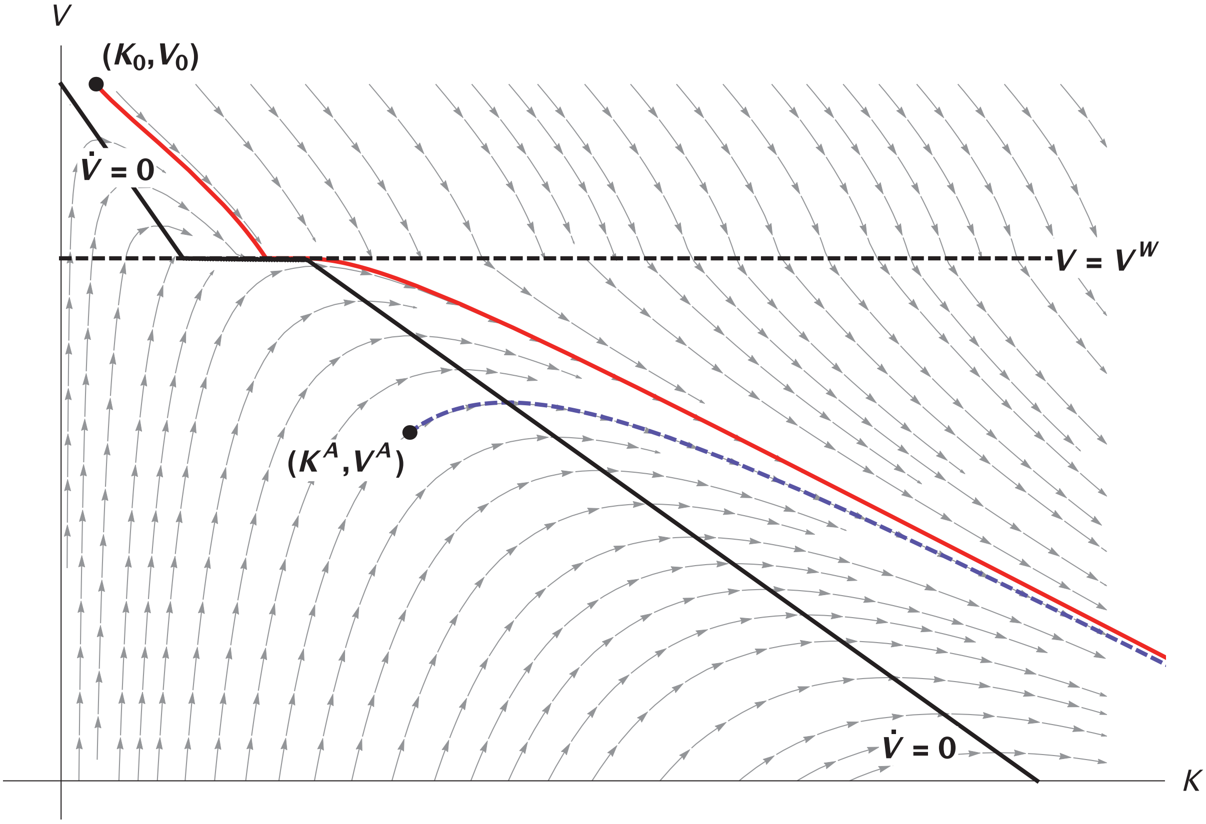


Figure 4.3: Transition dynamics in an MA type SOE

4.4.5 AM Type SOE

From (4.24) and the definition of AM type SOE, it follows that $\dot{K} < 0$ if $V \leq V^W$. On the other hand, the assumption on $\bar{V}$, equation (4.19), implies that $BP^{b-1}q_a(\bar{V}) > \rho + \delta$.[12] Since $BP^{b-1}q_a\left(V^W\right) = BP^b q_m < \rho + \delta$, there exists $V^S \in \left(V^W, \bar{V}\right)$ such that $BP^{b-1}q_a\left(V^S\right) = \rho + \delta$. Let $\dot{K} = \dot{V} = 0$ in (4.24) and (4.25); now, solving for (K, V), I obtain

$$K^S = \frac{g}{\omega_a}\left(\bar{V} - V^S\right), \quad V^S = q_a^{-1}\left(\frac{\rho + \delta}{BP^{b-1}}\right), \tag{4.26}$$

which is clearly uniquely determined. We can also use (4.18) and (4.23) to obtain

$$q_a\left(V^S\right) = \left(q_a\left(V^A\right)\right)^b \left(q_a\left(V^W\right)\right)^{1-b}, \tag{4.27}$$

which implies that $V^S \in \left(V^W, V^A\right)$. The consumption is, using (4.11),

$$C^S = \rho K^S. \tag{4.28}$$

Since $\omega_a < \Omega$ in the AM type SOE, $V^S < V^A$ implies that $K^S > K^A$ and consequently $C^S > C^A$.

[12] First, $\bar{V} > V^A$ implies that $BP^{b-1}q_a(\bar{V}) > BP^{b-1}\left(q_a(\bar{V})\right)^b\left(q_a\left(V^A\right)\right)^{1-b} = BP^{b-1}\left(q_a(\bar{V})\right)^b\left(P^A q_m\right)^{1-b}$. Second, $P^A > P$ gives that $BP^{b-1}\left(q_a(\bar{V})\right)^b\left(P^A q_m\right)^{1-b} > BP^{b-1}\left(q_a(\bar{V})\right)^b\left(P q_m\right)^{1-b} = B\left(q_a(\bar{V})\right)^b q_m^{1-b}$, which is greater than $\rho + \delta$ according to (4.19). Therefore, $BP^{b-1}q_a(\bar{V}) > \rho + \delta$.

We can check the stability by calculating the Jacobian of (4.24) and (4.25) at $\left(K^S, V^S\right)$:

$$J^S = \begin{bmatrix} 0 & J_{12}^S \\ -\omega_a & -g \end{bmatrix}, \tag{4.29}$$

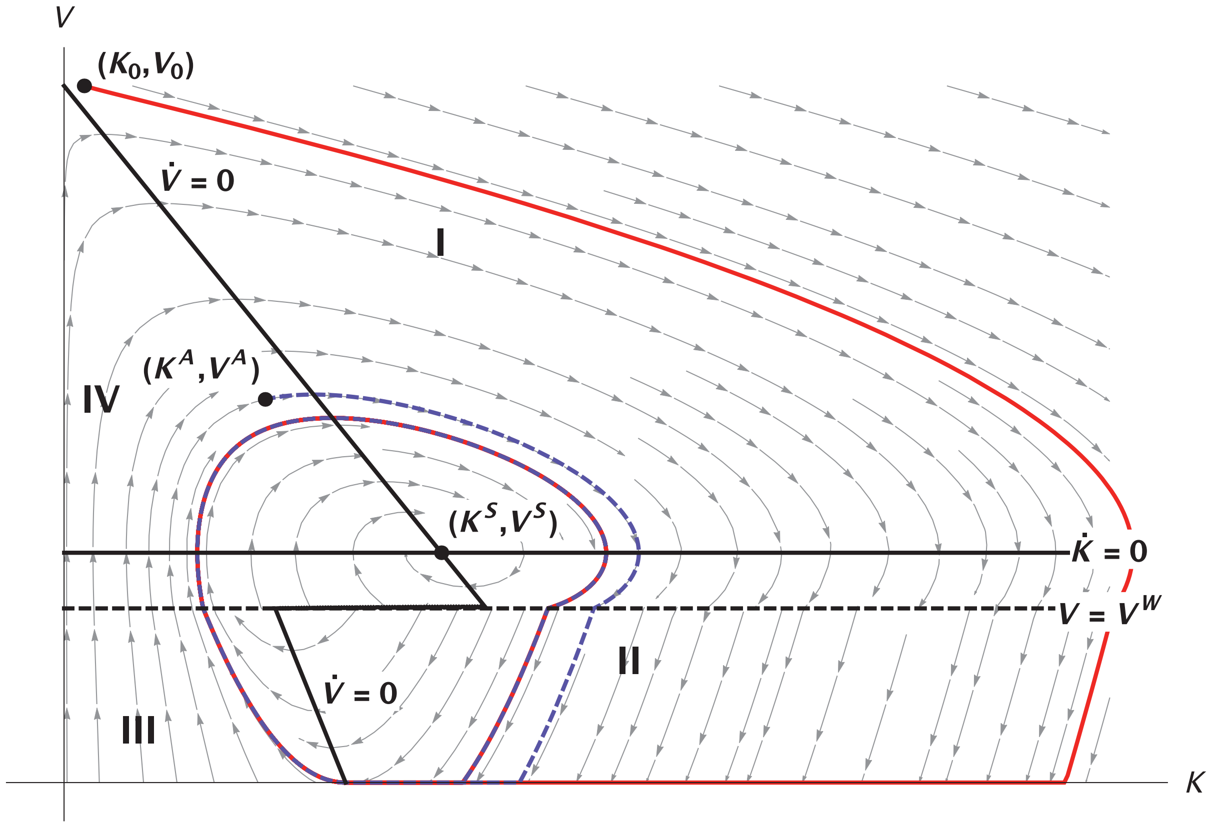
where $J_{12}^S \equiv BP^{b-1}q_a'\left(V^S\right)K^S > 0$. The local stability around $\left(K^S, V^S\right)$ follows directly from $\det J^S = \omega_a J_{12}^S > 0$ and $\mathrm{tr} J^S = -g < 0$. We can also easily check the local saddle stability of $\left(K^S, V^S, C^S\right)$.

Figure 4.4 illustrates the dynamics in an AM type SOE. Note that, depending on parameters, a periodic orbit could exist to which the trajectory of (K, V) can converge. Figure 4.4a corresponds to the case in which there exists a periodic orbit, and Figure 4.4b corresponds to the case of no such orbit. In both cases, the state plane is divided into four quadrants by the horizontal $\dot{K} = 0$ line and the downward-sloping $\dot{V} = 0$ line. The trajectory of (K, V) goes toward the lower-right, lower-left, upper-left, and upper-right directions in quadrant I, II, III, and IV, respectively. Since $V^S > V^W$, the $V = V^W$ line lies in quadrants II and III.

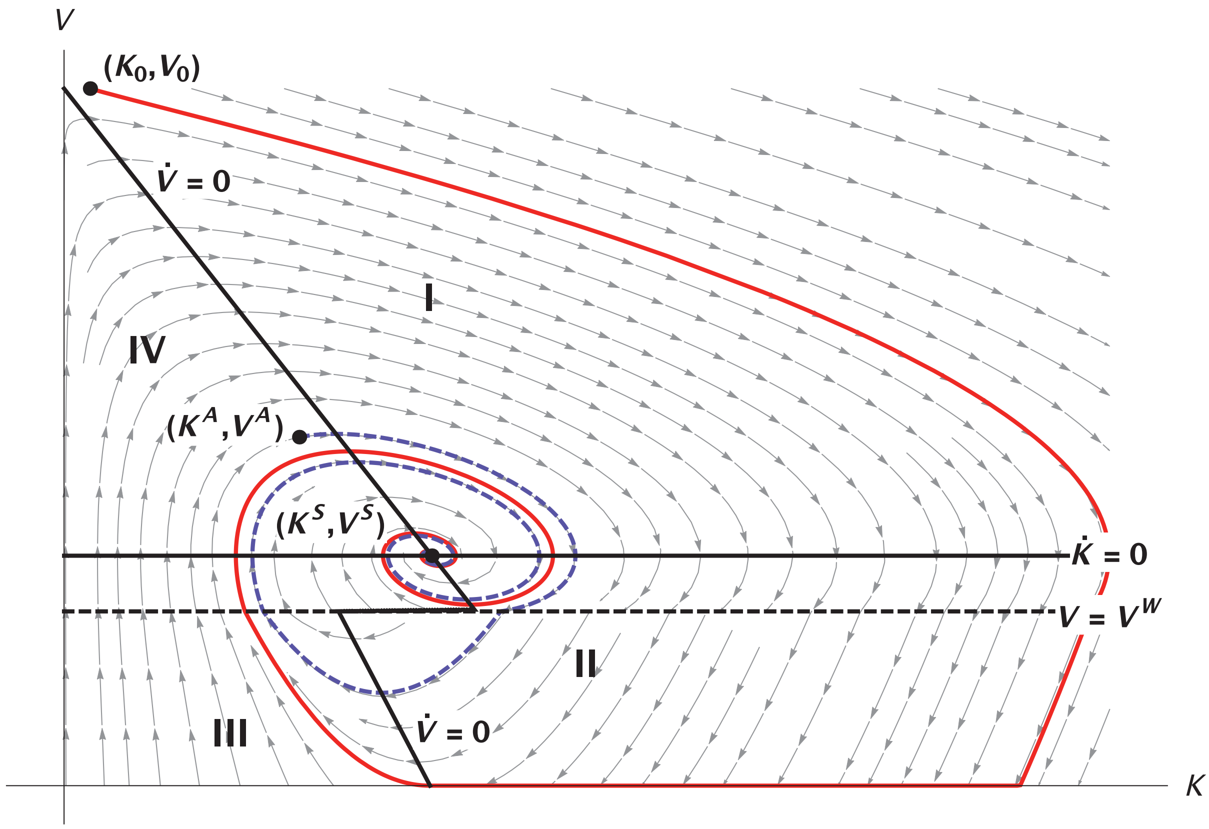
Starting from a point in quadrant I, say (K_0, V_0) in Figure 4.4a or Figure 4.4b, the small economy has a comparative advantage in agriculture and specializes in it since V_0 is relatively high. The rental is relatively high and therefore sustains capital accumulation. On the other hand, the natural growth of the environment is too slow to surpass the flow of pollution, and so the trajectory of (K, V) goes in the lower-right direction in quadrant I. Again, depending on the distance to $\left(K^S, V^S\right)$ and parameters, it may converge to $\left(K^S, V^S\right)$ or enter quadrant II by crossing the $\dot{K} = 0$ line. If the latter is the case, as illustrated in both Figure 4.4a and Figure 4.4b, the SOE continues to specialize in agriculture, but by now private capital starts to decline, because environmental deterioration cancels out the trade premium of capital rental such that it becomes too low to attract sufficient investment to sustain capital accumulation. Thus, the trajectory moves in the lower-left direction, and depending on the position, it might cross the $V = V^W$ line or the $\dot{V} = 0$ line. If the trajectory crosses the $V = V^W$ line, as illustrated in both Figure 4.4a and Figure 4.4b, the economy specializes in manufacturing and the trajectory continues to move in the lower-left direction, and eventually crosses the $\dot{V} = 0$ line and enters quadrant III.[13] Thereafter, the environment improves over time and the trajectory crosses the $V = V^W$ line and, consequently, the $\dot{K} = 0$ line, and enters quadrant IV. In quadrant IV, private capital as well as environmental capital increases over time, and the trajectory moves in the upper-right direction until it enters quadrant I.

If starting from the autarky steady state $\left(K^A, V^A\right)$, the SOE has a pre-trade comparative advantage in agriculture and specializes in it. Because agriculture is cleaner,

[13]The trajectory may first hit the horizontal K axis (thus, $V = 0$), then go left along the axis until it hits the $\dot{V} = 0$ line.



(a) The case with a periodic orbit



(b) The case with no periodic orbit

Figure 4.4: Transition dynamics in an AM type SOE

the composition effect enhances the environment right after trade liberalization. On the other hand, trade increases the rental of private capital and stimulates capital accumulation, and the trajectory of (K, V) moves in upper-right direction and eventually enters quadrant I. Thereafter, it proceeds similar to the manner discussed in the previous paragraph.

Note that although both transition paths converge to a periodic orbit in Figure 4.4a, this need not necessarily be the case. The result depends on parameters, and clearly on the initial condition too. For example, if (K_0, V_0) is close to $\left(K^S, V^S\right)$, local stability ensures that the economy converges to $\left(K^S, V^S\right)$.

The welfare effect of trade is ambiguous since consumption during the transition dynamics in free trade need not necessarily always be higher than that in the autarky steady state. However, since consumption increases initially when starting from the autarky steady state, there are welfare gains in trade given that the time preference ρ is sufficiently large. On the other hand, since the steady-state consumption in free trade is higher than that in autarky, there are welfare gains from trade if ρ is sufficiently small and the economy converges to the steady state.

4.4.6 AA Type SOE

An AA type SOE bears some resemblances to an AM type SOE. Furthermore, a unique, locally saddle stable steady state exists, in which the economy specializes in agriculture and satisfies (4.26) and (4.28). Moreover, the environment in trade steady state is worse than that in autarky steady state. The main differences are the following. First, the $\dot{V} = 0$ line has a steeper segment above the $V = V^W$ line since now agriculture is dirtier. Second, starting from the autarky steady state, the environment degrades when trade is opened, as illustrated in Figure 4.5.

Note that although $K^S < K^A$ in Figure 4.5, this need necessarily be so. From (4.18) and (4.26), it follows that the sign of $K^S - K^A$ is determined by

$$\Omega\left(V^A - V^S\right) - (1-b)\left(\omega_a - \omega_m\right)\left(\bar{V} - V^A\right). \tag{4.30}$$

In an AM type SOE, the first term in (4.30) is positive whereas the second term is negative. However, in an AA type SOE, $\omega_a > \omega_m$ and so the second term becomes positive, rendering the sign of (4.30) indeterminate. Since $V^S \in \left(V^W, V^A\right)$, V^S is close to V^A if V^W is close to V^A (small pre-trade comparative advantage). Thus, the second term in (4.30) tends to dominate, and therefore $K^S < K^A$ is likely to hold. In contrast, if ω_a is close to ω_m (similar pollution intensities), the first term tends to dominate, and therefore $K^S > K^A$ is likely to hold. The intuition comes from realizing two forces working in opposite directions. The first force is the decline in the capacity of hosting private capital by specializing in dirtier agriculture. The second force is the increase in

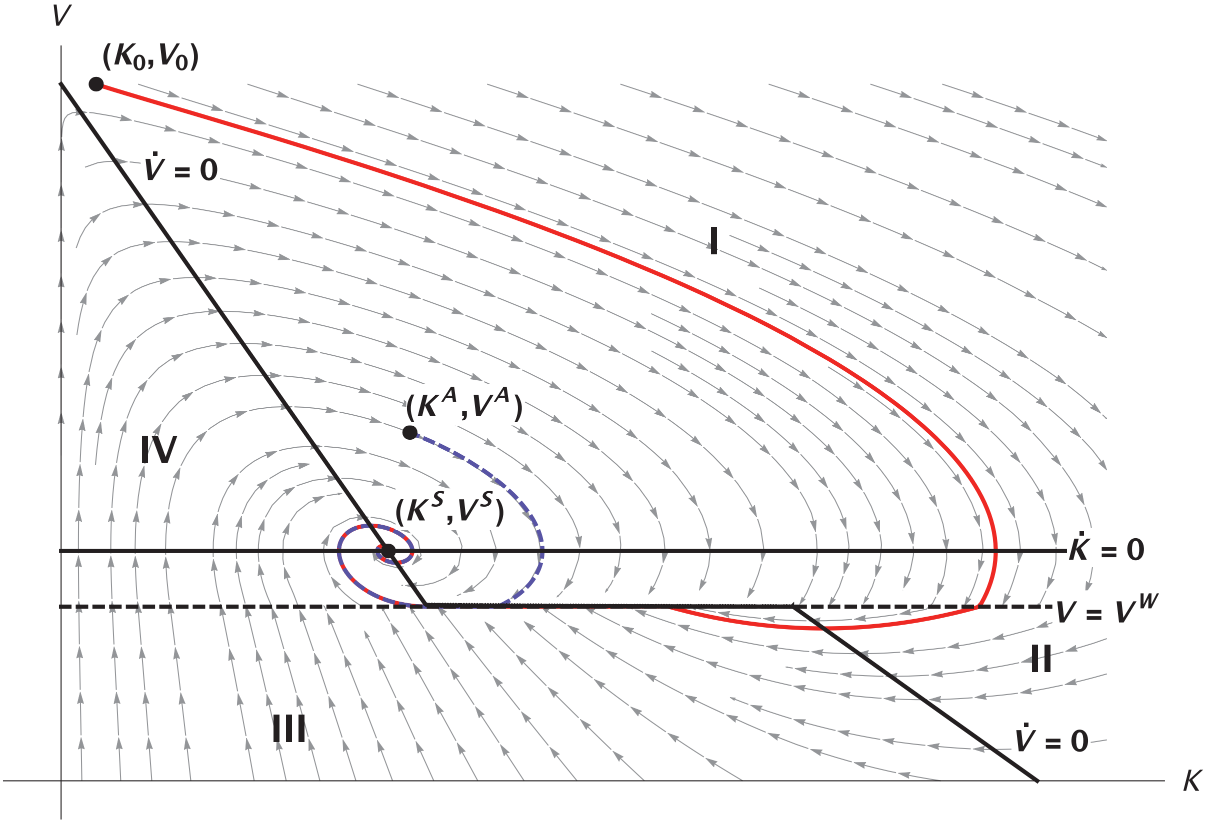


Figure 4.5: Transition dynamics in an AA type SOE

investment driven by the increase in capital rental. If agriculture and manufacturing are similarly dirty, the first force is weak. If the pre-trade comparative advantage is small, the second force is weak.

The following proposition summarizes these discussions on the AM type and AA type SOEs:

Proposition 4.3. *In free trade, an SOE with pre-trade comparative advantage in agriculture (AM type or AA type) adopts the following characteristics:*

(i) The economy specializes in either sector right after trade liberalization, depending on the initial condition, but will specialize in agriculture in the steady state, which is unique, locally stable, and satisfies (4.26) and (4.28); there could be a periodic orbit in the AM type SOE.

(ii) Environmental capital increases or decreases right after trade liberalization, depending on the initial condition, but will degrade in the steady state (compared to the autarky steady state);

(iii) Private capital and consumption can increase or decrease right after trade liberalization, but will increase in the steady state in the AM type SOE (ambiguous in the AA type SOE); the welfare gains from trade are ambiguous.

4.5 Optimal Policy in Autarky

Our analysis so far focuses on laissez faire, which is clearly not socially optimal because of two sources of externalities. First, numerous intermediate goods firms maximize their profit by taking the stock of environmental capital as given. Second, numerous households make investment decisions by taking the rental as given. The former results in overproduction of the dirtier goods, whereas the latter leads to excessive investment. To achieve the social optimum, government interventions are expected. In this section, I characterize the optimal policy in autarky by considering the social planner problem and replicating it in a market-based economy.

4.5.1 Social Planner Problem in Autarky

The social planner chooses consumption C, investment I, and private capital allocation $K_i \geq 0$ ($i = a, m$) to maximize $\int_0^\infty \ln C e^{-\rho t} dt$, subject to private capital accumulation $\dot{K} = I - \delta K$; environmental capital evolution (4.10); pollution flow (4.9); technologies (4.3), (4.6) and (4.7); and material constraints $C + I \leq Y$, $X_i \leq Y_i$ ($i = a, m$) and $K_a + K_m \leq K$.

Since consumption is always valuable, $C + I \leq Y$ and $X_i \leq Y_i$ ($i = a, m$) bind in optimum. Now, substitute $C + I = Y$ into $\dot{K} = I - \delta K$ to eliminate I and obtain $\dot{K} = Y - C - \delta K$. The Hamiltonian can then be written as[14]

$$
\begin{aligned}
H = \ln C &+ \gamma \left(Y - C - \delta K\right) + \lambda \left(g\left(\bar{V} - V\right) - Z\right) \\
&- \tau\gamma\left(\omega_a K_a + \omega_m K_m - Z\right) - r\gamma\left(K_a + K_m - K\right) - p\gamma\left(Y - X_a^b X_m^{1-b}\right) \\
&- p_a\gamma\left(Y_a - q_a\left(V\right)K_a\right) - p_a^d\gamma\left(X_a - Y_a\right) - p_m\gamma\left(Y_m - q_m K_m\right) - p_m^d\gamma\left(X_m - Y_m\right),
\end{aligned}
\tag{4.31}
$$

where γ and λ are costate variables of K and V, measuring respectively the values (in terms of utility) of the increases in private capital and environmental capital. Lagrange multipliers τ, r, p, p_a, p_a^d, p_m, and p_m^d measure respectively the shadow prices of pollution flow, private capital service, final good, agriculture supply, agriculture demand, manufacturing supply, and manufacturing demand. They are multiplied by γ such

[14] The first-order condition includes $\gamma = 1/C$ (from $\partial H/\partial C = 0$), $p = 1$ (from $\partial H/\partial Y = 0$), $\lambda = \tau\gamma$ (from $\partial H/\partial Z = 0$), $p_a = p_a^d = b X_a^{b-1} X_m^{1-b}$ (from $\partial H/\partial Y_a = 0$ and $\partial H/\partial X_a = 0$), $p_m = p_m^d = (1-b) X_a^b X_m^{-b}$ (from $\partial H/\partial Y_m = 0$ and $\partial H/\partial X_m = 0$), $p_a q_a(V) = r + \tau\omega_a$ (from $\partial H/\partial K_a = 0$), $p_m q_m = r + \tau\omega_m$ (from $\partial H/\partial K_m = 0$), the Kuhn–Tucker condition $r \geq 0$, $K_a + K_m - K \leq 0$, $r(K_a + K_m - K) = 0$, and $\dot{\gamma} = (\rho + \delta - r)\gamma$ (from the Euler equation $\partial H/\partial K = \rho\gamma - \dot{\gamma}$), $\dot{\lambda} = (\rho + g)\lambda - p_a\gamma q_a'(V) K_a$ (from the Euler equation $\partial H/\partial V = \rho\lambda - \dot{\lambda}$). The non-negative constraint of K_i ($i = a, m$) is not explicitly considered because both the intermediate goods are essential, and thus in optimum $K_i > 0$ must hold in autarky. In addition to the first-order condition, the transversality conditions $\lim_{t\to\infty} \gamma K e^{-\rho t} = 0$ and $\lim_{t\to\infty} \lambda V e^{-\rho t} = 0$ are required to pin down the optimal path.

that the shadow price of final good $p = 1$ and other shadow prices are measured in terms of the final good. In doing so, the final good plays the role of the numeraire.

4.5.2 Optimal Policy: A Dynamic Pigouvian Tax on Pollution

The Euler equation for K suggests that (4.2) still holds, whereas the Euler equation for V, using $\lambda = \tau\gamma$ from the first-order condition, yields

$$\dot{\tau} = \left(\rho + g - \frac{\dot{\gamma}}{\gamma}\right)\tau - p_a q_a'(V) K_a. \tag{4.32}$$

It follows that $\tau = (\gamma(0)/\gamma(t)) e^{(\rho+g)t}\left(\tau_0 - \int_0^t (\gamma(s)/\gamma(0)) p_a q_a'(V) K_a e^{-(\rho+g)s} ds\right)$, where τ_0 is the initial value of τ. The time index follows γ to prevent confusion. Plug the expression of τ into the transversality condition $\lim_{t\to\infty} \lambda V e^{-\rho t} = 0$, again using $\lambda = \tau\gamma$ from the first-order condition, to obtain $\tau_0 = \int_0^\infty (\gamma(s)/\gamma(0)) p_a q_a'(V) K_a e^{-(\rho+g)s} ds$, which can be plugged back to derive

$$\tau^* = \frac{1}{\gamma(t)} \int_t^\infty \gamma(s) p_a q_a'(V) K_a e^{-(\rho+g)(s-t)} ds. \tag{4.33}$$

The economic meaning behind (4.33) is as follows. An additional unit of pollution flow at time t, according to (4.10), leads to a unit of reduction in environmental capital, which could harm future agriculture productivities. If the environment cannot recover by itself, the consequence of the reduction remains for the future, leading to $q_a'(V(s))$ units of decline in agriculture productivity at time s, and, in turn, $p_a(s) q_a'(V(s)) K_a(s)$ units of income loss. However, the environment can heal itself (by converging to the capacity level at speed g), and so a unit of pollution flow at time t leads to $p_a(s) q_a'(V(s)) K_a(s) e^{-g(s-t)}$ units of income loss at time s. Its present value can be obtained in three steps. First, multiply it by $\gamma(s)$ to convert its unit from the final good to utility. Second, multiply the result from the first step by $e^{-\rho(s-t)}$ to obtain its present value (still in utility unit). Third, divide the result from the second step by $\gamma(t)$ to get back the final good unit. The three steps yield $\gamma(s) p_a(s) q_a'(V(s)) K_a(s) e^{-(\rho+g)(s-t)}/\gamma(t)$. The social damage can be obtained by integrating this term from time t to infinity, which would exactly give the right-hand side of (4.33). Therefore, according to (4.33), the shadow price of pollution flow should be measured by the social damage of an additional unit of pollution flow. Pigou's idea remains valid in this dynamic framework, although the Pigouvian tax now adopts a dynamic version of (4.33).

The easiest way to replicate the social optimum in a market-based economy is to impose a pollution tax satisfying (4.33) and redistribute the tax revenue to households in a lump sum to clear the market.[15] The following proposition summarizes these

[15] It follows from the first-order condition $p_a = (r + \omega_a\tau)/q_a(V)$ and $p_m = (r + \omega_m\tau)/q_m$ that $Y = p_a Y_a + p_m Y_m = rK + \tau Z$.

results.

Proposition 4.4. *In autarky, the social optimum can be achieved in a market-based economy by imposing a dynamic Pigouvian tax (4.33) on pollution, with lump-sum transfers of tax revenue to households.*

4.5.3 Autarky Dynamic System under Optimal Policy

To obtain the dynamic system in autarky, I show how other variables, including the rental, the MRT, intermediate prices, and the allocation of private capital, are determined by (τ, V) at every point in time.

As in a planned economy, perfect competition yields $p_a = (r + \omega_a \tau) / q_a(V)$ and $p_m = (r + \omega_m \tau) / q_m$, which when plugged into (4.5) gives

$$B = \left(\frac{r + \omega_a \tau}{q_a(V)}\right)^b \left(\frac{r + \omega_m \tau}{q_m}\right)^{1-b}. \tag{4.34}$$

This gives the rental as a function of (τ, V): $r = r(\tau, V)$, which satisfies

$$\frac{\partial r(\tau, V)}{\partial \tau} < 0, \quad \frac{\partial r(\tau, V)}{\partial V} > 0, \quad r(0, V) = B(q_a(V))^b q_m^{1-b}. \tag{4.35}$$

The MRT facing private firms is defined as

$$\zeta \equiv \frac{q_a(V)(r + \omega_m \tau)}{q_m(r + \omega_a \tau)}. \tag{4.36}$$

Since r is a function of (τ, V) in autarky, so is ζ:

$$\zeta = \zeta(\tau, V) = \frac{q_a(V)(r(\tau, V) + \omega_m \tau)}{q_m(r(\tau, V) + \omega_a \tau)}. \tag{4.37}$$

From (4.35), it follows that

$$\frac{\partial \zeta(\tau, V)}{\partial \tau} \gtreqless 0 \text{ if } \omega_a \lesseqgtr \omega_m, \quad \frac{\partial \zeta(\tau, V)}{\partial V} > 0, \quad \zeta(0, V) = \frac{q_a(V)}{q_m}. \tag{4.38}$$

That is, an increase in pollution tax more than proportionately raises the cost of the dirtier good.

Note that $\zeta(\tau, V) = p_m / p_a$ in autarky, which along with (4.5) gives

$$p_a = B(\zeta(\tau, V))^{b-1}, \quad p_m = B(\zeta(\tau, V))^b. \tag{4.39}$$

From (4.39), it follows that

$$\frac{\partial p_i}{\partial \tau} > 0 \text{ and } \frac{\partial p_j}{\partial \tau} < 0 \text{ if } \omega_i > \omega_j. \tag{4.40}$$

That is, by choosing the final good as the numeraire, an increase in pollution tax raises (reduces) the price of the dirtier (cleaner) good.

The allocation of private capital follows from $p_a X_a / p_m X_m = b / (1-b)$, (4.6), and (4.39):

$$K_a = l(\tau, V) K, \quad K_m = (1 - l(\tau, V)) K, \tag{4.41}$$

where

$$l(\tau, V) \equiv \frac{b(r(\tau, V) + \omega_m \tau)}{b(r(\tau, V) + \omega_m \tau) + (1-b)(r(\tau, V) + \omega_a \tau)}. \tag{4.42}$$

From (4.35), it follows that

$$\frac{\partial l(\tau, V)}{\partial \tau} \gtreqless 0 \text{ and } \frac{\partial l(\tau, V)}{\partial V} \lesseqgtr 0 \text{ if } \omega_a \lesseqgtr \omega_m, \quad l(0, V) = b \tag{4.43}$$

That is, an increase in pollution tax shifts private capital from the dirtier sector to the cleaner sector and reduces the flow of pollution (given the stock of private capital). In contrast, an increase in environmental capital has the opposite effect. We can formulate this as

$$Z = l(\tau, V) \omega_a + (1 - l(\tau, V)) \omega_m \equiv \Psi(\tau, V) K. \tag{4.44}$$

Then, (4.43) yields

$$\frac{\partial \Psi(\tau, V)}{\partial \tau} < 0 \text{ and } \frac{\partial \Psi(\tau, V)}{\partial V} > 0 \text{ if } \omega_a \neq \omega_m, \tag{4.45}$$

implying $\partial Z / \partial \tau < 0$ and $\partial Z / \partial V > 0$.

By substituting these results into private capital accumulation (4.1), environmental capital evolution (4.10), and the Euler equations (4.2) and (4.32), we obtain the dynamic system in autarky under optimal policy:[16]

$$\dot{K} = r(\tau, V) K + \tau \Psi(\tau, V) K - C - \delta K, \tag{4.46}$$

$$\dot{V} = g(\bar{V} - V) - \Psi(\tau, V) K, \tag{4.47}$$

$$\frac{\dot{C}}{C} = r(\tau, V) - \delta - \rho, \tag{4.48}$$

$$\dot{\tau} = (g + r(\tau, V) - \delta) \tau - B(\zeta(\tau, V))^{b-1} q_a'(V) l(\tau, V) K, \tag{4.49}$$

4.5.4 Autarky Steady State and Phase Diagram under Optimal Policy

In the steady state, γ, p_a, V, and K_a are constant. The expression of the optimal pollution tax (4.33) can be simplified as[17]

$$\tau^{*A} = \frac{p_a q_a'(V^{*A}) K_a^{*A}}{\rho + g}, \tag{4.50}$$

[16]To obtain (4.46), the income rK in (4.1) should be replaced by $rK + \tau Z = r(\tau, V) K + \tau \Psi(\tau, V) K$; to obtain (4.49), $\gamma = 1/C$ and (4.41) are used.

[17]Letting $\dot{\tau} = \dot{\gamma} = 0$ in (4.32) gives the same result.

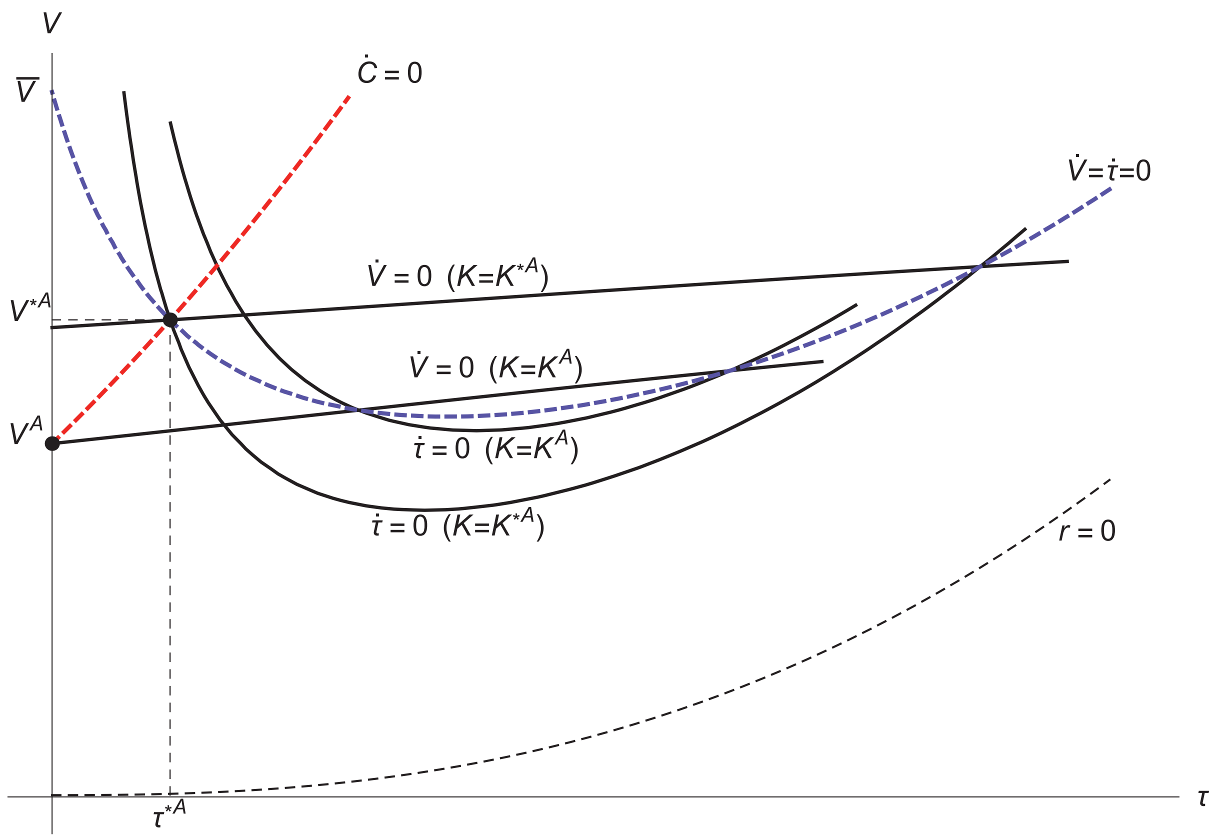


Figure 4.6: Autarky environment and pollution tax under optimal policy

where the superscript $*A$ denotes the autarky steady-state value in social optimum.

The dynamics and the steady state can be further analyzed by using the phase diagram. I focus on the phase diagram on the (τ, V) plane, as illustrated in Figure 4.6, since it provides relatively rich information. The $\dot{V}=0$ curve can be obtained by letting $\dot{V}=0$ in (4.47):

$$g\left(\bar{V}-V\right)=\Psi\left(\tau,V\right)K. \tag{4.51}$$

Given each K, from (4.45), we obtain an upward-sloping locus, above (below) which $\dot{V}<0$ ($\dot{V}>0$). Furthermore, the curve shifts downward as K increases, as illustrated in the figure (where $K^{*A}<K^{A}$). The $\dot{\tau}=0$ curve can be obtained by letting $\dot{\tau}=0$ in (4.49):

$$\left(g+r\left(\tau,V\right)-\delta\right)\tau=B\left(\zeta\left(\tau,V\right)\right)^{b-1}q_a'\left(V\right)l\left(\tau,V\right)K, \tag{4.52}$$

Now, assuming that

$$q_a''\left(V\right)\leq 0, \tag{4.53}$$

and given each K, it can be shown that the $\dot{\tau}=0$ curve is U-shaped, above (below) which $\dot{\tau}>0$ ($\dot{\tau}<0$). Moreover, the curve shifts upward as K increases.

Changing K continuously, the intersections of the $\dot{V}=0$ curve and the $\dot{\tau}=0$ curve constitute the $\dot{V}=\dot{\tau}=0$ curve. Its expression can be simply obtained by combining (4.51) and (4.52) to eliminate K:

$$\left(g+r\left(\tau,V\right)-\delta\right)\tau\Psi\left(\tau,V\right)=B\left(\zeta\left(\tau,V\right)\right)^{b-1}q_a'\left(V\right)l\left(\tau,V\right)g\left(\bar{V}-V\right), \tag{4.54}$$

which is also U-shaped.

To pin down the steady state, draw the $\dot{C} = 0$ curve, which is upward-sloping according to (4.35). The steady state, denoted by (τ^{*A}, V^{*A}), is given by the intersection of the $\dot{C} = 0$ curve and the $\dot{V} = \dot{\tau} = 0$ curve. The uniqueness of the steady state is ensured by the opposite signs in the slopes of the two curves. Moreover, we can verify that the steady state is saddle stable. Thus, we have the following proposition:

Proposition 4.5. *The autarky steady state under optimal policy is unique and saddle stable, and the optimal pollution tax satisfies (4.50).*

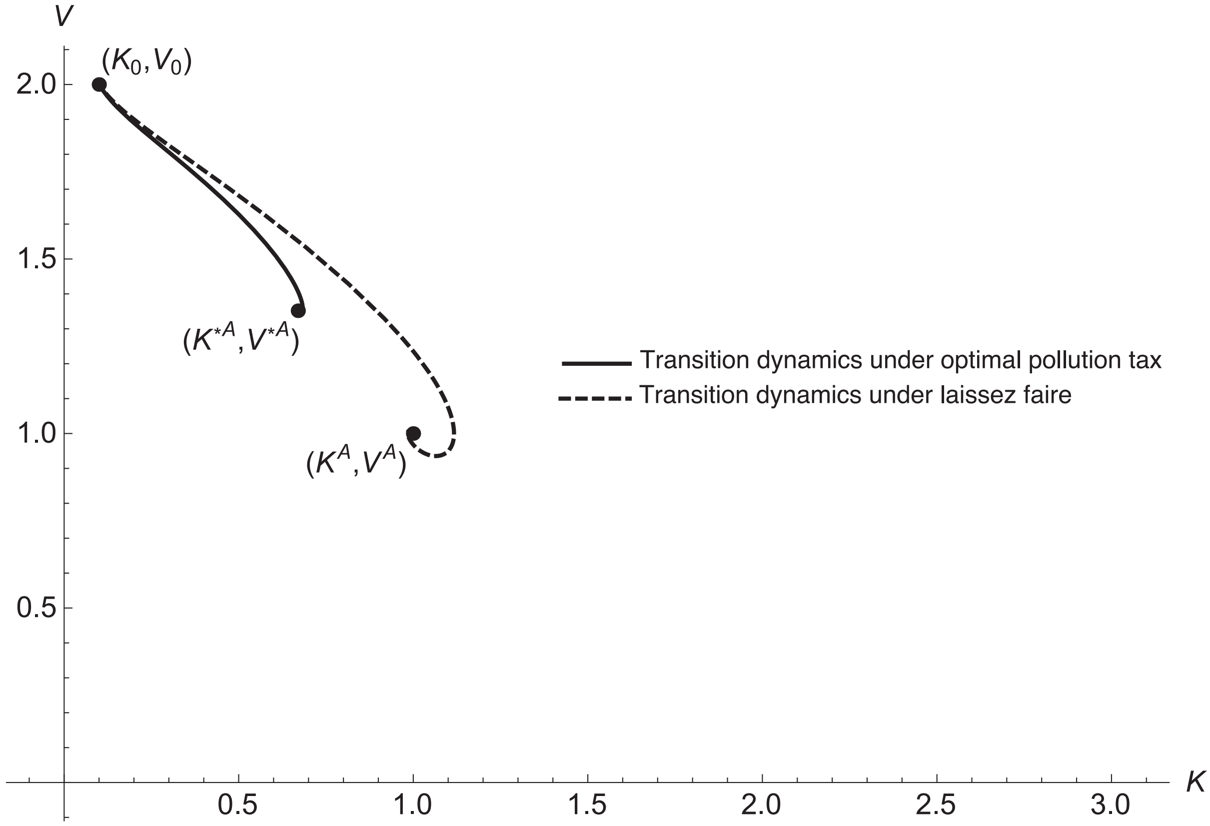
4.5.5 Transition Dynamics under Optimal Policy: A Numerical Example

Differing from laissez faire, there is no closed-form consumption function under the optimal policy. To compare the transition dynamics observed under the optimal policy and under laissez faire, I consider a simple numerical example. The appendix gives the numerical specification, and the results are illustrated in Figure 4.7.

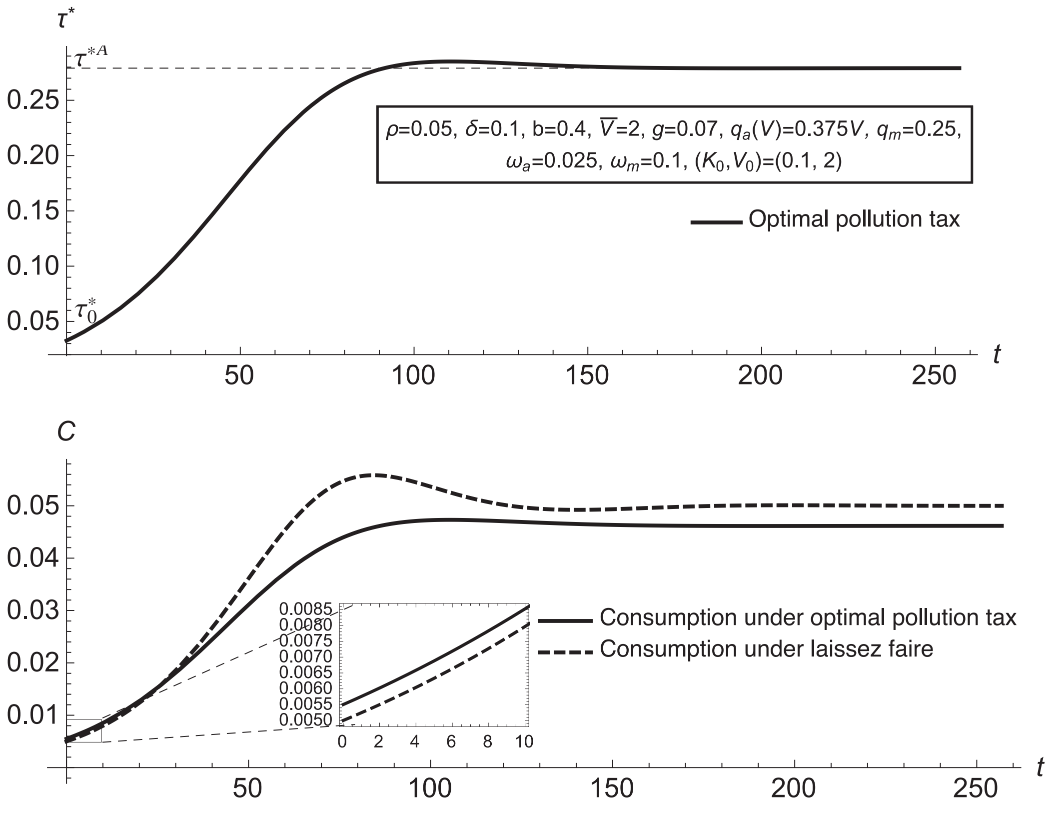
Figure 4.7a depicts (K, V) trajectory under the optimal policy and under laissez faire. Note that although the figure shows that $K^{*A} < K^{A}$, this is not necessarily the case. Two forces work in opposite directions. On one hand, a pollution tax shifts out some private capital from the dirtier sector, and thereby reduces, on average, the per unit private capital emission. Thus, more private capital can be brought in without pulling down the quality of the environment and hence the rental of private capital. On the other hand, from (4.34), a pollution tax directly reduces the rental of private capital, thereby discouraging investment. The final outcome will depend on which force dominates.

Figure 4.7b illustrates the optimal pollution tax and consumption level during the transition period. It shows that the optimal pollution tax is low at the starting point but increases gradually. This is because initially the environment is good ($V_0 = \bar{V} = 2$) and the stock of private capital is relatively low ($K_0 = 0.1$), implying a relatively small p_a and K_a. However, the environment degrades and private capital accumulates over time, which, given the pollution tax, leads to an increase in both p_a and K_a. Therefore, the pollution tax is likely to increase, from (4.33). Of course, the property of $q_a(\cdot)$ also affects the result, but in the numerical example its effect is eliminated because $q_a'(\cdot) = 1$.

The steady-state consumption under the optimal policy could be higher or lower, depending on the parameters and functional forms. Our numerical example indicates a lower steady-state consumption, as Figure 4.7b shows. By the definition of optimal policy, welfare under the optimal policy is necessarily higher. To see this in the example, note that consumption under the optimal policy is slightly higher in the early



(a) Autarky transition dynamics: (K, V)



(b) Autarky transition dynamics: τ^* and C

Figure 4.7: Compare autarky transition dynamics

period of the time span, as prominently shown in the figure. Since the value of consumption is exponentially discounted, this initially small difference leads to a higher lifetime discounted utility.

4.6 Optimal Policy in Small Open Economy

Now, consider the optimal policy in an SOE. Given the world relative price P, the social planner determines the volume of trade M_i ($i = a, m$), as well as those variables in autarky.

4.6.1 Social Planner Problem and Optimal Policy in SOE

With trade, the material constraint for intermediate goods is $X_i + M_i \leq Y_i$ ($i = a, m$). A positive sign of M_i denotes exports and a negative sign denotes imports. Assume that the balance of trade, $M_a + PM_m = 0$, holds at every point in time. For the same reason in autarky, $C + I \leq Y$ and $X_i + M_i \leq Y_i$ bind in optimum. The Hamiltonian can then be written as

$$\begin{aligned} H = \ln C &+ \gamma\left(Y - C - \delta K\right) + \lambda\left(g\left(\bar{V} - V\right) - Z\right) \\ &- \tau\gamma\left(\omega_a K_a + \omega_m K_m - Z\right) - r\gamma\left(K_a + K_m - K\right) - p\gamma\left(Y - X_a^b X_m^{1-b}\right) \\ &- p_a\gamma\left(Y_a - q_a\left(V\right)K_a\right) - p_a^d\gamma\left(X_a - PM_m - Y_a\right) \\ &- p_m\gamma\left(Y_m - q_m K_m\right) - p_m^d\gamma\left(X_m + M_m - Y_m\right). \quad (4.55) \end{aligned}$$

The first-order condition is similar to that in autarky (see footnote 14). The basic scheme of the optimal pollution tax in autarky remains valid in the SOE. There is a new first-order condition in an SOE, $p_m/p_a = P$, which states that in optimum, the relative (shadow) price is equalized to the world relative price. This simply means that in order to achieve the social optimum in a market-based SOE, no tariff should be imposed. Thus, we have the following lemma:

Lemma 4.6. *In an SOE, the optimal policy includes a zero tariff and a tax-transfer system as in autarky.*

4.6.2 SOE Dynamic System under Optimal Policy

Another difference from autarky arises in the first-order condition from the fact that trade introduces complete specializations. Thus, we need to explicitly consider the non-negative constraint of K_i ($i = a, m$): $K_a \geq 0$, $\partial H/\partial K_a \leq 0$, $K_a \partial H/\partial K_a = 0$, and $K_m \geq 0$, $\partial H/\partial K_m \leq 0$, $K_m \partial H/\partial K_m = 0$. This yields the following lemma.[18]

[18] The following briefly gives the derivation. The Kuhn–Tucker condition requires that any one of the following four holds: (i) $\partial H/\partial K_a < 0$ and $\partial H/\partial K_m < 0$, (ii) $\partial H/\partial K_a = 0$ and $\partial H/\partial K_m < 0$, (iii)

Lemma 4.7. *In social optimum, the trade patterns in an SOE can be determined by its comparative advantage, which is revealed by comparing the MRT ζ and the world relative price P.*

To eliminate indeterminacy, I assume no trade if $\zeta = P$. The rental of private capital and the flow of pollution can be expressed as a piecewise function of (τ, V). Specifically, if $\zeta > P$, then $r = BP^{b-1}q_a(V) - \omega_a\tau$ and $Z = \omega_a K$. If $\zeta < P$, then $r = BP^b q_m - \omega_m\tau$ and $Z = \omega_m K$. If $\zeta = P$, then $r = BP^b q_m - \omega_m\tau$ and $Z = \Psi(\tau, V) K$. From definition (4.36),

$$\zeta = \begin{cases} \frac{BP^{b-1}q_a(V)+(\omega_m-\omega_a)\tau}{BP^{b-1}q_m} & \text{if } \zeta > P, \\ \zeta(\tau, V) & \text{if } \zeta = P, \\ \frac{BP^b q_a(V)}{BP^b q_m-(\omega_m-\omega_a)\tau} & \text{if } \zeta < P. \end{cases} \tag{4.56}$$

Note that if $\zeta = P$, the MRT can be determined in the same fashion as in autarky. This is not because there is no trade when $\zeta = P$, but because $p_a = (r + \omega_a\tau)/q_a(V)$ and $p_m = (r + \omega_m\tau)/q_m$ if $\zeta = P$, which implies that (4.34) and thus (4.37) holds as in autarky. Therefore, $\zeta(\tau, V) = P$ also means that $\zeta = P$ in an SOE.

Substituting these results into (4.1) (noting that the income is $rK + \tau Z$ rather than

$\partial H/\partial K_a < 0$ and $\partial H/\partial K_m = 0$, and (iv) $\partial H/\partial K_a = \partial H/\partial K_m = 0$. The first gives $K_a = K_m = 0$, which is clearly not optimal. The second gives $K_m = 0$, which implies that $p_a = (r + \omega_a\tau)/q_a(V)$ and $p_m < (r + \omega_m\tau)/q_m$, and thus $\zeta > p_m/p_a = P$. The third gives $K_a = 0$, which implies that $p_a < (r + \omega_a\tau)/q_a(V)$ and $p_m = (r + \omega_m\tau)/q_m$, and thus $\zeta < p_m/p_a = P$. The fourth gives $\zeta = p_m/p_a = P$, which implies that there is no difference in how to allocate private capital between sectors. Therefore, in social optimum, the SOE should specialize in agriculture (manufacturing) if the MRT ζ is greater (less) than the world relative price P.

rK), (4.10), (4.2), and (4.32) yields the following dynamic system in an SOE:

$$\dot{K} = \begin{cases} BP^{b-1}q_a(V)K - C - \delta K & \text{if } \zeta > P, \\ \left(BP^b q_m - \omega_m \tau + \Psi(\tau, V)\tau\right)K - C - \delta K & \text{if } \zeta = P, \\ BP^b q_m K - C - \delta K & \text{if } \zeta < P, \end{cases} \tag{4.57}$$

$$\dot{V} = \begin{cases} g(\bar{V} - V) - \omega_a K & \text{if } \zeta > P, \\ g(\bar{V} - V) - \Psi(\tau, V)K & \text{if } \zeta = P, \\ g(\bar{V} - V) - \omega_m K & \text{if } \zeta < P, \end{cases} \tag{4.58}$$

$$\frac{\dot{C}}{C} = \begin{cases} BP^{b-1}q_a(V) - \omega_a \tau - \delta - \rho & \text{if } \zeta > P, \\ BP^b q_m - \omega_m \tau - \delta - \rho & \text{if } \zeta = P, \\ BP^b q_m - \omega_m \tau - \delta - \rho & \text{if } \zeta < P, \end{cases} \tag{4.59}$$

$$\dot{\tau} = \begin{cases} \left(g + BP^{b-1}q_a(V) - \omega_a \tau - \delta\right)\tau - BP^{b-1}q_a'(V)K & \text{if } \zeta > P, \\ \left(g + BP^b q_m - \omega_m \tau - \delta\right)\tau - BP^{b-1}q_a'(V)\,l(\tau, V)K & \text{if } \zeta = P, \\ \left(g + BP^b q_m - \omega_m \tau - \delta\right)\tau & \text{if } \zeta < P. \end{cases} \tag{4.60}$$

4.6.3 Small Open Economies under Optimal Policy: Six Types

To characterize the dynamics and the steady state under optimal policy, I categorize SOEs into six types:

$$\bar{P} < P, \quad \omega_a < \omega_m, \qquad \text{(SMM type)}$$

$$\bar{P} < P, \quad \omega_a > \omega_m, \qquad \text{(SMA type)}$$

$$P^A < P \leq \bar{P}, \quad \omega_a < \omega_m, \qquad \text{(SDM type)}$$

$$P^A < P \leq \bar{P}, \quad \omega_a > \omega_m, \qquad \text{(SDA type)}$$

$$P^A > P \quad \omega_a < \omega_m, \qquad \text{(AM type)}$$

$$P^A > P, \quad \omega_a > \omega_m. \qquad \text{(AA type)}$$

Note that the categorization AM type and AA type under laissez faire remain valid here as well. However, the MM type and MA type SOEs are further divided into the SMM type, SMA type, SDM type, and SDA type SOEs according to whether the world relative price P is greater than $\bar{P}$ or not, where $\bar{P}$ satisfies that when $P = \bar{P}$, the $\zeta = P$ curve, the $\dot{V} = \dot{\tau} = 0$ curve, and the $\dot{C} = 0$ curve pass through the same point.[19]

[19]Formally, $\bar{P}$ (together with τ and V) can be solved from the following three equations: $(\rho + g)\tau = B\bar{P}^{b-1}q_a'(V)g(\bar{V} - V)/\omega_a$, $\rho + \delta = B\bar{P}^{b-1}q_a(V) - \omega_a\tau$, and $\rho + \delta = B\bar{P}^b q_m - \omega_m\tau$.

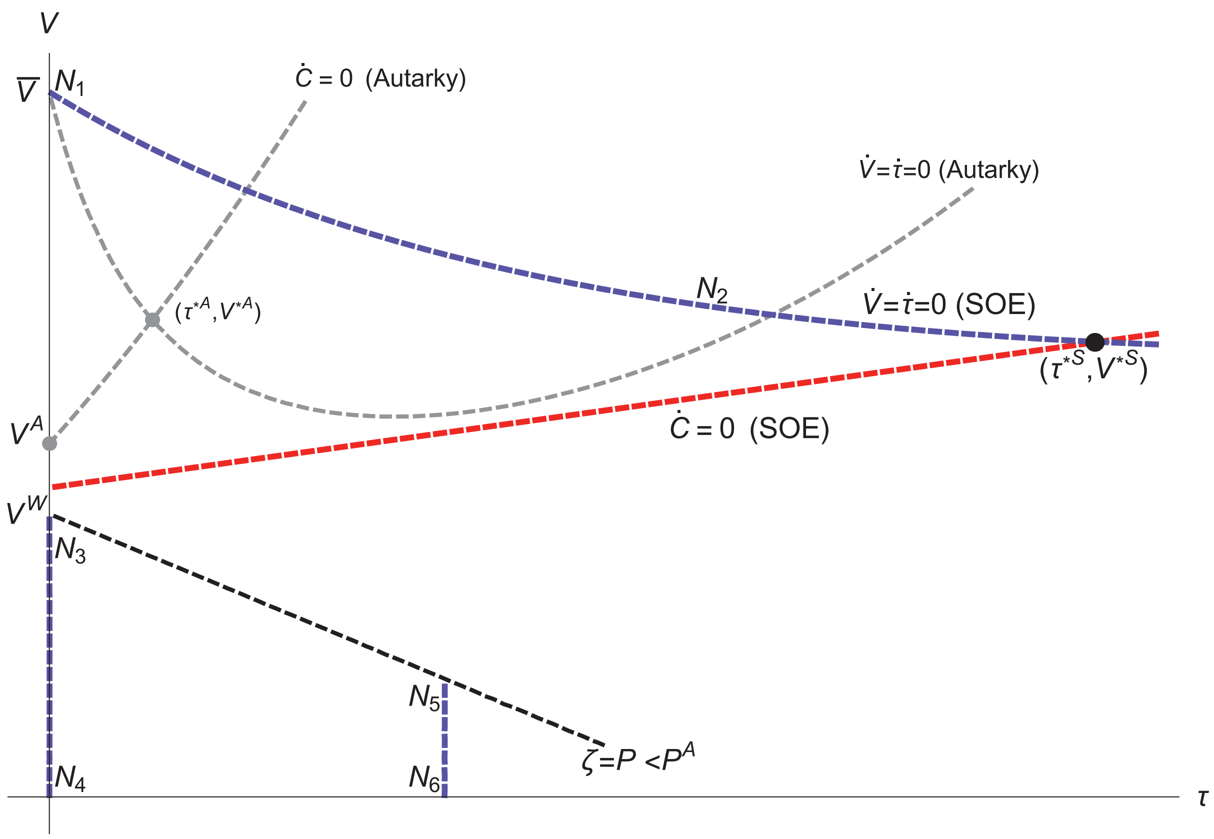


Figure 4.8: Environment and pollution tax in an AM type SOE under optimal policy

4.6.4 AM Type and AA Type SOEs

From (4.33) and (4.59), it follows that an AM type or AA type SOE cannot remain specializing in manufacturing. However, there exists a steady state where the economy can specialize in agriculture. Here, the optimal pollution tax is

$$\tau^{*S} = \frac{BP^{b-1}q_a'\left(V^{*S}\right)K^{*S}}{\rho + g}, \tag{4.61}$$

where the superscript $*S$ denotes the SOE steady-state value in the social optimum. K^{*S}, V^{*S} and C^{*S} can be solved from equations (4.57) to (4.60).

This can be seen clearly from the phase diagram on the (τ, V) plane. Figure 4.8 depicts the case of an AM type SOE. The figure for the AA type SOE is similar, except that the $\zeta(\tau, V) = P$ curve is upward-sloping. Since $\zeta\left(0, V^W\right) = P$, the $\zeta(\tau, V) = P$ curve starts from $\left(0, V^W\right)$ and, following (4.56), goes in the lower-right direction. Above (below) the $\zeta(\tau, V) = P$ curve is the agriculture (manufacturing) regime. Note that $r < \rho + \delta$ at $\left(0, V^W\right)$ and $r > \rho + \delta$ at $\left(0, V^A\right)$, and the $\dot{C} = 0$ curve (on which $r = \rho + \delta$) starts from somewhere between $\left(0, V^W\right)$ and $\left(0, V^A\right)$, and goes in the upper-right direction.

The $\dot{V} = \dot{\tau} = 0$ curve has three parts: N_1N_2 in the agriculture regime, and N_3N_4 and N_5N_6 in the manufacturing regime. N_3N_4 corresponds with $\tau = 0$ in (4.60), and N_5N_6 corresponds with $g + BP^b q_m - \omega_m \tau - \delta = 0$. N_1N_2 starts from $(0, \bar{V})$ and goes

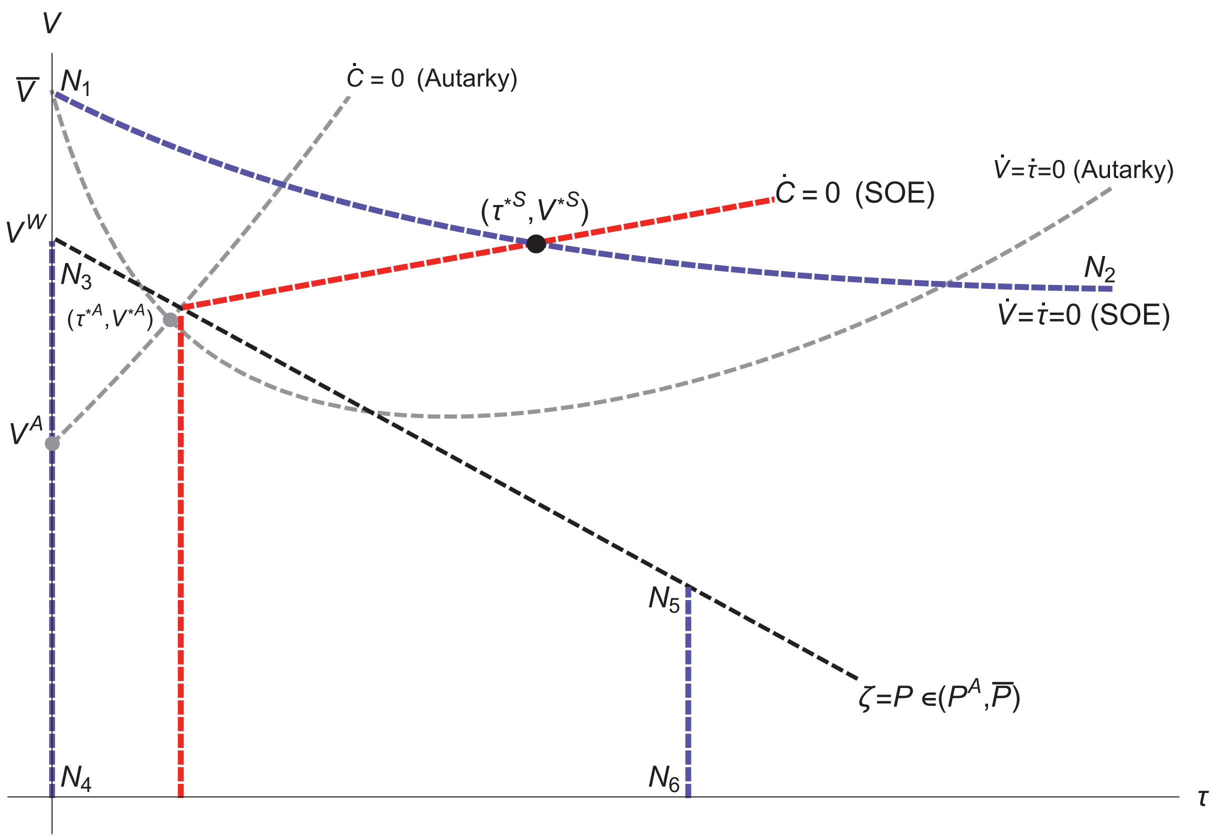


Figure 4.9: Environment and pollution tax in an SDM type SOE under optimal policy

in the lower-right direction first. Because N_1N_2 lies below the $V = \bar{V}$ line, it must intersect the $\dot{C} = 0$ curve somewhere, which is the steady state, denoted as $\left(\tau^{*S}, V^{*S}\right)$ in the figure. Thus, we have the following proposition:

Proposition 4.8. *Under optimal policy, if the world relative price P is low (AM type or AA type), the SOE specializes in agriculture in the steady state.*

4.6.5 SDM Type and SDA Type SOEs

Note that $\bar{P} > P^{*A}$. To verify this, consider the special case of $P = P^{*A}$, where the $\zeta = P$ curve passes through $\left(\tau^{*A}, V^{*A}\right)$, like the $\dot{C} = 0$ curve in autarky. From (4.59), the $\dot{C} = 0$ curve in an SOE also passes through $\left(\tau^{*A}, V^{*A}\right)$, from where it goes in the upper-right direction in the agriculture regime and becomes a vertical line in the manufacturing regime. In contrast, a comparison of (4.54) with (4.58) and (4.60) shows that the $\dot{V} = \dot{\tau} = 0$ curve in an SOE lies above $\left(\tau^{*A}, V^{*A}\right)$. Thus, the $\dot{C} = 0$ curve and the $\dot{V} = \dot{\tau} = 0$ curve must intersect somewhere in the agriculture regime. This holds true when P is slightly higher than P^{*A}, as illustrated in Figure 4.9, corresponding to an SDM type SOE. The figure for the SDA type SOE is similar except that the slope of the $\zeta = P$ curve is positive.

Although the SDM type and SDA type SOEs have a steady state in the agriculture regime, they may also converge to the growth path by specializing in manufacturing with zero pollution tax. If this is the case, the consumption function (4.11) holds and the transition dynamics becomes the same as under laissez faire. Whether the optimal path converges to the steady state or to the growth path depends on the starting point of the economy.

Moreover, if an SDM type or SDA type SOE starts from the agriculture regime but finally specializes in manufacturing, the optimal path must pass through $\left(0, V^W\right)$, because, from (4.33), the optimal pollution tax is positive as long as the economy still enters the agriculture regime, otherwise it becomes zero. On the other hand, since τ does not jump, the only way to enter the manufacturing regime from the agriculture regime is to pass through $\left(0, V^W\right)$, as clearly shown in Figure 4.9. To summarize,

Proposition 4.9. *Under the optimal policy, if the world relative price P is in-between (SDM type or SDA type), the SOE can specialize in agriculture or manufacturing in the long run, depending on the initial condition. If the economy first specializes in agriculture and then specializes in manufacturing, (τ, V) must pass through $\left(0, V^W\right)$.*

4.6.6 SMM Type and SMA Type SOEs

If P is too large, the $\dot{C} = 0$ curve in the agriculture regime may start from some point above the $\dot{V} = \dot{\tau} = 0$ curve, and so the $\dot{C} = 0$ curve cannot intersect the $\dot{V} = \dot{\tau} = 0$ curve. Therefore, there is no steady state in the SMM type and SMA type SOEs. Figure 4.10 illustrates an SMM type SOE. The figure for the SMA type SOE is similar, except for the slope of the $\zeta = P$ curve. We have the following proposition:

Proposition 4.10. *Under the optimal policy, if the world relative price P is high (SMM type or SMA type), the SOE eventually remains specializing in manufacturing, where pollution tax is zero, and private capital and consumption grow at the same rate.*

It is noteworthy that unlike under laissez faire, where trade always harms the environment in the long run (compared to the autarky steady state), under the optimal policy, trade does not necessarily harm the environment in the long run because of the existence of pollution tax. In Figure 4.9, the environment actually becomes better after trade liberalization ($V^{*S} > V^{*A}$).

4.7 Conclusion

Featuring the Ramsey style investment and agricultural production externalities, my two-sector dynamic model helps us understand the close nexus between trade, economic development, and the environment. I find that trade has a scale effect once capital accumulation is introduced, and that, under laissez faire, it necessarily harms the

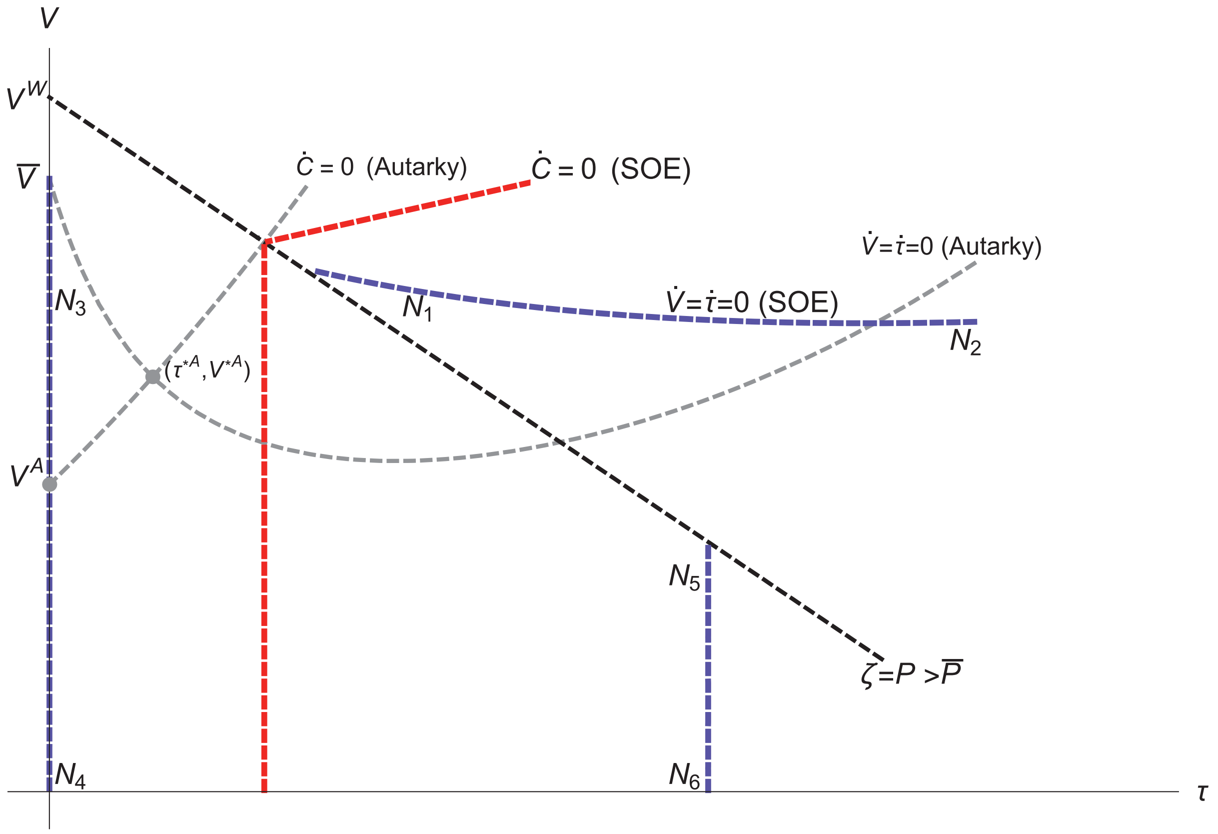


Figure 4.10: Environment and pollution tax in an SMM type SOE under optimal policy

environment in the long run. The implication is significant. Trade liberalization can be catastrophic to an economy that lacks in appropriate environmental regulations, especially if environmental degradation also causes disutility. This study further shows that the social optimum can be achieved through a pollution tax with a lump-sum transfer of tax revenue to households. The optimal pollution tax can be thought of as a dynamic version of the Pigouvian tax. I also find that although the specialization pattern is endogenously determined at every point in time, the long-run specialization pattern basically can be predicted from the parameters.

In this study, pollution is local, but the transboundary type of pollution, such as greenhouse gases, is also crucial. Moreover, abatement behavior is another significant aspect in reality, but I exclude it in this study for simplicity. It will be interesting to incorporate these aspects into this framework in a future research.

4.A Appendix

4.A.1 Numerical Specification

In all cases, $(K_0, V_0) = (0.1, 2)$. For the parameters, see Table 4.1.

ρ	Time preference	0.05
δ	Private capital depreciation rate	0.1
b	Share of agriculture good	0.4
$\bar{V}$	Environmental carrying capacity	2
g	Environmental recovery rate	0.07
$q_a(V)$	Agriculture productivity	$0.375V$
q_m	Manufacturing productivity	0.25
ω_a	Agriculture pollution intensity	0.025^{*}, 0.1^{**}
ω_m	Manufacturing pollution intensity	0.1^{*}, 0.05^{**}
P	World relative price	$2.25^{\dagger}$, $0.75^{\dagger\dagger}$

Notes: * and ** correspond with the case of dirtier manufacturing and the case of dirtier agriculture; † and †† denote a pre-trade comparative advantage in manufacturing and that in agriculture.

Table 4.1: Parameter specification

Both sets of pollution intensities yield $\Omega \equiv b\omega_a + (1-b)\,\omega_m = 0.07$, implying identical autarky dynamics under laissez faire. Given the specification above, the two world relative prices are equivalent to $V^W = 1.5$ and $V^W = 0.5$, respectively.

Chapter 5

A Generalized Model of Trade with Resource-use and Pollution[1]

5.1 Introduction

The natural environment play many roles in economic activities, but economists usually focus on the aspect of the environment that directly relates to the issue of interest.[2] In the analysis of the interaction between trade and the environment, there are two influential theoretical models. The Brander–Taylor model (Brander and Taylor, 1997, 1998) focuses on renewable resources in trade context and explains why resource exporting country may lose from trade. The Copeland–Taylor model (Copeland and Taylor, 1999) considers production externalities of pollution and shows how pollution can provide an incentive to trade. Both models formulate a single, though different, role of the environment: a place growing resources in the Brander–Taylor model and a pollution sink in the Copeland–Taylor model. These simplifications do help highlight the essence, but also come at prices. In both models, the economic usage of the environment–the impact of economic activities on the environment–arises from only one sector among the two.[3] Thus, the Brander–Taylor model ignores, say, the impact of industrial wastes on fishery resources, whereas the Copeland–Taylor model fails to formulate the impact of chemical fertilizers on the soil quality, which is crucial to farming itself.

[1] This chapter is joint work with Akihiko Yanase. Earlier versions of this chapter were presented at the 4th Spring Meeting of JSIE, the 10th Asia Pacific Trade Seminars, CIREQ, Summer Workshop on Economic Theory 2014, RICF (DBJ), IEFS Japan Annual Meeting 2014, JAAE Fall Meeting 2014.

[2] The major functions of the environment include supplying resources, absorbing pollutants and wastes, creating amenity values, and functioning as life support system. See, e.g., Smulders and Gradus (1996) and Hanley et al. (1996).

[3] In the Brander–Taylor model, harvesting reduces the stock of renewable resources, while manufacturing has no impact on resources. In the Copeland–Taylor model, pollution from manufacturing lowers the quality of the environment, while farming is totally clean.

In this study, we try to move one step forward and develop a two-sector dynamic model that allows activities in both sectors to harm the environment. The magnitude of a sector's economic usage of the environment is assumed to be proportionate to the sector's output with a sector-specific coefficient. The capacity of the environment is measured by a stock variable, whose dynamics is characterized by the difference between its natural growth and the economic usage. The primary sector is vulnerable to environmental degradation, whereas the manufacturing sector is not.

From one-dimensional to two-dimensional source of the economic usage of the environment, our model includes the Brander–Taylor model and the Copeland–Taylor model as special cases, and provides a unified framework for renewable resources extraction and pollution emission. It is a small change in the structure but a significant improvement in the explanation ability of the model. In our generalized model, the environment can be interpreted in a broader sense: either fishery resources as in the Brander–Taylor model, or air and water as in the Copeland–Taylor model, or a mix of both. With our model, one can consider fishery resources without ignoring the impact of toxic wastes from manufacturing. One can also think about agriculture while taking into account the influence of pollution from agriculture.

Our model yields several important insights. First, we ask which sector is dirtier in the long run in terms of per labor economic usage. The question matters because its answer determines a country's long-run supply curve, and in turn the response to trade liberalization. We show that, under certain conditions, the answer depends on the country's intrinsic nature, namely the parameters associated with the country. As a result, a country's long-run supply curve is either entirely upward-sloping if the primary sector is dirtier, or entirely downward-sloping if the manufacturing sector is dirtier. Accordingly, countries can be categorized into two types, called respectively in this chapter the Brander–Taylor type and the Copeland–Taylor type.[4]

Second, to facilitate the analysis of the two-country case, we construct the world production pattern diagram, which is a useful tool in providing a whole picture for how the world production patterns are distributed on the plane of the comparative advantage index (depending on two countries' environments) and the relative effective size (depending on their labor endowments). In the long run, the environment can evolve and so the comparative advantage index is an endogenous variable. We carefully characterize the long-run world production patterns for two countries of the same type and of different types. We show that trade between two countries of different types may harm both in the long run. This scenario is of special interest since it captures the trade between emerging industrial nations and resource countries, which

[4]We ignore the third case in which both sectors are equally dirty in the long run and the long-run supply curve is entirely flat.

cannot be analyzed in the Brander–Taylor model nor in the Copeland–Taylor model.[5]

Third, since the type of a country depends on the parameter values and functional forms. A change in parameters may change a country's type, leading to dramatic changes in the trade pattern. Using a simple example, we show that when the labor endowment crosses a critical value, two ex-ante identical countries of Brander–Taylor type become two of Copeland–Taylor type, leading to a jump in trade volume.

The rest of the chapter is organized as follows. Section 2 describes the basic model. Section 3 analyzes the supply side, including the long-run supply curve and the production possibility frontier (PPF). Section 4 considers the autarky case. Section 5 and Section 6 examine small open economy and two-country trade, respectively. Section 7 provides some discussion and Section 8 concludes.

5.2 The Basic Model

There are two production sectors under perfect competition with technologies

$$X_p = A(S)\,L_p, \quad X_m = aL_m, \tag{5.1}$$

where the subscript p and m denote respectively the primary sector and the manufacturing sector, X_i and L_i $(i = p, m)$ are the output from and the labor allocated in the corresponding sector. The environmental stock S measures the capacity of the environment, which is interpreted as the stock of renewable resources in Brander and Taylor (1997, 1998), and as the quality of the environment in Copeland and Taylor (1999). The primary sector is environmentally sensitive in the sense that $A'(S) > 0$. The productivity in the manufacturing sector, a, is fixed. Labor market is perfect, thus

$$L_p + L_m = L. \tag{5.2}$$

The environmental stock evolves according to

$$\dot{S} = G(S) - E, \tag{5.3}$$

where $G(S)$ is the natural growth of the environment, E the economic usage of the environment (the environmental impact of economic activities). To formulate the idea that the maximum capacity of the environment is finite, impose an assumption on $G(S)$ as follows.

[5]Many emerging nations impose relatively low environmental standards and often adopt cheap but dirty technologies in manufacturing industries. These emerging nations are likely to be categorized into Copeland–Taylor type. On the other hand, some countries with abundant renewable resources have resource industries as the pillar of the economy. These resource countries are likely to be of Brander–Taylor type. Some developing countries still use traditional methods of subsistence agriculture such as slash-and-burn cultivation, which cause deforestation, erosion and nutrient loss, and in turn harm agriculture itself. These developing countries are also likely to be of Brander–Taylor type.

Assumption 5.1. *There exists $K > 0$ such that $G(K) = 0$ and $G'(K) < 0$.*

Therefore, if there is no economic activities, namely that $E = 0$, the capacity of the environment in steady state is K, which is often called the carrying capacity of the environment. E is assumed to be positively correlated to the scale of the economy in the following manner:

$$E = l_p X_p + l_m X_m, \tag{5.4}$$

where non-negative parameters l_p and l_m reflect the cleanness of production technology. We exclude the trivial case of $l_p = l_m = 0$. Note that our model becomes the Brander–Taylor model if letting $l_p = 1$, $l_m = 0$, and becomes the Copeland–Taylor model if letting $l_p = 0$, $l_m > 0$.

According to (5.4), per labor economic usage of the environment is $l_p A(S)$ in the primary sector and $l_m a$ in the manufacturing sector. Since their relative magnitude is crucial in determining the behavior of the economy, as we can see in what follows, it is convenient to define the type of sector as follows.

Definition 5.2 (Sector type)**.** The dirtier (cleaner) sector has the higher (lower) per labor economic usage of the environment.

So, the primary (manufacturing) sector is dirtier if $l_p A(S) > l_m a$ ($l_p A(S) < l_m a$). Note that by assuming $l_m = 0$, the Brander–Taylor model focuses on the case of the dirtier primary sector. In contrast, by assuming $l_p = 0$, the Copeland–Taylor model focuses on the case of the cleaner primary sector. Define S_0 by

$$l_p A(S_0) \equiv l_m a,$$

then the primary (manufacturing) sector is dirtier if $S > S_0$ ($S < S_0$), as illustrated in Figure 5.1.

To finish the setup, assume the preference can be described by the representative household with instantaneous utility

$$u\left(C_p, C_m\right) = b \ln C_p + (1-b) \ln C_m. \tag{5.5}$$

5.3 Supply Side

5.3.1 Short-run Properties

At every point in time, using (5.1) and (5.2) gives

$$\frac{X_p}{A(S)} + \frac{X_m}{a} = L. \tag{5.6}$$

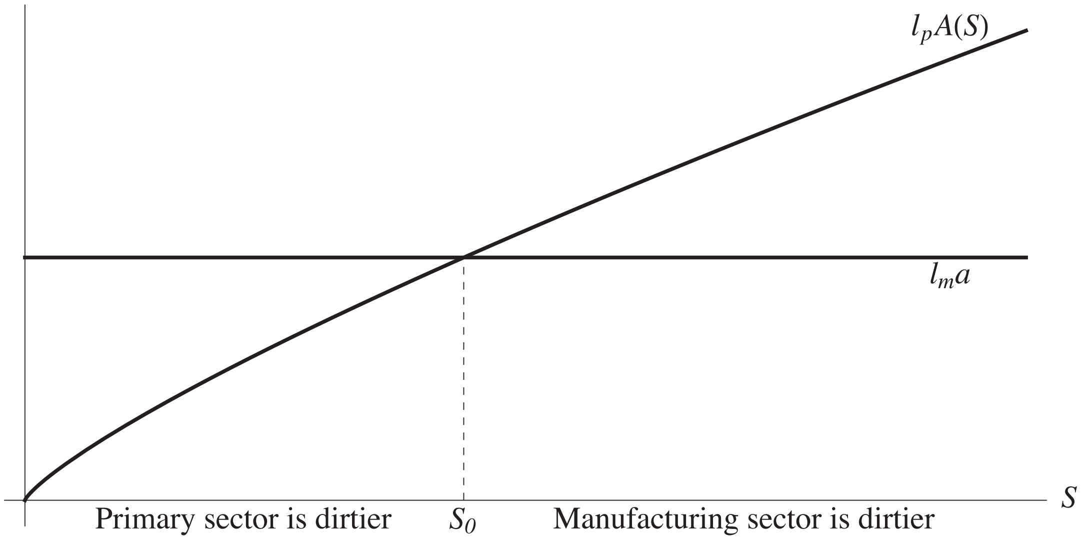


Figure 5.1: Sector type varies with the environmental stock

Firms under perfect competition make decisions by taking the environmental stock S as given. For firms, the marginal rate of transformation (MRT) from the primary good to the manufacturing good is

$$\text{MRT} = \frac{A(S)}{a}. \tag{5.7}$$

Therefore, the model behaves like a Ricardian economy in the short run. Equation (5.6) can be seen as the expression for the short-run production possibility frontier (PPF), which gives a straight line on the (X_p, X_m) plane through $(0, aL)$ with the slope -$A(S)/a$.

In the long run, however, the environment can change. A production schedule feasible in the short run is not necessarily sustainable in the long run. To understand the long-run properties of the supply side, it is convenient to define

$$\beta \equiv \frac{L_p}{L},$$

then the outputs can be rewritten into

$$X_p = A(S)\beta L, \quad X_m = a(1-\beta)L. \tag{5.8}$$

In equilibrium, β will endogenously determined by harmonizing supply and demand. But in this section, we treat β as exogenously given to examine the long-run properties of the supply side.

Without loss of generality, let the primary good be the numeraire, and P denote the price of the manufacturing good. Since labor is freely and costlessly mobile across sectors, wages are always equalized within the country. The necessary condition for both sectors to be active is $w = A(S) = aP$, or equivalently,

$$\frac{A(S)}{a} = P. \tag{5.9}$$

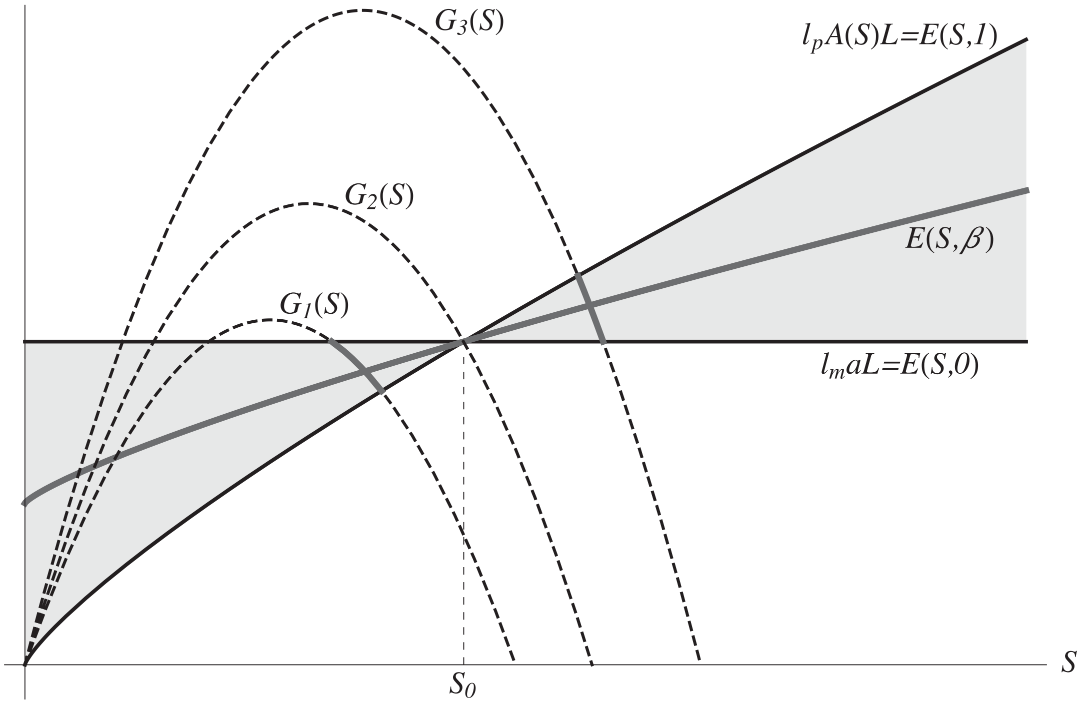


Figure 5.2: Labor allocation and the environment in steady state

If (5.9) fails to hold, say, in the fashion that $A(S)/a > P$, the primary sector pays higher wage and has all labor working in it.

It follows directly from (5.4) and (5.8) that

$$E = E(S, \beta) = \left[l_p A(S)\beta + l_m a(1-\beta)\right] L. \tag{5.10}$$

Recalling that $A'(S) > 0$, the economic usage of the environment $E(S,\beta)$ is an increasing function of S. Moreover, as a linear combination, $E(S,\beta)$ must be bound by $l_p A(S) L$ and $l_m aL$. As illustrated in Figure 5.2, the locus of $E(S,\beta)$ lies within the shadow area between $l_p A(S) L$ and $l_m aL$, and rotates counter-clockwise on point $(S_0, l_m aL)$ as β increases.

5.3.2 Long-run Supply Curve and Country Types

Given β, the steady state requires $\dot{S} = 0$ in (5.3), yielding

$$G(S) = \left[l_p A(S)\beta + l_m a(1-\beta)\right] L. \tag{5.11}$$

The stability requires $G(S)$ intersects $E(S,\beta)$ from above, namely that

$$G'(S) < l_p A'(S)\beta L. \tag{5.12}$$

Given β, let $S(\beta)$ denote the corresponding set of the levels of S that satisfy (5.11) and (5.12). Taking the total differential of (5.11) yields

$$S'(\beta) = \frac{\left[l_p A(S(\beta)) - l_m a\right] L}{G'(S(\beta)) - l_p A'(S(\beta))\beta L}. \tag{5.13}$$

The denominator of the right-hand side of (5.13) is negative due to the stability condition. If the primary (manufacturing) sector is dirtier, a marginal increase in β lowers (enhances) the steady-state level of the environmental stock. Given any parameter values and functional forms, $S(\beta)$ can be empty or multi-valued. To eliminate these complexities, assume that

Assumption 5.3. *Parameters a, l_p, l_m, L and functions $G(\cdot)$, $A(\cdot)$ satisfy that $S(\beta)$ is a positive and continuous function in* $[0, 1]$.

It is worth emphasizing that the constraint imposed by Assumption 5.3 is not as restrictive as it seems. For example, if $G(0) > l_m a$ and $G'(S) < 0$, which is assumed in the Copeland–Taylor model, Assumption 5.3 will holds. Alternatively, if $G(S)$ takes the form of logistic function and has the maximum greater than $l_m a$, which is assumed in the Brander–Taylor model, Assumption 5.3 holds as well. In Figure 5.2, $G_1(S)$, $G_2(S)$ and $G_3(S)$ are featured by the typical shape of logistic function, and Assumption 5.3 holds for all of them. Therefore, Assumption 5.3 is less restrictive than the constraints specified in the Brander–Taylor model and the Copeland–Taylor model, as well as those in many previous studies.

Substituting $S(\beta)$ into (5.9) for S gives $A(S(\beta))/a = P$. Take the total differentials to obtain, using (5.8),

$$\frac{dP}{dX_m} = -\frac{A'(S(\beta))\, S'(\beta)}{a^2 L}. \tag{5.14}$$

The following lemma follows directly:

Lemma 5.4. *An expansion of the dirtier (cleaner) sector lowers (enhances) the steady-state environmental stock. If the manufacturing sector is dirtier (cleaner), its steady-state price increases (decreases) with its output.*

Note that $S(\beta)$ changes with β and the type of sector is determined the relative magnitude of $S(\beta)$ and S_0. The question naturally arising is whether a change in β can change the type of a sector. The answer is given by the following lemma.

Lemma 5.5. *Given Assumption 5.3, the dirtier sector stays dirtier in all steady states.*

Proof. Assume to the contrary that the sign of $l_p A(S(\beta)) - l_m a$ can change. By the continuity of $S(\beta)$ from Assumption 5.3, there exists $\beta_0 \in [0, 1]$ satisfying $l_p A(S(\beta_0)) = l_m a$. This implies that $G(S)$, as $G_2(S)$ in Figure 5.2, passes through $(S(\beta_0), l_m a L) = (S_0, l_m a L)$ and $S(\beta) = S(\beta_0)$ holds for all $\beta \in [0, 1]$. As a result, $l_p A(S(\beta)) - l_m a = 0$ for all $\beta \in [0, 1]$, leading to a contradiction. □

Therefore, which sector is dirtier in the long run is an intrinsic nature of the economy. It depends only on the parameter values and functional forms. Figure 5.2 shows that, with others remaining the same, how different shapes of $G(S)$ determine which sector is dirtier in the long run. A country endowed with $G_1(S)$ has $S(\beta) < S_0$ for all $\beta \in [0,1]$. As a result, the manufacturing sector is dirtier in all steady states. In contrast, a country endowed with $G_3(S)$ has $S(\beta) > S_0$ for all $\beta \in [0,1]$, implying the primary sector is dirtier in all steady states. As for $G_2(S)$, it happens to pass through $(S_0, l_m aL)$, resulting $S(\beta) = S_0$ for all $\beta \in [0,1]$. This knife-edge case is of no interest and thus ignored from the analysis.[6] For convenience, define the type of country as follows.

Definition 5.6 (Country Type)**.** The country of Brander–Taylor (Copeland–Taylor) type has a dirtier (cleaner) primary sector in all steady states.

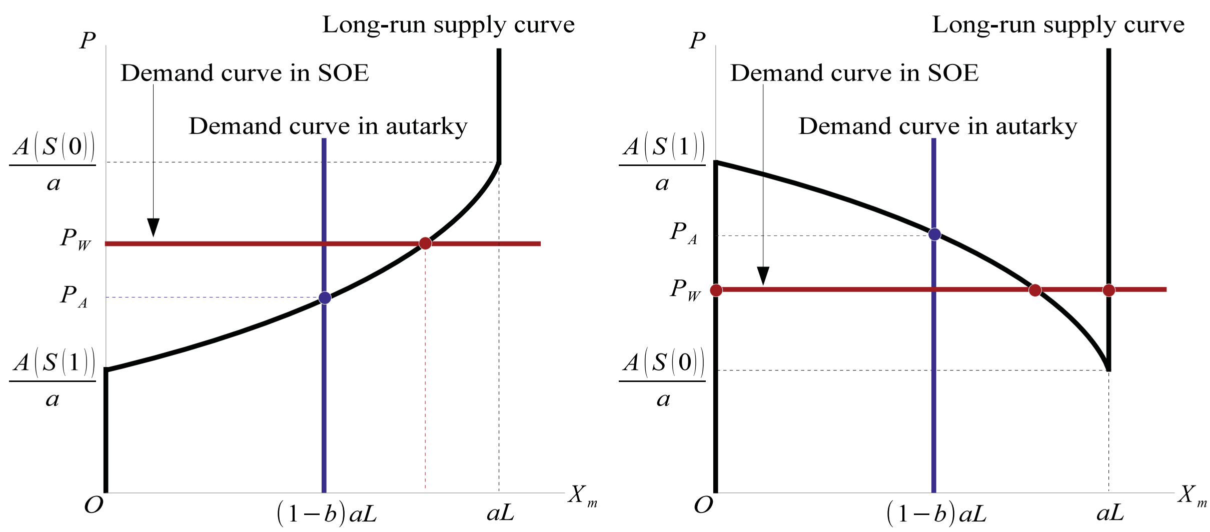
The following proposition summarizes the discussion above.

Proposition 5.7. *Given Assumption 5.3, a country either belongs to the Brander–Taylor type or belongs to the Copeland–Taylor type. A country of Brander–Taylor type has a regular entirely upward-sloping long-run supply curve, while a country of Copeland–Taylor type has an irregular entirely downward-sloping one.*

The intuition comes by realizing the key difference is whether the environmentally sensitive sector, namely the primary sector, is dirtier or cleaner than the manufacturing sector. If the environmentally sensitive sector is dirtier (Brander–Taylor type), more labor in manufacturing leads to better environment in steady state and thus higher productivity in the environmentally sensitive sector, which means higher relative price of manufacturing good. In contrast, if the environmentally sensitive sector is cleaner (Copeland–Taylor type), more labor in manufacturing leads to worse environment in steady state and thus lower productivity in the environmentally sensitive sector, which means lower relative price of manufacturing good.

The long-run supply curve consists of the two endpoints: $X_m = 0$ where $\beta = 1$ and $X_m = aL$ where $\beta = 0$. Supposing $X_m = 0$ in steady state, the wage in the primary sector is $A(S(1))$ and the potential wage in the manufacturing sector Pa. We must have $A(S(1)) \geq Pa$ for $X_m = 0$ to hold in steady state, implying that $P \leq A(S(0))/a$. Similarly, if $X_m = aL$ in steady state, then we must have $A(S(1)) \leq Pa$, implying that $P \geq A(S(0))/a$. With these in hand, now we can draw the long-run supply curves for both types of countries, as illustrated in Figure 5.3.

[6] If countries are producing the same goods, they seem having the "same" environment. So, how can the growth functions of the environment be different between countries? A possible justification is that, even the contents of the environment, say fishery resources, are the same, the "size" of the environment, say the area of the lake, can vary among countries. If letting the growth function take the logistic form $G(S) = gS(\bar{S} - S)$, then g can be seen as the intrinsic recovery rate of the environment, while $\bar{S}$ represents the size. Given the same g, the difference in $\bar{S}$ still leads to different shapes of $G(S)$. This is actually the case of $G_1(S)$, $G_2(S)$ and $G_3(S)$ in Figure 5.2.



(a) Brander–Taylor type country (b) Copeland–Taylor type country

Figure 5.3: Long-run supply curve

5.3.3 Long-run PPF

The long-run PPF is important for two reasons. First, it is useful in policy analysis. In the model, firms take the environmental stock as given and measure their opportunity costs by the MRT (5.7). This deviates from the social marginal rate of transformation (SMRT), which can be measured by the slope of the long-run PPF. Second, the long-run PPF, together with the short-run PPF, facilitates the analysis of the welfare effect of trade.

To derive the long-run PPF, substitute $S(\beta)$ into the first equation in (5.8) to obtain $X_p = A(S(\beta))\beta L$. Since the second equation in (5.8) gives $\beta = 1 - X_m/aL$, the long-run PPF can be written as

$$X_p = T(X_m) = A\left(S\left(1 - \frac{X_m}{aL}\right)\right)\left(L - \frac{X_m}{a}\right). \tag{5.15}$$

The SMRT follows directly:

$$\text{SMRT} \equiv -T'(X_m) = A'(S(\beta))\,S'(\beta)\,\frac{\beta}{a} + \frac{A(S(\beta))}{a}, \tag{5.16}$$

where β instead of X_m is used to simplify the expression. Note that $\frac{A(S(\beta))}{a} = \text{MRT}$, which equals the price of the manufacturing good P if both goods are produced. Therefore,

Proposition 5.8. *In a country of Brander–Taylor (Copeland–Taylor) type, the MRT (from the manufacturing good to the primary good) facing private firms is greater (less) than the SMRT for all $X_m > 0$.*

Proof. In a country of Brander–Taylor type, $S'(\beta) < 0$ and according to (5.16), MRT $>$ SMRT for all $\beta \in (0, 1]$. In contrast, if the country is of Copeland–Taylor type, we have $S'(\beta) > 0$ and thus MRT $<$ SMRT for all $\beta \in (0, 1]$. □

The intuition is straightforward. In a country of Brander–Taylor (Copeland–Taylor) type, the primary (manufacturing) sector is dirtier, so, from the perspective of the whole economy, the cost of the primary (manufacturing) good is underestimated by private firms.

Proposition 5.8 suggests that, to correct externalities in a country of Brander–Taylor type, a production tax can be imposed on the primary good to reduces the MRT facing firms. If the tax rate is $\tau \in (0,1)$, firms face the new MRT: $\text{MRT}(\tau) = (1-\tau)\,A\,(S\,(\beta))\,/a$. The optimal tax rate can be easily derived by equalizing the $\text{MRT}(\tau)$ to the SMRT. Similarly, in a country of Copeland–Taylor type, firms underestimate the cost of the manufacturing good, thus a tax on the manufacturing sector helps to correct externalities. To summarize,

Corollary 5.9. *In a country of Brander–Taylor (Copeland–Taylor) type, a possible optimal policy is the production tax on the primary (manufacturing) good that equalizes the MRT and the SMRT.*

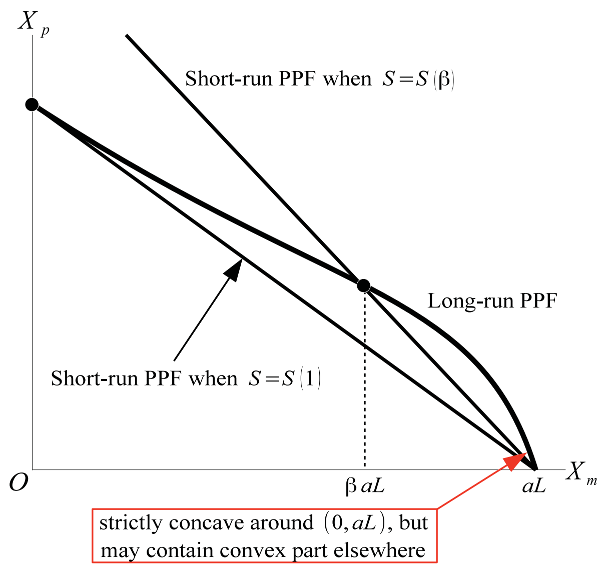
Proposition 5.8 also implies some restrictions on the shape of the long-run PPF. Suppose a point $\left(X_m', X_p'\right)$ on the long-run PPF, which corresponds to the steady-state environmental stock $S\,(\beta')$. According to (5.6), the straight line starting from $(0, aL)$ and passing through $\left(X_m', X_p'\right)$ is the short-run PPF that corresponds with $S = S\,(\beta')$. If the country is of Brander–Taylor (Copeland–Taylor) type, the MRT is greater (less) than the SMRT at $\left(X_m', X_p'\right)$, and the short-run PPF intersects the long-run PPF from above (below). According to Proposition 5.8, this is true for all other points on the long-run PPF, which requires the long-run PPF to be strictly concave (convex) around $(0, aL)$. Apart from $(0, aL)$, the long-run PPF may have convex or concave part, but it cannot be too convex (concave) since the short-run PPF connecting $(0, aL)$ and any point on the long-run PPF must intersect it from above (below), as illustrated in Figure 5.4a and Figure 5.4b. The following corollary summarizes these observations.

Corollary 5.10. *A country of Brander–Taylor (Copeland–Taylor) type has a long-run PPF strictly concave (convex) around $(0, aL)$, and any straight line passing through $(0, aL)$ must intersect, if any, the long-run PPF from above (below).*

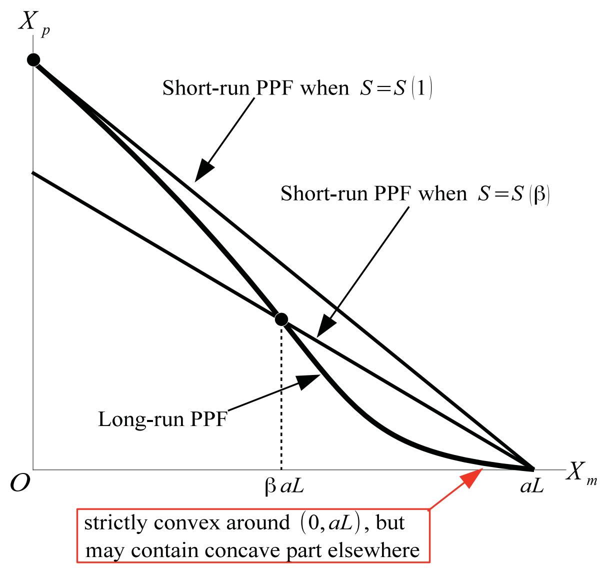
Proof. See Appendix 5.A.1 for detailed calculation. □

5.4 Autarky Equilibrium

After characterizing the supply side, now we can introduce the demand side to close the model. In this section, we consider autarky, which servers as the benchmark for the comparison with free trade.



(a) Brander–Taylor type country



(b) Copeland–Taylor type country

Figure 5.4: Long-run PPF

In autarky, all demands are fulfilled by domestic supplies: $C_i = X_i$ $(i = p, m)$. The preference (5.5) ensures both goods produced. As a result, the manufacturing price is $P = A(S)/a$, and the wage is $w = A(S)$, and thus the income is $A(S)L$. The maximization of utility requires that the share of income spent on the primary good is b, which gives

$$C_p = bA(S)L, \quad C_m = (1-b)aL.$$

Therefore, in autarky the demand for the manufacturing good is a vertical line on the (X_m, P) plane as shown in Figure 5.3. It follows from (5.1) that $L_p = bL$ and $L_m = (1-b)L$. That is, in autarky

$$\beta = b.$$

Therefore, the autarky steady-state environmental stock, denoted S_A, and the autarky steady-state price of the manufacturing good, denoted P_A, can be written into

$$S_A = S(b), \quad P_A = \frac{A(S(b))}{a},$$

which are unique and stable according to Assumption 5.3. These results are the same as Proposition 1 in Brander and Taylor (1997) and in Copeland and Taylor (1999).

5.5 Small Open Economy

Consider free trade in a small open economy. Let P_W denote the world price of the manufacturing good. In the short run, the environment S is given, so production patterns can be obtained by comparing MRT $= A(S)/a$ with P_W. If $A(S)/a > P_W$

$(A(S)/a < P_W)$, the economy has a comparative advantage in the primary (manufacturing) good, thus completely specializing in it.

In the long run, however, the environmental stock may change, thus affecting the MRT and consequently production patterns. The long-run supply curve, as illustrated in Figure 5.3, is very useful to show the long-run response of an economy to free trade.

Let us first see what happens in a small open economy of Brander–Taylor type. There are three situations. First, if $P_W \in (A(S(1))/a, A(S(0))/a)$, as shown in Figure 5.3a, the economy cannot completely specialize in either good in steady state, because the economy eventually loses the comparative advantage due to environmental changes. Trade patterns in steady state can be revealed by comparing P_W and P_A: exporting the primary (manufacturing) good if $P_W < P_A$ $(P_W > P_A)$ and no trade if $P_W = P_A$. Second, if the world price is low enough such that $P_W \leq A(S(1))/a$, in steady state the economy will completely specialize in the primary good. Finally, if the world price is high enough such that $P_W \geq A(S(0))/a$, in steady state the economy will completely specialize in the manufacturing good. As long as the economy produces the manufacturing good, the wage is $w = P_W a$ at every point in time, implying the same consumption of the manufacturing good as in autarky: $C_m = (1-b)wL/P_W = (1-b)aL$. However, if the small economy only produces the primary good, the wage is $w = A(S)$ at every point in time. In steady state, $w = A(S(1))$ and $C_m = (1-b)wL/P_W > (1-b)aL$ since $P_W \leq A(S(1))/a$. These results are the analogue to Proposition 7 in Brander and Taylor (1997).

The long-run welfare effect of free trade can be intuitively analyzed by comparing budget lines and by using its relation with the short-run PPF and the long-run PPF. In Figure 5.5, B_A is the budget line in autarky (also the short-run PPF for $S = S(b)$), while B_2 is the budget line in trade steady state given the world price P_{W2} (also the corresponding short-run PPF). Note that the budget line can be represented by the short-run PPF only when the manufacturing good is produced. If $P_W < A(S(1))/a$, only the primary good will be produced in steady state and the budget line is a line starting from point $(A(S(1))L, 0)$ with the slope P_W, as the budget line B_1 given in Figure 5.5. When the world price is P_{W2}, the budget line B_2 is higher than B_A, so the steady-state utility is higher in free trade than in autarky.

The total welfare effect of trade can be decomposed into two components: the TOT effect (gains from terms of trade improvement) and the green effect (gains from productivity changes in the primary sector). The TOT effect is a static effect, and is always positive. In contrast, the green effect is a dynamic effect, which comes from environmental changes, and can be negative. This happens if $P_W < P_A$, in which the economy uses more labor than in autarky to produce the primary good and export it. Since the primary good is dirtier in an small open economy of Brander–Taylor type, this leads to environmental degradation and consequently a decline in the productivity in the pri-

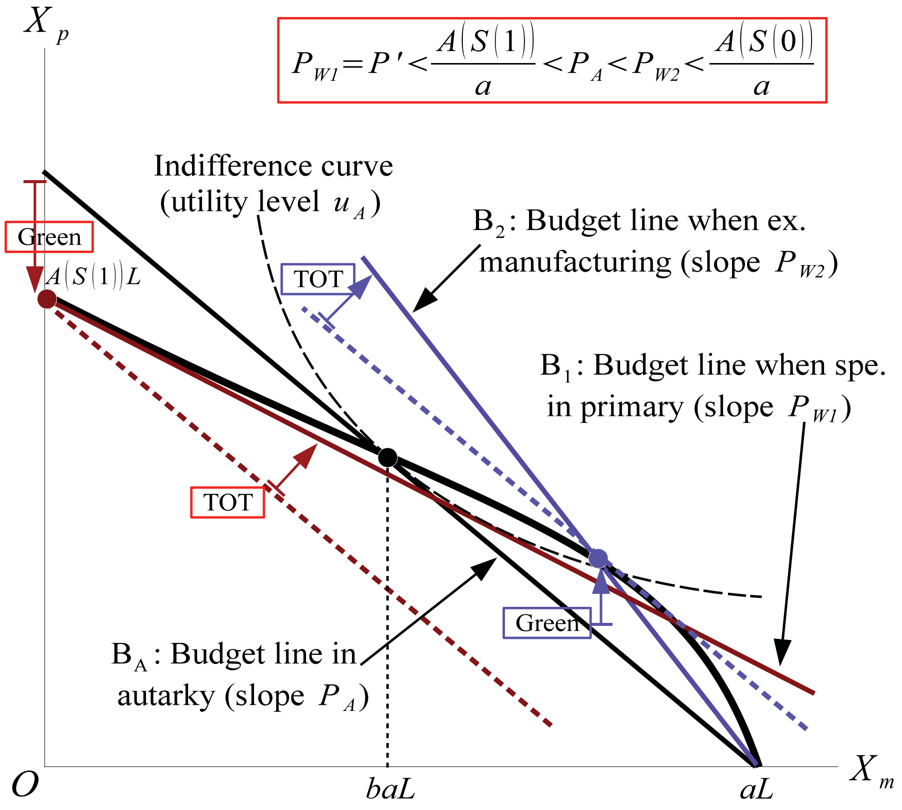


Figure 5.5: Welfare effect in Brander–Taylor type SOE

mary sector, which means a negative green effect. As a result, the steady-state utility in free trade can be lower or higher than in autarky, depending on which effect dominates. Note that there is a threshold level of the world price, denoted P', such that the budget line is tangent to the indifference curve in autarky if $P_W = P'$, as shown in Figure 5.5. If $P_W < P'$, the TOT effect dominates and so the long-run welfare effect of trade is positive. These arguments can be summarized as follows.

Proposition 5.11. *A small open economy of Brander–Taylor type*

(i) remains diversified in the long run if $P_W \in (A(S(1))/a, A(S(0))/a)$*, otherwise specializes,*

(ii) exports the cleaner manufacturing good in the long run if $P_W > P_A$*, and gains from trade,*

(iii) exports the dirtier primary good in the long run if $P_W < P_A$*, and loses from trade if the terms of trade (TOT) effect is small* $(P' < P_W < P_A)$*, and gains if the TOT effect is large* $(P_W < P')$*, where* P' *satisfies* $P' < A(S(1))/a$*.*

Now we move on to a small economy of Copeland–Taylor type, which has quit different long-run responses to free trade. If $P_w \in (A(S(0))/a, A(S(1))/a)$, there are three steady states as shown in Figure 5.3b. The two specialized steady states are stable, while the middle diversified one is unstable. In the long run, the small economy completely specializes in either good. In contrast, if $P_W < A(S(0))/a$ or

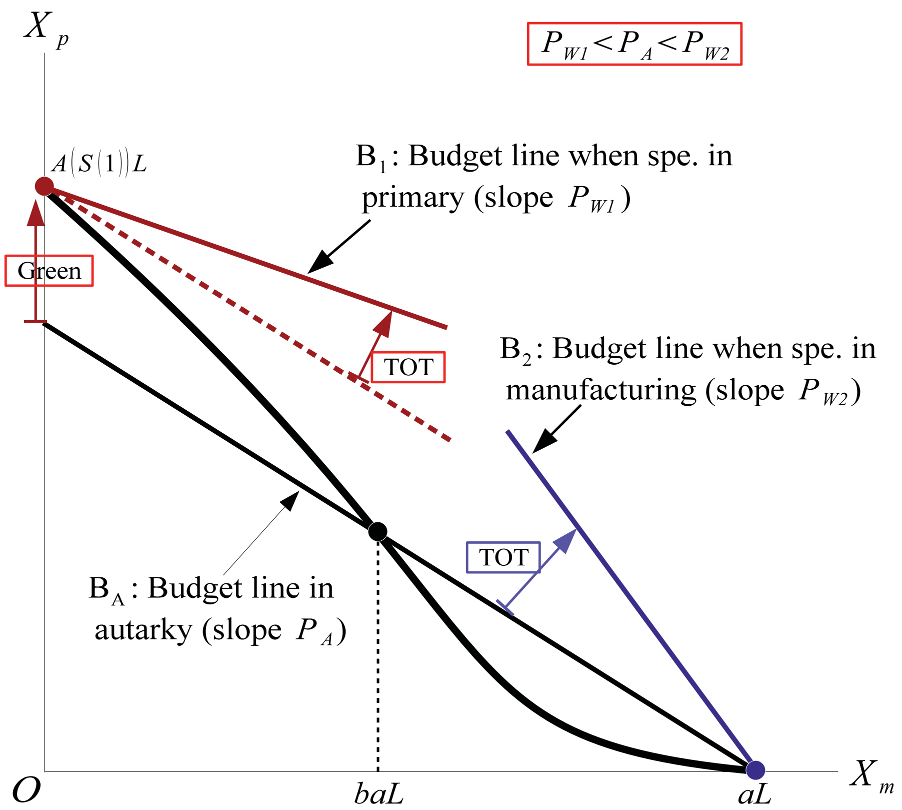


Figure 5.6: Welfare effect in Copeland–Taylor type SOE

$P_W > A(S(1))/a$, there is only one steady state, the economy must specialize in the primary good or the manufacturing good respectively. This result is similar with Proposition 2 in Copeland and Taylor (1999).

If there is no other force, the long-run production pattern in free trade is the same as the production pattern right after trade liberalization. For instance, suppose an economy in autarky steady state before trade liberalization and $P_W < P_A$, as illustrated in Figure 5.3b. When opened to trade, the economy will specialize in the primary good. Since the primary good is cleaner, the environment improves gradually, which reinforces the comparative advantage in the primary good. This self-reinforcing process ensures that the trade pattern right after trade remains unchanged in the long run.

The long-run welfare effect in the small economy of Copeland–Taylor type is relatively simple. If the economy specializes in the cleaner primary good, the green effect is positive. Together with the positive TOT effect, the new budget line, as shown by B_1 in Figure 5.6, will be higher than that in autarky, as shown by B_A in the figure. If the economy specializes in the dirtier manufacturing good, there is no green effect since the manufacturing productivity is fixed. But there are still gains due to the TOT effect. As a result, the budget line, as given by the line B_2 in Figure 5.6, is also higher than that in autarky. In both cases, a small open economy of Copeland–Taylor type enjoys the long-run gains from trade. The following proposition summarizes these results.

Proposition 5.12. *A small open economy of Copeland–Taylor type specializes in the good that* *has lower relative price than the world when opened to trade, and gains in the long run.*

5.6 Two-country World

In this section, we consider free trade between two countries, Home and Foreign. The analysis proceeds mainly in two steps. First, we focus on the short run and construct the *world production pattern diagram* to show how production patterns in the short run are related to environmental stocks and labor endowments in both countries. Second, we characterize the long-run behaviors with the aid of the diagram.

5.6.1 Short-run World Production Patterns

Let Foreign variables denoted with an asterisk *, and Foreign related functions denoted with the subscript f. Home related functions are denoted with h. For simplicity, assume the identical preference in two countries. In the short run, S and S^* are given, so trade patterns can be readily revealed by comparing the MRTs in two countries, namely $A_h(S)/a$ in Home and $A_f(S^*)/a^*$ in Foreign. It is convenient to define the *comparative advantage index* by

$$v \equiv \frac{A_f(S^*)\,a}{A_h(S)\,a^*}. \tag{5.17}$$

If $v < 1$ ($v > 1$), Home (Foreign) has a comparative advantage in the primary good, thus exporting it in free trade.

World production patterns in the short run also depend on the country size (L and L^*), the manufacturing productivity (a and a^*) and the preference (b). It is also convenient to define the *relative effective size* by

$$z \equiv \frac{aL}{a^*L^*}. \tag{5.18}$$

If $z < 1$ ($z > 1$), Home (Foreign) is smaller in terms of the manufacturing production capability.

Let us consider first the situation of $v < 1$, in which Home exports the primary good and Foreign exports the manufacturing good. The short-run Ricardian structure of the model ensures that at least one country completely specializes. So, there are three possible world production patterns.

Pattern (p,d): Home specializes in the primary good, while Foreign produces both goods.

Pattern (p,m): Home specializes in the primary good, while Foreign specializes in the manufacturing good.

Pattern (d,m): Home produces both goods, while Foreign specializes in the manufacturing good.

In what follows, we shall check which pattern arises under what condition.

In production pattern (p,d), we have $X_p = A_h(S)L$, $X_p^* = A_f(S^*)\beta^* L^*$ and $X_m^* = a^*(1-\beta^*)L^*$. Since Foreign produces both goods, the world manufacturing price depends only on Foreign condition:

$$P_W = \frac{A_f(S^*)}{a^*}. \tag{5.19}$$

Since both countries produce the primary good, which is the numeraire, the wage in Home is $w = A_h(S)$, while the wage in Foreign is $w^* = A_f(S^*)$. Since two countries have the same preference and the manufacturing good is only supplied by Foreign, we have $(1-b)(wL + w^*L^*) = P_W X_m^*$, which gives, using (5.19),

$$\beta^* = b - (1-b)\frac{z}{v}. \tag{5.20}$$

Since Foreign produces both, we must have $\beta^* > 0$, this requires

$$z < \frac{b}{1-b}v \tag{5.21}$$

In production pattern (p,m), we have $X_p = A_h(S)L$ and $X_m^* = a^*L^*$. Home wage is $w = A_h(S)$ while Foreign wage is $w^* = P_W a^*$. Because both countries completely specialize, P_W is determined such that the world supply equals the world demand. This requires $b(wL + w^*L^*) = X_p$, which gives

$$P_W = \frac{(1-b)A_h(S)L}{ba^*L^*}. \tag{5.22}$$

Notting that $P_W \in \left[A_f(S^*)/a^*, A_h(S)/a\right]$ must hold for two countries to completely specialize, it follows from (5.22) that

$$\frac{b}{1-b}v \leq z \leq \frac{b}{1-b}. \tag{5.23}$$

In production pattern (d,m), we have $X_p = A_h(S)\beta L$, $X_m = a(1-\beta)L$ and $X_m^* = a^*L^*$. Since Home produces both goods, the world manufacturing price depends only on Home condition:

$$P_W = \frac{A_h(S)}{a}. \tag{5.24}$$

Home wage is $w = A_h(S)$, while Foreign wage becomes $w^* = P_W a^* = A_h(S)a^*/a$. Since the primary good is only supplied by Home, we have $b(wL + w^*L^*) = X_p$, which gives

$$\beta = b\left(1 + \frac{1}{z}\right). \tag{5.25}$$

Since Home produces both, we must have $1-\beta>0$, which requires

$$z>\frac{b}{1-b}. \tag{5.26}$$

Let us consider now the situation of $v>1$, in which Home has a comparative advantage in the manufacturing good and exports it in free trade. The possible world production patterns include:

Pattern (d,p): Home produces both goods, while Foreign specializes in the primary good.

Pattern (m,p): Home specializes in the manufacturing good, while Foreign specializes in the primary good.

Pattern (m,d): Home specializes in the manufacturing good, while Foreign produces both goods.

The conditions for these patterns to arise can be similarly obtained. Specifically, production pattern (d,p) requires

$$z>\frac{1-b}{b}v. \tag{5.27}$$

Given (z,v), the labor allocation in Home is

$$\beta=b-(1-b)\frac{v}{z}. \tag{5.28}$$

Production pattern (m,p) requires

$$\frac{1-b}{b}\leq z\leq\frac{1-b}{b}v. \tag{5.29}$$

The condition for pattern (m,d) is

$$z<\frac{1-b}{b}. \tag{5.30}$$

Given (z,v), the labor allocation in Foreign is

$$\beta^*=b(1+z). \tag{5.31}$$

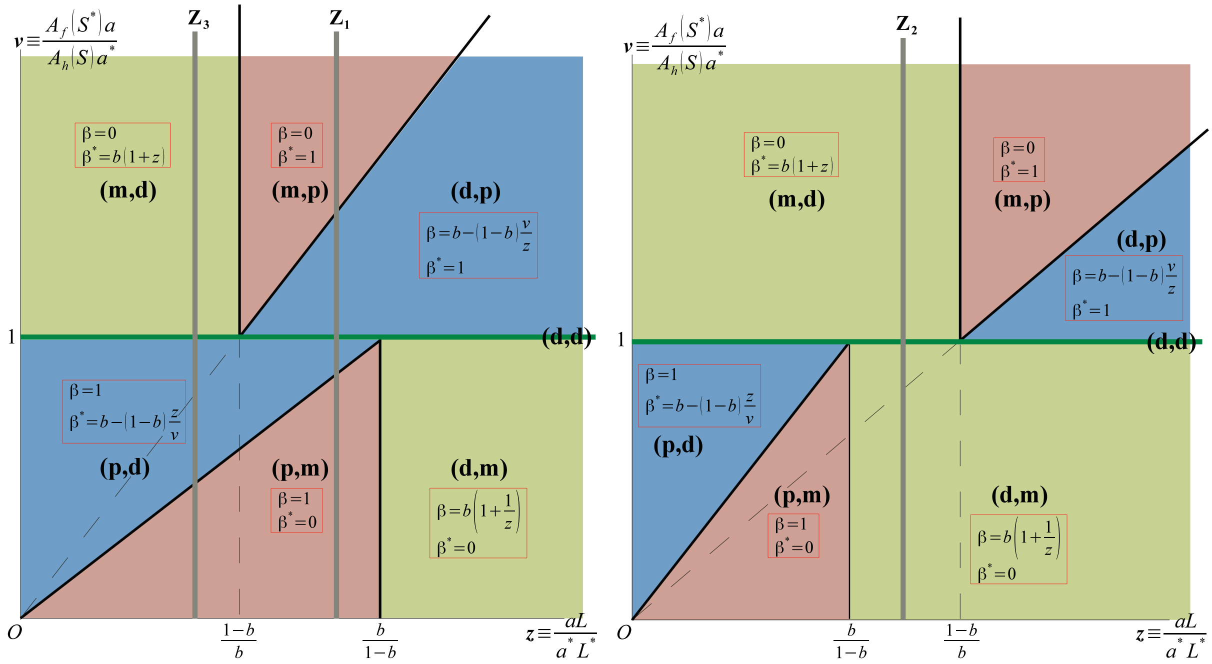
Finally, consider the situation of $v=1$, denoted by (d,d). In this situation, wages are $w=A_h(S)$ and $w^*=A_f(S^*)$. The world equilibrium is achieved as long as

$$b\left(A_h(S)L+A_f(S^*)L^*\right)=A_h(S)\beta L+A_f(S^*)\beta^*L^*,$$

which gives

$$\beta z+\beta^*=b(z+1). \tag{5.32}$$

Note that β can vary from 1 to $b-(1-b)/z$, implying an indeterminacy in pattern (d,d).



(a) $b > 1/2$ (b) $b < 1/2$

Figure 5.7: World production pattern diagram

5.6.2 World Production Pattern Diagram

Given (z, v), conditions (5.21), (5.23), (5.26), (5.27), (5.29), and (5.30) help identify what production pattern arises. With these results in hand, we can go further by constructing a diagram to show how world production patterns are distributed on the (z, v) plane, as shown in Figure 5.7.

The construction of world production pattern diagram is just a visualization of the conditions for each pattern on the (z, v) plane. Specifically, (5.21) is the condition for pattern (1,D) to arise, which requires that (z, v) lies above the ray with the slope $(1-b)/b$. Together with $v < 1$ (and of course $z > 0$), we obtain a triangle area labeled (p,d) in Figure 5.7. Given (z, v) in this area, pattern (p,d) arises. As for pattern (p,m), (5.23) requires (z, v) lying below the ray with the slope $(1-b)/b$ and on the left of the vertical line $z = b/(1-b)$. This, together with $v < 1$, gives the area labelled (p,m) in the figure. As for pattern (d,m), it follows from (5.26) that (z, v) must be on the right of the vertical line $z = b/(1-b)$ and below the horizontal line $v = 1$, as illustrated by the corresponding area in the figure.

The distribution of patterns (m,d), (m,p) and (d,p) can be similarly obtained. Note that world production patterns are affected by the preference parameter b. For example, if $b < 1/2$ and two countries are similar in the effective size ($b/(1-b) < z < (1-b)/b$), neither country can specializes in the primary good since the world demand is too small. This difference caused by the preference b can be clearly seen by comparing Figure 5.7a and Figure 5.7b.

With the aid of the world production pattern diagram, we can easily pin down the short-run world production pattern given z and v. The reader may have noticed that the diagram is nothing more than an exposition of the conventional two-country Ricardian trade model. But it does provide a new and intuitive way to show how different world production patterns are related with each other. The diagram is also very useful for comparative statics exercises. For example, an increase in Home's labor endowment (L) causes an increase in z, thus the new world production pattern can be simply obtained by moving right from the original point. An increase in Home's manufacturing productivity (a) has two effects. It raises Home's effective size aL and consequently z, implying a horizontal movement to the right. At the same time, an increase in a reduces Home's MRT $A(S)/a$ and thus increases v, implying a vertical movement to the upper. The total effect is a upper-right movement in the diagram.

5.6.3 Labor Allocation and Comparative Advantage Index

In the long run, the comparative advantage index v is endogenously determined because Home environment S and Foreign environment S^* can evolve over time. Using (5.3) and (5.10), we have

$$\dot{S} = G_h(S) - \left[l_p A_h(S)\beta + l_m a(1-\beta)\right] L, \tag{5.33}$$

$$\dot{S}^* = G_f(S^*) - \left[l_p^* A_f(S^*)\beta^* + l_m^* a^*(1-\beta^*)\right] L^*. \tag{5.34}$$

To close the dynamic system, we need to express β and β^* in terms of S and S^*, which is our focus in this section.

Given the relative effective size z, in the world production pattern diagram, the candidates of the long-run world production patterns lie on a vertical line. Moving along the vertical line, v changes and thus, according to (5.20), (5.25), (5.28), (5.31) and (5.32), β and β^*also changes. But the way β and β^* change varies with b and z. For example, if $b > 1/2$ and $(1-b)/b \leq z \leq b/(1-b)$, we obtain a vertical line Z_1 as shown in Figure 5.7a. Moving along Z_1 from the bottom, the world production pattern changes from (p,m) to (p,d) and further to (d,d), (d,p) and (m,p). In contrast, if $b < 1/2$ and $b/(1-b) \leq z \leq (1-b)/b$, we have a vertical line Z_2 as shown in Figure 5.7b. Moving along Z_2 from the bottom, the world production pattern changes from (d,m) to (d,d) and to (m,d). Table 5.1 summarizes all six cases categorized by the ranges of b and z, each of which corresponds with a particular way the world production pattern changes with v.

However, we do not have to discuss all the six cases. First, we can ignore the two cases related to small Foreign. This is because there are only two countries. Once we derive the results for small Home, the results for small Foreign can be immediately obtained by switching "Home" and "Foreign" in the results. Second, we can ignore

	$b > 1/2$	$b < 1/2$
Similar sizes	(p,m), (p,d), (d,d), (d,p), (m,p)	(d,m), (d,d), (m,d)
Small Home	(p,m), (p,d), (d,d), (m,d)	(p,m), (p,d), (d,d), (m,d)
Small Foreign	(d,m), (d,d), (d,p), (m,p)	(d,m), (d,d), (d,p), (m,p)

Table 5.1: Possible world production patterns

the case of small Home with $b < 1/2$. This is because the same changing pattern appears to be same in the two cases related to small Home. It is enough to discuss one instead of both cases.

Therefore, it is sufficient to analyze the following three cases, corresponding with the three shadowed in Table 5.1. For each case, we can write β and β^* as piecewise functions of v, denoted by $\beta_h(v)$ and $\beta_f(v)$.

Case 1. High demand for the primary good and similar effective sizes: $b > 1/2$, $(1-b)/b \leq z \leq b/(1-b)$.

Case 2. Low demand for the primary good and similar effective sizes: $b < 1/2$, $b/(1-b) \leq z \leq (1-b)/b$.

Case 3. High demand for the primary good and small Home: $b > 1/2, z < (1-b)/b$.

High demand for primary good and similar effective sizes: Case 1 Note that Case 1 corresponds with Z_1 in Figure 5.7a. As v increases, the world production pattern shifts from (p,m) to (p,d) and further to (d,d), (d,p) and (m,p). Home labor allocation β is

$$\beta_h(v) = \begin{cases} 1 & \text{if } v \leq \frac{1-b}{b}z, \\ 1 & \text{if } \frac{1-b}{b}z < v < 1, \\ \left[b - \frac{1-b}{z}, 1\right] \text{ s.t. (5.32)} & \text{if } v = 1, \\ b - (1-b)\frac{v}{z} & \text{if } 1 < v < \frac{b}{1-b}z, \\ 0 & \text{if } v \geq \frac{b}{1-b}z. \end{cases} \tag{5.35}$$

Foreign labor allocation β^* is

$$\beta_f(v) = \begin{cases} 0 & \text{for } v \leq \frac{1-b}{b}z, \\ b - (1-b)\frac{z}{v} & \text{for } \frac{1-b}{b}z < v < 1, \\ \left[b - (1-b)z, 1\right] \text{ s.t. (5.32)} & \text{for } v = 1, \\ 1 & \text{for } 1 < v < \frac{b}{1-b}z, \\ 1 & \text{for } v \geq \frac{b}{1-b}z. \end{cases} \tag{5.36}$$

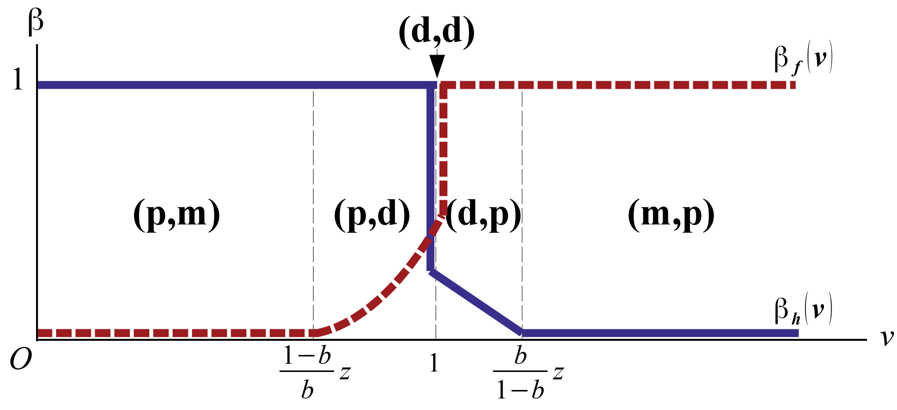


Figure 5.8: Labor allocations in Case 1

The functions (5.35) and (5.36) are nothing but a summary of previous results including (5.20) and (5.28). Figure 5.8 illustrates the shapes of $\beta_h(v)$ and $\beta_f(v)$. Note that both are continuous functions of v except at $v = 1$, where both countries are diversified, and $\beta_h(1)$ and $\beta_f(1)$ satisfy (5.32). When crossing $v = 1$, $\beta_h(v)$ drops from 1 to $b - (1-b)/z$ and $\beta_f(v)$ jumps from $b - (1-b)z$ to 1 as illustrated.

Low demand for primary good and similar effective sizes: Case 2 Note that Case 2 corresponds with Z_2 in Figure 5.7b. As v increases, the world production pattern changes from (d,m) to (d,d) and to (m,d). It follows from (5.25) and (5.31) that

$$\beta_h(v) = \begin{cases} b\left(1+\frac{1}{z}\right) & \text{if } v < 1, \\ \left[0, b\left(1+\frac{1}{z}\right)\right] \text{ s.t. (5.32)} & \text{if } v = 1, \\ 0 & \text{if } v > 1, \end{cases} \tag{5.37}$$

and

$$\beta_f(v) = \begin{cases} 0 & \text{if } v < 1, \\ [0, b(1+z)] \text{ s.t. (5.32)} & \text{if } v = 1, \\ b(1+z) & \text{if } v > 1. \end{cases} \tag{5.38}$$

Figure 5.9 illustrates the shapes of $\beta_h(v)$ and $\beta_f(v)$ in Case 2. They are constant over $[0,1)$ and $(1,\infty)$. As in Case 1, both countries are diversified at $v = 1$, and $\beta_h(1)$ and $\beta_f(1)$ satisfy (5.32). When crossing $v = 1$, $\beta_h(v)$ drops while $\beta_f(v)$ jumps.

High demand for primary good and small Home: Case 3 Note that Case 3 corresponds with Z_3 in Figure 5.7a. As v increases, the world production pattern changes

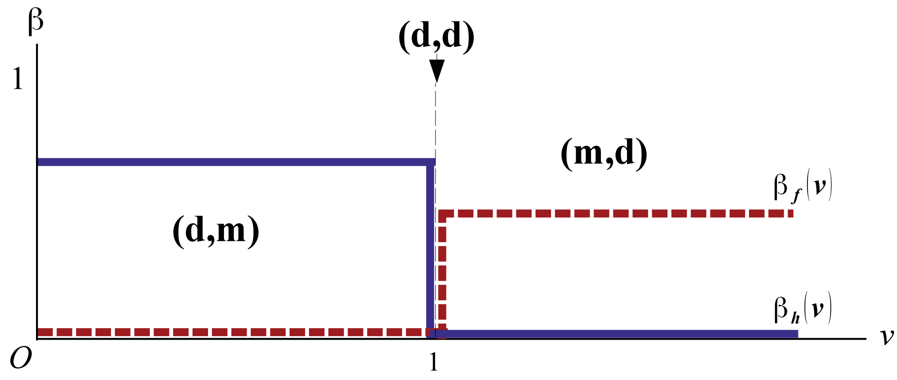


Figure 5.9: Labor allocations in Case 2

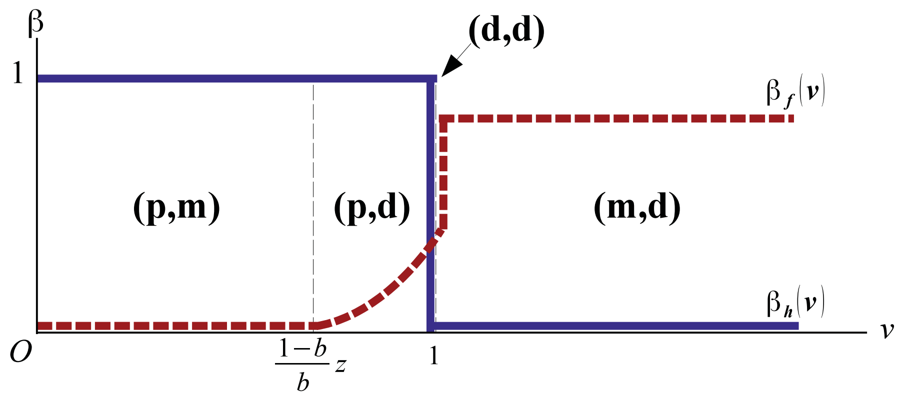


Figure 5.10: Labor allocations in Case 3

from (p,m) to (p,d) and further to (d,d) and (m,d). Specifically, we have

$$\beta_h(v) = \begin{cases} 1 & \text{if } v \leq \frac{1-b}{b}z, \\ 1 & \text{if } \frac{1-b}{b}z < v < 1, \\ [0,1] \text{ s.t. (5.32)} & \text{if } v = 1, \\ 0 & \text{if } v > 1, \end{cases} \tag{5.39}$$

and

$$\beta_f(v) = \begin{cases} 0 & \text{if } v \leq \frac{1-b}{b}z \\ b - (1-b)\frac{z}{v} & \text{if } \frac{1-b}{b}z < v < 1, \\ [b-(1-b)z, b(1+z)] \text{ s.t. (5.32)} & \text{if } v = 1, \\ b(1+z) & \text{if } v > 1. \end{cases} \tag{5.40}$$

Their shapes are illustrated in Figure 5.10.

5.6.4 Characterize Long-run Production Patterns: Preparation

Using the definition of v, and substituting $\beta_h(v)$ and $\beta_f(v)$ into the dynamic equations (5.33) and (5.34) for β and β^*, we can obtain

$$\dot{S} = G_h(S) - \left[l_p A_h(S)\beta_h\left(\frac{A_f(S^*)a}{A_h(S)a^*}\right) + l_m a\left(1-\beta_h\left(\frac{A_f(S^*)a}{A_h(S)a^*}\right)\right)\right]L, \tag{5.41}$$

$$\dot{S}^* = G_f(S^*) - \left[l_p^* A_f(S^*)\beta_f\left(\frac{A_f(S^*)a}{A_h(S)a^*}\right) + l_m^* a^*\left(1-\beta_f\left(\frac{A_f(S^*)a}{A_h(S)a^*}\right)\right)\right]L^*. \tag{5.42}$$

Together with the expressions of $\beta_h(\cdot)$ and $\beta_f(\cdot)$, (5.41) and (5.42) provide a complete description of the dynamics of S and S^*.

Letting $\dot{S} = \dot{S}^* = 0$ in (5.41) and (5.42), we can obtain two equations for the steady-state environmental stocks $\tilde{S}$ and $\tilde{S}^*$:

$$\tilde{S} = S_h\left(\beta_h\left(\frac{A_f(\tilde{S}^*)a}{A_h(\tilde{S})a^*}\right)\right), \quad \tilde{S}^* = S_f\left(\beta_h\left(\frac{A_f(\tilde{S}^*)a}{A_h(\tilde{S})a^*}\right)\right), \tag{5.43}$$

where $S_h(\beta)$ is the solution for S given β and $\dot{S} = 0$ in (5.33), and $S_f(\cdot)$ is the Foreign counterpart.

Provided specific parameter values and functional forms, $\tilde{S}$ and $\tilde{S}^*$ can be solved out from (5.43). We can also conduct some qualitative analysis. For this purpose, note that (5.43) implies

$$\frac{A_f(\tilde{S}^*)a}{A_h(\tilde{S})a^*} = \frac{A_f\left(S_f\left(\beta_h\left(\frac{A_f(\tilde{S}^*)a}{A_h(\tilde{S})a^*}\right)\right)\right)a}{A_h\left(S_h\left(\beta_h\left(\frac{A_f(\tilde{S}^*)a}{A_h(\tilde{S})a^*}\right)\right)\right)a^*}. \tag{5.44}$$

Let $\tilde{v} = A_f(\tilde{S}^*)a/(A_h(\tilde{S})a^*)$ and define

$$g(v) \equiv \frac{A_f(S_f(\beta_f(v)))a}{A_h(S_h(\beta_h(v)))a^*}. \tag{5.45}$$

Then (5.44) can be rewritten into

$$\tilde{v} = g(\tilde{v}). \tag{5.46}$$

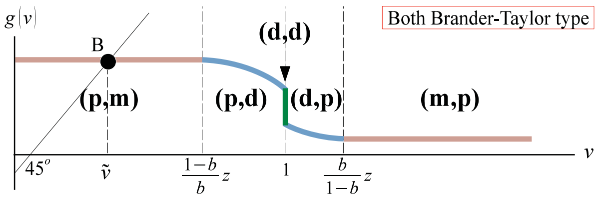
Therefore, a necessary condition for being in steady state is that the comparative advantage index satisfies equation (5.46). Furthermore, Appendix 5.A.2 shows that the steady state is stable if $g'(\tilde{v}) < 1$. To summarize,

Proposition 5.13. *Let $\tilde{v}$ denote the value of v in a stable steady state, then*

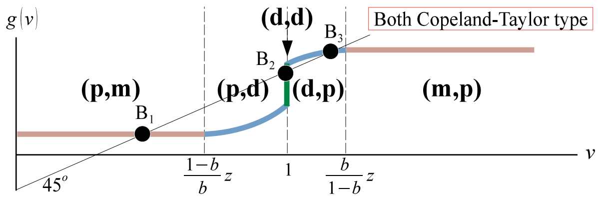
$$g(\tilde{v}) = \tilde{v}, \tag{5.47}$$

$$g'(\tilde{v}) < 1. \tag{5.48}$$

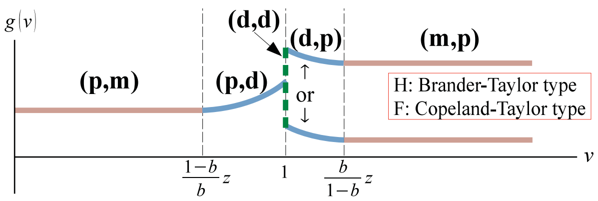
Having these features, we can examine the properties of the steady state by focusing on the shapes of $g(v)$. Let us first look at Case 1.



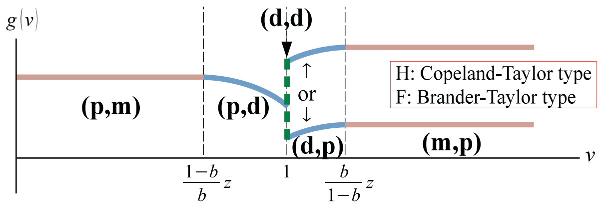
(a) Both countries Brander–Taylor type



(b) Both countries Copeland–Taylor type



(c) H: Brander–Taylor type, F: Copeland–Taylor type



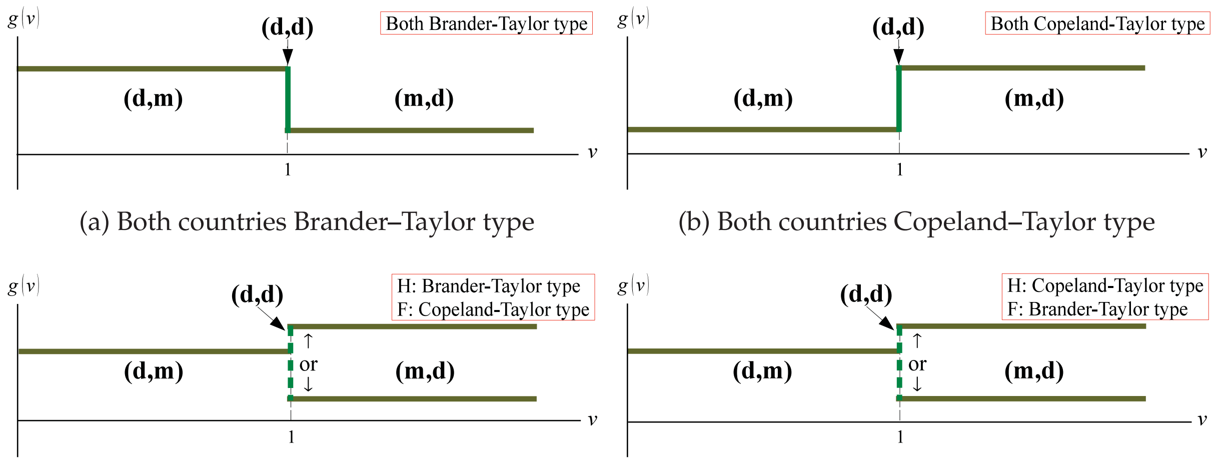
(d) H: Copeland–Taylor type, F: Brander–Taylor type

Figure 5.11: Shapes of $g(v)$ in Case 1

Shapes of $g(v)$ in Case 1 As analyzed in details in Appendix 5.A.3, $g(v)$ is constant on the interval $[0, (1-b)z/b]$ and $[bz/(1-b), \infty)$, since $\beta_h(v)$ and $\beta_f(v)$ are constant on the interval. For v in-between, labor allocation β or β^* changes with v, and so does $S_h(\beta)$ or $S_f(\beta^*)$. The shape of $g(v)$ can be obtained by using the fact that $S_h'(\beta)$ is negative (positive) if Home is of Brander–Taylor (Copeland–Taylor) type. The same applies to Foreign. Therefore, if both countries are of Brander–Taylor type, $g(v)$ is strictly decreasing on $((1-b)z/b, 1)$ and drops at $v = 1$. If both countries are Copeland–Taylor type, $g(v)$ jumps at $v = 1$ and is strictly increasing on $(1, bz/(1-b))$. If Home is of Brander–Taylor type and Foreign is of Copeland–Taylor type, $g(v)$ strictly increases with v on $((1-b)z/b, 1)$ and decreases with v on $(1, bz/(1-b))$. The opposite holds if Home is of Copeland–Taylor type and Foreign is of Brander–Taylor type. For a world with two countries of different types, whether $g(v)$ drops or jumps at $v = 1$ is ambiguous, depending on specific parameter values and functional forms. Figure 5.11 provides the possible shapes of $g(v)$.

Shapes of $g(v)$ in Case 2 As shown previously, $\beta_h(v)$ and $\beta_f(v)$ are simpler in Case 2, thus yielding simpler properties of $g(v)$, compared to Case 1. Following the similar steps as in Case 1, It is easy to shown that $g(v)$ is constant over $[0, 1)$ and $(1, \infty)$, and jumps (drops) at $v = 1$ if both countries are of Brander–Taylor (Copeland–Taylor) type. If two countries are of different types, it is ambiguous whether $g(v)$ jumps or drops at $v = 1$. Figure 5.12 provides the possible shapes of $g(v)$ in Case 2.

Shapes of $g(v)$ in Case 3 It is easy to see that $g(v)$ in Case 3 has the same feature with Case 1 if $v < 1$ and the same feature with Case 2 if $v > 1$. That is, $g(v)$ remains



(a) Both countries Brander–Taylor type (b) Both countries Copeland–Taylor type

(c) H: Brander–Taylor type, F: Copeland–Taylor type (d) H: Copeland–Taylor type, F: Brander–Taylor type

Figure 5.12: Shapes of $g(v)$ in Case 2

constant on $[0, (1-b)z/b]$ and $(1, \infty)$. If both countries are of Brander–Taylor type, $g(v)$ is strictly decreasing on $((1-b)z/b, 1)$ and drops at $v = 1$. In contrast, if both countries are of Copeland–Taylor type, $g(v)$ is strictly increasing on $((1-b)z/b, 1)$ and jumps at $v = 1$. If two countries are of different types, the behavior at $v = 1$ is also ambiguous. Figure 5.13 gives the possible shapes of $g(v)$ in Case 3.

5.6.5 Long-run Production Patterns and Welfare Effects

Since the steady state is the fixed point of $g(v)$, with the shapes of $g(v)$ in hand, we can easily pin down the steady state by looking for its intersection with the 45 degree ray. For example, in Figure 5.11a, the intersection point B determines a steady-state value of v, denoted $\tilde{v}$ in the figure. Because $\tilde{v}$ locates in the range of pattern (p,m), indicating that the world production pattern in steady state.

Although the model is general, we can still derive some interesting insights. We begin with two countries of Brander–Taylor type.

Proposition 5.14. *If two countries of Brander–Taylor type are in free trade, then*

(i) there exists a unique $\tilde{v}$ in steady state;

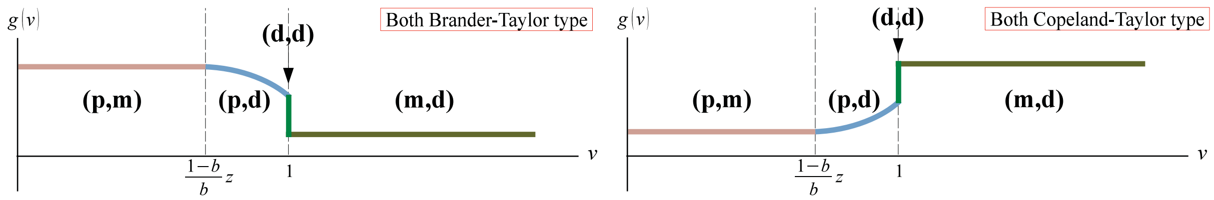
(ii) neither country can specialize in the manufacturing good in steady state if

$$\frac{A_f(S_f(0))\, a}{A_h(S_h(0))\, a^*} = 1; \tag{5.49}$$

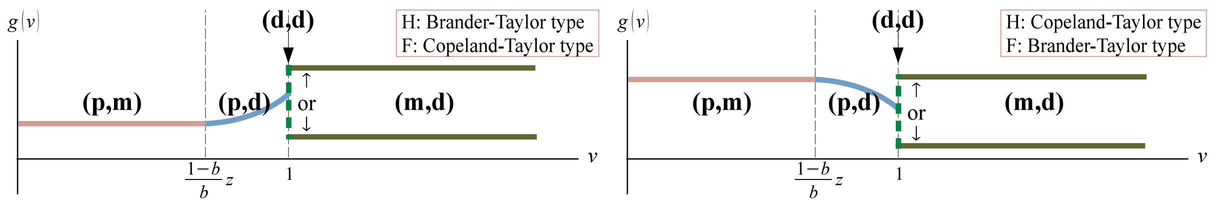
(iii) neither country can specialize in the primary good in steady state if

$$\frac{A_f(S_f(1))\, a}{A_h(S_h(1))\, a^*} = 1, \tag{5.50}$$

and the country exporting the primary good loses from trade in the long run.



(a) Both countries Brander–Taylor type (b) Both countries Copeland–Taylor type



(c) H: Brander–Taylor type, F: Copeland–Taylor type (d) H: Copeland–Taylor type, F: Brander–Taylor type

Figure 5.13: Shapes of $g(v)$ in Case 3

Proof. (i) In all three cases, $g(v)$ decreases with v and drops at $v = 1$. Clearly, $g(v)$ intersects the 45 degree ray once and only once. This gives the existence and uniqueness. The stability follows directly from Proposition 5.13.

(ii) Suppose to the contrary that Home specializes in the manufacturing good. Then Foreign must produce some primary good, implying that $\beta^* > 0$. On the other hand, it follows from (5.49) that

$$\frac{A_h(S_h(0))}{a} = \frac{A_f(S_f(0))}{a^*} > \frac{A_f(S_f(\beta^*))}{a^*}$$

holds in steady state. This says that Home has a comparative advantage in the primary good, implying that Home will not specialize in the manufacturing good in steady state. The similar argument applies to Foreign, too.

(iii) Suppose to the contrary that Home specializes in the primary good. Then Foreign must produce some manufacturing good, implying that $\beta^* < 1$. On the other hand, it follows from (5.50) that

$$\frac{A_h(S_h(1))}{a} = \frac{A_f(S_f(1))}{a^*} < \frac{A_f(S_f(\beta^*))}{a^*}$$

holds in steady state. This says that Home has a comparative advantage in the manufacturing good, implying that Home will not specialize in the primary good in steady state. Supposing that Home exports the primary good, it must produce both goods since it cannot specialize in the primary good. As a result, the world price of the manufacturing good depends only on Home condition, which will be higher than $A_h(S_h(1))/a$ in steady state. It follows directly from Proposition 5.11 (iii) that Home loses from trade in the long run. The similar arguments apply to Foreign, too. □

Note that (5.50) is implicitly assumed in Brander and Taylor (1998), which leads to their Proposition 4 (i). The long-run welfare effect in two countries of Brander–Taylor type is similar to that in the case of small open economy. In contrast, if two countries are of Copeland–Taylor type, something that does not arise in the case of small open economy happens.

Proposition 5.15. *If two countries of Copeland–Taylor type are in free trade, then*

(i) there are odd number of $\tilde{v}$, among which $(n+1)/2$ are stable while the remaining are unstable;

(ii) one country must specialize and gains from trade in the long run;

(iii) the other country may specialize or not; it loses from trade in the long run if remaining diversified and exports the manufacturing good, otherwise gains.

Proof. (i) In all three cases, $g(v)$ approaches a positive value as $v \to 0$ and approaches another larger value as $v \to \infty$. Clearly, $g(v)$ must intersect the 45 degree ray odd number of times. At the first intersection point, we must have $g'(v) < 1$, thus the first one is stable according to Proposition 5.13. At the second intersection point, if any, we must have $g'(v) > 1$, imply that the second one is unstable. At the third intersection point, if any, we must have $g'(v) < 1$, and so on.

(ii) This is equivalent to prove the steady state in which both countries are diversified, if any, is unstable. In this situation, $v = 1$ is a fixed point of $g(v)$. Because both countries are of Copeland–Taylor type, $g(v)$ jumps at $v = 1$, implying $g'(1) = \infty$. According to Proposition 5.13, the steady state is unstable. The gains from trade follows directly from Proposition 5.12.

(iii) If Foreign exports the manufacturing good in steady state and remains diversified, then according to (ii), Home must specializes in the primary good. Compared to autarky, Foreign environmental stock degrades, leading to declines in the productivity in the primary sector. This produces a negative green effect in Foreign since it produces the primary good in steady state. For the same reason, the world price of manufacturing good decreases, causing a negative TOT effect. Therefore, the total welfare effect of trade must be negative in the long run. □

As illustrated in Figure 5.11b, the 45 degree ray intersects $g(v)$ three times. At B_1 and B_3, corresponding to pattern (p,m) and pattern (d,p), $g(v)$ cross the 45 degree ray from above, implying that B_1 and B_3 are stable. At point B_2, corresponding to pattern (d,d), $g(v)$ cross the 45 degree ray from below, implying that the diversified steady state is unstable.

Unlike the case of small open economy, in a two-country world, a country of Copeland–Taylor type may remain diversified in steady state due to the limitation of the world market size. It is this difference brings about the possibility that a country of Copeland–Taylor type loses from trade.

So far, at most one country may lose from trade in the long run. However, the following proposition shows that this is not always the case if two countries are of different types.

Proposition 5.16. *If two countries are of different types and the country of Copeland–Taylor type exports the manufacturing good and remains diversified, both countries may lose from trade in the long run.*

Proof. Without loss of generality, let Home be of Brander–Taylor type and Foreign be of Copeland–Taylor type. Foreign exports the manufacturing good, so Home exports the primary good. That is, both countries exports their own dirtier goods. Since Foreign stays diversified, it loses from trade according to Proposition 5.15 (iii). If the world price of manufacturing good not low enough, Home also loses according to Proposition 5.11 (iii). □

An direct corollary of Proposition 5.16 is that, if two countries of different types remain diversified in free trade, both countries lose from trade in the long run. A question naturally arises: under what condition two countries of different types stay diversified in trade steady state? An example is provided as follows. If the country of Brander–Taylor type is relatively small and

$$\frac{A_f\left(S_f\left(1\right)\right)a}{A_h\left(S_h\left(0\right)\right)a^*} = \frac{A_f\left(S_f\left(0\right)\right)a}{A_h\left(S_h\left(1\right)\right)a^*} = 1 \tag{5.51}$$

holds, then two countries of different types remain diversified in free trade. To see this, assume without loss of generality that Home is of Brander–Taylor type and relatively small, then we can use the results derived from Case 3. By (5.51), $g\left(v\right) = 1$ on the interval $[0, (1-b)\,z/b]$, $g\left(v\right) > 1$ on $((1-b)\,z/b, 1)$, and $g\left(v\right) < A_f\left(S_f\left(1\right)\right)a/\left(A_h\left(S_h\left(0\right)\right)a^*\right) = 1$ on $(1, \infty)$. This implies that $\tilde{v} = 1$, namely in steady state two countries stay diversified.

5.7 Discussion

5.7.1 Parameter-induced Regime Change

As shown previously, the type of country is crucial in determining trade patterns. This can be seen more clearly by considering two ex-ante identical countries. If both countries are of Brander–Taylor type, there is no trade between them. In contrast, if both are of Copeland–Taylor type, the two countries will trade with each other and at least one country completely specializes.

Remember that the type of country is determined by parameters (l_p, l_m, a, L) and functions ($A\left(S\right)$, $G\left(S\right)$). Therefore, technological progress or population growth may

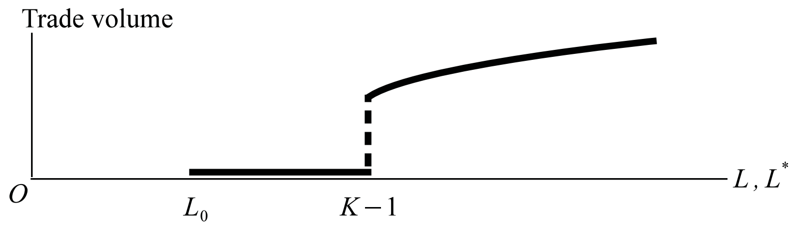


Figure 5.14: Trade volume and labor endowment

change the type of a country, causing dramatic change in trade patterns. To see this parameter-induced regime change as simply as possible, we use the following specific functions

$$A(S) = S, \quad G(S) = K - S. \tag{5.52}$$

Especially, we focus on a change in labor endowment and let $l_p = l_m = a = 1$. Hence,

$$X_1 = S\beta L,$$
$$X_2 = (1-\beta)L,$$
$$E = X_1 + X_2.$$

Note that the primary sector is cleaner than the manufacturing sector if $S < 1$ and dirtier if $S > 1$. Given β, the steady-state environmental stock $S(\beta)$ can be solved from $G(S) = E$, namely $K - S = S\beta L + (1-\beta)L$, which gives

$$S(\beta) = \frac{K - (1-\beta)L}{1 + \beta L}.$$

It follows that

$$S'(\beta) = \frac{L(L+1-K)}{(1+\beta L)^2}.$$

Therefore, the type of the country depends on the size of L. If $L < K - 1$, we have $S'(\beta) < 0$ and the country is of Brander–Taylor type. If $L > K - 1$, $S'(\beta) > 0$ and the country is of Copeland–Taylor type.

Consider now two identical countries with labor endowment L. Suppose the initial labor endowment is $L_0 < K - 1$. So, there is no trade between two identical Brander–Taylor countries. Suppose that labor in both countries increases gradually at the same speed. As long as $L < K - 1$, there is still no trade. But when L cross $K - 1$, trade volume suddenly jumps, as illustrated in Figure 5.14. This is because the type of two countries changes from the Brander–Taylor type to the Copeland–Taylor type.

5.7.2 Long-run Effect of Policy on Production Patterns

Function $g(\cdot)$ also provides a tool to see the long-run effect of a tax on world production patterns. This is because the effect of tax can be represented by a shift of $g(\cdot)$.

To see this, suppose that Foreign imposes a production tax $\tau \in (0,1)$ on the primary good, then the new $\mathrm{MRT}^* (\tau) = (1-\tau) A_f (S^*) / a^*$. The comparative advantage index becomes

$$v = (1-\tau) \frac{A_f (S^*) a}{A_h a^*}.$$

Given this new expression, all results derived in Section 5.6.3 hold, implying that we have the same world production pattern diagram. As long as tax revenues are transferred back to households, we also have the same expressions of $\beta_h (v)$ and $\beta_f (v)$. On the other hand, the expression of $g(\cdot)$ becomes

$$g(v) \equiv (1-\tau) \frac{A_f \left(S_f \left(\beta_f (v) \right) \right) a}{A_h \left(S_h \left(\beta_h (v) \right) \right) a^*}.$$

Compared to the original expression (5.45), the locus of $g(\cdot)$ shifts up. Therefore, say, if we are in Case 1 and both countries are of Copeland–Taylor type as illustrated in Figure 5.11b, a tax on the primary sector will raise the steady-state values of v, which is intuitive. Moreover, among two stable steady-state patterns (p,m) and (d,p), pattern (p,m) may change to (p,d), or vanish. Pattern (d,p) may change to (m,p). If the tax rate is high enough, the only world production pattern left would be pattern (m,p).

5.8 Conclusion

By allowing environment impacts from both sectors, this study synthesizes two strands of literature in trade and environmental economics: one related to renewable resources extraction and the other to pollution emission. We carefully examine the supply side, autarky, small open economy and two-country world. One of the interesting implications obtained from the model is that both countries trading with each other may lose from trade by exporting their own dirty goods. Focusing on the interaction between environmental change and trade, we do not consider policies in this study. Moreover, labor endowment and manufacturing productivity are exogenous. These extensions are interesting topics and can be discussed by using our model.

5.A Appendix

5.A.1 Calculation of $T''(X_m)$

Using (5.16) and $\beta = 1 - X_m/aL$ we have

$$\begin{aligned}
T''(X_m) &= \frac{d}{dX_m}\left(\frac{dT(X_m)}{dX_m}\right) = \frac{d}{\beta}\left[-A'(S(\beta))\,S'(\beta)\,\frac{\beta}{a} - \frac{A(S(\beta))}{a}\right]\frac{d\beta}{dX_m} \\
&= \frac{1}{a^2L}\frac{d}{\beta}\left[A'(S(\beta))\,S'(\beta)\,\beta + A(S(\beta))\right] \\
&= \frac{1}{a^2L}\left\{A''(S(\beta))\,S'^2(\beta)\,\beta + A'(S(\beta))\,S''(\beta)\,\beta + 2A'(S(\beta))\,S'(\beta)\right\} \\
&\equiv \frac{1}{a^2L}\Delta.
\end{aligned}$$

To be concise, henceforth use, say, A' instead of $A'(S(\beta))$. It follows from (5.13) that

$$\begin{aligned}
S''(\beta) &= \frac{l_pA'S'LB_2 - B_1L\left[G''S' - l_pA''S'\beta L - l_pA'L\right]}{B_2^2} \\
&= \frac{l_pA'\frac{B_1L}{B_2}LB_2 - B_1L\left[G''\frac{B_1L}{B_2} - l_pA''\frac{B_1L}{B_2}\beta L - l_pA'L\right]}{B_2^2} \\
&= \frac{l_pA'B_1L^2 - B_1L^2\left[G''\frac{B_1}{B_2} - l_pA''\frac{B_1}{B_2}\beta L - l_pA'\right]}{B_2^2} \\
&= \frac{B_1L^2}{B_2^2}\left[l_pA' - G''\frac{B_1}{B_2} + l_pA''\frac{B_1}{B_2}\beta L + l_pA'\right] \\
&= \frac{B_1L^2}{B_2^2}\left[2l_pA' - G''\frac{B_1}{B_2} + l_pA''\frac{B_1}{B_2}\beta L\right] \\
&= \frac{B_1L^2}{B_2^2}\left\{2l_pA' + \frac{B_1}{B_2}\left[l_pA''\beta L - G''\right]\right\},
\end{aligned}$$

where $B_1 \equiv l_pA - l_ma$, $B_2 \equiv G' - l_pA'\beta L$. Thus we have

$$A'S''\beta = \frac{B_1L}{B_2^2}A'\beta L\left\{2l_pA' + \frac{B_1}{B_2}\left[l_pA''\beta L - G''\right]\right\}.$$

Similarly, we can obtain

$$A''S'^2\beta = \frac{B_1L}{B_2^2}B_1A''\beta L,$$

and

$$2A'S' = \frac{B_1L}{B_2^2}2B_2A'.$$

Using these results we can calculate

$$\begin{aligned}
\Delta &= \frac{B_1 L}{B_2^2}\left\{A'\beta L\left\{2l_p A' + \frac{B_1}{B_2}\left[l_p A''\beta L - G''\right]\right\} + B_1 A''\beta L + 2B_2 A'\right\} \\
&= \frac{B_1 L}{B_2^2}\left\{\frac{B_1}{B_2}\beta L\left[l_p A'' A'\beta L - G'' A' + B_2 A''\right] + 2G'A'\right\} \\
&= \frac{B_1 L}{B_2^2}\left\{\frac{B_1}{B_2}\beta L\left[-G''A' + G'A''\right] + 2G'A'\right\} \\
&= \frac{B_1 L}{B_2^2}A'^2\left[2\frac{G'}{A'} - \frac{B_1}{B_2}\beta L\frac{d}{dS}\left(\frac{G'}{A'}\right)\right].
\end{aligned}$$

Therefore,

$$T''\left(X_m\right) = \frac{1}{a^2 L}\Delta = \frac{B_1 A'^2}{a^2 B_2^2}\left[2\frac{G'}{A'} - \frac{B_1}{B_2}\beta L\frac{d}{dS}\left(\frac{G'}{A'}\right)\right].$$

Suppose that $X_m \to aL$, then we have $\beta \to 0$ and thus

$$\lim_{X_m \to aL} T''\left(X_m\right) = \frac{2B_1 A' G'}{a^2 B_2^2}.$$

The stability condition requires $G' < 0$ to hold as $\beta \to 0$, which means that $T''\left(X_m\right)$ has the opposite sign to B_1 as $X_m \to aL$, which is negative (positive) in a country of Brander–Taylor (Copeland–Taylor) type.

5.A.2 Stability Condition

The Jacobian of (5.41) and (5.42) is

$$\begin{aligned}
J &= \begin{bmatrix} \frac{\partial \dot{S}}{\partial S} & \frac{\partial \dot{S}}{\partial S^*} \\ \frac{\partial \dot{S}^*}{\partial S} & \frac{\partial \dot{S}^*}{\partial S^*} \end{bmatrix} \\
&= \begin{bmatrix} G_h' - l_p A_h' \beta_h L - \left(l_p A_h - l_m a\right) L\beta_h' \frac{\partial v}{\partial S} & -\left(l_p A_h - l_m a\right) L\beta_h' \frac{\partial v}{\partial S^*} \\ -\left(l_p^* A_f - l_m^* a\right) L^* \beta_f' \frac{\partial v}{\partial S} & G_f' - l_p^* A_f' \beta_f L^* - \left(l_p^* A_f - l_m^* a\right) L^* \beta_f' \frac{\partial v}{\partial S^*} \end{bmatrix}.
\end{aligned}$$

Note that $\partial v/\partial S < 0$ and $\partial v/\partial S^* > 0$. The sufficient condition for local stability around $\left(\tilde{S}, \tilde{S}^*\right)$ is

$$\begin{aligned}
J_1 &\equiv G_h' - l_p A_h' \beta_h L - \left(l_p A_h - l_m a\right) L\beta_h' \frac{\partial v}{\partial S} < 0, \\
J_2 &\equiv |J| > 0.
\end{aligned}$$

Note that

$$\frac{|J|}{\left(G_h' - l_p A_h' \beta_h L\right)\left(G_f' - l_p^* A_f' \beta_f L^*\right)} = \begin{vmatrix} 1 - \frac{\left(l_p A_h - l_m a\right) L}{G_h' - l_p A_h' \beta_h L} \beta_h' \frac{\partial v}{\partial S} & -\frac{\left(l_p A_h - l_m a\right) L}{G_h' - l_p A_h' \beta_h L} \beta_h' \frac{\partial v}{\partial S^*} \\ -\frac{\left(l_p^* A_f - l_m^* a\right) L}{G_f' - l_p^* A_f' \beta_f L^*} \beta_f' \frac{\partial v}{\partial S} & 1 - \frac{\left(l_p^* A_f - l_m^* a\right) L^*}{G_f' - l_p^* A_f' \beta_f L^*} \beta_f' \frac{\partial v}{\partial S^*} \end{vmatrix}$$

$$= \begin{vmatrix} 1 - S_h' \beta_h' \frac{\partial v}{\partial S} & -S_h' \beta_h' \frac{\partial v}{\partial S^*} \\ -S_f' \beta_f' \frac{\partial v}{\partial S} & 1 - S_f' \beta_f' \frac{\partial v}{\partial S^*} \end{vmatrix}$$

$$= 1 - S_h' \beta_h' \frac{\partial v}{\partial S} - S_f' \beta_f' \frac{\partial v}{\partial S^*}$$

$$= 1 + S_h' \beta_h' v \frac{A_h'}{A_h} - S_f' \beta_f' v \frac{A_f'}{A_f},$$

where (5.13) is used. By (5.12), we have $G_h' - l_p A_h' \beta_h L < 0$ and $G_f' - l_p^* A_f' \beta_f L^* < 0$. Hence, $|J| > 0$ is equivalent to

$$S_f' \beta_f' v \frac{A_f'}{A_f} - S_h' \beta_h' v \frac{A_h'}{A_h} < 1.$$

On the other hand, it follows from the expression of $g(v)$ that

$$g'(v) = \frac{A_f' S_f' \beta_f' a}{A_h a^*} - \frac{A_f a A_h' S_h' \beta_h'}{A_h^2 a^*}$$

$$= S_f' \beta_f' v \frac{A_f'}{A_f} - S_h' \beta_h' v \frac{A_h'}{A_h}.$$

Therefore, $|J| > 0$ is further equivalent to $g'(v) < 1$.

5.A.3 Properties of $g(v)$ in Case 1

For pattern (p,m), we have labor allocation $(\beta, \beta^*) = (1, 0)$. Using the definition of $g(v)$, we have

$$g(v) = \frac{A_f\left(S_f(0)\right) a}{A_h\left(S_h(1)\right) a^*} \text{ if } v \leq \frac{1-b}{b} z,$$

which is a constant. For pattern (p,d), we can obtain, using $\beta = 1$ and (5.20),

$$g(v) = \frac{A_f\left(S_f\left(b - (1-b)\frac{z}{v}\right)\right) a}{A_h\left(S_h(1)\right) a^*} \text{ if } \frac{1-b}{b} z < v < 1.$$

Therefore, if Foreign is of Brander–Taylor type, we have $S_f'(\cdot) < 0$ and thus $g(v)$ is a strictly decreasing function on $((1-b)z/b, 1)$. If Foreign is of Copeland–Taylor type, we have $S_f'(\cdot) > 0$ and thus $g(v)$ is a strictly increasing function. Moreover, noting that $g(v) \to A_f\left(S_f(0)\right) a / \left(A_h\left(S_h(1)\right) a^*\right)$ as $v \to (1-b)z/b$, $g(v)$ is a continuous function on $[0, 1)$.

Similarly, for pattern (d,p), we can obtain, using (5.28),

$$g(v) = \frac{A_f\left(S_f(1)\right) a}{A_h\left(S_h\left(b - (1-b)\frac{v}{z}\right)\right) a^*} \text{ if } 1 < v < \frac{b}{1-b} z,$$

which is a strictly decreasing function if Home is of Brander–Taylor type ($S_h'(\cdot) < 0$), and a strictly increasing function if Home is of Copeland–Taylor type.

For pattern (m,p), we have

$$g(v) = \frac{A_f\left(S_f(1)\right)a}{A_h\left(S_h(0)\right)a^*} \text{ if } v \geq \frac{b}{1-b}z,$$

which is also a constant. Note that $g(v) \to A_f\left(S_f(1)\right)a/\left(A_h\left(S_h(0)\right)a^*\right)$ as $v \to bz/(1-b)$, $g(v)$ is a continuous function on $(1,\infty)$.

Finally, note that

$$\lim_{v\to 1^-} g(v) = \frac{A_f\left(S_f(b-(1-b)z)\right)a}{A_h\left(S_h(1)\right)a^*},$$
$$\lim_{v\to 1^+} g(v) = \frac{A_f\left(S_f(1)\right)a}{A_h\left(S_h\left(b-\frac{1-b}{z}\right)\right)a^*}.$$

If both countries are of Brander–Taylor type, we have $A_f\left(S_f(b-(1-b)z)\right) > A_f\left(S_f(1)\right)$ and $A_h(S_h(1)) < A_h(S_h(b-(1-b)/z))$. Hence, $\lim_{v\to 1^-} g(v) > \lim_{v\to 1^+} g(v)$, which means $g(v)$ jumps down at $v=1$. In contrast, if both countries are of Copeland–Taylor type, we have $\lim_{v\to 1^-} g(v) > \lim_{v\to 1^+} g(v)$, implying that $g(v)$ jumps up at $v=1$.

However, if two countries are of different types, say, Brander–Taylor type Home and Copeland–Taylor type Foreign, we have $A_h(S_h(1)) < A_h(S_h(b-(1-b)/z))$ and $A_f\left(S_f(b-(1-b)z)\right) < A_f\left(S_f(1)\right)$. Hence, unless specific parameter ranges and functional forms are given, it is ambiguous about which is larger among $\lim_{v\to 1^-} g(v)$ or $\lim_{v\to 1^+} g(v)$.

Chapter 6

Conclusion

6.1 Externalities and Economic Activities

Economic activities can be categorized into production process, utilizing process, and allocation process according to their purposes. In the production process, inputs are transformed into outputs under certain technologies. In the utilizing process, the utility value of outputs is exploited through consumption, investment, or trade. In the allocation process, inputs and outputs are allocated among economic agents by the social planner, or through exchange in the market with or without government interventions.

Externalities can be better understood through its relationship with the three basic processes. As illustrated in Figure 6.1, externalities can affect the production process. For example, toxic industrial wastes can severely harm the interest of farmers near the polluting spot. Externalities can also cause disutility. The health consequences of air and water pollution are significant examples.

Externalities can arise from any process among the three. In the production process, both inputs and outputs may lead to externalities. An example of input-generated externalities is the congestion phenomena like traffic jams. Deforestation caused by timber harvests is a good example of output-generated externalities. In the utilizing process, either consumption, investment, or trade can bring about externalities. A good example of consumption-generated externalities is automobile emissions. As for investment, there are two channels through which externalities can arise. On one hand, capital formation itself often comes at the expense of land and other resources. On the other hand, investment increases the amount of factors of production. As for trade, it separates consumption from production and often stimulates investment, which can aggravate externalities. Moreover, the allocation process can aggravate or relieve externalities by affecting the allocation of inputs and outputs, as well as the choice of technology.

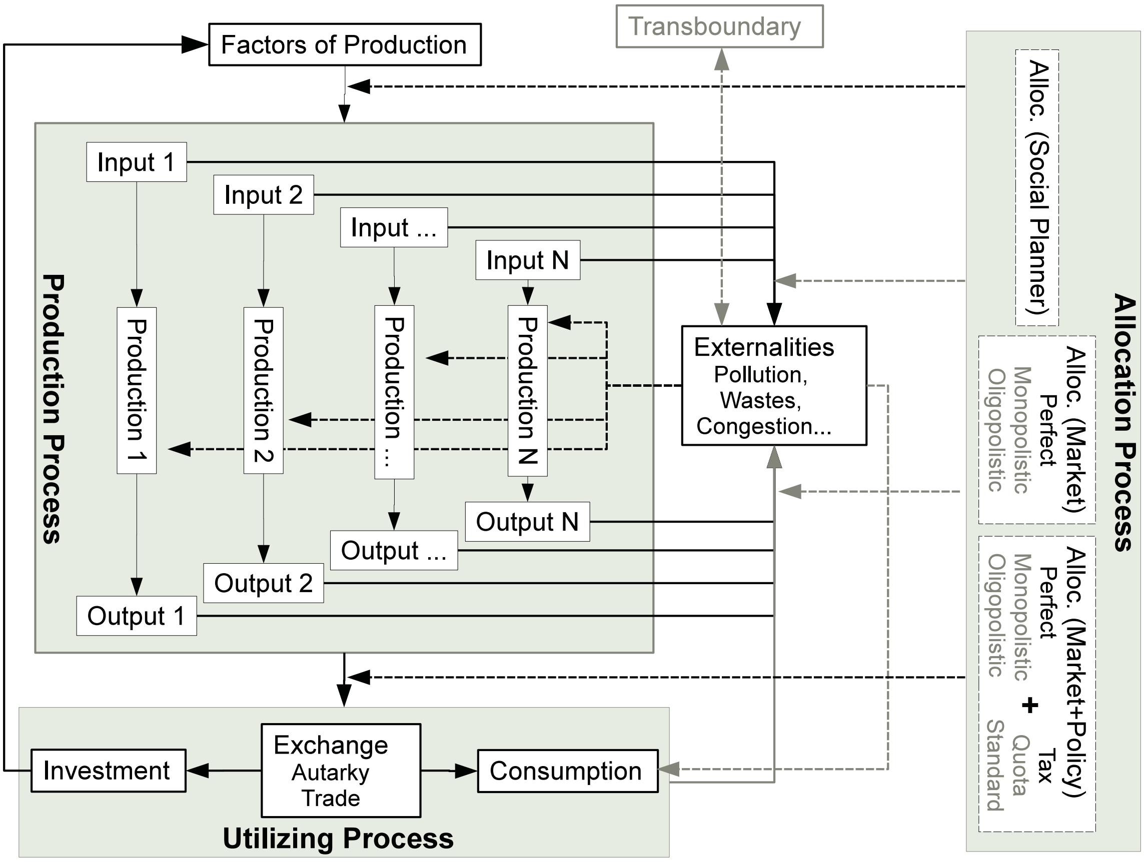


Figure 6.1: Externalities and economic activities

My focus in the dissertation is production externalities that affect the production activities. I consider the case that production externalities arise from the production process, and the case that production externalities arise from investment and trade in the utilizing process.

6.2 Contribution

Having the big picture above, the contribution of the dissertation can be summarized as follows. In a static framework, Chapter 2 and Chapter 3 focus on the production possibility frontier (PPF) and examine the impacts of externalities on the production process. In doing this, both chapters assume a social planner in the allocation process and ignore the utilizing process. Then, Chapter 2 examines whether the social planner uses all factors in the presence of production externalities in a general multi-factor multi-goods model for both input-generated and output-generated cases. The analysis shows that the two kinds of production externalities are very different in the sense that using all factors may be inefficient if externalities are input-generated but is efficient if externalities are output-generated.

Based on this result, Chapter 3 examines the properties of the PPF under strong input-generated production externalities in a single-factor two-goods model. Here "strong" denotes the situation that using all factor is inefficient. I focus on this seemingly special case for two reasons. First, it is an important phenomenon in reality. For example, a worker has 24 hours but it is clearly inefficient to work for 24 hours without rest. The public intermediate goods can be also seen as a special case of strong input-generated production externalities. Second, the PPF tends to be "irregular" (convex) in this case. Especially when there is only a single factor, the PPF is convex if the generation function of by-product is quasi-concave. This provides an explanation for how comparative advantages arise among ex-ante identical agents.

Rather than focusing on the impacts on the production process in a static framework, Chapter 4 and Chapter 5 turn to the context of trade and the environment in a dynamic framework. The environment is measured by a stock variable, whose change depends on the difference between the natural growth and the flow of environmental impacts. For simplicity, both chapters deal with the single-factor two-goods case and assume that externalities affect only one sector among the two. Then, Chapter 4 starts off to highlight the role of investment in the interaction between trade and the environment. The analysis shows that trade stimulates investment and necessarily harms the environment in the long run under laissez faire. The policy analysis shows that although there are two misallocations–too much factor in dirty sector and too much investment, the social optimum can be achieved by using just one instrument: the pollution tax. Chapter 4 explains why the optimal pollution tax can be interpreted as a dynamic version of the Pigouvian tax.

In contrast, Chapter 5 emphasizes the importance of modeling the multi-dimensional functions of the environment. By allowing environmental impacts from both sectors, the model can formulate both the impact of toxic wastes from manufacturing and the damage from fish harvests on fishery resources. This small step in the model structure opens up a new agenda. For example, the model indicates the possibility that both countries export their own dirty goods and lose from trade in the long run. This sharply contrasts previous studies. In addition, Chapter 5 carefully investigates the behaviors of the model, such as the production patterns in a two-country world.

6.3 Future Research

Because of my focus in the dissertation, I consider only local externalities that do not cross the borders. So, the results cannot apply to, say, greenhouse gases. It is a promising work to extend the models to transboundary externalities.

I assume perfect competition in Chapter 4 and Chapter 5. But other market structures such as monopolistic competition and oligopolistic competition are also important. It is interesting to see the implications of different market structures.

The policy is not discussed in Chapter 5, which is another promising direction. The policy analysis can proceed in three steps. First, given the policy in a country, consider the optimal policy response of the other country. Second, consider both countries behaving like the second country in the first step and see what is the outcome of the Nash equilibrium. Third, consider how to improve from the Nash equilibrium since it is usually not optimal owing to externalities.

The abatement activities and the choice of technologies are two important topics and ignored in the dissertation. Moreover, the disutility of externalities and consumption-generated externalities are also significant in reality but missing here. It is worth introducing these elements into the models in Chapter 4 and Chapter 5.

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