Essays on Risk Premiums in Higher-Order Moments of Financial Asset Returns

Thesis by

Hiroshi Sasaki

Advisor

Associate Professor Hidetoshi Nakagawa

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy,
Graduate School of International Corporate Strategy (ICS),
Hitotsubashi University,
Tokyo, Japan

July 31, 2014

This Thesis is Dedicated to My Wife, Yukiko, and My Son, Yu.

Acknowledgements

I would like to give heartful thanks to Associate Prof. Hidetoshi Nakagawa whose enormous support and insightful comments were invaluable during the course of my study. Without his encouragement and guidance, this dissertation would not have materialized.

Special thanks also go to Prof. Kazuhiko Ohashi, Prof. Ryozo Miura, Prof. Toshiki Honda, Prof. Fumio Hayashi, Prof. Nobuhiro Nakamura, Prof. Tokuo Iwaisako, Associate Prof. Tatsuyoshi Okimoto, Associate Prof. Daisuke Yokouchi, whose opinions and information have helped me very much throughout the production of this thesis.

I would also like to express the deepest appreciation to Prof. Shigeo Kusuoka, Prof. Masayuki Ikeda, Prof. Jun Sekine, Associate Prof. Masaaki Fukasawa, Associate Prof. Koichiro Takaoka, whose advice and comments given by them have been a great help in studying the problems focused in this thesis.

Moreover, Dr. Takanori Adachi gives me constructive comments and warm encouragement, and I am particularly grateful for the assistance given by him.

I have had the support and encouragement of superiors and colleagues at work in my company for proceeding to the degree, and finally, I would also like to express my gratitude to my family for their moral support and warm encouragements for this thesis.

Contents

1 Introduction **8**

1.1 Motivation for this study . . . 8

1.2 Abstract and focus for each chapter . . . 11

2 The Skewness Risk Premium in Equilibrium and Stock Return Predictability **15**

2.1 Introduction . . . 15

2.2 Model Framework . . . 19

2.2.1 Model Setup and Assumptions . . . 19

2.2.2 The Model Solution in Equilibrium . . . 21

2.2.3 Risk Premiums in Higher-Order Moments in Equilibrium . . . 24

2.2.4 An Equity Risk Premium Representation . . . 30

2.3 Model Implications . . . 33

2.4 Empirical Measurements . . . 37

2.4.1 Measurements for the Higher-Order Moments . . . 37

2.4.2 Data Description . . . 38

2.4.3 Main Empirical Findings . . . 41

2.5 Concluding Remarks . . . 45

Appendix 2.A Proof of Proposition 2 . . . 46

Appendix 2.B The Risk-Free Rate . . . 47

3 An Approach to the Option Market Model Based on End-user Net Demand **48**

3.1 Introduction . . . 48

3.2 The Model . . . 52

3.2.1 Assumptions of a Financial Option Market . . . 52

3.2.2 Assumptions for Option Prices . . . 55

3.3 Optimality and Equilibrium . . . 56

3.3.1 Optimization for the Market-maker and the End-user 56
3.3.2 The Pricing Kernel in Equilibrium 63
3.4 Concluding Remarks 74
Appendix 3.A Expected Delta-Hedged Gain and Loss 75
Appendix 3.B Proof of Proposition 7 76
Appendix 3.C Proof of Proposition 8 77
Appendix 3.D Monte Carlo Simulation Results 78

4 Understanding Delta-hedged Option Returns in Stochastic Volatility Environments 80
4.1 Introduction 80
4.2 The Model and the Methodology 83
4.2.1 An explicit representation for delta-hedged option returns 83
4.2.2 Estimation Strategy for the Volatility Risk Premium 89
4.2.3 An Analytical Process for a Contribution Analysis 92
4.3 Data and Methodology for an Empirical Implementation 92
4.3.1 Description of the OTC Currency Option Market and Data 92
4.3.2 Parameter Estimation
for the Heston[1993] Stochastic Volatility Model 94
4.3.3 Estimation of the Volatility Risk Premium 94
4.3.4 Estimation of the Expected DHGL 95
4.4 An Empirical Analysis 96
4.4.1 Estimation Results for the Model Parameters 96
4.4.2 A Contribution Analysis on the Expected DHGL 100
4.5 Concluding Remarks 107
Appendix 4.A Proof of Proposition 13 107
Appendix 4.B Time Series Data 109
Appendix 4.C Estimation Results of the Heston Model 111

5 Concluding Remarks 113

List of Figures

2.1 The Risk Premiums in Higher-Order Moments and the Equity Risk Premium . . . 32
2.2 The Factor Loading $\beta_{vp,q}$ . . . 34
2.3 The Factor Loading $\beta_{vp,\lambda}$ . . . 34
2.4 The Factor Loading $\beta_{sp,q}$ . . . 35
2.5 The Factor Loading $\beta_{sp,\lambda}$ . . . 35
2.6 The Factor Loading to $Var_t^{\mathbb{P}}$:π_{Var} . . . 35
2.7 The Factor Loading to vp_t:π_{vp} . . . 35
2.8 The Factor Loading to skp_t:π_{skp} . . . 36
2.9 The VIX and The Current Volatility . . . 39
2.10 The Risk-Neutral Skewness and The Current Skewness . . . 39

3.1 The end-user net demand δ_2^* and the option price $p_2(\delta_2^*)$ in equilibrium . 62
3.2 The end-user net demand δ_3^* and the option price $p_3(\delta_3^*)$ in equilibrium . 63
3.3 The parameter c_1^* in equilibrium . . . 66
3.4 The shape of terminal wealth for the end-user : $W_T^E(\delta^*; S_T)$ . . . 73
3.5 The Distributions of Delta-Neutral Hedging Errors ϵ_T^i, $i = 1, 2, 3$ . . . 78
3.6 The End-User Net Demand δ_i^* and the Option Prices $p_i(\delta^*)$ in Equilibrium 79

4.1 The time series of the volatility risk premium parameter λ . . . 97
4.2 Risk aversion parameter γ . . . 100
4.3 USD-JPY WM/Reuter Closing Spot Rate . . . 110
4.4 USD-JPY 1M ATM Implied Volatility（Mid Price） . . . 110
4.5 USD-JPY 1M ATM Mid-Bid Spread . . . 111
4.6 The time-series of the estimated parameter：$\tilde{k}$ . . . 111
4.7 The time-series of the estimated parameter：$\tilde{v}$ . . . 112
4.8 The time-series of the estimated parameter：$\tilde{\rho}$ . . . 112

List of Tables

2.1 The Set of Model Parameters 33
2.2 Summary statistics for the monthly returns and predictor variables . . . 41
2.3 The Monthly and Quarterly Return Regressions 43
2.4 The Univariate Regressions with Traditional Predictor Variables 44
2.5 Summary statistics for the CAY 45
2.6 The Univariate Regressions with the CAY 45

3.1 Summary Statistics of Hedging Error Distributions 61

4.1 Summary statistics on the volatility risk premium parameter λ 98
4.2 Contribution Analysis for the Expected Delta-hedged Gain Loss 101
4.3 Summary Statistics of the DHGL for the ATM Straddle Short Strategy . 102
4.4 Summary Statistics of the DHGL for the OTM Put Short Strategy . . . 103
4.5 Relative Contribution Comparison of the Expected DHGL between Pre- and Post Lehman Crisis : Based on a time-series of overlapping results . 105
4.6 Relative Contribution Comparison of the Expected DHGL between Pre- and Post Lehman Crisis : Based on a time-series of non-overlapping results 106
4.7 Summary Statistics for Implied Volatilities 109

Chapter 1

Introduction

1.1 Motivation for this study

It is well known that market prices of financial options reflect common assessment of the probability distribution of the underlying asset on the expiration day, adjusted for the investors' tolerance for bearing risk. Recent studies suggest that there is a difference between the probability distribution of the underlying asset implied by financial option prices and that realized in the underlying asset market. For practitioners such as risk managers in the investment banks or speculators in the hedge funds industry, it is very important to understand why this difference between the risk neutral and physical probability distributions arise from the theoretical point of view. What are the theoretical determinants of this difference? What role does investor behavior play in explaning dynamic movements in this difference?

In this study, we investigate risk premiums in higher-order moments, such as variance and skewness, of financial asset returns and some recent topics related to those risk premiums. In the area of financial economics, the risk premiums in higher-order moments have recently received broad attention due to a rich body of implications for financial asset pricing mechanism and asset return predictability, and a lot of efforts have been put into examining the existence and the characteristics of those risk premiums. The purpose of this study is to provide the way of a deeper understanding on the risk premiums in higher-order moments examined by recent academic studies through both theoretical and empirical approaches.

The concern with risk premiums in higher-order moments has been growing for the last several years in terms of the practical point of view as well as the academic viewpoints. In particular, there has been a growing interest in the information of the difference between option-implied and realized distributions, which is usually recognized as a risk premium required by the representative agent, in terms of the financial risk management or the asset pricing implications. This study especially focuses on the variance and skewness risk premiums in financial asset returns and investigates their existence in financial markets and their implications for financial asset pricing mechanism in detail. The variance risk premium is defined as the difference between option-implied and realized variance and the skewness risk premium is also defined as the same with the variance risk premium, that is, the difference between option-implied and realized skewness.

Recent investigations on the risk premiums in higher-order moments of financial asset returns have focused on three major aspects, that is, empirical and theoretical aspects on the existence of those risk premiums and an aspect on the asset return predictability. Bakshi and Kapadia[2003], Low and Zhang[2005], Carr and Wu[2009], and Broadie, Chernov, and Johannes[2009] examine the existence of the volatility risk premiums based on empirical manners and they show that the risk premiums in higher-order moments of financial asset returns significantly exist in actual financial market. Bansal and Yaron[2004], Eraker[2008], Branger and Völkert[2010], Drechsler and Yaron[2011], and Bollerslev, Sizova, and Tauchen[2012] provide general equilibrium models that are consistent with the existence of the risk premiums in higher-order moments of financial asset returns and they suggest theoretical approaches to prove the existence of those risk premiums. Bali and Hovakimian[2009], Goyal and Saretto[2009], Bollerslev, Tauchen, and Zhou[2010], and Drechsler and Yaron[2011] investigate the asset return predictability of the risk premiums in higher-order moments of financial asset returns empirically and they find that those risk premiums have superior predictive power for future asset returns.

In this study, we aim to advance these previous studies with theoretical and empirical manners in order to shed light on the nature of the risk premiums in higher-order moments of financial asset returns. Specifically, we provide three major topics on those risk premiums.

First, in Chapter 2, we develop the model which is consistent with the existence of the skewness risk premium. The variance risk premium is recognized as the compensation for the risk induced by stochastic volatility in financial asset price processes. It is well known that this risk premium is essentially related to delta-hedged option returns (e.g., Bakshi and Kapadia[2003]). And moreover, in recent years, there remains an ever-increasing interest and challenge to develop an entirely self-contained equilibrium-based explanation for the nonzero variance risk premium and its predictability for stock index returns (e.g., Bollerslev, Tauchen, and Zhou[2009]). Thus, recent studies have mainly focused on the variance risk premium as one of the risk premiums in higher-order moments. However, as far as we know, there have been few reports about the risk premium which compensates for uncertainty of the third moment, that is, the skewness, of asset returns in financial markets. Therefore, first of all, we begin to demonstrate that the skewness risk premium exists in equilibrium and captures attitudes toward economic uncertainty as well as the variance risk premium. Among recent studies on self-contained equilibrium-based models for the nonzero variance risk premium referred above, almost all the studies model the asset price process as conditional normal, so that the one-step-ahead conditional distribution of the market return is also normal and, as a result, the skewness of that distribution is zero. Therefore, the models proposed by these recent studies can not explain the negative risk-neutral skewness, which is found by the previous studies such as Aït-Sahalia and Lo[1998] and Aït-Sahalia, Wang, and Yared[2001]. Extending the models proposed in the previous studies by introducing a stochastic jump intensity into the consumption growth rate process, we will provide a representation of the skewness risk premium based on a general equilibrium model and prove that this skewness risk premium should be non-zero in equilibrium.

Second, in Chapter 3, we investigate the reason why the variance risk premium exists in financial markets theoretically and empirically in more detail under a partial equilibrium setting. We consider a theoretical relationship between the net demand of end-users for financial options and the variance risk premiums in a stochastic volatility environment with the demand-based pricing kernel, which is the pricing kernel related to the amount of the net demand of end-users for financial options in equilibrium, and provide an explicit explanation of the existence of the variance risk premium in terms of the net option demand of end-users. To the best of our knowledge, this approach is the first to provide some implications in the well-known empirical evidence, that is, the negative variance risk premium, in terms of the end-user net demand for financial options.

Third and finally, in Chapter 4, the effect of parameter estimation risk on the results of empirical studies for the existence and the sign of the variance risk premium is examined. Theoretical financial models often assume that the economic agent who makes an optimal financial decision knows the true parameters of the model. However, the true parameters are rarely if ever known to the decision maker. In reality, model parameters have to be estimated based on historical information and, hence, the model's usefulness depends partly on how good the estimates are. This gives rise to estimation risk in virtually all financial valuation models. Bakshi and Kapadia[2003] and Low and Zhang[2005] find a theoretical relationship between delta-hedged option returns in financial options and the variance risk premium. They study delta-hedged option returns in a stock index option market and currency option markets, respectively, and relying on a theoretical relationship, they provide evidence that the variance risk premiums are not zero and negative because of non-zero expected delta-hedged option returns with hedging-based empirical tests on the variance risk premium. We show that the effect of parameter estimation risk on option prices quoted in actual financial option markets is significant, and therefore, suggest that the standard hedging-based empirical tests on the existence of the variance risk premiums, which is explored and examined by, for example, Bakshi and Kapadia[2003] and Low and Zhang[2005], may be unreliable because of that significant effect of parameter estimation risk on option prices.

These analyses in this study lead to the conclusion that the uncertainty of the variance and skewness of financial asset returns are consistently priced in equilibrium and we also find that these prices of the risks have broad implications for financial asset pricing mechanism, which is one of the most important problems to be solved in the financial economics.

1.2 Abstract and focus for each chapter

Let us provide the abstract and focus for each chapter in this study in the following.

Chapter 2

In this chapter, we study risk premiums in higher-order moments of financial asset returns in a general equilibrium setting. To the best of our knowledge, the first attempt to demonstrate the existence of the volatility risk premium based on a general equilibrium market model is made by Eraker[2008]. Eraker[2008] develop an equilibrium explanation for the volatility risk premium based on the long-run risks (LRR) model [1] which emphasizes the role of long-run risks, that is, low-frequency movements in consumption growth rates and volatility, in accounting for a wide range of asset pricing puzzles. The LRR model features an Epstein and Zin[1989] utility function with an investor preference for early resolution of uncertainty and contains (i) a persistent expected consumption growth component and (ii) long-run variation in consumption volatility. On the basis of the LRR model, Eraker[2008] studies the volatility risk premium through the framework of a general equilibrium model.

The recent studies on the risk premiums in higher-order moments focus mainly on the risk premium of the second moment, that is, the volatility or the variance. Conversely,

[1] The long-run risks model is pioneered by Bansal and Yaron[2004], which is a stylized self-contained general equilibrium model incorporating the effects of time-varying economic uncertainty.

as far as we know, there are few reports about the risk premium which compensates for uncertainty of the third moment, that is, the skewness, of asset returns. In this chapter, we demonstrate that the skewness risk premium, defined by the difference between two expected values of the skewness under the risk-neutral and physical probability measures, respectively, also captures attitudes toward economic uncertainty as well as the variance risk premium.

Extending the long-run risks (LRR) model proposed by Bansal and Yaron[2004] by introducing a stochastic jump intensity for jumps in the LRR factor and the variance of consumption growth rate, we provide an explicit representation for the skewness risk premium, as well as the volatility risk premium, in equilibrium. On the basis of the representation for the skewness risk premium, we propose a possible reason for the empirical fact of time-varying and negative risk-neutral skewness. Moreover, we also provide an equity risk premium representation of a linear factor pricing model with the variance and skewness risk premiums. The empirical results prove that the skewness risk premium, as well as the variance risk premium, has superior predictive power for future aggregate stock market index returns. Compared with the variance risk premium, the results show that the skewness risk premium plays an independent and essential role for predicting the market index returns.

Chapter 3

In this chapter, we study financial option prices in terms of demand pressure effects based on the preferences of the representative market-maker and the representative end-user under a partial equilibrium setting. We assume that there are two types of agents in the option market: market-makers and end-users. Market-makers play a key role in providing liquidity to end-users by taking the other side of end-user net demand. We examine a demand-based option market model in which market-makers hedge their option positions based on the delta-neutral hedging strategy with futures or forwards in order to control the risks induced by taking the other side of end-user net demand. If we can assume a dynamically complete financial market that can be governed by standard market models such as the Black-Scholes-Merton model[1973], the no-arbitrage theory determines derivative prices uniquely and independently of investor demand for options because market-makers can hedge their option positions perfectly through continuous time trading with the underlying assets and cash. However, the assumption of a dynamically complete market would not apply to the actual financial market. Under the assumption of market incompleteness induced by additional risk factors such as stochastic volatility and/or jumps, market-makers cannot hedge their option positions perfectly and are exposed to the risk of significant losses in the processes of making markets and managing their options portfolios.

Assuming an incomplete market governed by a stochastic volatility factor in underlying asset price processes, we demonstrate that the demand pressure for an option contract directly impacts traded option prices due to the covariance of the unhedgeable parts of a demanded option and the other traded options. Whereas Gârleanu et al.[2009] also provide a result which is similar to the fact stated above, but they assume that the aggregate option demand of end-users is provided exogeneously and independent of the preferences of market-makers and end-users. In contrast, as mentioned by Green and Figlewski[1999], we assume that the preference of market-makers to the background risks induced by the net demand of end-users affects option prices directly and study the supply-demand balance for option contracts by considering the preferences of both market-makers and end-users. Considering each of optimization problems for the representative market-maker and the representative end-user independently, we derive the equilibrium demand pressures for traded option contracts and provide an explicit representation for the pricing kernel in equilibrium as a function of the equilibrium demand pressures. Moreover, we provide some interesting implications in the existence of the variance risk premium and the shape of the implied risk aversion function with the pricing kernel derived above.

Chapter 4

In developing risk management strategies for financial option portfolios in incomplete markets, it is necessary to specify the risk factors in the markets and select an option pricing model which is consistent with those specified risks. In particular, for the practitioners it is essential to consider the matters mentioned above for their risk management processes. For this reason, in this chapter, we study the features on empirical option prices and delta-hedged option returns in a stochastic volatility environment. A rich body of studies on empirical option prices and delta-hedged option returns in financial option markets has developed in recent years with some stylized empirical analyses. In particular, Bakshi and Kapadia[2003] and Low and Zhang[2005] study delta-hedged option returns in a stock index option market and currency option markets, respectively, and they provide evidence of the existence of the negative stochastic variance risk premiums based on non-zero expected delta-hedged option returns.

While we also explore an empirical study on the existence of the variance risk premiums in this chapter, our study is different from that explored by Bakshi and Kapadia[2003] in that we explicitly consider the effect of parameter estimation risk on financial option prices. Theoretical models often assume that the economic agent who makes an optimal financial decision knows the true parameters of the model. But the true parameters are rarely if ever known to the decision maker. In reality, model parameters have to be estimated based on historical information and, hence, the model's usefulness depends partly on how good the estimates are. This gives rise to estimation risk in virtually all option valuation models.

Considering the effect of parameter estimation risk on financial option prices, we provide a novel representation of delta-hedged option returns in a stochastic volatility environment. The representation of delta-hedged option returns provided in this chapter consists of two terms; volatility risk premium and parameter estimation risk. Based on the representation for delta-hedged option returns, we explore an empirical simulation. Examining the delta-hedged option returns of the USD-JPY currency options with a historical simulation from October 2003 to June 2010, we find that the delta-hedged option returns for OTM put options are strongly affected by parameter estimation risk as well as the volatility risk premium, especially in the post-Lehman shock period. In particular, we find that approximately 13 % of the value of the OTM currency option premium is generated by the existence of parameter estimation risk in the post-Lehman crisis period, and this effect induced by parameter estimation risk on option prices is more significant than the effect of the volatility risk premium. One of the most important implications of this chapter is that the sign and the level of the expected delta-hedged option returns do not necessarily explain the existence of volatility risk premiums.

Chapter 5

Conclusion.

Chapter 2

The Skewness Risk Premium in Equilibrium and Stock Return Predictability

2.1 Introduction

The concern with the information content in option-implied distributions has been growing for the last several years. In particular, there has been a growing interest in the information of the difference between option-implied and realized distributions, which is usually recognized as a risk premium required by the representative agent, in terms of the financial risk management and asset pricing implications. In this chapter we investigate the risk premiums in higher order moments, especially in the skewness, of financial asset returns under a general equilibrium setting. In this study, each of the risk premiums in higher order moments of financial asset returns is defined by the difference between two expected values of the moment under the risk-neutral and physical probability measures, respectively.

In recent years, there remains an ever-increasing interest and challenge to develop an entirely self-contained equilibrium-based explanation for the nonzero volatility (or variance) risk premium [1] and its predictability for stock index returns. To the best of our knowledge, the first attempt to demonstrate the existence of the volatility risk premium based on a general equilibrium market model is made by Eraker[2008]. Eraker[2008] develop an equilibrium explanation for the volatility risk premium based on the long-run

[1] The volatility (variance) risk premium is defined by the difference between two expected values of the volatility (variance) under the risk-neutral and physical probability measures, respectively.

risks (LRR) model [2] which emphasizes the role of long-run risks, that is, low-frequency movements in consumption growth rates and volatility, in accounting for a wide range of asset pricing puzzles. The LRR model features an Epstein and Zin[1989] utility function with an investor preference for early resolution of uncertainty and contains (i) a persistent expected consumption growth component and (ii) long-run variation in consumption volatility. On the basis of the LRR model, Eraker[2008] studies the volatility risk premium through the framework of a general equilibrium model.

In addition to the development of an entirely self-contained equilibrium-based explanation for the risk premiums in higher order moments, several academic studies related to those risk premiums are provided in recent years. For example, motivated by fruitful implications from the LRR model pioneered by Bansal and Yaron[2004], Bollerslev, Tauchen, and Zhou[2009] investigate the stock return predictability of the variance risk premium in terms of a general equilibrium setting based on the LRR model framework. They show that the difference between implied and realized variation, or the variance risk premium, is able to explain a nontrivial fraction of the time-series variation in post-1990 aggregate stock market returns, with high (low) premia predicting high (low) future returns. The magnitude of the predictability is particularly strong at the intermediate quarterly return horizon, where it dominates that afforded by other popular predictor variables, such as the P/E ratio, the default spread, and the consumption-wealth ratio.

Drechsler and Yaron[2011] also show the predictability of the variance risk premium for stock index returns based on an extended LRR model with jumps in uncertainty and the long-run component of cash-flows. They demonstrate that a risk aversion greater than one and a preference for early resolution of uncertainty correctly signs the variance risk premium and the coefficient from a predictive regression of returns on the variance risk premium.

All of the studies cited above focus only on the variance risk premium required by a representative investor due to the stochastic nature of asset return variance. Conversely, as far as we know, there are few reports about the risk premium which compensates for uncertainty of the third moment, that is, the skewness, of asset returns. In this chapter, we demonstrate that the skewness risk premium, defined by the difference between two expected values of the skewness under the risk-neutral and physical probability measures, respectively, also captures attitudes toward economic uncertainty as well as the variance risk premium. Among recent studies on self-contained equilibrium-based models for the nonzero variance risk premium referred above, all of the studies except for Drechsler and

[2] The long-run risks model is pioneered by Bansal and Yaron[2004], which is a stylized self-contained general equilibrium model incorporating the effects of time-varying economic uncertainty.

Yaron[2011] model the processes of both the variance of consumption growth rate and the LRR factor as conditional normal, so that the one-step-ahead conditional distribution of the market return is also normal and, as a result, the skewness of that distribution is zero. Therefore, the models proposed by those studies can not explain the *negative* risk-neutral skewness, which is found by the previous studies such as those by Aït-Sahalia and Lo[1998] and Aït-Sahalia, Wang, and Yared[2001]. They document several empirical features of the state price density for the S & P500 index option market over time, including the term structures of mean returns, volatility, skewness, and kurtosis, that are implied by option-implied distributions. In particular, They show that the nonparametric state price densities are negatively skewed, have fatter tails and the amount of skewness and kurtosis both increase with maturity.

We show that jump components in the LRR factor and/or the variance of consumption growth rate can explain the nonzero (or negative) skewness of the one-step-ahead asset return distribution. To the best of our knowledge, Drechsler and Yaron[2011] is the first paper that indicates an important role for transient non-Gaussian shocks (jumps) to fundamentals such as the LRR-factor and the variance of consumption growth rate for understanding how perceptions of economic uncertainty and cash-flow risk manifest themselves in asset prices. However, the assumption of an affine structure on the jump intensity process λ_t, that is, $\lambda_t = l_0 + l_1\sigma_t^2$ where $l_0, l_1 > 0$ and σ_t^2 is the variance of consumption growth rate, in Drechsler and Yaron[2011] can not explain an empirical fact on a simultaneous relation between monthly stock returns and monthly changes of the option-implied skewness:

$$
\begin{aligned}
r_{m,t+1} &= 0.006 - 0.019 \times \triangle ISkew_{t+1}, \\
&\quad\ (2.46) \quad (-3.46) \\
r_{m,t+1} &= 0.006 - 0.007 \times \triangle VIX_{t+1} - 0.016 \times \triangle ISkew_{t+1}, \\
&\quad\ (3.33) \quad (-16.00) \qquad\qquad (-3.94)
\end{aligned}
\tag{2.1}
$$

where $r_{m,t+1}$ is the monthly return of the S & P500 Total Return Index from time t to $t+1$, $\triangle VIX_{t+1}$ is the monthly change of implied volatility calculated with the CBOE's VIX from time t to $t+1$, and $\triangle ISkew_{t+1}$ is the monthly change of implied skewness calculated with the CBOE's Skew Index from time t to $t+1$. These results are obtained based on the monthly data from Jan-1990 to Aug-2012. Under the assumption on the jump intensity process in Drechsler and Yaron[2011], however, we can confirm that the regression parameters to $\triangle ISkew_{t+1}$ in the above regression models should be positive.

In this chapter, we propose an extension of the LRR models developed by Bansal and Yaron[2004] and Drechsler and Yaron[2011]. Our model contains a rich set of transient dynamics and can quantitatively account for the time variation and asset return predictability of the skewness premium as well as the variance risk premium. In particular, we introduce a stochastic jump intensity for transient jumps to fundamentals such as the LRR factor and the variance of consumption growth rate, and show that this additional introduction of a stochastic jump intensity enables our model to capture the various empirical aspects of the stock index returns and its option implied moments including the result of (2.1). Christoffersen et al.[2012] find very strong support for time-varying jump intensities for S & P500 index returns, and they show that, compared to the risk premium on dynamic volatility, the risk premium on the dynamic jump intensity has a much larger impact on option prices. We find that the existence of the negative skewness and the skewness risk premium have a close relationship with the existence of the jumps and the jump risk premium, respectively.

This chapter also shows that the skewness of asset return distribution and the skewness risk premium, which compensates for the stochastic nature of the skewness, are both time-varying due to the stochastic nature of the jump intensity for transient jumps in both the LRR factor and the variance of consumption growth rate. Providing an equity risk premium representation of a linear factor pricing model with time-varying variance and skewness risk premiums, we find that those risk premiums can explain a nontrivial fraction of the time series variation in the aggregate stock market returns and show an empirical evidence in which the skewness risk premium, as well as the variance risk premium, has superior predictive power for future aggregate stock market index returns. Compared with the variance risk premium, the results show that the skewness risk premium plays an independent and essential role for predicting the market index returns.

The remainder of this chapter is organized as follows. Section 2 outlines the basic theoretical model with jumps in consumption growth rate and its volatility, shows how equilibrium is derived for our model economy, and highlights its key features. In particular, we provide an equity risk premium representation of a linear factor pricing model with time-varying variance and skewness risk premiums. Section 3 provides the implications from a calibrated version of the theoretical equity risk premium representation of a linear factor pricing model derived in Section 2 to help guide and interpret our subsequent empirical reduced form predictability regressions. Section 4 describes the data used for examining the equity risk premium representation empirically and discusses the results from the predictive regressions on the stock returns to the variance and the skewness risk premiums with historical data. Section 5 provides concluding remarks.

2.2 Model Framework

2.2.1 Model Setup and Assumptions

The underlying environment is a discrete time endowment economy. The representative agent's preferences on the consumption stream are of the Epstein and Zin[1989] form, allowing for the separation of risk aversion and the intertemporal elasticity of substitution (IES). Thus, the agent maximizes his lifetime utility, which is defined recursively as

$$V_t = \left[(1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta\Big(\mathbb{E}_t[V_{t+1}^{1-\gamma}]\Big)^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}, \tag{2.2}$$

where C_t is consumption at time t, $0 < \delta < 1$ reflects the agent's time preference, γ is the coefficient of risk aversion, $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$, and ψ is the intertemporal elasticity of substitution (IES). This preference structure collapses to a standard CRRA utility representation if $\gamma = \frac{1}{\psi}$, that is, $\theta = 1$, and in this case, only innovations to consumption are priced. In the following, based on the result provided by Bansal and Yaron[2004] we assume that both γ and ψ are larger than one. It then holds that $\gamma > \frac{1}{\psi}$, which implies $\theta < 0$. With this choice, the investor has a preference for early resolution of uncertainty. Then, not only consumption risk is priced, but state variables carry risk premia, too. The parameter restrictions also ensure that the signs of the risk premia are in line with economic intuition, and that a worsening of economic conditions leads to a decrease in asset prices.

Utility maximization is subject to the budget constraint:

$$W_{t+1} = (W_t - C_t)R_{c,t+1},$$

where W_t is the wealth of the agent and $R_{c,t}$ is the return on all invested wealth. As shown in Epstein and Zin[1989], for any asset j, the first-order condition yields the following Euler condition:

$$\mathbb{E}_t\Big[\exp(m_{t+1} + r_{j,t+1})\Big] = 1, \tag{2.3}$$

where $r_{j,t+1}$ is the log of the gross return on asset j and m_{t+1} is the log of the intertemporal marginal rate of substitution (IMRS), which is given by $m_{t+1} = \theta \log\delta - \frac{\theta}{\psi} \triangle c_{t+1} + (\theta - 1)r_{c,t+1}$. Here, $r_{c,t+1}$ is $\log R_{c,t+1}$ and $\triangle c_{t+1}$ is the change in $\log C_t$, that is, $\log\left(\frac{C_{t+1}}{C_t}\right)$.

We model consumption and dividend growth rates, $g_{t+1} \equiv \log(\frac{C_{t+1}}{C_t})$ and $g_{d,t+1} \equiv \log(\frac{D_{t+1}}{D_t})$ where D_t is dividend at time t, respectively, as containing a small persistent predictable component x_t, which determines the conditional expectation of consumption growth,

$$\begin{aligned} x_{t+1} &= \rho_x x_t + \varphi_e \sigma_t e_{t+1} + J_{x,t+1}, \\ g_{t+1} &= \mu_g + x_t + \varphi_\eta \sigma_t \eta_{t+1}, \\ g_{d,t+1} &= \mu_d + \rho_d x_t + \varphi_\zeta \sigma_t \zeta_{t+1}, \end{aligned} \tag{2.4}$$

where $\varphi_e, \varphi_\eta, \varphi_\zeta, \rho_x, \rho_d > 0$, $\mu_g, \mu_d \in \mathbb{R}$, e_t, η_t, and ζ_t are mutually independent $i.i.d.N(0,1)$ processes, and $J_{x,t+1}$ is a compound-Poisson process represented by $J_{x,t+1} \equiv \sum_{j=1}^{N_{t+1}^x} \epsilon_x^j$ where N_{t+1}^x is the Poisson counting process for that jump component whose the intensity process is $\lambda_{x,t+1} \equiv l_x \lambda_{t+1}, l_x > 0$, and $\epsilon_x^j \sim i.i.d.\ N(0, \delta_x^2)$, $\delta_x > 0$, is the size of the jump that occurs upon the N_{t+1}^x.

Furthermore, we also model the dynamics of the volatility as follows:

$$\begin{aligned} \sigma_{t+1}^2 &= \mu_\sigma + \rho_\sigma \sigma_t^2 + \sqrt{q_t} w_{t+1} + J_{\sigma^2,t+1}, \\ q_{t+1} &= \mu_q + \rho_q q_t + \varphi_\xi \sqrt{q_t} \xi_{t+1}, \end{aligned} \tag{2.5}$$

where the parameters satisfy $\mu_\sigma > 0$, $\mu_q > 0$, $|\rho_\sigma| < 1$, $|\rho_q| < 1$, $\varphi_\xi > 0$, and w_t and ξ_t are mutually independent $i.i.d.N(0,1)$ processes and are independent of each of e_t, η_t, and ξ_t. $J_{\sigma^2,t+1}$ is a compound-Poisson process, which is represented by $J_{\sigma^2,t+1} \equiv \sum_{j=1}^{N_{t+1}^{\sigma^2}} \epsilon_{\sigma^2}^j$ where $N_{t+1}^{\sigma^2}$ is the Poisson counting process for that jump component whose the intensity process is $\lambda_{\sigma^2,t+1} \equiv l_{\sigma^2} \lambda_{t+1}, l_{\sigma^2} > 0$, and $\epsilon_{\sigma^2}^j \sim i.i.d.\ N(0, \delta_{\sigma^2}^2)$, $\delta_{\sigma^2} > 0$, is the size of the jump that occurs upon the $N_{t+1}^{\sigma^2}$. We assume that N_{t+1}^x and $N_{t+1}^{\sigma^2}$ are mutually independent and ϵ_x^j and $\epsilon_{\sigma^2}^j$ are too. The stochastic variance process σ_t^2 represents time-varying economic uncertainty in consumption growth with the variance-of-variance process q_t in effect inducing an additional source of temporal variation in that same process. We also model the variance-of-variance process q_t in the same fashion as Bollerslev et al.[2009].

Importantly, we introduce the jump intensity dynamics in the economy which is represented by the following discrete-time stochastic process,

$$\lambda_{t+1} = \mu_\lambda + \rho_\lambda \lambda_t + \varphi_u \sqrt{q_t}(\rho \xi_{t+1} + \sqrt{1-\rho^2} u_{t+1}), \tag{2.6}$$

where $\mu_\lambda > 0$, $|\rho_\lambda| < 1$, $|\rho| \leq 1$, and u_t is an $i.i.d.N(0,1)$ process, which is independent of each of e_t, η_t, ζ_t, w_t, and ξ_t.

One of the notable features of our model setup is this introduction for the jump intensity process (2.6). Christoffersen et al.[2012] also find very strong support for time-varying jump intensities for S & P500 index returns, and they show that, compared to the risk premium on dynamic volatility, the risk premium on the dynamic jump intensity has a much larger impact on option prices. In the previous studies, Drechsler and Yaron[2011] is the first paper that introduces transient jumps to fundamentals such as the LRR-factor x_t and the variance of consumption growth rate σ_t^2. However, it assumes that the jump intensity process λ_t is represented by an affine structure of $\lambda_t = l_0 + l_1\sigma_t^2$ where $l_0, l_1 > 0$. As mentioned in the introduction of this chapter, such assumption for the jump intensity process can not explain the empirical fact of regression (2.1). We extend the LRR models of Bansal and Yaron[2004] and Drechsler and Yaron[2011] so as to introduce a stochastic jump intensity of (2.6) into the economy. As shown in the following, this introduction enables our model to have a consistency with the empirical fact shown in (2.1) and plays a key role in describing the characteristics of asset return distributions.

2.2.2 The Model Solution in Equilibrium

We distinguish between the unobservable return on a claim to aggregate consumption, $R_{c,t+1}$, and the observable return on the market portfolio, $R_{m,t+1}$: the latter is the return on the aggregate dividend claim. Solving our model numerically, we demonstrate the mechanisms working in our model via approximate analytical solutions in the same fashion as the previous studies such as those by Bansal and Yaron[2004], Bollerslev et al.[2009], Drechsler and Yaron[2011], etc. To derive these solutions for our model, we use the standard approximation utilized in Campbell and Shiller[1988],

$$r_{c,t+1} = \kappa_0 + \kappa_1 v_{t+1} - v_t + g_{t+1}, \tag{2.7}$$

where lowercase letters refer to logs, so that $r_{c,t+1} = \log(R_{c,t+1})$ is the continuous return, $v_t = \log(\frac{P_t}{C_t})$ is the log price-consumption ratio of the asset that pays the consumption endowment, $\{C_{t+i}\}_{i=1}^{\infty}$, and κ_0 and κ_1 are approximating constants that both depend only on the average level of v [3]. Analogously, $r_{m,t+l}$ and $v_{m,t+1}$ correspond to the market return and its log price-dividend ratio and the similar approximation presented below can also be derived:

$$r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m} v_{m,t+1} - v_{m,t} + g_{d,t+1}. \tag{2.8}$$

The standard solution method for finding the equilibrium in a model like the one defined above then consists in conjecturing solutions for v_t and $v_{m,t}$ as an affine function of the state variables, x_t, σ_t^2, q_t, and λ_t,

$$v_t = A_0 + A_x x_t + A_\sigma \sigma_t^2 + A_q q_t + A_\lambda \lambda_t, \tag{2.9}$$

[3] Note that $\kappa_1 = \frac{\exp(\bar{v})}{1+\exp(\bar{v})}$ and this value is approximately 0.997 (cf) Bansal and Yaron[2004]), which is also consistent with magnitudes used in Campbell and Shiller[1988].

$$v_{m,t} = A_{0,m} + A_{x,m}x_t + A_{\sigma,m}\sigma_t^2 + A_{q,m}q_t + A_{\lambda,m}\lambda_t, \tag{2.10}$$

respectively, solving for the coefficients A_0, A_x, A_σ, A_q, and A_λ in v_t and for the coefficients $A_{0,m}$, $A_{x,m}$, $A_{\sigma,m}$, $A_{q,m}$, and $A_{\lambda,m}$ in $v_{m,t}$.

Substituting (2.9) for (2.7), we have a temporal representation for $r_{c,t+1}$ with the state variables, x_t, σ_t^2, q_t, and λ_t, and furthermore, substituting this $r_{c,t+1}$ for the Euler equation (2.3), we can derive an identity with those state variables. Solving the identity in the same manner as Bansal and Yaron[2004], Bollerslev et al.[2009], Drechsler and Yaron[2011], etc., we can derive the equilibrium solutions for the four parameters as follows:

$$\begin{aligned}
A_x &= \frac{\gamma - 1}{\theta(\kappa_1\rho_x - 1)}, \\
A_\sigma &= -\frac{1}{2}\frac{(1-\gamma)^2\varphi_\eta^2 + \theta^2\kappa_1^2 A_x^2\varphi_e^2}{\theta(\kappa_1\rho_\sigma - 1)}, \\
A_\lambda &= \frac{2 - \exp(\frac{1}{2}\theta^2\kappa_1^2 A_x^2\delta_x^2) - \exp(\frac{1}{2}\theta^2\kappa_1^2 A_\sigma^2\delta_\sigma^2)}{\theta(\kappa_1\rho_\lambda - 1)},
\end{aligned} \tag{2.11}$$

A_q is a solution of the quadratic equation presented below:

$$\theta A_q(\kappa_1\rho_q - 1) + \frac{\theta^2\kappa_1^2}{2}\Big[A_\sigma^2 + A_q^2\varphi_\xi^2 + 2A_qA_\lambda\varphi_\xi\varphi_u\rho + A_\lambda^2\varphi_u^2\Big] = 0.$$

Considering the expressions of (2.11), the following proposition can be proven easily:

Proposition 1 *If $\gamma > 1$ and $\psi > 1$, then, $A_x > 0$, $A_\sigma < 0$, $A_q < 0$, and $A_\lambda < 0$.*

The above proposition suggests that if the IES and risk aversion are higher than 1, a rise in each of the state variables of σ_t^2, q_t, and λ_t lowers the price-consumption ratio.

Having solved for $r_{c,t+1}$ with the four parameters derived above, we can substitute it (and $\triangle c_{t+1} = g_{t+1}$) into m_{t+1} to obtain an expression for the conditional innovation to the log pricing kernel at time $t+1$:

$$\begin{aligned}
&m_{t+1} - \mathbb{E}_t[m_{t+1}] \\
&\quad= \theta\log\delta - \frac{\theta}{\psi}\triangle c_{t+1} + (\theta-1)r_{c,t+1} - \mathbb{E}_t\Big[\theta\log\delta - \frac{\theta}{\psi}\triangle c_{t+1} + (\theta-1)r_{c,t+1}\Big] \\
&\quad= \Big(-\frac{\theta}{\psi} + \theta - 1\Big)\varphi_\eta\sigma_t\eta_{t+1} + (\theta-1)\kappa_1A_x\varphi_e\sigma_te_{t+1} + (\theta-1)\kappa_1A_\sigma\sqrt{q_t}w_{t+1} \\
&\qquad+ (\theta-1)\kappa_1(A_q\varphi_\xi + A_\lambda\varphi_u\rho)\sqrt{q_t}\xi_{t+1} + (\theta-1)\kappa_1A_\lambda\varphi_u\sqrt{1-\rho^2}\sqrt{q_t}u_{t+1} \\
&\qquad+ (\theta-1)\kappa_1A_x(J_{x,t+1} - \mathbb{E}_t[J_{x,t+1}]) + (\theta-1)\kappa_1A_\sigma(J_{\sigma^2,t+1} - \mathbb{E}_t[J_{\sigma^2,t+1}]) \\
&\quad= -\Lambda^t(G_tz_{t+1} + J_{t+1} - \mathbb{E}_t[J_{t+1}]),
\end{aligned} \tag{2.12}$$

where

$$\Lambda \equiv \begin{pmatrix} \gamma & (1-\theta)\kappa_1 A_x & (1-\theta)\kappa_1 A_\sigma & (1-\theta)\kappa_1 A_q & (1-\theta)\kappa_1 A_\lambda & 0 \end{pmatrix}^t,$$

$$G_t \equiv \begin{pmatrix} \varphi_\eta \sigma_t & 0 & 0 & 0 & 0 & 0 \\ 0 & \varphi_e \sigma_t & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{q_t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \varphi_\xi \sqrt{q_t} & 0 & 0 \\ 0 & 0 & 0 & \rho \varphi_u \sqrt{q_t} & \varphi_u \sqrt{1-\rho^2}\sqrt{q_t} & 0 \\ 0 & 0 & 0 & 0 & 0 & \varphi_\zeta \sigma_t \end{pmatrix}, \tag{2.13}$$

$$z_{t+1} \equiv \begin{pmatrix} \eta_{t+1} & e_{t+1} & w_{t+1} & \xi_{t+1} & u_{t+1} & \zeta_{t+1} \end{pmatrix}^t,$$

$$J_{t+1} \equiv \begin{pmatrix} 0 & J_{x,t+1} & J_{\sigma^2,t+1} & 0 & 0 & 0 \end{pmatrix}^t,$$

$$\mathbb{E}_t[J_{t+1}] \equiv \begin{pmatrix} 0 & \mathbb{E}_t[J_{x,t+1}] & \mathbb{E}_t[J_{\sigma^2,t+1}] & 0 & 0 & 0 \end{pmatrix}^t.$$

Λ can be interpreted as the price of risk for Gaussian shocks and also the sensitivity of the IMRS to the jump shocks. From the expression of Λ, one can see that the prices of risks are determined by the A coefficients, that is, A_x, A_σ, A_q, and A_λ. The expression of Λ also shows that the signs of the risk prices depend on the signs of the A coefficients and $(1-\theta)$. In particular, when $\gamma = \frac{1}{\psi}$, $\theta = 1$, and we are in the case of constant relative risk aversion (CRRA) preferences, it is clear that only the transient shock to consumption $z_{c,t+1} \equiv \eta_{t+1}$ is priced, and prices do not separately reflect the risk of shocks to x_t (long-run risk), σ_t^2 (volatility-related risk), q_t (variance-of-variance-related risk), and λ_t (jump intensity-related risk).

In the discussion and calibrations explored below, we especially focus on the case in which the agent's risk aversion γ and the IES ψ are both greater than 1, which implies that $A_x > 0$, $A_\sigma < 0$, $A_q < 0$, and $A_\lambda < 0$ by the proposition provided above. Thus, positive shocks to long-run growth decrease the IMRS, while positive shocks to the levels of the other state variables, σ_t^2, q_t, and λ_t, increase the IMRS. Note that in this case, since $(1-\theta) > 0$, each of the A coefficients has the same sign as the corresponding price of risk.

To study the risk premiums in higher-order moments of the market returns, we first need to solve for the market return. A share in the market is modeled as a claim to a dividend with growth process given by $g_{d,t}$. To solve for the price of a market share, we proceed along the same lines as for the consumption claim and solve for $v_{m,t+1}$, the log price-dividend ratio of the market, by using the the conjecture (2.10), Campbell and

Shiller[1988]-approximation (2.8), and the Euler equation (2.3) [4] .

With the equilibrium solutions for the parameters of $A_{x,m}$, $A_{\sigma,m}$, $A_{q,m}$, and $A_{\lambda,m}$ in (2.10), we can obtain an expression for $r_{m,t+1}$ in terms of the state variables and its innovations by substituting the expression for $v_{m,t(+1)}$ into (2.8):

$$\begin{aligned} r_{m,t+1} = {} & \kappa_{0,m} + \kappa_{1,m}A_{0,m} + \kappa_{1,m}A_{\sigma,m}\mu_d + \kappa_{1,m}A_{q,m}\mu_g + \kappa_{1,m}A_{\lambda,m}\mu_\lambda - A_{0,m} + \mu_d \\ & + (\kappa_{1,m}A_{x,m}\rho_x - A_{x,m} + \rho_d)x_t \\ & + (\kappa_{1,m}A_{\sigma,m}\rho_\sigma - A_{\sigma,m}){\sigma_t}^2 \\ & + (\kappa_{1,m}A_{q,m}\rho_q - A_{q,m})q_t \\ & + (\kappa_{1,m}A_{\lambda,m}\rho_\lambda - A_{\lambda,m})\lambda_t \\ & + \kappa_{1,m}A_{x,m}\varphi_e\sigma_t e_{t+1} + \kappa_{1,m}A_{\sigma,m}\sqrt{q_t}w_{t+1} \\ & + \kappa_{1,m}(A_{q,m}\varphi_\xi + A_{\lambda,m}\varphi_u\rho)\sqrt{q_t}\xi_{t+1} \\ & + \kappa_{1,m}A_{\lambda,m}\varphi_u\sqrt{1-\rho^2}\sqrt{q_t}u_{t+1} + \varphi_\zeta\sigma_t\zeta_{t+1} \\ & + \kappa_{1,m}A_{x,m}J_{x,t+1} + \kappa_{1,m}A_{\sigma,m}J_{\sigma^2,t+1} \\ = {} & r_0 + (B_r^t F - A_m^t)Y_t + B_r^t G_t z_{t+1} + B_r^t J_{t+1}, \end{aligned} \tag{2.14}$$

where

$$r_0 \equiv \kappa_{0,m} + (\kappa_{1,m} - 1)A_{0,m} + (\kappa_{1,m}A_{\sigma,m} + 1)\mu_d + \kappa_{1,m}A_{q,m}\mu_g + \kappa_{1,m}A_{\lambda,m}\mu_\lambda,$$
$$B_r \equiv \kappa_{1,m}A_m + e_d \quad ,$$

$$A_m \equiv \begin{pmatrix} 0 \\ A_{x,m} \\ A_{\sigma,m} \\ A_{q,m} \\ A_{\lambda,m} \\ 0 \end{pmatrix}, \quad e_d \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad F \equiv \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \rho_x & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_\sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_q & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_\lambda & 0 \\ 0 & \rho_d & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_t \equiv \begin{pmatrix} g_t \\ x_t \\ {\sigma_t}^2 \\ q_t \\ \lambda_t \\ g_{d,t} \end{pmatrix}. \tag{2.15}$$

2.2.3 Risk Premiums in Higher-Order Moments in Equilibrium

Before proceeding to investigating the risk premiums in higher-order moments in equilibrium, we need to provide some further explanation on the jump dynamics and the features of the pricing kernel introduced above.

[4] Because the datails of the four parameters, $A_{x,m}$, A_{σ_m}, $A_{q,m}$, and $A_{\lambda,m}$, are insignificant and do not affect the discussion explored in the following at all, for simplicity, we express the parameters, $A_{x,m}$, A_{σ_m}, $A_{q,m}$, and $A_{\lambda,m}$, as they are and do not show explicit representations of those parameters in this study.

To handle the jumps, we introduce some notation. $\psi_k(u_k) = \mathbb{E}[\exp(u_k\epsilon_k)]$ (k is x or σ^2) denotes the moment-generating function (mgf) of the jump size ϵ_k. The mgf for the jump component of k, $\mathbb{E}[\exp(u_k J_{k,t+1})]$, then equals $\exp(\Psi_{t,k}(u_k))$, where $\Psi_{t,k}(u_k) = \lambda_{k,t}(\psi_k(u_k) - 1)$. $\Psi_{t,k}$ is called the cumulant-generating function (cgf) of $J_{k,t+1}$ and is a very helpful tool for calculating asset pricing moments. The reason is that its n-th derivative evaluated at 0 equals the n-th central moment of $J_{k,t+1}$.

Regarding the features of the pricing kernel, we can show what described below in line with Drechsler and Yaron[2011]. Let us set the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{M_{t+1}}{\mathbb{E}_t[M_{t+1}]}$, where $\mathbb{P}$ is the physical probability measure and $\mathbb{Q}$ is the risk-neutral probability measure in our economy. From (2.12), we have $\frac{M_{t+1}}{\mathbb{E}_t[M_{t+1}]} \propto \exp(-\Lambda^t(G_t z_{t+1} + J_{t+1}))$. Since z_{t+1} and J_{t+1} are independent, we can treat their measure transformations between $\mathbb{P}$ and $\mathbb{Q}$ separately. As a consequence, Drechsler and Yaron[2011] show that

$$z_{t+1} \overset{\mathbb{Q}}{\sim} N(-G_t'\Lambda, I), \tag{2.16}$$

where I is the identity matrix in $\mathbb{R}^{6\times 6}$. That is to say that, under $\mathbb{Q}$, z_{t+1} is still a vector of independent normals with unit variances, but with a shift in the mean.

For the case of J_{t+1}, we could also proceed by transforming the probability density function directly. As guided in Drechsler and Yaron[2011], Proposition (9.6) in Cont and Tankov[2004] shows that under $\mathbb{Q}$, the $J_{t+1,k}$ are still compound Poisson processes, but with cgf given by

$$\Psi^{\mathbb{Q}}_{t,k}(u_k) = \lambda_{k,t}\psi_k(-\Lambda_k)\Big(\frac{\psi_k(u_k - \Lambda_k)}{\psi_k(-\Lambda_k)} - 1\Big), \tag{2.17}$$

where $k = x$ or $k = \sigma^2$ and Λ_x denotes the price of risk for the LRR-factor x_t, that is, $(1-\theta)\kappa_1 A_x$, and Λ_{σ^2} denotes the price of risk for the variance of consumption growth rate, that is, $(1-\theta)\kappa_1 A_\sigma$. (see (2.13)) In the following discussion, we use the facts mentioned above to calculate the higher-order moments of the market return and to investigate the risk premiums in the moments.

The Variance Risk Premium in Equilibrium

According to Bollerslev et al.[2009] and Drechsler and Yaron[2011], the variance risk premium in equilibrium, vp_t, is defined by

$$vp_t \equiv \mathbb{E}^{\mathbb{Q}}_t[\mathrm{Var}^{\mathbb{Q}}_{t+1}(r_{m,t+2})] - \mathbb{E}^{\mathbb{P}}_t[\mathrm{Var}^{\mathbb{P}}_{t+1}(r_{m,t+2})], \tag{2.18}$$

where $\mathrm{Var}^{\mathbb{P}}_{t+1}$ ($\mathrm{Var}^{\mathbb{Q}}_{t+1}$) is the variance operator under the physical (risk-neutral) probability measure at time $t+1$. From (2.14), the conditional variance of the market return

$r_{m,t+2}$ at time $t+1$ under $\mathbb{P}$ can be obtained as follows:

$$\begin{aligned} \mathrm{Var}^{\mathbb{P}}_{t+1}(r_{m,t+2}) &= B_r^t G_{t+1} G_{t+1}^t B_r + \sum_i B_r^2(i) \mathrm{Var}^{\mathbb{P}}_{t+1}(J_{i,t+2}) \\ &= B_r^t G_{t+1} G_{t+1}^t B_r + B_r^{2^t} \Psi^{(2)}_{t+1}(0), \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} B_r &= \kappa_{1,m} A_m + e_d \quad (\because (2.15)) \\ &\equiv \begin{pmatrix} B_r(1) & B_r(2) & B_r(3) & B_r(4) & B_r(5) & B_r(6) \end{pmatrix}^t \in \mathbb{R}^6, \\ B_r^2 &\equiv \begin{pmatrix} B_r^2(1) & B_r^2(2) & B_r^2(3) & B_r^2(4) & B_r^2(5) & B_r^2(6) \end{pmatrix}^t \in \mathbb{R}^6, \\ \Psi^{(2)}_{t+1}(0) &\equiv \begin{pmatrix} 0 & \Psi^{(2)}_{t+1,x}(0) & \Psi^{(2)}_{t+1,\sigma^2}(0) & 0 & 0 & 0 \end{pmatrix}^t \in \mathbb{R}^6, \end{aligned}$$

and $\Psi^{(2)}_{t+1,x}(0)$ and $\Psi^{(2)}_{t+1,\sigma^2}(0)$ are respectively the second derivative of the cgf (cumulant-generating function) for $J_{x,t+1}$ and $J_{\sigma^2,t+1}$ evaluated at 0, that is,

$$\begin{aligned} \Psi^{(2)}_{t+1,x}(0) &\equiv \frac{\partial^2}{\partial u^2} \Psi_{t+1,x}(u) \mid_{u=0} = \frac{\partial^2}{\partial u^2} \lambda_{x,t+1}(\psi_x(u) - 1) \mid_{u=0}, \\ \Psi^{(2)}_{t+1,\sigma}(0) &\equiv \frac{\partial^2}{\partial u^2} \Psi_{t+1,\sigma^2}(u) \mid_{u=0} = \frac{\partial^2}{\partial u^2} \lambda_{\sigma^2,t+1}(\psi_{\sigma^2}(u) - 1) \mid_{u=0} . \end{aligned}$$

Thus the expression of (2.19) is rearranged to the following representation,

$$\begin{aligned} \mathrm{Var}^{\mathbb{P}}_{t+1}(r_{m,t+2}) &= B_r^t G_{t+1} G_{t+1}^t B_r + B_r^{2^t} \Psi^{(2)}_{t+1}(0) \\ &= B_r^t (H_{\sigma^2} \sigma^2_{t+1} + H_q q_{t+1}) B_r + B_r^{2^t} \mathrm{diag}\Big(\psi^{(2)}(0)\Big) \Pi_{t+1}, \end{aligned} \tag{2.20}$$

where

$$H_{\sigma^2} \equiv \begin{pmatrix} \varphi_\eta^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varphi_e^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varphi_\zeta^2 \end{pmatrix}, \quad H_q \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varphi_\xi^2 & \rho\varphi_\xi\varphi_u & 0 \\ 0 & 0 & 0 & \rho\varphi_\xi\varphi_u & \varphi_u^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{diag}\Big(\psi^{(2)}(0)\Big) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \psi_x^{(2)}(0) & 0 & 0 & 0 & 0 \\ 0 & 0 & \psi_{\sigma^2}^{(2)}(0) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Pi_{t+1} \equiv \begin{pmatrix} 0 \\ \lambda_{x,t+1} \\ \lambda_{\sigma^2,t+1} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Under the risk-neutral probability measure $\mathbb{Q}$, the conditional variance of the market return $r_{m,t+2}$ at time $t+1$ also can be obtained in the same manner demonstrated above. As a consequence, we can show the following proposition based on the definition of the variance risk premium (2.18).

Proposition 2 (The Variance Risk Premium in Equilibrium) *In equilibrium, the variance risk premium at time t, vp_t, is linear to the variance-of-variance, q_t, and the jump intensity, λ_t, and the representation of it is provided as follows:*

$$vp_t = \beta_{vp,c} + \beta_{vp,q} q_t + \beta_{vp,\lambda}\lambda_t, \tag{2.21}$$

where

$$\beta_{vp,c} \equiv \Big[l_x B_r^2(2)(\psi_x^{(2)}(-\Lambda_x) - \psi_x^{(2)}(0)) + l_{\sigma^2} B_r^2(3)(\psi_{\sigma^2}^{(2)}(-\Lambda_{\sigma^2}) - \psi_{\sigma^2}^{(2)}(0))\Big]\mu_\lambda,$$

$$\begin{aligned}\beta_{vp,q} \equiv & -B_r^t\Big[\Lambda_{\sigma^2} H_{\sigma^2} + \varphi_\xi(\varphi_\xi \Lambda_q + \rho\varphi_u \Lambda_\lambda) H_q\Big] B_r \\ & - \varphi_u(\rho\varphi_\xi\Lambda_q + \varphi_u\Lambda_\lambda)(l_x B_r^2(2)\psi_x^{(2)}(-\Lambda_x) + l_{\sigma^2} B_r^2(3)\psi_{\sigma^2}^{(2)}(-\Lambda_{\sigma^2})),\end{aligned}$$

$$\begin{aligned}\beta_{vp,\lambda} \equiv & B_r^t H_{\sigma^2} B_r \psi_{\sigma^2}^{(1)}(-\Lambda_{\sigma^2}) \\ & + \Big[l_x B_r^2(2)(\psi_x^{(2)}(-\Lambda_x) - \psi_x^{(2)}(0)) + l_{\sigma^2} B_r^2(3)(\psi_{\sigma^2}^{(2)}(-\Lambda_{\sigma^2}) - \psi_{\sigma^2}^{(2)}(0))\Big]\rho_\lambda.\end{aligned}$$

Proof *See the Appendix.*

A number of interesting implications arise from the expression (2.21). In particular, any temporal variation in the endogenously generated variance risk premium is solely due to the variance-of-variance q_t and the jump intensity λ_t. Moreover, provided that $\theta < 0$, $\Lambda_x > 0$, and $\Lambda_{\sigma^2} < 0$, as would be implied by $\gamma > 1$ and $\psi > 1$, the factor loading to the jump intensity, that is, $\beta_{vp,\lambda}$, is guaranteed to be positive, but that to the variance-of-variance, that is, $\beta_{vp,q}$, can be both positive and negative in general. However, if the correlation between the dynamics of the variance-of-variance and that of the jump intensity, that is , ρ, is positive, then $\beta_{vp,q}$ is also guaranteed to be positive due to the facts that $\Lambda_q < 0$ and $\Lambda_\lambda < 0$.

The Skewness Risk Premium in Equilibrium

On the basis of the same manner used to derive the expression (2.20) in the previous subsection, we can also derive the representations for the skewness of the market return under $\mathbb{P}$ and $\mathbb{Q}$, respectively, as follows:

$$\begin{aligned}\text{Skew}_t^{\mathbb{P}}(r_{m,t+1}) &= B_r^{3^t}\text{diag}(\psi^{(3)}(0))\Pi_t, \\ \text{Skew}_t^{\mathbb{Q}}(r_{m,t+1}) &= B_r^{3^t}\text{diag}(\psi^{(3)}(-\Lambda))\Pi_t,\end{aligned} \tag{2.22}$$

where

$$B_r^3 \equiv \begin{pmatrix} B_r^3(1) \\ B_r^3(2) \\ B_r^3(3) \\ B_r^3(4) \\ B_r^3(5) \\ B_r^3(6) \end{pmatrix}, \quad \mathrm{diag}\Big(\psi^{(3)}(0)\Big) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \psi_x^{(3)}(0) & 0 & 0 & 0 & 0 \\ 0 & 0 & \psi_{\sigma^2}^{(3)}(0) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{diag}\Big(\psi^{(3)}(-\Lambda)\Big) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \psi_x^{(3)}(-\Lambda_x) & 0 & 0 & 0 & 0 \\ 0 & 0 & \psi_{\sigma^2}^{(3)}(-\Lambda_{\sigma^2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this study, we define the skewness risk premium in equilibrium at time t, skp_t, as the following expression, which is the same manner with the case of the variance risk premium (2.18):

$$skp_t \equiv \mathbb{E}_t^{\mathbb{Q}}[\mathrm{Skew}_{t+1}^{\mathbb{Q}}(r_{m,t+2})] - \mathbb{E}_t^{\mathbb{P}}[\mathrm{Skew}_{t+1}^{\mathbb{P}}(r_{m,t+2})]. \tag{2.23}$$

Substituting (2.22) into (2.23), the explicit representation for the skewness risk premium can be obtained as follows:

$$skp_t \equiv {B_r^3}^t \mathrm{diag}(\psi^{(3)}(-\Lambda))\mathbb{E}_t^{\mathbb{Q}}[\Pi_{t+1}] - {B_r^3}^t \mathrm{diag}(\psi^{(3)}(0))\mathbb{E}_t^{\mathbb{P}}[\Pi_{t+1}]. \tag{2.24}$$

We also find a number of interesting implications from the expressions of (2.22) and (2.24). First, in the case that there is no jump to fundamentals in the economy, that is, in the case of $\Pi_t \equiv 0$ ($\in \mathbb{R}^6$), it is clear that the conditional skewness of the market return should be zero due to (2.22). Thus, the existence of the nonzero skewness of the market return crucially depend on the existence of the jumps to fundamentals in the economy. Second, any temporal variation in endogenously generated skewness and skewness risk premium are solely due to the temporal variation in the jump intensity process λ_t. For example, if the jump intensity is constant, then it is clear that the skewness (under $\mathbb{P}$ and $\mathbb{Q}$) and skewness risk premium should be constant by (2.22) and (2.24). Third, since we have the fact of $A_\sigma < 0$ by the proposition 1, then in the case that the jump to the variance of consumption growth rate exists, that is, in the case that $\lambda_{\sigma^2,t} > 0$, we can show easily by (2.22) that the risk-neutral skewness at time t, $\mathrm{Skew}_t^{\mathbb{Q}}(r_{m,t+1})$, should be negative. Finally, we can also find via (2.24) that in the case that either $\lambda_{x,t} > 0$ or $\lambda_{\sigma^2,t} > 0$ is satisfied, the skewness risk premium at time t, skp_t, in equilibrium also should be negative due to the facts of $A_x > 0$ and $A_\sigma < 0$.

On the basis of the definition of (2.23), let us provide the proposition for the representation of the skewness risk premium in equilibrium.

Proposition 3 (The Skewness Risk Premium in Equilibrium) *In equilibrium, the skewness risk premium at time t, skp_t, is linear to the variance-of-variance, q_t, and the jump intensity, λ_t, and the representation of it is provided as follows:*

$$\begin{aligned} skp_t &= \beta_{sp,c} + \beta_{sp,q} q_t + \beta_{sp,\lambda}\lambda_t, \\ &where \\ \beta_{sp,c} &\equiv \Big[l_x B_r^3(2){\psi_x}^{(3)}(-\Lambda_x) + l_{\sigma^2} B_r^3(3){\psi_{\sigma^2}}^{(3)}(-\Lambda_{\sigma^2})\Big]\mu_\lambda, \\ \beta_{sp,q} &\equiv \Big[l_x B_r^3(2){\psi_x}^{(3)}(-\Lambda_x) + l_{\sigma^2} B_r^3(3){\psi_{\sigma^2}}^{(3)}(-\Lambda_{\sigma^2})\Big]\varphi_u(-\rho\varphi_\xi\Lambda_q - \varphi_u\Lambda_\lambda), \\ \beta_{sp,\lambda} &\equiv \Big[l_x B_r^3(2){\psi_x}^{(3)}(-\Lambda_x) + l_{\sigma^2} B_r^3(3){\psi_{\sigma^2}}^{(3)}(-\Lambda_{\sigma^2})\Big]\rho_\lambda. \end{aligned} \tag{2.25}$$

Proof *Considering (2.6), (2.16), and the definition of the moment-generating function, it is trivial to derive the above expression.* □

From the above proposion, we find that any temporal variation in endogenously generated skewness risk premium is also solely due to the variance-of-variance q_t and the jump intensity λ_t as well as the volatility risk premium. Moreover, provided that $\Lambda_x > 0$ and $\Lambda_{\sigma^2} < 0$, the factor loading to the jump intensity, that is, $\beta_{sp,\lambda}$, is guaranteed to be negative, but that to the variance-of-variance, that is, $\beta_{sp,q}$, can be both positive and negative in general. However, if the correlation between the dynamics of the variance-of-variance and that of the jump intensity, that is , ρ, is positive, then $\beta_{sp,q}$ is also guaranteed to be negative due to the facts that $\Lambda_q < 0$ and $\Lambda_\lambda < 0$.

Before we turn to the next discussion, it will be useful to mention about some features of the higher-order moments of the market return and the risk premiums in them.

First, as mentioned in the introduction in this chapter, the usual assumption of an affine structure on the jump intensity process λ_t, that is, $\lambda_t = l_0 + l_1\sigma_t^2$ where $l_0, l_1 > 0$ and σ_t^2 is the variance of consumption growth rate, in the previous studies such as Drechsler and Yaron[2011] can not explain an empirical fact on a simultaneous relation between monthly stock returns and monthly changes of the option-implied skewness shown by (2.1). It is because, under such assumption, we can show analytically that the regression parameters to $\triangle ISkew_{t+1}$ in (2.1) should be positive. However, based on our model provided above, the correlation between the one-step-ahead market return, $r_{m,t+1}$, and the one-step-ahead change of risk-neutral skewness, $\triangle Skew^{\mathbb{Q}}_{t+1} \equiv Skew^{\mathbb{Q}}_{t+1} - Skew^{\mathbb{Q}}_{t}$, at time t can be derived with (2.14) and (2.22) as follows:

$$\begin{aligned}\text{Corr}\Big(r_{m,t+1}, \triangle Skew^{\mathbb{Q}}_{t+1}\Big) &= K\varphi_u \kappa_{1,m}(\rho A_{q,m} + \varphi_u A_{\lambda,m})q_t, \\ \text{where} \quad K &\equiv l_x B_r^3(2)\psi_x{}^{(3)}(-\Lambda_x) + l_{\sigma^2} B_r^3(3)\psi_{\sigma^2}{}^{(3)}(-\Lambda_{\sigma^2}).\end{aligned}$$

From the above expression, we can show that, when $\rho < -\varphi_u \frac{A_{\lambda,m}}{A_{q,m}}$, the correlation between $r_{m,t+1}$ and $\triangle Skew^{\mathbb{Q}}_{t+1} \equiv Skew^{\mathbb{Q}}_{t+1} - Skew^{\mathbb{Q}}_{t}$ should be negative because, in the case of $\gamma > 1$ and $\psi > 1$, it is proven that K is negative . This observation is consistent with the empirical fact of (2.1) shown in the introduction of this chapter. Thus, we would like to emphasize that there is considerable validity in our model setting with the stochastic jump intensity compared with the previous studies such as Drechsler and Yaron[2011], etc.

Second, although both the variance risk premium and the skewness risk premium are linear to the variance-of-variance q_t and the jump intensity λ_t, we can show that they are mutually independent because of the fact that $det \equiv \beta_{vp,q}\beta_{sp,\lambda} - \beta_{vp,\lambda}\beta_{sp,q}$ is not zero, which will be proven in Section 3 with a model calibration result.

2.2.4 An Equity Risk Premium Representation

In this subsection, let us show an equity risk premium representation with the variance and skewness risk premiums in equilibrium. In the beginning, we start with an expression for the equity risk premium provided by Drechsler and Yaron[2011] as follows:

$$\log \mathbb{E}_t(R_{m,t+1}) - r_{f,t} = B_r^t G_t G_t^t \Lambda + \Pi_t^t(\psi(B_r) - 1 - \psi(B_r - \Lambda) + \psi(-\Lambda)),$$

where

$$\psi(B_r) \equiv \begin{pmatrix} 0 \\ \psi_x(B_r(2)) \\ \psi_{\sigma^2}(B_r(3)) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \psi(B_r - \Lambda) \equiv \begin{pmatrix} 0 \\ \psi_x(B_r(2) - \Lambda_x) \\ \psi_{\sigma^2}(B_r(3) - \Lambda_{\sigma^2}) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \psi(-\Lambda) \equiv \begin{pmatrix} 0 \\ \psi_x(-\Lambda_x) \\ \psi_{\sigma^2}(-\Lambda_{\sigma^2}) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

As mentioned in Drechsler and Yaron[2011], the first term, $B_r^t G_t G_t^t \Lambda$, represents the contributions of the Gaussian shocks to the equity risk premium. In particular, according to the expression of $G_t G_t^t = H_{\sigma^2}\sigma_t^2 + H_q q_t$ (see (2.20)), this term aggregates both the risk-return tradeoff relationship and a true premium for variance risk. The next terms, $\Pi_t^t(\psi(B_r) - 1 - \psi(B_r - \Lambda) + \psi(-\Lambda))$, represents the contributions from the jump processes. The derivation of this expression is presented in the Appendix in Drechsler and Yaron[2011].

The $r_{f,t}$ is the risk-free rate at time t in the economy and the explicit expression of this $r_{f,t}$ is provided in the Appendix.

With the expression of $G_t G_t^t = H_{\sigma^2}\sigma_t^2 + H_q q_t$ and $\Pi_t \equiv \begin{pmatrix} 0 & \lambda_{x,t} & \lambda_{\sigma^2,t} & 0 & 0 & 0 \end{pmatrix}^t$, the following representation can be obtained via the expression for the equity risk premium shown above:

$$\begin{aligned}
\log \mathbb{E}_t(R_{m,t+1}) - r_{f,t} &= \beta_{er,\sigma}\sigma_t^2 + \beta_{er,q}q_t + \beta_{er,\lambda}\lambda_t, \\
\textit{where} \quad & \beta_{er,\sigma} \equiv B_r^t H_\sigma \Lambda, \\
& \beta_{er,q} \equiv B_r^t H_q \Lambda, \\
& \beta_{er,\lambda} \equiv l_x \Big[\psi_x(B_r(2)) - 1 - \psi_x(B_r(2) - \Lambda_x) + \psi_x(-\Lambda_x)\Big] \\
& \qquad + l_\sigma \Big[\psi_{\sigma^2}(B_r(3)) - 1 - \psi_\sigma(B_r(3) - \Lambda_{\sigma^2}) + \psi_\sigma(-\Lambda_{\sigma^2})\Big].
\end{aligned} \tag{2.26}$$

As shown in (2.26), the equity risk premium is driven by the state variables of σ_t^2, q_t, and λ_t and have a time-varying nature essentially because each of those variables has the stochastic nature. In particular, in the case of $\gamma > 1$ and $\psi > 1$, it is proven that $\beta_{er,\sigma} > 0$, $\beta_{er,q} > 0$, and $\beta_{er,\lambda} > 0$ because of the facts that $\Lambda_x > 0$, $\Lambda_{\sigma^2} < 0$, $\Lambda_q < 0$, and $\Lambda_\lambda < 0$, which are provided in Proposition 1, so that if each of the state variables increases, then the equity risk premium also increases, and vice versa.

The conditional variance of the equity return at time t, σ_t^2, is also expressed by

$$\begin{aligned}
\sigma_t^2 = \mathrm{Var}_t^{\mathbb{P}}(r_{m,t+1}) &= B_r^t G_t G_t^t B_r + B_r^{2^t} \mathrm{diag}\Big(\psi^{(2)}(0)\Big)\Pi_\mathrm{t} \\
&= B_r^t H_{\sigma^2} B_r \sigma_t^2 + B_r^t H_q B_r q_t + (l_x B_r^2(2)\psi_x^{(2)}(0) + l_{\sigma^2} B_r^2(3)\psi_{\sigma^2}^{(2)}(0))\lambda_t \\
&\equiv \beta_{var,\sigma}\sigma_t^2 + \beta_{var,q}q_t + \beta_{var,\lambda}\lambda_t,
\end{aligned}$$

so that with (2.21), (2.25), (2.26), and the above expression for the conditional variance of the equity return we can derive an explicit equity risk premium representation of a linear factor pricing model with the variance and skewness risk premiums and the conditional variance of the equity return.

Proposition 4 (An Explicit Representation for the Equity Risk Premium)

$$\log \mathbb{E}_t(R_{m,t+1}) - r_{f,t} = \pi_c + \pi_{var}\mathrm{Var}_t^{\mathbb{P}}(r_{m,t+1}) + \pi_{vp}vp_t + \pi_{sp}skp_t, \tag{2.27}$$

where

$$\pi_c \equiv \Big(- \frac{\beta_{er,\sigma}\beta_{var,q}}{\beta_{var,\sigma}} + \beta_{er,q}\Big)\frac{-\beta_{sp,\lambda}\beta_{vp,c} + \beta_{vp,\lambda}\beta_{sp,c}}{det} + \Big(- \frac{\beta_{er,\sigma}\beta_{var,\lambda}}{\beta_{var,\sigma}} + \beta_{er,\lambda}\Big)\frac{\beta_{sp,q}\beta_{vp,c} - \beta_{vp,q}\beta_{sp,c}}{det},$$

$$\pi_{var} \equiv \frac{\beta_{er,\sigma}}{\beta_{var,\sigma}},$$

$$\pi_{vp} \equiv \Big(- \frac{\beta_{er,\sigma}\beta_{var,q}}{\beta_{var,\sigma}} + \beta_{er,q}\Big)\frac{\beta_{sp,\lambda}}{det} - \Big(- \frac{\beta_{er,\sigma}\beta_{var,\lambda}}{\beta_{var,\sigma}} + \beta_{er,\lambda}\Big)\frac{\beta_{sp,q}}{det},$$

$$\pi_{sp} \equiv -\Big(- \frac{\beta_{er,\sigma}\beta_{var,q}}{\beta_{var,\sigma}} + \beta_{er,q}\Big)\frac{\beta_{vp,\lambda}}{det} + \Big(- \frac{\beta_{er,\sigma}\beta_{var,\lambda}}{\beta_{var,\sigma}} + \beta_{er,\lambda}\Big)\frac{\beta_{vp,q}}{det},$$

$$det \equiv \beta_{vp,q}\beta_{sp,\lambda} - \beta_{vp,\lambda}\beta_{sp,q}.$$

This representation of (2.27) suggests that the skewness risk premium, as well as the variance risk premium and the conditional variance of the market return, constitutes the dominant source of the variation in the equity risk premium. In the following section, we can show that *det* in (2.27) is not zero under the suitable parameter condition, so that the skewness risk premium has an essential source of the variation in the equity risk premium, which is different from that of the variance risk premium (see Fig.2.1).

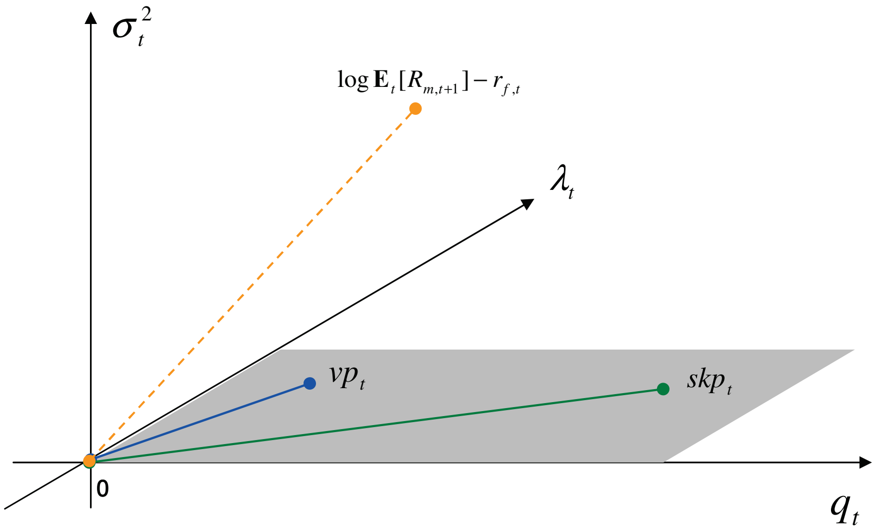

Fig. 2.1: The Risk Premiums in Higher-Order Moments and the Equity Risk Premium

Some recent studies such as those by Bali and Hovakimian[2009], Yan[2009], Chang et al.[2012], Driessen et al.[2012], and Rehman and Vilkov[2012] focus on a significant relationship between skewness or jump risks and expected stock returns, and they provide empirical evidence for a significantly positive link between the expected stock returns and the jump or skewness risks. To the best of our knowledge, this result of (2.27), which suggests an explicit relationship between the skewness risk premium and the expected equity excess return, is the first to provide a theoretical implication in their empirical evidence in terms of the LRR model approach pioneered by Bansal and Yaron[2004].

2.3 Model Implications

Before proceeding to an empirical analysis based on the representation of (2.27), we show the implications from a calibrated version of the theoretical model (2.27) to help guide and interpret our subsequent empirical reduced form predictability regressions.

Table 2.1: The Set of Model Parameters

Parameter	Source	(Calibrated) Values
(1) Preference		
ψ	BST	2.5
(2) Consumption Growth		
φ_η	BY	1.0
(3) Long Run Risk		
ρ_x	BY	0.979
φ_e	BY	0.044
(4) Variance		
ρ_σ	BTZ	0.978
(5) Variance-of-Variance		
ρ_q	BTZ	0.8
φ_ξ	BTZ	0.001
(6) Campbell-Shiller		
κ_1	BTZ	0.9
(7) Jump Intensity		
ρ_λ	CM	0.9
φ_u	-	0.01
$\delta_x, \delta_{\sigma^2}$	-	0.01
μ_λ	-	1.0
l_x, l_{σ^2}	-	1.0

This table reports the parameter values used in the calibration of the factor loadings in the theoretical model (2.27). CM, BY, BTZ, and BST in this table denote values taken directly from Chan and Maheu[2002], Bansal and Yaron[2004], Bollerslev et al.[2009], and Bollerslev et al.[2012], respectively.

Table 2.1 reports the parameter values used in the calibration of the factor loadings in the theoretical model (2.27). CM, BY, BTZ, and BST in this table denote values taken directly from Chan and Maheu[2002], Bansal and Yaron[2004], Bollerslev et al.[2009], and Bollerslev et al.[2012], respectively. Those previous studies refer to the unit time interval in the calibrated equilibrium models as a month, and we also refer to the unit time as the same. On the basis of the parameters exhibited in Table 2.1, we calibrate the factor loadings for the variance risk premium, which appear in the representation of (2.21), and for the skewness risk premium, which appear in the representation of (2.25), in equilibrium.

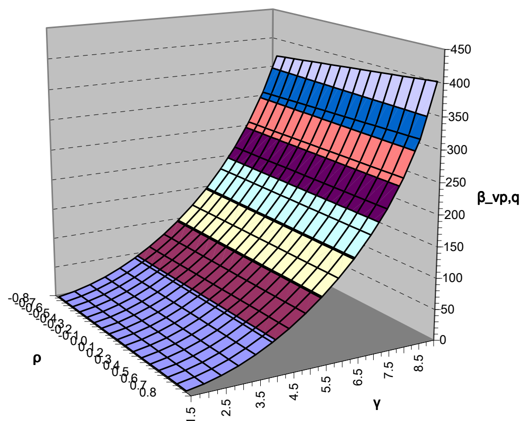

Fig. 2.2: The Factor Loading $\beta_{vp,q}$

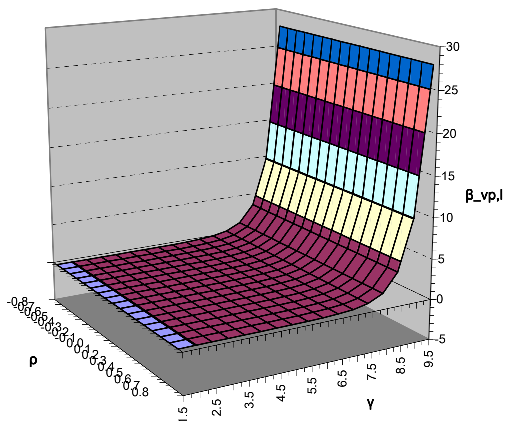

Fig. 2.3: The Factor Loading $\beta_{vp,\lambda}$

The figures from Fig.2.2 to Fig.2.5 show the factor loadings, $\beta_{vp,q}$, $\beta_{vp,\lambda}$, $\beta_{sp,q}$, and $\beta_{sp,\lambda}$, corresponding to the parameters of the risk aversion parameters γ and the correlation ρ between the volatility of volatility q_t and the jump intensity λ_t. As is shown in the previous section, $\beta_{vp,\lambda}$, which is the factor loading to the jump intensity λ_t in the variance risk premium representation (2.21), is essentially positive, and under the parameter values exhibited in Table 2.1, $\beta_{vp,q}$, which is the factor loading to the variance-of-variance q_t in (2.21), also seems to be positive. These results indicate that when the variance-of-variance and (or) the jump intensity rise(s), the level of the variance risk premium also increases. In contrast, $\beta_{sp,\lambda}$, which is the factor loading to the jump intensity in the skewness risk premium representation (2.25), is essentially negative and this result is consistent with the discussion explored in the previous section. However, interestingly, $\beta_{sp,q}$, which is the factor loading to the variance-of-variance in (2.25), can be both positive and negative corresponding to the parameters of γ and ρ. These results on the $\beta_{sp,\lambda}$

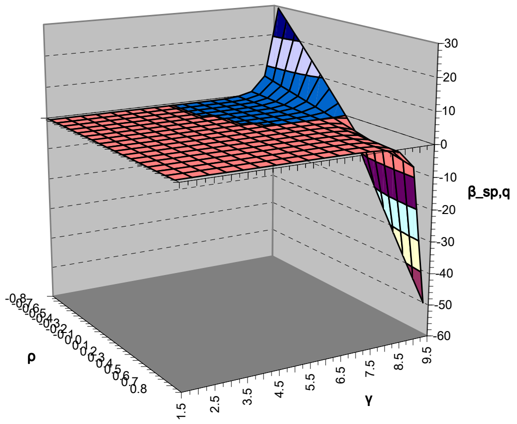

Fig. 2.4: The Factor Loading $\beta_{sp,q}$

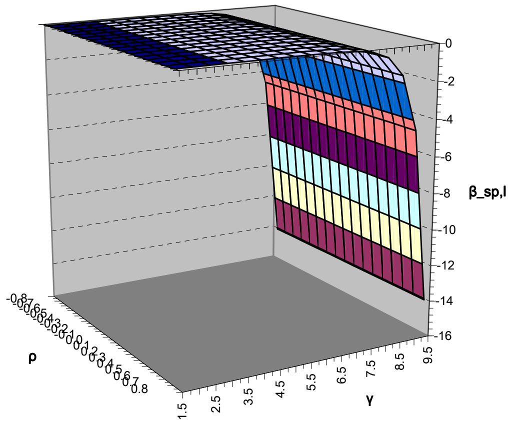

Fig. 2.5: The Factor Loading $\beta_{sp,\lambda}$

and the $\beta_{sp,q}$ indicate that although an increase in the jump intensity reduces the level of the skewness risk premium essentially, but an increase in the variance-of-variance will raise or reduce the level of the skewness risk premium corresponding to the values of γ and ρ.

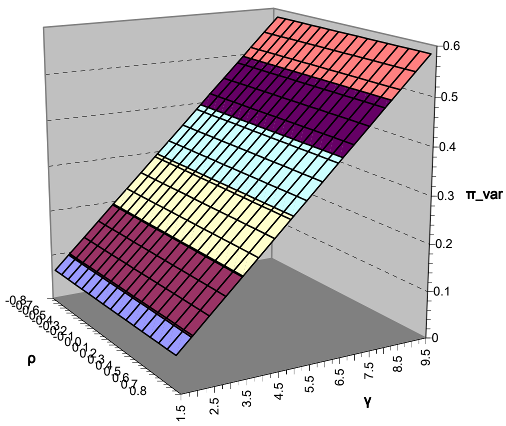

Fig. 2.6: The Factor Loading to $Var_t^{\mathbb{P}}$:π_{Var}

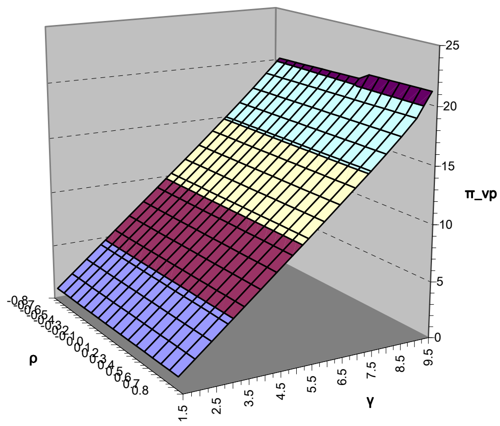

Fig. 2.7: The Factor Loading to vp_t:π_{vp}

The figures from Fig.2.6 to Fig.2.8 show the factor loadings to the variance of the market return, the variance risk premium, and the skewness risk premium in the equity risk premium representation (2.27). It is interesting that both of the π_{Var} and π_{vp} are essentially positive and these results are irrelevant to the values of γ and ρ. Moreover, these results are consistent with the previous studies such as those by Bollerslev et

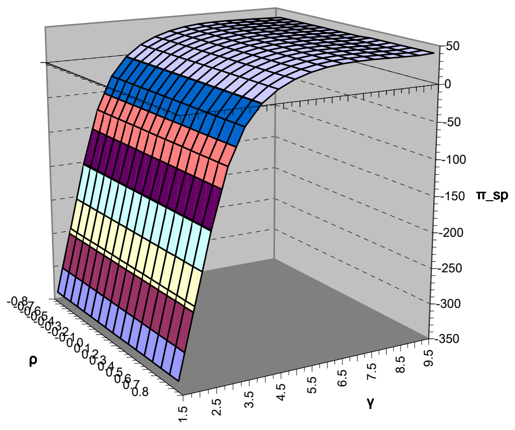

Fig. 2.8: The Factor Loading to skp_t:π_{skp}

al.[2009] and Drechsler and Yaron[2012]. An important point to emphasize is that the factor loading of π_{skp}, which is the loading to the skewness risk premium in (2.27), can be positive corresponding to the risk aversion parameter γ. In particular, when the γ is over 4, it is clear from Fig.2.8 that the π_{skp} is strictly positive. This result indicates that a decrease in the skewness risk premium, which is the case that the risk-neutral skewness is going to be much smaller than the skewness under the physical measure, reduces the equity risk premium when γ is over 4. This implication is interesting as it shows the essential contribution of the skewness risk premium to the equity risk premium explicitly implying the sign of the π_{skp} corresponding to the values of γ and ρ. As mentioned above, some recent studies such as those by Bali and Hovakimian[2009], Yan[2011], Chang et al.[2012], Driessen et al.[2012], and Rehman and Vilkov[2012] focus on a significant relationship between skewness or jump risks and expected stock returns, and they provide empirical evidence for a significantly positive link between the expected stock returns and the jump or skewness risks. In particular, Bali and Hovakimian[2009] and Yan[2011] provide evidence for a significantly positive link between expected returns and the call-put options' implied volatility spread that can be considered as a proxy for jump risk. Moreover, using data on individual stock options, Rehman and Vilkov[2012] show that the currently observed option-implied ex ante skewness is positively related to future stock returns. There has been no study that tried to provide the theoretical equilibrium model which is consistent with the empirical results cited above. To the best of our knowledge, this is the first paper that demonstrates what mentioned above with a stylized model that accounts for a close relationship between the skewness risk premium and the equity risk premium.

2.4 Empirical Measurements

The theoretical model outlined in the previous section suggests that the variance and skewness risk premiums, as well as the variance of the market return, may serve as useful predictor variables for the future market returns. To examine this suggestion empirically, we plan for running some statistical tests based on simple linear regressions of the S & P500 excess returns on different sets of lagged predictor variables including the variance and skewness risk premiums. We always rely on monthly and quarterly observations and focus our discussion on the estimated slope coefficients and their statistical significance as determined by the t-statistics. We also report the forecasts' accuracy of the regressions as measured by the corresponding adjusted R^2s.

Before showing the results of the predictive regressions of the S & P500 excess returns, let us note some key points on the measurements for the variance and skewness risk premiums and describe the data used in our analysis explored in the following subsection.

2.4.1 Measurements for the Higher-Order Moments

Our method for measuring the risk premiums in higher-order moments is similar to that in Bollerslev et al.[2009] and Drechsler and Yaron[2011]. As mentioned above, we formally define the variance risk premium as the difference between the risk-neutral and physical expectations of the variance of the market return and also define the skewness risk premium in the same manner. We focus on the one-month- and three-month-forward predictability of those risk premiums and use the squared VIX and the SKEW index from the Chicago Board of Options Exchange (CBOE) as our measures for the risk-neutral expected variance and skewness, respectively. The VIX is calculated by the CBOE using the model-free approach to measure 30-day expected volatility of the S & P500 return. The components of the VIX are near- and next-term put and call options, usually in the first and second SPX (S & P500 index) contract months. The model-free approach used to calculate the VIX is provided by, for example, Demeterfi et al.[1999]. The SKEW index from the CBOE is also calculated from the S & P 500 option prices based on the method similar to that used to calculate the VIX, which is obtained by a portfolio of S & P 500 index options that mimics an exposure to the skewness payoff of one-step-ahead cumulative return distribution of the index. The Skew index is also calculated by the model-free approach provided, for example, Bakshi et al.[2003]. [5]

For the measures of the expected variance and skewness under the physical measure, we use the current variance and skewness of the S & P500 index return, which are respectively defined as the historical 22 days actual variance estimated based on daily return data of the index and the historical 12 months actual skewness estimated based on monthly return data of the index. To match the definition of those historical moments of the index return distribution with the risk-neutral expected moments mentioned above, we use the annualized current variance, while the current skewness, which is estimated based on historical 12 months monthly return data, is used as it is. Bollerslev et al.[2009] suggest that, for highly persistent variance dynamics, or $\rho_\sigma \approx 1$, the objective expected future variance will obviously be close to the value of the current variance so that the same qualitative implications hold true for the variance difference obtained by replacing $\mathbb{E}_t^{\mathbb{P}}[Var_{t+1}^{\mathbb{P}}(r_{m,t+2})]$ in Equation (2.18) with the current variance. In a similar point of view, the same will be said for the objective expected future skewness. Moreover, compared to the variance and skewness risk premiums defined by (2.18) and (2.23), respectively, the usage of the historical return moments in order to substitute for the objective expected future moments has the advantage that those risk premiums are directly observable at time t. This is obviously important from a forecasting perspective. It is for these reasons mentioned above that we use the current variance and skewness of the index return to measure the expected variance and skewness under the physical measure.

2.4.2 Data Description

Our data series for the VIX, SKEW index, and expected variance and skewness under $\mathbb{P}$ covers the period from January 1990 to August 2012. The main limitation on the length of our sample comes from the VIX and SKEW index, since the time series published by the CBOE begins in January 1990. As mentioned in the previous subsection, we rely on the monthly and quarterly data for the squared VIX and SKEW index for quantifying $\mathbb{E}_t^{\mathbb{Q}}[Var_{t+1}^{\mathbb{Q}}(r_{m,t+2})]$ in (2.18) and $\mathbb{E}_t^{\mathbb{Q}}[Skew_{t+1}^{\mathbb{Q}}(r_{m,t+2})]$ in (2.23), respectively, and purposely rely on the readily available squared VIX as our measure for the risk-neutral expected variance and the value of $\frac{1}{10}(100 - SKEW\ \ index)$ as our measure for the risk-neutral expected skewness. The expected variance $\mathbb{E}_t^{\mathbb{P}}[Var_{t+1}^{\mathbb{P}}(r_{m,t+2})]$ and the expected skewness $\mathbb{E}_t^{\mathbb{P}}[Skew_{t+1}^{\mathbb{P}}(r_{m,t+2})]$ at time t are respectively calculated based on the historical

[5] According to the description of the CBOE's SKEW index, we have the proxy for the risk-neutral expected skewness, $\mathbb{E}_t^{\mathbb{Q}}[Skew_{t+1}^{\mathbb{Q}}(r_{m,t+2})]$, as $\frac{1}{10}(100 - SKEW\ \ index)$.

index returns as described in the previous subsection.

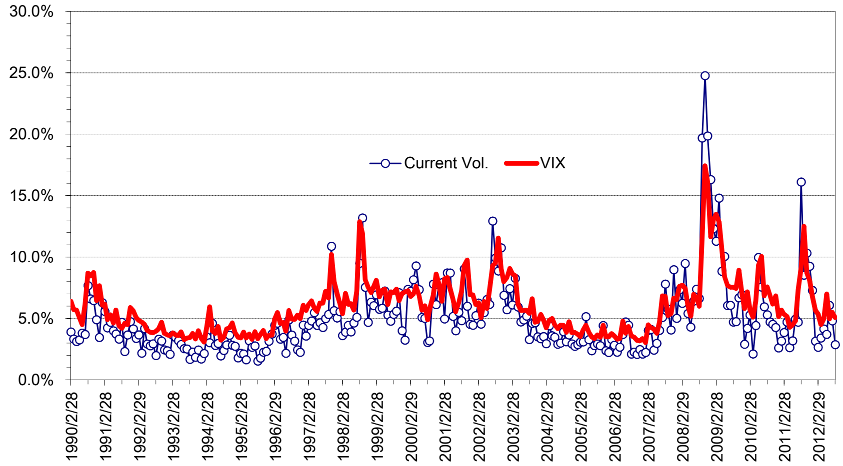

Fig. 2.9: The VIX and The Current Volatility

This figure shows the time-series data of the VIX and the current volatility (the square root of the current variance defined in the main paper). The current volatility is the historical 22 days actual volatility estimated based on daily return data of the S & P500 index.

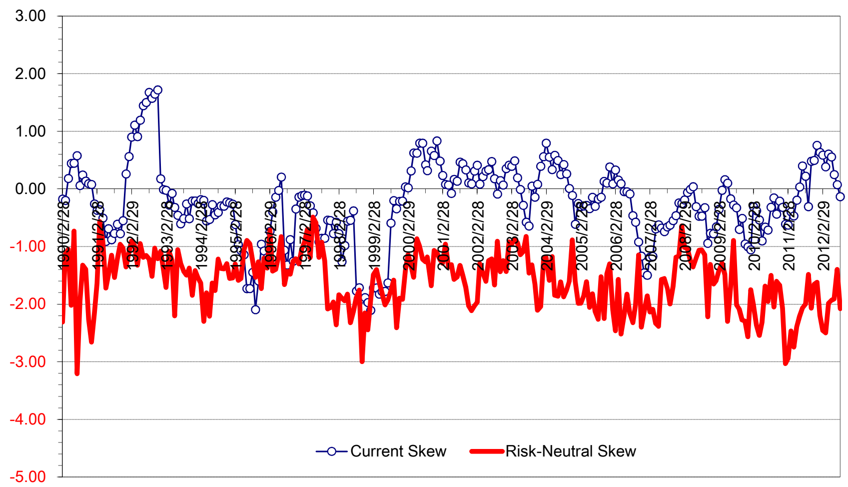

Fig. 2.10: The Risk-Neutral Skewness and The Current Skewness

This figure shows the time-series data of the risk-neutral expected skewness extracted from the SKEW index and the current skewness. The current skewness is the historical 12 months actual skewness estimated based on monthly return data of the S & P500 index.

To illustrate the data, Fig.2.9 and Fig.2.10 plot the monthly time-series of the risk-neutral expected volatility (VIX), the current volatility (historical 22 days annualized actual volatility), the risk-neutral expected skewness, and the current skewness (historical 12 months actual skewness). Consistent with the theoretical model developed in the previous section and the earlier empirical evidence, the spread between the risk-neutral expected variance (the squared VIX) and the current variance is almost always positive and the spread between the risk-neutral expected skewness and the current skewness is almost always negative. Moreover, it is clear that these spreads have a time-varying nature. It is interesting that, although the value of the VIX reaches an outstanding peak at the period of the Rehman crisis in 2008, the risk-neutral skewness seems to be more negative at the period of the European financial crisis in 2011 than at the period of the Rehman crisis.

In addition to the variance and skewness risk premiums, we also consider a set of other more traditional predictor variables for the predictive regressions examined in the following subsection. Specifically, we obtain monthly P/E ratios and dividend yields for the S & P 500 directly from Standard & Poor's. Data on the three-month T-bill, the high-yield spread (hys) (between Moody's BAA and AAA corporate bond spreads), and the term spread (ts) (between the ten-year T-bond and the three-month T-bill yields) are taken from the Thomson Reuters Data Stream. The CAY, which represents the aggregate consumption-wealth ratio, defined in Lettau and Ludvigson[2001] is downloaded from Lettau and Ludvigson's Web site.

Basic summary statistics for the monthly excess returns of the S & P500 index and predictor variables are exhibited in Table 2.2. The sample period extends from January 1990 to August 2012. All variables are reported in monthly-based percentage form whenever appropriate. The $r_{m,t} - r_{f,t}$ denotes the logarithmic return on the S & P 500 index in excess of the three-month T-bill rate. VIX^2 denotes the squared VIX index. ISKew refers to the risk-neutral expected skewness extracted from the CBOE SKEW index by the formula of $ISkew = \frac{1}{10}(100 - \text{Skew index})$. CVar and CSkew refer to the current variance, which is the annualized actual variance based on historical 22 days daily return data, and the current skewness, which is the actual skewness based on historical 12 months monthly return data, respectively. vp and skp respectively refer to the variance and skewness risk premiums, that is, $vp \equiv VIX^2 - CVar$ and $skp \equiv ISkew - CSkew$. The predictor variables include the log price-earning ratio ln(pe), the log dividend yield ln(dy), the high yield spread (hys) defined as the difference between Moody's BAA and AAA bond yield indices, and the term spread (ts) defined as the difference between the ten-year and three-month Treasury yields.

Table 2.2: Summary statistics for the monthly returns and predictor variables

	$r_{m,t} - r_{f,t}$	VIX^2	ISkew	CVar	CSkew	vp	skp	ln(pe)	ln(dy)	hys	ts
(A) Summary Statistics											
(1) Mean	0.3 %	6.0 %	-1.6	5.0 %	-0.2	1.0 %	-1.4	3.1	0.7	1.0 %	1.9 %
(2) Std. Dev.	4.4 %	2.3 %	0.5	3.1 %	0.7	1.5 %	0.8	0.4	0.3	0.4 %	1.2 %
(3) Skewness	-0.6	1.6	-0.4	2.7	-0.2	-2.2	0.2	2.3	0.1	3.2	-0.2
(4) Kurtosis	1.1	3.9	-0.2	10.7	0.7	9.9	0.0	7.1	-0.6	12.9	-1.1
(5) AR(1)	0.07	0.85	0.56	0.75	0.91	0.22	0.74	0.95	0.99	0.96	0.97
(B) Correlation Matrix											
$r_{m,t} - r_{f,t}$	1										
VIX^2	0.02	1									
ISkew	0.06	-0.00	1								
CVar	-0.11	0.87	0.03	1							
CSkew	-0.10	-0.06	0.19	-0.03	1						
vp	0.25	-0.25	-0.07	-0.69	-0.03	1					
skp	0.13	0.05	0.48	0.04	-0.77	-0.01	1				
ln(pe)	0.01	0.25	0.00	0.15	0.00	0.08	0.00	1			
ln(dy)	0.08	-0.14	0.11	-0.10	0.09	-0.02	-0.01	-0.35	1		
hys	-0.05	0.64	-0.04	0.67	0.07	-0.38	-0.09	0.16	0.21	1	
ts	-0.02	0.06	0.08	0.05	0.33	-0.02	-0.24	0.28	0.38	0.26	1

The sample period extends from January 1990 to August 2012. All variables are reported in monthly-based percentage form whenever appropriate. The $r_{m,t} - r_{f,t}$ denotes the logarithmic return on the S & P 500 in excess of the three-month T-bill rate. VIX^2 denotes the squared VIX index. ISKew refers to the risk-neutral expected skewness extracted from the CBOE SKEW index by the formula of $ISkew = \frac{1}{10}(100 - \text{Skew index})$. CVar and CSkew refer to the current variance, which is the annualized actual variance based on historical 22 days daily return data, and the current skewness, which is the actual skewness based on historical 12 months monthly return data, respectively. vp and skp respectively refer to the variance and skewness risk premiums, that is, $vp \equiv VIX^2 - CVar$ and $skp \equiv ISkew - CSkew$. The predictor variables include the log price-earning ratio ln(pe), the log dividend yield ln(dy), the high yield spread (hys) defined as the difference between Moody's BAA and AAA bond yield indices, and the term spread (ts) defined as the difference between the ten-year and three-month Treasury yields.

The mean excess return on the S & P 500 index over the sample equals 0.3 % monthly. The sample means for the VIX^2 and the current (historical 22 days) annualized variance are 6.0 % and 5.0 %, respectively, and the sample means for the risk-neutral expected skewness and the current (historical 12 months) skewness are -1.6 and -0.2, respectively. The numbers for the traditional forecasting variables are all directly in line with those reported in previous studies. In particular, all of the variables are highly persistent with first-order autocorrelations ranging from 0.95 to 0.99.

2.4.3 Main Empirical Findings

Table 2.3 provides the results of return predictability regressions with the variance and skewness risk premiums. All of our forecasts are based on simple linear regressions of the S & P500 excess returns on different sets of lagged predictor variables. There are two sets of columns with regression estimates. The first set of columns shows OLS estimates by monthly return regressions, that is, one-month-ahead forecasts, and the second set shows OLS estimates by non-overlapped quarterly return regressions, that is, one-quarter-ahead forecasts. These regressions are examined in the period from January 1990 to August 2012 and, in particular, each of the monthly return regressions is examined by 270-month samples and each of the quarterly return regressions is examined by 88-quarter samples. Each of the sets of columns consists of five regression results. The first two regressions are one-factor regression models using the variance risk premium (vp-model) or the skewness risk premium (skp-model) as a univariate regressor, while the third regression is two-factor regression model using both the variance and skewness risk premiums (vp+skp-model). The fourth regression model, which is denoted by 3-factor-model, represents the theoretical linear model of (2.27) derived in the previous section. Finally, we also provide the stepwise-selection model (Stepwise-model) of which the universe of independent variables consists of the risk premiums in higher-order moments, changes of those risk premiums, and one of the traditional predictor variables, that is, the log price-earning ratio ln(pe). The variables such as $\triangle VIX^2$ and $\triangle ISkew$ exhibited in this table are monthly or quarterly changes of the VIX^2 and $ISkew$, respectively.

From the monthly return regression results in this table, we can find that the slope coefficients of the vp- and skp-model are both significant at 5 % level and, in particular, the slope coefficient of the vp-model is significant at 1 % level. Moreover, the slope coefficients of the vp+skp-model are also significant at the same level as mentioned above and this model can account for about 7.2 % of the monthly return variation. The 3-factor-model represents the theoretical implication of (2.27) and this model has a superior predictive power in the adjusted R^2 than the vp+skp-model due to the additional variable of the current variance (CVar). Although the stepwise-model is not equivalent to the theoretical implication of (2.27), that is, the 3-factor-model, all of the independent variables of CVar, vp, and skp in (2.27) are significant at 5 % or 1 % level. These results indicate that the theoretical model of (2.27) and, in particular, the variance and skewness risk premiums have superior predictive power for future aggregate stock market index returns and this indication is consistent with the theory provided in the previous section in this chapter.

The quarterly regressions reported in this table further underscore the significance of the monthly return regressions and, in contrast to the monthly return regressions, all of the t-statistics for the skewness risk premium are insignificant at conventional levels. However, interestingly, we can find that the stepwise-model is perfectly equivalent to the theoretical implication of (2.27), that is, the 3-factor-model, and this model can account for about 14.7 % of the quarterly return variation. Although the slope coefficient to the skewness risk premium is not significant as mentioned above, the coefficients to the variance risk premium and the current variance are both significant at 5 % level and, in particular, at 1 % level for the variance risk premium.

Table 2.3: The Monthly and Quarterly Return Regressions

	(A) Monthly Return Regression Models					(B) Quarterly Return Regression Models				
	vp	skp	vp+skp	3-factor	Stepwise	vp	skp	vp+skp	3-facor	Stepwise
Constant	-0.004	0.013	0.006	-0.004	-0.015	-0.007	0.027	0.016	-0.042	-0.042
(t-stat)	-1.18	2.43**	1.18	-0.42	-1.57	-0.68	1.48	0.91	-1.44	-1.44
VIX^2										
ISkew										
CVar				0.158	0.312				0.909	0.909
				1.37	2.54**				2.43**	2.43**
CSkew										
$\triangle$ VIX^2					-0.883					
					-3.35***					
$\triangle$ ISkew										
$\triangle$ CVar					0.369					
					2.11**					
$\triangle$ CSkew										
vp	0.699		0.704	0.921	1.416	1.529		1.646	2.845	2.845
	4.19***		4.25***	4.02***	4.85***	2.95***		3.17***	4.03***	4.03***
skp		0.007	0.007	0.007	0.008		0.013	0.019	0.018	0.018
		2.11**	2.22**	2.16**	2.42**		1.11	1.60	1.56	1.56
$\triangle$ vp										
$\triangle$ skp										
ln(pe)										
Adj.R^2	5.8 %	1.3 %	7.2 %	7.5 %	10.6 %	8.1 %	0.3 %	9.8 %	14.7 %	14.7 %

The sample period extends from January 1990 to August 2012. VIX^2 denotes the squared VIX index. ISKew refers to the risk-neutral expected skewness extracted from the CBOE SKEW index by the formula of $ISkew = \frac{1}{10}(100-\text{Skew index})$. CVar and CSkew refer to the current variance, which is the annualized actual variance based on historical 22 days daily return data, and the current skewness, which is the actual skewness based on historical 12 months monthly return data, respectively. vp and skp respectively refer to the variance and skewness risk premiums, that is, $vp \equiv VIX^2 - CVar$ and $skp \equiv ISkew - CSkew$. The variables such as $\triangle VIX^2$ and $\triangle ISkew$ exhibited in this table are monthly or quarterly changes of the VIX^2 and the ISkew, respectively. The predictor variables include the log price-earning ratio ln(pe).

Let us show the other results to emphasize the superiority of the skewness risk premium, as well as the variance risk premium, as a predictor variable for the equity excess return. Table 2.4 reports monthly- and quarterly-based predictive regression results for the S & P500 index excess return with each of the traditional predictor variables exhibited in this table, that is, the price-earning ratio (pe), dividend yield (dy), high-yield spread (hys), and term spread (ts) defined in the previous subsection and the changes of those variables. As shown in this table, we can find that, in the case of the monthly return regressions, none of the predictor variables are superior in the adjusted R^2 to the variance and skewness risk premiums (see Table 2.3). In the case of the quarterly return regressions in this table, it seems that only $\triangle$ln(pe) and $\triangle$hys have superior adjusted R^2 in comparison with the skewness risk premium, but, none of the variables are superior in the adjusted R^2 to the variance risk premium. (see Table 2.3)

Table 2.4: The Univariate Regressions with Traditional Predictor Variables

(A) Monthly Return Regressions								
	ln(pe)	$\triangle$ ln(pe)	ln(dy)	$\triangle$ ln(dy)	hys	$\triangle$ hys	ts	$\triangle$ ts
Slope Coeff.	0.001	-0.002	0.011	-0.090	-0.516	-1.822	-0.084	-0.513
p-Value (%)	86.7	93.1	22.4	15.1	40.0	38.4	71.2	56.6
Adj.R^2 (%)	-0.4	-0.4	0.2	0.4	-0.1	-0.1	-0.3	-0.3
(B) Quarterly Return Regressions								
	ln(pe)	$\triangle$ ln(pe)	ln(dy)	$\triangle$ ln(dy)	hys	$\triangle$ hys	ts	$\triangle$ ts
Slope Coeff.	0.008	0.059	0.038	-0.093	-1.118	-7.402	-0.132	-1.661
p-Value (%)	73.8	10.6	20.6	41.3	58.8	2.5**	86.4	31.6
Adj.R^2 (%)	-1.0	1.9	0.7	-0.4	-0.8	4.6	-1.1	0.0

The sample period extends from January 1990 to August 2012. We obtain monthly P/E ratios (pe) and dividend yields (dy) for the S & P 500 directly from Standard & Poor's. Data on the three-month T-bill, the high-yield spread (hys) (between Moody's BAA and AAA corporate bond spreads), and the term spread (ts) (between the ten-year T-bond and the three-month T-bill yields) are taken from the Thomson Reuters Data Stream.

Table 2.6 reports monthly- and quarterly-based predictive regression results for the S & P500 index excess return with the CAY, the aggregate-consumption wealth ratio defined in Lettau and Ludvigson[2001]. The CAY is quarterly-based data and downloaded from Lettau and Ludvigson's web site. The downloaded data covers January 1990 to January 2012. Table 2.5 shows summary statistics for the CAY as well as the variance and skewness risk premiums under the period from January 1990 to January 2012. For the monthly return regressions, we define a monthly CAY series from the most recent quarterly observation.

As shown in Table 2.6, we can find that the CAY does not seem to be superior predictor variable in comparison with the variance and skewness risk premiums. This result is similar to the results in Table 2.4 and also suggests that the skewness risk premium, as well as the variance risk premium, has superior predictive power for future aggregate stock market index returns.

Table 2.5: Summary statistics for the CAY

	$r_{m,t} - r_{f,t}$	cay	vp	skp
		(A) Summary Statistics		
(1) Mean	0.29 %	0.21 %	0.94 %	-132.51 %
(2) Std. Dev.	4.38 %	2.38 %	1.55 %	75.83 %
(3) Skewness	-0.59	-0.11	-2.27	0.18
(4) Kurtosis	1.12	-1.40	10.02	0.08
(5) AR(1)	0.08	0.98	0.23	0.75
		(B) Correlation Matrix		
$r_{m,t} - r_{f,t}$	1			
cay	0.09	1		
vp	0.24	0.17	1	
skp	0.14	0.26	0.01	1

The sample period extends from January 1990 to January 2012. The CAY is the aggregate-consumption wealth ratio defined in Lettau and Ludvigson[2001], which is quarterly-based data and downloaded from Lettau and Ludvigson's web site.

Table 2.6: The Univariate Regressions with the CAY

		(A) Monthly Return Regressions	
	cay	vp	skp
Slope Coeff.	0.157	0.683	0.008
p-Value (%)	17.2	0.0***	2.3**
Adj.R^2 (%)	0.3	5.5	1.6
		(B) Quarterly Return Regressions	
	cay	vp	skp
Slope Coeff.	0.418	1.618	0.016
p-Value (%)	27.4	0.2***	21.5
Adj.R^2 (%)	0.2	9.3	0.7

The sample period extends from January 1990 to January 2012. The CAY is the aggregate-consumption wealth ratio defined in Lettau and Ludvigson[2001], which is quarterly-based data and downloaded from Lettau and Ludvigson's web site. For the monthly return regressions, we define a monthly CAY series from the most recent quarterly observation.

2.5 Concluding Remarks

In this chapter, we investigate the skewness risk premium in the financial market under a general equilibrium setting. Extending the long-run risks (LRR) model proposed by

Bansal and Yaron[2004] by introducing a stochastic jump intensity for jumps in the LRR factor and the variance of consumption growth rate, we provide an explicit representation for the skewness risk premium, as well as the volatility risk premium, in equilibrium.

On the basis of the representation for the skewness risk premium, we propose a possible reason for the empirical fact of time-varying and negative risk-neutral skewness. Moreover, we also provide an equity risk premium representation of a linear factor pricing model with the variance and skewness risk premiums. The empirical results prove that the skewness risk premium, as well as the variance risk premium, has superior predictive power for future aggregate stock market index returns. Compared with the variance risk premium, the results show that the skewness risk premium plays an independent and essential role for predicting the market index returns.

Some recent studies such as those by Bali and Hovakimian[2009], Yan[2011], Chang et al.[2013], Driessen et al.[2012], and Rehman and Vilkov[2012] focus on a significant relationship between skewness or jump risks and expected stock returns and they provide empirical evidence for a significantly positive link between the expected stock returns and the jump or skewness risks. To the best of our knowledge, this study is the first to provide a theoretical implication in their empirical evidence in terms of the LRR model approach pioneered by Bansal and Yaron[2004]. It remains some challenges for future research on providing more explicit theoretical explanation for the results presented by the recent studies cited above with the theoretical implication shown in this chapter. And moreover, it also needs a detailed analysis on the reasons why the skewness and variance risks are priced differently and, in particular, independently of each other. Further insight into this aspect is left to further work.

Appendix 2.A Proof of Proposition 2

From the definition of the variance risk premium (2.18) and the expressions of the conditional variance of the market return $r_{m,t+2}$ at time $t+1$ under each of the probability measures, we can derive the following expression,

$$
\begin{aligned}
vp_t = & -B_r^t\Big[\Lambda_{\sigma^2}H_{\sigma^2} + \varphi_\xi(\varphi_\xi\Lambda_q + \rho\varphi_u\Lambda_\lambda)H_q\Big]B_r q_t \\
& + B_r^t H_{\sigma^2} B_r\Big[\mathbb{E}_t^{\mathbb{Q}}[J^{\mathbb{Q}}_{\sigma^2,t+1}] - \mathbb{E}_t^{\mathbb{P}}[J^{\mathbb{P}}_{\sigma^2,t+1}]\Big] \\
& + B_r^{2t}\Big[\mathrm{diag}(\psi^{(2)}(-\Lambda))\mathbb{E}_{\mathrm{t}}^{\mathbb{Q}}[\Pi_{\mathrm{t+1}}] - \mathrm{diag}(\psi^{(2)}(0))\mathbb{E}_{\mathrm{t}}^{\mathbb{P}}[\Pi_{\mathrm{t+1}}]\Big],
\end{aligned} \tag{2.28}
$$

where $\Lambda_q \equiv (1-\theta)\kappa_1 A_q$, $\Lambda_\lambda \equiv (1-\theta)\kappa_1 A_\lambda$ (See (2.13)), and

$$\mathrm{diag}\Big(\psi^{(2)}(-\Lambda)\Big) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \psi_x^{(2)}(-\Lambda_x) & 0 & 0 & 0 & 0 \\ 0 & 0 & \psi_{\sigma^2}^{(2)}(-\Lambda_{\sigma^2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbb{E}_t^{\mathbb{Q}}[\Pi_{t+1}] \equiv \Big(0 \quad \mathbb{E}_t^{\mathbb{Q}}[\lambda_{x,t+1}] \quad \mathbb{E}_t^{\mathbb{Q}}[\lambda_{\sigma^2,t+1}] \quad 0 \quad 0 \quad 0\Big)^t,$$

$$\mathbb{E}_t^{\mathbb{P}}[\Pi_{t+1}] \equiv \Big(0 \quad \mathbb{E}_t^{\mathbb{P}}[\lambda_{x,t+1}] \quad \mathbb{E}_t^{\mathbb{P}}[\lambda_{\sigma^2,t+1}] \quad 0 \quad 0 \quad 0\Big)^t,$$

Substituting the following facts,

$$\mathbb{E}_t^{\mathbb{Q}}[J_{\sigma^2,t+1}^{\mathbb{Q}}] = \lambda_{\sigma^2,t}\psi_{\sigma^2}^{(1)}(-\Lambda_{\sigma^2}),$$
$$\mathbb{E}_t^{\mathbb{P}}[J_{\sigma^2,t+1}^{\mathbb{P}}] = \lambda_{\sigma^2,t}\psi_{\sigma^2}^{(1)}(0),$$

into (2.28) and considering (2.6) and (2.16), we can obtain the representation (2.21). □

Appendix 2.B The Risk-Free Rate

The explicit expression of the risk-free rate can be obtained by substituting $r_{f,t}$ into $r_{j,t+1}$ in (2.3). We finally provide the following proposition on the risk-free rate $r_{f,t}$.

Proposition 5 (The Risk-Free Rate) *The risk free rate is expressed as follows with the state variables of σ_t^2, q_t, and λ_t.*

$$r_{f,t} = \beta_{rf,c} + \beta_{rf,x}x_t + \beta_{rf,\sigma}\sigma_t^2 + \beta_{rf,q}q_t + \beta_{rf,\lambda}\lambda_t,$$

where

$$\beta_{rf,c} \equiv -\theta\log\delta + \gamma\mu_g - (\theta-1)(\kappa_0 - A_0) - (\theta-1)\kappa_1(A_0 + A_\sigma\mu_\sigma + A_q\mu_q + A_\lambda\mu_\lambda),$$
$$\beta_{rf,x} \equiv \gamma - (\theta-1)A_x(\kappa_1\rho_x - 1),$$
$$\beta_{rf,\sigma} \equiv (1-\theta)A_\sigma(\kappa_1\rho_\sigma - 1) - \frac{1}{2}\Big[\gamma^2\varphi_\eta^2 + (\theta-1)^2\kappa_1^2A_x^2\varphi_e^2\Big],$$
$$\beta_{rf,q} \equiv (1-\theta)A_q(\kappa_1\rho_q - 1) - \frac{1}{2}(\theta-1)^2\kappa_1^2\Big[A_\sigma^2 + A_q^2\varphi_\xi^2 + 2A_qA_\lambda\varphi_\xi\varphi_u\rho + A_\lambda^2\varphi_u^2\Big],$$
$$\beta_{rf,\lambda} \equiv (1-\theta)A_\lambda(\kappa_1\rho_\lambda - 1) - l_x\Big[\exp(\frac{1}{2}(\theta-1)^2\kappa_1^2A_x^2) - 1\Big] - l_\sigma\Big[\exp(\frac{1}{2}(\theta-1)^2\kappa_1^2A_\sigma^2) - 1\Big].$$

Chapter 3

An Approach to the Option Market Model Based on End-user Net Demand

3.1 Introduction

We study financial option prices and their related topics in terms of option demand pressure. The starting point of our analysis is that options are traded because they are useful and therefore cannot be redundant for all investors. In the actual financial option market, we can classify market participants into two types of agents: market-makers and end-users. Market-makers play a key role in providing liquidity to end-users by taking the other side of end-user net demand. [1] The positions of market-makers are controlled by risk management strategies such as option writing and holding, cash flow matching, or delta-neutral hedging with futures or forwards according to their trading constraints and market conditions (Green and Figlewski[1999]).

In this study, we consider a demand-based option market model in which market-makers employ the delta-neutral hedging strategy with futures or forwards in order to control the risks induced by taking the other side of end-user net demand. Green and Figlewski[1999] prove that simply writing and holding options as if they are ordinary risky assets entails a very large exposure to risk. They note that cash flow matching is not a strategy that can generally be followed by market-makers because the nature of the

[1] Some investment strategies such as portfolio insurance and speculative trading will induce the end-user net demand for financial options (e.g., Bollen and Whaley[2004]). Gârleanu et al.[2009] provide the descriptive statistics on the end-user net demand for the S&P500 index options.

derivatives business is that the public wants to be long options, so that the market-maker community must be short on balance. Therefore, market-makers as a group must hold exposed and unmatched option positions. In light of these facts, the authors emphasize that delta-neutral hedging based on a valuation model is the only viable trading and risk management strategy for most market-makers.

If we can assume a dynamically complete financial market that can be governed by standard market models such as the Black-Scholes-Merton model, the no-arbitrage theory uniquely determines derivative prices that are independent of end-users net demand for options because market-makers can hedge their option positions perfectly through continuous time trading with underlying assets and cash. However, there is a rich body of studies that provides empirical evidence of jumps and stochastic volatilities in underlying asset processes, so the assumption of a dynamically complete market would not apply to the actual financial market. Under an incomplete market governed by additional risk factors such as stochastic volatilities and jumps, market-makers cannot hedge their option positions perfectly and are exposed to the risk of significant losses in the process of making markets and managing their options portfolios.

In light of these facts, we investigate how options are priced through optimized trading strategies for market-makers and end-users assuming a market in which market-makers cannot hedge their option positions perfectly due to risk factors such as stochastic volatilities and/or jumps. We provide a pricing kernel representation in equilibrium between supply and demand for options as an explicit function of end-users net demand for options and investigate the characteristics of equilibrium option prices that are consistent with that pricing kernel.

In recent years, there has been a renewal of interest in demand-pressure effects on financial option prices. On the basis of empirical analyses, several recent studies have shown that option demand has a significant impact on option prices. Green and Figlewski[1999] note that option market-makers cannot perfectly hedge their inventories of options and are consequently motivated to increase (decrease) their volatility forecasts by some suitable amount in pricing options in order to compensate for bearing various types of risks and to provide an expected profit above the risk-free rate. Bollen and Whaley[2004] demonstrate that changes in implied volatility are correlated with signed option volume. They show that changes in option demand lead to changes in option prices, while leaving open the question of whether the level of option demand impacts the overall level (i.e., expensiveness) of option prices or the overall shape of implied-volatility curves. Han[2007] provides evidence that investor sentiment helps explain both the shape of the S&P500's option volatility smile and the risk-neutral skewness of the index return extracted from index option prices. The author indicates that one channel for investor sentiment to affect option prices is through a demand-pressure effect and notes that the relative demand pressure of options helps explain time-series variations in index risk-neutral skewness. Kang and Park[2008] and Shiu, et al.[2010] also empirically investigate the effects of net-buying pressure for options on option prices and the shape of the implied volatility curve, and find significant effects of demand pressure on option prices.

Although the studies cited above relate option prices to demand-pressure effects in terms of various empirical aspects, Gârleanu et al.[2009] complement these studies by providing a theoretical model that can explicitly consider the demand-pressure effects on option prices and testing the precise quantitative implications of their model. They provide a novel representation of the pricing kernel with end-users net demand for options under the dynamic optimality of option market-makers. However, they unfortunately assume that the inelastic option demand of end-users is given exogeneously and that the option demand is independent of the preferences of both market-makers and end-users. In contrast, in this chapter, we consider a model in which the preference of market-makers for the background risks induced by the net demand of end-users affects option prices and, as a result, affects the net demand of end-users directly in an incomplete market. This aspect of our model is also empirically pointed out by Green and Figlewski[1999]. Considering the preferences of both market-makers and end-users explicitly, we examine the equilibrium between supply and demand for options that is implicit in our model and derive a pricing kernel representation with their preferences and end-users net demand in equilibrium. End-users net demand is induced *endogeneously* in our model, and this aspect differentiates our study from that of Gârleanu et al.[2009]. We investigate the effects of end-users net demand on option prices with the pricing kernel derived from our model setup and demonstrate that the demand pressure for financial options induced by end-users critically affects equilibrium option prices according to the preferences of both the representative market-maker and end-user.

Moreover, consistent with the pricing kernel in equilibrium between supply and demand for options, we provide some important implications for the problems on the features of financial option prices quoted in actual markets. First, we investigate the theoretical relationship between the net demand of end-users for financial options and the variance risk premium in a stochastic volatility environment with the demand-based pricing kernel derived from our model setup. The variance risk premium is the compensation for the risk induced by stochastic volatility in financial asset price processes, and it is well known that this risk premium is essentially related to delta-hedged option returns (e.g., Bakshi and Kapadia[2003]). We derive the variance risk premium with the covariance between variance changes and changes in the pricing kernel and show that this variance risk premium can be represented explicitly with end-users net demand for options. We find that if end-users net demand for options in the market is zero, the variance risk premium is also zero because, in this case, option market-makers are not faced with unhedgeable risks that are induced by stochastic volatility environments. However, we also find that the variance risk premium is not zero but a positive or negative value under the condition in which end-users net demand for options differs from zero and that the sign of the variance risk premium is determined by the sign of end-users net demand. In particular, under suitable conditions, the negative variance risk premium, which is shown empirically by, for example, Bakshi and Kapadia[2003], Low and Zhang[2005], and Carr and Wu[2009], can be explained by a positive demand pressure for options of convex payoffs. Moreover, consistent with our model setup, it is also found that a positive demand pressure for convex payoff options can be induced in equilibrium between supply and demand for options under such suitable conditions. To the best of our knowledge, this study is the first to provide a theoretical implication of the effect of end-users net demand for options on the variance risk premium in equilibrium.

The second implication is related to the pricing kernel puzzle or, equivalently, the implied risk aversion smile found by, for example, Aït-Sahalia and Lo[2000], Jackwerth[2000], Rosenberg and Engle[2002], and Ziegler[2007]. In a representative agent economy, the equilibrium pricing kernel based on the standard consumption-based framework should be monotone and decreasing in the aggregate price of equity. However, the recent studies cited above have presented puzzling evidence to the contrary: the pricing kernel (or the implied risk aversion function), when plotted against the market return, is not monotonically decreasing but instead exhibits an upward-sloping and/or negative region. Ziegler[2007] explores different potential explanations for these unexpected results within the standard consumption-based framework, but none of the potential explanations is able to account for these unexpected results. The author notes that, to explain the unexpected results, it seems necessary to go beyond the standard consumption-based framework and analyze the impact of factors such as market incompleteness and/or market frictions. In this study, we prove that the net demand of end-users plays a key role in explaining these unexpected results and present a numerical simulation result in which the implied risk aversion function in equilibrium between supply and demand for options has an upward-sloping region due to the net demand of end-users for options.

This article is organized as follows. Section 3.2 presents an option market model assumed in this analysis. The model reveals that both the representative option market-maker and the representative option end-user play key roles in determining option prices in equilibrium and that the prices are directly affected by the preferences of these market participants. Section 3.3 explores associated implications for the features of option market prices, especially on the variance risk premium and the implied risk aversion smile, with the equilibrium pricing kernel which is a function of the net demand of end-users. Section 3.4 concludes.

3.2 The Model

3.2.1 Assumptions of a Financial Option Market

We start with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\geq 0})$. Time is indexed by $t \in [0, T]$. We consider a two-dimensional asset price process that allows return volatility to be stochastic under the physical probability measure $\mathbb{P}$:

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu_t dt + \sigma_t \sqrt{1-\rho_t^2} dB_t^1 + \sigma_t \rho_t dB_t^2, \\ d\sigma_t &= \theta_t dt + \eta_t dB_t^2, \end{aligned} \tag{3.1}$$

where μ_t, θ_t, η_t and ρ_t are $\{\mathcal{F}_t\}_{t\geq 0}$-adapted stochastic processes, which allow the above equations to have a strong solution, and these processes are independent of S_t. (B_t^1, B_t^2) denotes a standard 2-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and we give the information set $\mathcal{F}_t$ as a sigma-algebra $\sigma\{B_s^1, B_s^2 | s \leq t\} \vee \mathcal{N}$, where $\mathcal{N}$ is the null set. We assume that there is a risk-free asset in the financial market and the annualized risk-free rate is $r \in \mathbb{R}_+^1$. In this chapter, we make an important assumption in line with Gârleanu et al.[2009] that there are two representative agents in the financial option market: a market-maker and an end-user. The market-maker is recognized as a liquidity provider for financial options, and this role is mainly performed by dealers in financial institutions. The end-user represents option market participants such as speculators and portfolio insurers. The market-maker quotes option prices that are optimized in accordance with the end-user net demand for options, and the end-user plays with the prices as a price taker to hold an optimal option position in his investment portfolio. We will describe their optimization problems in detail in the following section. In this chapter, we also assume that the option market-maker employs the delta-neutral hedging strategy with underlying assets for controlling the risks caused by the positions of the other side of the end-user net demand for options. For simplicity, we do not account for any transaction costs and liquidity constraints in the delta-neutral hedging strategy and suppose that the market-maker can continuously trade underlying assets.

It is well known that the absence of arbitrage opportunities is essentially equivalent to the existence of a probability $\mathbb{Q}$, equivalent to the physical probability measure $\mathbb{P}$, under which the discounted prices process is a $\mathcal{F}_t$-adapted martingale; such a probability will be called an equivalent martingale measure. Any equivalent martingale measure $\mathbb{Q}$ is characterized by a continuous version of its density process with respect to $\mathbb{P}$, which can be written from the integral form of martingale representation:

$$M_t(\nu., \lambda.) \equiv \frac{d\mathbb{Q}}{d\mathbb{P}} \mid_{\mathcal{F}_t} \equiv \exp\Big(- \int_0^t \nu_u dB_u^1 - \int_0^t \lambda_u dB_u^2 - \frac{1}{2}\int_0^t \nu_u^2 du - \frac{1}{2}\int_0^t \lambda_u^2 du \Big), \quad (3.2)$$

where (ν_t, λ_t) is adapted to $\mathcal{F}_t$ and satisfies the integrability conditions $\int_0^T \nu_u^2 du < \infty$ and $\int_0^T \lambda_u^2 du < \infty$ a.s.. Each process of ν_t and λ_t is interpreted as the price of risk relative respectively to the two sources of uncertainty B_t^1 and B_t^2. In particular, if $\Lambda_t \equiv M_t B_t$ denotes the discount factor process where $B_t \equiv \exp\big(- rt\big)$, then the price of volatility risk λ_t is given by $\lambda_t \equiv -\mathrm{Cov}_t(\frac{d\Lambda_t}{\Lambda_t}, d\sigma_t)$ (see, e.g., Cochrane[2001]). It can be seen that a positive correlation between the discount factor process Λ_t and the volatility process σ_t implies a negative λ_t. $\mathbb{Q}$ is also denoted by $\mathbb{Q}(\nu., \lambda.)$ when we need to specify the price of risk implied by $\mathbb{Q}$.

Assuming the model of a financial market given above, an explicit representation of delta-neutral hedged option returns for the market-maker can be derived as follows.

$C_t \equiv F(t, S_t, \sigma_t)$ denotes the time-t theoretical price of an European-type call option [2] , which is consistent with (3.1) and the market preference, written on S_t, struck at K, and expiring at time T for a $C^{1,2,2}$-function $F(t, S, \sigma)$. Under the equivalent martingale measure $\mathbb{Q}$ (which is also consistent with (3.1) and the market preference), the C_t defined above can be expressed as the following manner:

$$C_t \equiv F(t, S_t, \sigma_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\max(S_T - K, 0) \mid \mathcal{F}_t]. \quad (3.3)$$

On the basis of the approach explored by Bakshi and Kapadia[2003], let us derive the expression of a theoretical delta-hedged option return for $C_t = F(t, S_t, \sigma_t)$ under the above assumptions.

When $0 \leq \tau \leq T - t$, we can derive a following equation via Ito's lemma:

$$\begin{aligned} C_{t+\tau} = C_t &+ \int_t^{t+\tau} \frac{\partial F}{\partial S}(u, S_u, \sigma_u) dS_u \\ &+ \int_t^{t+\tau} \frac{\partial F}{\partial \sigma}(u, S_u, \sigma_u) d\sigma_u + \int_t^{t+\tau} \mathcal{D}F(u, S_u, \sigma_u) du, \end{aligned} \quad (3.4)$$

[2] In this section, we focus on an European call option, but the discussion and results we provide here can apply more generally to other options such as puts and straddle options.

where

$$\begin{aligned}\mathcal{D}F(t,S_t,\sigma_t) = \frac{\partial F}{\partial t}(t,S_t,\sigma_t) &+ \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 F}{\partial S^2}(t,S_t,\sigma_t) \\ &+ \frac{1}{2}\eta_t^2 \frac{\partial^2 F}{\partial \sigma^2}(t,S_t,\sigma_t) + \rho_t\eta_t\sigma_t S_t \frac{\partial^2 F}{\partial S\partial\sigma}(t,S_t,\sigma_t).\end{aligned} \tag{3.5}$$

On the other hand, standard assumptions also show that the call option price C_t is a solution to the Black-Scholes valuation equation,

$$\begin{aligned}\frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 F}{\partial S^2}(t,S_t,\sigma_t) + \frac{1}{2}\eta_t^2 \frac{\partial^2 F}{\partial \sigma^2}(t,S_t,\sigma_t) + \rho\eta_t\sigma_t S_t \frac{\partial^2 F}{\partial S\partial\sigma}(t,S_t,\sigma_t) + rS_t\frac{\partial F}{\partial S}(t,S_t,\sigma_t) \\ + (\theta_t - \lambda_t)\frac{\partial F}{\partial \sigma}(t,S_t,\sigma_t) + \frac{\partial F}{\partial t}(t,S_t,\sigma_t) - rC_t = 0.\end{aligned} \tag{3.6}$$

Thanks to (3.6),

$$\begin{aligned}\frac{\partial F}{\partial t}(t,S_t,\sigma_t) = -\frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 F}{\partial S^2}(t,S_t,\sigma_t) - \frac{1}{2}\eta_u^2 \frac{\partial^2 F}{\partial \sigma^2}(t,S_t,\sigma_t) - \rho\eta_t\sigma_t S_t \frac{\partial^2 F}{\partial S\partial\sigma}(t,S_t,\sigma_t) \\ - rS_t\frac{\partial F}{\partial S}(t,S_t,\sigma_t) - (\theta_t - \lambda_t)\frac{\partial F}{\partial \sigma}(t,S_t,\sigma_t) + rC_t.\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.5), we have

$$\begin{aligned}C_{t+\tau} = C_t + \int_t^{t+\tau} \frac{\partial F}{\partial S}(u,S_u,\sigma_u)dS_u + \int_t^{t+\tau}\Big(rC_u - rS_u\frac{\partial F}{\partial S}(u,S_u,\sigma_u)\Big)du \\ + \int_t^{t+\tau}\lambda_u\frac{\partial F}{\partial \sigma}(u,S_u,\sigma_u)du + \int_t^{t+\tau}\eta_u\frac{\partial F}{\partial \sigma}(u,S_u,\sigma_u)dB_u^2.\end{aligned} \tag{3.8}$$

If we define the delta-hedged gain and loss (hereinafter, DHGL) $\Pi_{t,t+\tau}$ for C_t in the period of $[t, t+\tau]$ by

$$\Pi_{t,t+\tau} \equiv C_{t+\tau} - C_t - \int_t^{t+\tau} \frac{\partial F}{\partial S}(u,S_u,\sigma_u)dS_u - \int_t^{t+\tau}\Big(rC_u - rS_u\frac{\partial F}{\partial S}(u,S_u,\sigma_u)\Big)du,$$

then, from (3.8) it follows

$$\Pi_{t,t+\tau} = \int_t^{t+\tau}\lambda_u\frac{\partial F}{\partial \sigma}(u,S_u,\sigma_u)du + \int_t^{t+\tau}\eta_u\frac{\partial F}{\partial \sigma}(u,S_u,\sigma_u)dB_u^2. \tag{3.9}$$

Thus, taking expectation to (3.9) under the physical measure $\mathbb{P}$, the expected delta-hedged option return for $C_t = F(t,S_t,\sigma_t)$ can be derived as follows:

$$\mathbb{E}\big[\Pi_{t,t+\tau} \mid \mathcal{F}_t\big] = \int_t^{t+\tau}\mathbb{E}\big[\lambda_u\frac{\partial F}{\partial \sigma}(u,S_u,\sigma_u)\big]du. \tag{3.10}$$

The implication of equation (3.10) is that if the volatility risk is not priced (i.e., $\lambda_u = 0$), then on average, the delta-hedged option return should be zero. In contrast, if the volatility risk is priced (i.e., $\lambda_u \neq 0$), then on average, the delta-hedged option return must not be zero. Because the Vega of the call option, $\frac{\partial F}{\partial \sigma}(u, S_u, \sigma_u)$, is not zero, the sign of the volatility risk premium λ_u determines whether the average delta-hedged option return is positive or negative.

It is clear that the problem of the determination of a λ_u in actual financial market governed by stochastic volatility is equivalent to the problem of the determination of fair option prices in that market. In this chapter, we suppose that option prices in equilibrium are determined through the determination of λ_u, which is also affected by the net demand of end-users for options.

To consider optimization problems of portfolio construction for the market-maker and the end-user, let us introduce wealth processes and utility functions for these representative agents. Let W_t^O and W_t^E denote $\mathcal{F}_t$-adapted wealth processes for the market-maker and the end-user, respectively, and the utility function of the market-maker at time T, $U^O(\cdot) : \mathbb{R} \to \mathbb{R}$, is denoted by $U^O(W_T^O) \equiv -A\exp(-W_T^O/A)$, which is an exponential utility function. $A > 0$ is a risk tolerance parameter for the market-maker. This assumption of the utility function for the market-maker is similar to the assumption in Gârleanu, et al.[2009]. In contrast, the utility function of the end-user at time T is denoted by $U^E(W_T^E) \in C^2(\mathbb{R})$ and we do not assume any specific types for the end-user's utility function in the following discussion in which we develop a theoretical background.

Based on the assumptions stated above, we examine one-period optimization problems for the market-maker and the end-user. The end-user constructs an optimized portfolio that consists of cash, a single stock of which the price process is represented by (3.1), and options. The end-user holds the optimized portfolio to the maturity time of T. We assume that there are N tradable options in the market and each of payoff functions of N options is denoted by $g_i(S_T) : \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, N$, respectively. Each optimal amount of N options for the end-user at time $t = 0$ is denoted by $\delta_i^* \in \mathbb{R}$, $i = 1, \cdots, N$.

3.2.2 Assumptions for Option Prices

Suppose that prices of traded N options are functions of the vector $\delta \equiv (\delta_1, \delta_2, \cdots, \delta_N)^t \in \mathbb{R}^N$, where each of δ_i, $i = 1, \cdots, N$, is the net demand of end-users for the i-th option contract, and these functions are denoted by $p_i(\cdot) \in C^2(\mathbb{R}^N)$, $i = 1, \cdots, N$, respectively. If the vector of the net demand of end-users for each option contract, $\delta \equiv (\delta_1, \delta_2, \cdots, \delta_N)^t \in \mathbb{R}^N$, is given explicitly, by considering (3.9), (3.10), and the proposition presented in Appendix A it is intuitively clear that the delta-hedged option return at time T of the i-th option contract written by the market-maker in order to match the net demand of end-users for that option is described by the following expression,

$$\Pi_T^i(\delta) \equiv p_i(\delta) - {p_i}^0 + \epsilon_T^i, \tag{3.11}$$

where ϵ_T^i is a $\mathcal{F}_T$-adapted random variable whose the expected value is zero, that is, $\mathbb{E}[\epsilon_T^i] = 0$, and p_i^0 is the price of the i-th option contract whose the expected value of its delta-hedged return is zero. Considering (3.10), we find that this p_i^0 can be expressed as follows:

$$p_i^0 = e^{-rT}\mathbb{E}^{\mathbb{Q}(\nu.,0)}\Big[g_i(S_T)\Big] = e^{-rT}\mathbb{E}\Big[M_T(\nu.,0)g_i(S_T)\Big], \tag{3.12}$$

which is the price under zero volatility risk premium in the stochastic volatility environment of (3.1).

Let us introduce two Borel probability measures on $\mathbb{R}^1$, $\mathbb{P}^{S_T}(B)$ and $\mathbb{Q}^{S_T}(\nu.,\lambda.)(B)$, $B \in \mathcal{B}(\mathbb{R}^1)$, as $\mathbb{P}^{S_T}(B) \equiv (\mathbb{P}\circ S_T^{-1})(B)$ and $\mathbb{Q}^{S_T}(\nu.,\lambda.)(B) \equiv (\mathbb{Q}(\nu.,\lambda.)\circ S_T^{-1})(B)$, respectively. Introducing a Radon-Nikodym derivative process as $M_T^{S_T}(\nu.,\lambda.)(x) \equiv \frac{d\mathbb{Q}^{S_T}(\nu.,\lambda.)}{d\mathbb{P}^{S_T}}(x)$, $x \in \mathbb{R}^1$, we can also derive the following expression,

$$\begin{aligned} p_i^0 &= e^{-rT}\mathbb{E}^{\mathbb{Q}(\nu.,0)}\Big[g_i(S_T)\Big] = e^{-rT}\mathbb{E}^{\mathbb{Q}^{S_T}(\nu.,0)}\Big[g_i(x)\Big] = e^{-rT}\mathbb{E}^{\mathbb{P}^{S_T}}\Big[M_T^{S_T}(\nu.,0)(x)g_i(x)\Big] \\ &= e^{-rT}\mathbb{E}\Big[M_T^{S_T}(\nu.,0)(S_T)g_i(S_T)\Big]. \end{aligned} \tag{3.13}$$

An intuitive interpretation of the assumption (3.11) is that the market-maker requires the expected excess profit of $p_i(\delta) - {p_i}^0$ as a risk premium for unhedgeable risks in delta-neutral hedging that are essentially induced by the net demand of end-users, which is also pointed out in Green and Figlewski[1999].

3.3 Optimality and Equilibrium

3.3.1 Optimization for the Market-maker and the End-user

On the basis of the assumptions stated above, let us describe the optimization problems for the market-maker and the end-user. The wealth equation of the end-user is given as follows:

$$W_t^E(\delta_1,\cdots,\delta_N;S_t) = \Big(W_0^E - S_0 - \sum_{i=1}^N \delta_i p_i(\delta)\Big)e^{rt} + S_t + \sum_{i=1}^N \delta_i \frac{e^{-r(T-t)}}{M_t}\mathbb{E}\Big[g_i(S_T)M_T \mid \mathcal{F}_t\Big].$$

In particular, at time T,

$$W_T^E(\delta_1, \cdots, \delta_N; S_T) = \Big(W_0^E - S_0 - \sum_{i=1}^N \delta_i p_i(\delta)\Big)e^{rT} + S_T + \sum_{i=1}^N \delta_i g_i(S_T), \tag{3.14}$$

where $W_0^E \in \mathbb{R}$ denotes the initial endowment of the end-user. The wealth equation of the market-maker is given as follows:

$$W_t^O(\delta_1, \cdots, \delta_N; S_t) = W_0^O e^{rt} + \sum_{i=1}^N \delta_i \frac{e^{-r(T-t)}}{M_t} \mathbb{E}\Big[\Pi_T^i(\delta) M_T \mid \mathcal{F}_t\Big].$$

In particular, at time T,

$$W_T^O(\delta_1, \cdots, \delta_N; S_T) = W_0^O e^{rT} + \sum_{i=1}^N \delta_i \Pi_T^i(\delta) = W_0^O e^{rT} + \sum_{i=1}^N \delta_i(p_i(\delta) - {p_i}^0 + \epsilon_T^i), \tag{3.15}$$

where $W_0^O \in \mathbb{R}$ denotes the initial endowment of the market-maker. Under an equilibrium between supply and demand for options (defined below), the end-user and the market-maker make their own decision independently for trading and constructing their own optimal options positions according to their objective functions and the equilibrium option prices.

For clarity, let us describe a definition of equilibrium between supply and demand for options contracts.

Definition 1 *We call a pair of $(\delta^*, p^*) \in \mathbb{R}^N \times \mathbb{R}^N$ equilibrium if and only if the optimal options positions for the market-maker and the end-user, which are denoted by $\delta^{O*} \equiv \{\delta_i^{O*}\}_{i=1}^N \in \mathbb{R}^N$ and $\delta^{E*} \equiv \{\delta_i^{E*}\}_{i=1}^N \in \mathbb{R}^N$, respectively, are mutually symmetrical, that is to say that $\delta^{O*} + \delta^{E*} = 0$ under an option price vector $p^* \equiv \{p_i^*\}_{i=1}^N \in \mathbb{R}^N$. We also call this option price vector $p^* \equiv \{p_i^*\}_{i=1}^N \in \mathbb{R}^N$ the equilibrium price and $\delta^* \equiv \delta^{E*}(= -\delta^{O*})$ the equilibrium demand pressure.*

Let us describe each of optimization problems for the market-maker and the end-user as follows. First, an optimization problem for the market-maker's options position under a prespecified option price vector $p \equiv \{p_i\}_{i=1}^N \in \mathbb{R}^N$ at time 0 is described with the wealth process defined by (3.15) as follows:

$$\sup_{\delta_i, i=1,\cdots,N} \mathbb{E}\Big[U^O(W_T^O(\delta_1, \cdots, \delta_N; S_T))\Big] = \sup_{\delta_i, i=1,\cdots,N} \mathbb{E}\Big[U^O(W_0^O e^{rT} + \sum_{i=1}^N \delta_i \Pi_T^i(\delta))\Big]. \tag{3.16}$$

For simplicity, we assume $W_0^O = 0$ in the following discussion. Considering the first-order condition for (3.16), the following proposition can be derived with assumptions of $U^O(W_T^O) \equiv -A\exp(-W_T^O/A)$, $A > 0$, and (3.11).

Proposition 6 *Under a prespecified option price vector $p \equiv \{p_i\}_{i=1}^N \in \mathbb{R}^N$, the optimal option position vector $\delta = \{\delta_1, \delta_2, \cdots, \delta_N\}^t \in \mathbb{R}^N$ induced by the optimization problem (3.16) for the market-maker satisfies the following relationship:*

$$p_i = {p_i}^0 - \frac{\mathbb{E}\Big[\exp\Big(-\sum_{j=1}^N \frac{\delta_j}{A}\epsilon_T^j\Big)\epsilon_T^i\Big]}{\mathbb{E}\Big[\exp\Big(-\sum_{j=1}^N \frac{\delta_j}{A}\epsilon_T^j\Big)\Big]}, \quad i = 1, 2, \cdots, N. \tag{3.17}$$

In contrast, when the net option demand of end-users is explicitly given as $\delta = \{\delta_1, \delta_2, \cdots, \delta_N\}^t \in \mathbb{R}^N$, the option prices quoted by the market-maker should be expressed by (3.17) so as to satisfy the optimal condition described by (3.16). In this case, (3.17) is also recognized as an inverse supply function. The end-user solve an own optimization problem under the price condition of (3.17) induced by the net demand of their own, $\delta = \{\delta_1, \delta_2, \cdots, \delta_N\}^t \in \mathbb{R}^N$, for options contracts.

Obtaining the condition of (3.17) in the above proposition, we can also show the next proposition.

Proposition 7 *If we define a probability measure $\mathbb{P}^\delta$ on $(\Omega, \mathcal{F})$ which is absolutely continuous and equivalent to $\mathbb{P}$ as*

$$\frac{d\mathbb{P}^\delta}{d\mathbb{P}} \equiv \frac{\exp\Big(-\sum_{j=1}^N \frac{\delta_j}{A}\epsilon_T^j\Big)}{\mathbb{E}\Big[\exp\Big(-\sum_{j=1}^N \frac{\delta_j}{A}\epsilon_T^j\Big)\Big]},$$

then,

$$p_i(\delta) = {p_i}^0 - \mathbb{E}^{\mathbb{P}^\delta}\Big[\epsilon_T^i\Big], \quad \frac{\partial p_i(\delta)}{\partial \delta_k} = \frac{1}{A}\mathrm{Cov}^{\mathbb{P}^\delta}\Big(\epsilon_T^i, \epsilon_T^k\Big), \quad i = 1, 2, \cdots, N. \tag{3.18}$$

In particular,

$$p_i(0) = {p_i}^0, \quad \frac{\partial p_i(\delta)}{\partial \delta_i} = \frac{1}{A}\mathrm{Var}^{\mathbb{P}^\delta}\Big(\epsilon_T^i\Big) > 0, \quad i = 1, 2, \cdots, N.$$

Proof *See the Appendix.* □

(3.18) indicates some important points on the equilibrium prices of options in terms of the net demand of end-users:

1. When the end-user net demand for options is given by $\delta \in \mathbb{R}^N$, the representation of the i-th option premium, $p_i(\delta)$, consists of two terms: an option premium p_i^0, which is the expected replication cost of the delta-neutral replication strategy, and a risk premium term $-\mathbb{E}^{\mathbb{P}^\delta}\left[\epsilon_T^i\right]$ for unhedgeable risks of the market-maker's hedging strategy, which are induced by the net demand of end-users.

2. $p_i^0 = p_i(0)$, that is, when the net demand of end-users is zero (i.e. $\delta = 0 \in \mathbb{R}^N$), the equilibrium price $p_i(0)$ for the i-th option contract is equal to the p_i^0, which is the expected replication cost of the delta-neutral replication strategy.

3. The net demand of end-users increases (decreases) the price of any other option by an amount proportional to the covariance of the unhedgeable parts of the two options under the probability measure $\mathbb{P}^\delta$. In particular, the net demand of end-users in one option increases its price by an amount proportional to the variance of the unhedgeable part of the option under $\mathbb{P}^\delta$.

In contrast to the Black-Scholes-Merton framework, if options cannot be hedged perfectly and if intermediaries who take the other side of end-users' option demand are risk averse, then we can recognize by the proposition provided above that the net demand of end-users for options will essentially impact option prices.

The points cited above have parallel implication for the equilibrium prices of options in terms of the net demand of end-users when compared with Gârleanu et al.[2009], and these points are not novel. In fact, Gârleanu et al.[2009] show that demand pressure in one option contract increases its price by an amount proportional to the variance of the unhedgeable part of the option based on a model of competitive risk-averse intermediaries who cannot perfectly hedge their option positions. However, in the following discussion, we also consider an optimization problem for the end-user's portfolio that consists of options and an underlying asset with the result of (3.18) and, as a result, we derive the net demand of end-users for options and their prices in equilibrium between supply and demand. This aspect differentiates our study from that of Gârleanu et al.[2009] because Gârleanu et al.[2009] assume inelastic end-user demand, that is, the end-user's exogenous aggregate demand for options, in the optimization problem for the option market-maker. By endogeneously considering equilibrium between supply and demand for options, some interesting implications for evidence recognized as option pricing anomalies can be derived via the pricing kernel in equilibrium.

Let us proceed to describe the details of an optimization problem for the end-user. The end-user optimizes positions for options, $\delta = \{\delta_1, \delta_2, \cdots, \delta_N\}^t \in \mathbb{R}^N$, to maximize his own expected utility according to the set of prices quoted by the market-maker, which is also affected by the net demand of end-users:

$$\begin{aligned}
&\sup_{\delta_1,\cdots,\delta_N} \mathbb{E}[U^E(W_T^E(\delta_1,\cdots,\delta_N;S_T))] = \\
&\sup_{\delta_1,\cdots,\delta_N} \mathbb{E}\Big[U^E\Big(\Big(W_0^E - S_0 - \sum_{i=1}^N \delta_i p_i(\delta)\Big)e^{rT} + S_T + \sum_{i=1}^N \delta_i g_i(S_T)\Big)\Big] \\
&s.t. \quad p_i(\delta) = {p_i}^0 - \mathbb{E}^{\mathbb{P}^\delta}\Big[\epsilon_T^i\Big], \quad i = 1,2,\cdots,N,
\end{aligned} \tag{3.19}$$

where $U^E(\cdot) \in C^2(\mathbb{R}) : \mathbb{R} \to \mathbb{R}$ denotes the end-user's utility function which has a property of strictly increasing, that is, $(U^E)' > 0$. We can derive an equilibrium solution of the net demand of end-users, $\delta^* = (\delta_1^*, \delta_2^*, \cdots, \delta_N^*)^t \in \mathbb{R}^N$, by solving the optimization problem of (3.19). The following lemma can be proven easily by the first-order condition for (3.19).

Lemma 1 *$\delta^* = (\delta_1^*, \delta_2^*, \cdots, \delta_N^*)^t \in \mathbb{R}^N$ satisfies the following equation:*

$$\begin{aligned}
\frac{\partial}{\partial \delta_i}\Big(\sum_{j=1}^N \delta_j p_j(\delta)\Big)\Big|_{\delta=\delta^*} &= \mathbb{E}\Big[\frac{U^{E'}(W_T^E(\delta_1^*,\cdots,\delta_N^*;S_T))}{e^{rT}\mathbb{E}[U^{E'}(W_T^E(\delta_1^*,\cdots,\delta_N^*;S_T))]} g_i(S_T)\Big] \\
\textit{where} \quad p_i(\delta^*) &= {p_i}^0 - \mathbb{E}^{P^{\delta^*}}\Big[\epsilon_T^i\Big], \quad i = 1,2,\cdots,N,
\end{aligned} \tag{3.20}$$

and, by applying (3.18) to (3.20), we can derive the following expression:

$$\begin{aligned}
{p_i}^0 - \mathbb{E}^{P^{\delta^*}}\Big[\epsilon_T^i\Big] - \mathrm{Cov}^{P^{\delta^*}}\Big(\epsilon_T^i, -\sum_{j=1}^N \frac{{\delta_j}^*}{A}\epsilon_T^j\Big) &= \mathbb{E}\Big[\frac{U^{E'}(W_T^E(\delta_1^*,\cdots,\delta_N^*;S_T))}{e^{rT}\mathbb{E}[U^{E'}(W_T^E(\delta_1^*,\cdots,\delta_N^*;S_T))]} g_i(S_T)\Big], \\
\textit{where} \quad i &= 1,2,\cdots,N.
\end{aligned} \tag{3.21}$$

For an intuitive interpretation, let us show an example of an equilibrium solution of (3.21) based on a Monte Carlo simulation under simple assumptions. In this example, we assume the stochastic volatility model proposed by Heston[1993] for an underlying asset price process:

$$\begin{aligned}
\frac{dS_t}{S_t} &= \sigma_t\sqrt{1-\rho^2}dB_t^1 + \sigma_t\rho dB_t^2, \\
d{\sigma_t}^2 &= \kappa(\gamma - {\sigma_t}^2)dt + \eta\sigma_t dB_t^2.
\end{aligned} \tag{3.22}$$

where κ, γ, and η are positive constants. This assumption is correspondent to the assumption of (3.1) in the case that $\mu_t \equiv 0$, $\theta_t \equiv -\kappa\sigma_t$, and $\eta_t \equiv \eta$. For simplicity, suppose that $T \equiv 1$ and divide the time interval of $[0,T](\equiv [0,1])$ into 250 small intervals, that is, $t_0 = 0, t_1 = \frac{1}{250}, t_2 = \frac{2}{250}, \cdots, t_{250} = \frac{250}{250} = 1$, in order to run 10,000 times sample-path simulations for S_t based on the Euler approximation. The model parameters of (3.22) assumed in this simulation are $\kappa = 2.52$, $\gamma = 0.01$, $\eta = 0.23$ and $\rho = -0.3$. [3] Moreover, we simulate hedging errors, ϵ_T^i, of each delta-neutral hedged option position with $10,000$ simulated paths for S_t based on the Black-Scholes formula. In this example, we also assume $N = 3$, $S_0 = 1$, and consider three types of European options whose strike prices are $K_1 = 1.0$, $K_2 = 0.9$, and $K_3 = 1.1$, respectively. In particular, options of both $i = 1$ and $i = 2$ are put options and the option of $i = 3$ is a call option. Each price p_i^0 is calculated with the closed formula for option valuation proposed by Heston[1993], setting the volatility risk premium parameter as zero and based on the parameter condition defined above.

Table 3.1 shows each price p_i^0 and the distributions of delta-neutral hedging errors for options of $i = 1$, $i = 2$, and $i = 3$. (see Fig.3.5 in the Appendix for details on the distributions of these delta-neutral hedging errors) This table indicates that the distributions of delta-neutral hedging errors are fat-tailed and negatively skewed distributions in all cases of $i = 1$, $i = 2$, and $i = 3$.

Table 3.1: Summary Statistics of Hedging Error Distributions

(a) Strategy	(b) p_i^0	(c) $\mathbb{E}[S_T]$	(d) $\mathbb{E}[\epsilon_T^i]$	(e) $\sigma[\epsilon_T^i]$	(f) Skewness	(g) Kurtosis
(1) ATM Put	3.78909	1.00396	-0.01877	1.27998	-1.46150	4.11050
(2) OTM Put	0.86253	1.00396	-0.00092	0.93800	-2.92277	15.94547
(3) OTM Call	0.74167	1.00396	-0.00375	0.91430	-1.26386	6.40482

(1) In this example, we set $T = 1$, $S_0 = 1$, $\kappa = 2.52$, $\gamma = 0.01$, $\eta = 0.23$, and $\rho = -0.3$. Each of prices, $p_i^0 s$, are calculated based on the closed formula for option valuation proposed by Heston[1993] setting the volatility risk premium parameter as zero.

(2) The strike prices of K_1, K_2, and K_3 are respectively defined as 1.0, 0.9, and 1.1

(3) This Monte Carlo simulation is based on trials of 10,000 times.

We will now examine the explicit solution of (3.21) in detail based on the hedging errors of ϵ_T^i, $i = 1, 2, 3$, generated by the Monte Carlo simulation explained above. In this example, the utility function of the representative end-user, $U^E(W_T^E) \in C^2(\mathbb{R})$, is temporally assumed as a power utility function, which is represented as follows:

$$U^E(W_T^E(\delta^*; S_T)) \equiv \frac{W_T^E(\delta^*; S_T)^{1-\gamma}}{1-\gamma}, \quad \gamma \neq 1.$$

[3] These parameters are estimated with the USD-JPY exchange rate data in the period from October 2003 to June 2010. The estimation methodology is based on Aït-Sahalia and Kimmel[2007].

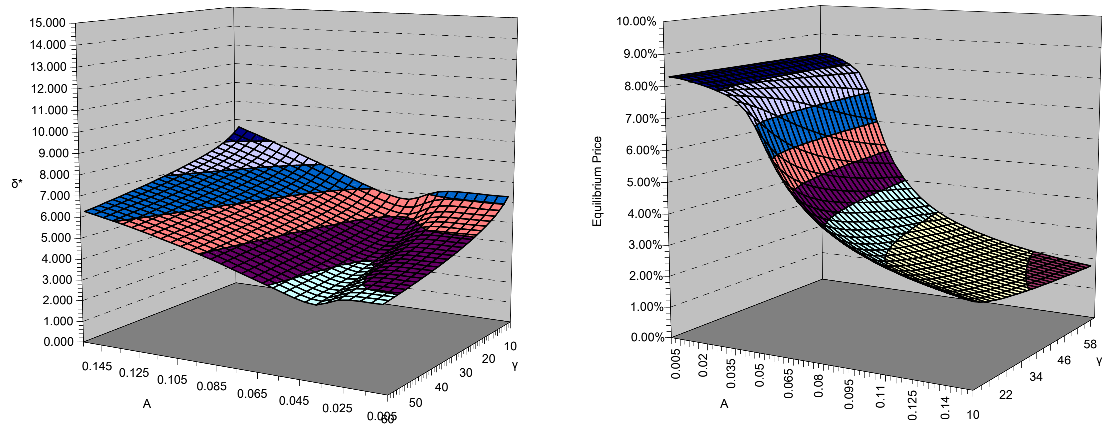

Fig. 3.1: The end-user net demand δ_2^* and the option price $p_2(\delta_2^*)$ in equilibrium

(1) This result shows the solutions in the case that the end-user is permitted only to hold a position in the out-of-the-money put of $K_2 = 0.9$ and $\delta^* = (\delta_2) \in \mathbb{R}^1$.

(2) The left side of above figures shows the end-user net demand, δ_2^*, and the right side shows the equilibrium price of the option contract, $p_2(\delta_2^*)$.

In the beginning, we will examine the optimal position of δ^* under the condition that the end-user is permitted to hold a position only in an option of $i = 2$, in which the end-user holds a position only in the out-of-the-money put of $K_2 = 0.9$ and $\delta^* = (\delta_2) \in \mathbb{R}^1$. Fig.3.1 shows the results on the end-user net demand δ_2^* and the option price in equilibrium $p_2(\delta_2^*)$ according to the market-maker's preference parameter (risk tolerance parameter) $A \in \mathbb{R}$ and the end-user's preference parameter (risk aversion parameter) $\gamma \neq 1$. A ranges from 0.005 to 0.15 and γ ranges from 10 to 60, which is consistent with the results shown in the previous studies such as those by Mehra and Prescott[1985], Cochrane and Hansen[1992], and Aït-Sahalia and Lo[2000]. This figure indicates that when the risk tolerance parameters of the market-maker (A) and the end-user ($\frac{1}{\gamma}$) increase, the net demand for the option of $i = 2$ in equilibrium will also increase and the option price will decrease simultaneously. In contrast, when the risk tolerance parameters decrease simultaneously, this figure also indicates that the net demand for that option in equilibrium will decrease and the equilibrium price of that option will increase dramatically.

Similarly, the optimal position and the equilibrium price under the condition that the end-user is permitted to hold a position only in an option of $i = 3$, in which the end-user holds a position only in the out-of-the-money call of $K_3 = 1.1$ and $\delta^* = (\delta_3) \in \mathbb{R}^1$, are shown in Fig.3.2 according to the market-maker's preference parameter (risk tolerance parameter) $A \in \mathbb{R}$ and the end-user's preference parameter (risk aversion parameter)

$\gamma \neq 1$. The indication of this figure is the same as that of Fig.3.1, which is that when the risk tolerance parameters of the market-maker (A) and the end-user ($\frac{1}{\gamma}$) increase, the net demand for the option of $i = 3$ in equilibrium will also increase, and vice versa.

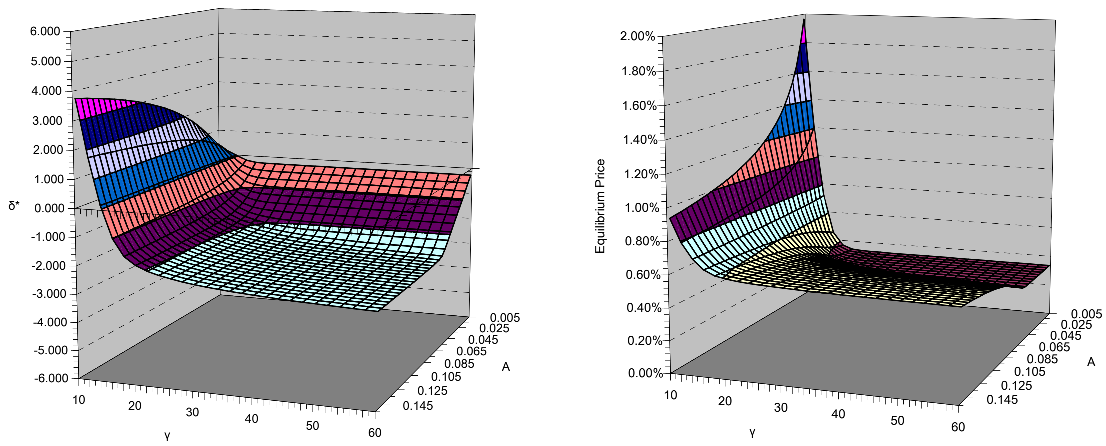

Fig. 3.2: The end-user net demand δ_3^* and the option price $p_3(\delta_3^*)$ in equilibrium

(1) This result shows the solutions in the case that the end-user is permitted only to hold a position in an out-of-the-money call of $K_3 = 1.1$ and $\delta^* = (\delta_3) \in \mathbb{R}^1$.

(2) The left side of above figures shows the end-user net demand, δ_3^*, and the right side shows the equilibrium price of the option contract, $p_3(\delta_3^*)$.

For the case that the end-user can hold positions in the options of both $i = 1$ and $i = 3$, Fig.3.6 in the Appendix shows the end-user net demand and the prices of those of options in equilibrium.

3.3.2 The Pricing Kernel in Equilibrium

On the basis of the discussion explored above, let us derive the equilibrium option price with a pricing kernel determined by the end-user net demand for options.

If an optimal solution of the end-user's optimization problem (3.19), $\delta^* = (\delta_1^*, \delta_2^*, \cdots, \delta_N^*)^t \in \mathbb{R}^N$, exists, then the following proposition can be derived.

Proposition 8 *Each option price in equilibrium is given by*

$$p_i(\delta^*) = \mathbb{E}\Big[\frac{1}{e^{rT}}\Big(\frac{-\mathbb{E}^{P^{\delta^*}}\Big[\epsilon_T^i\Big]}{-\mathbb{E}^{P^{\delta^*}}\Big[\epsilon_T^i\Big] + \mathrm{Cov}^{P^{\delta^*}}\Big(\epsilon_T^i, \sum_{j=1}^N \frac{\delta_j{}^*}{A}\epsilon_T^j\Big)}\frac{U^{E'}(W_T^E(\delta^*;S_T))}{\mathbb{E}[U^{E'}(W_T^E(\delta^*;S_T))]} + \frac{\mathrm{Cov}^{P^{\delta^*}}\Big(\epsilon_T^i, \sum_{j=1}^N \frac{\delta_j{}^*}{A}\epsilon_T^j\Big)}{-\mathbb{E}^{P^{\delta^*}}\Big[\epsilon_T^i\Big] + \mathrm{Cov}^{P^{\delta^*}}\Big(\epsilon_T^i, \sum_{j=1}^N \frac{\delta_j{}^*}{A}\epsilon_T^j\Big)} M_T^{S_T}(\nu.,0)(S_T)\Big) g_i(S_T)\Big], \tag{3.23}$$

and the pricing kernel (projected to the i-th hedging error ϵ_T^i) [4] $Z_T^{\delta^*}$ *is represented by*

$$Z_T^{\delta^*} = \frac{1}{e^{rT}}\Big(\frac{-\mathbb{E}^{P^{\delta^*}}\Big[\epsilon_T^i\Big]}{-\mathbb{E}^{P^{\delta^*}}\Big[\epsilon_T^i\Big] + \mathrm{Cov}^{P^{\delta^*}}\Big(\epsilon_T^i, \sum_{j=1}^N \frac{\delta_j{}^*}{A}\epsilon_T^j\Big)}\frac{U^{E'}(W_T^E(\delta^*;S_T))}{\mathbb{E}[U^{E'}(W_T^E(\delta^*;S_T))]} + \frac{\mathrm{Cov}^{P^{\delta^*}}\Big(\epsilon_T^i, \sum_{j=1}^N \frac{\delta_j{}^*}{A}\epsilon_T^j\Big)}{-\mathbb{E}^{P^{\delta^*}}\Big[\epsilon_T^i\Big] + \mathrm{Cov}^{P^{\delta^*}}\Big(\epsilon_T^i, \sum_{j=1}^N \frac{\delta_j{}^*}{A}\epsilon_T^j\Big)} M_T^{S_T}(\nu.,0)(S_T)\Big). \tag{3.24}$$

Moreover, if the following condition

$$\frac{\partial}{\partial A}\log\Big(- \mathbb{E}^{P^{\delta^*}}\Big[\epsilon_T^i\Big]\Big) = \exists_1 c(\delta^*, A) \in \mathbb{R}^1, \quad \forall i = 1, 2, \cdots, N, \tag{3.25}$$

is given for a constant $c(\delta^*, A)$, which depends on the end-user net demand δ^* and the risk tolerance parameter of the market-maker A but independent of i, then,

$$\begin{aligned} \frac{-\mathbb{E}^{P^{\delta^*}}\Big[\epsilon_T^i\Big]}{-\mathbb{E}^{P^{\delta^*}}\Big[\epsilon_T^i\Big] + \mathrm{Cov}^{P^{\delta^*}}\Big(\epsilon_T^i, \sum_{j=1}^N \frac{\delta_j{}^*}{A}\epsilon_T^j\Big)} &= \frac{1}{1 - Ac(\delta^*, A)} \in \mathbb{R}^1, \\ \frac{\mathrm{Cov}^{P^{\delta^*}}\Big(\epsilon_T^i, \sum_{j=1}^N \frac{\delta_j{}^*}{A}\epsilon_T^j\Big)}{-\mathbb{E}^{P^{\delta^*}}\Big[\epsilon_T^i\Big] + \mathrm{Cov}^{P^{\delta^*}}\Big(\epsilon_T^i, \sum_{j=1}^N \frac{\delta_j{}^*}{A}\epsilon_T^j\Big)} &= \frac{-Ac(\delta^*, A)}{1 - Ac(\delta^*, A)} \in \mathbb{R}^1, \end{aligned} \tag{3.26}$$

[4] For the projected pricing kernel, see Cochrane[2001] and Rosenberg and Engle[2002].

where $i = 1, 2, \cdots, N$ [5], and the following representation can be derived for all i:

$$p_i(\delta^*) = \mathbb{E}\Big[\frac{1}{e^{rT}}\Big(\frac{1}{1 - Ac(\delta^*, A)}\frac{U^{E'}(W_T^E(\delta^*; S_T))}{\mathbb{E}[U^{E'}(W_T^E(\delta^*; S_T))]} + \frac{-Ac(\delta^*, A)}{1 - Ac(\delta^*, A)}M_T^{S_T}(\nu., 0)(S_T)\Big)g_i(S_T)\Big],$$

$$Z_T^{\delta^*} = \frac{1}{e^{rT}}\Big(\frac{1}{1 - Ac(\delta^*, A)}\frac{U^{E'}(W_T^E(\delta^*; S_T))}{\mathbb{E}[U^{E'}(W_T^E(\delta^*; S_T))]} + \frac{-Ac(\delta^*, A)}{1 - Ac(\delta^*, A)}M_T^{S_T}(\nu., 0)(S_T)\Big).$$

Proof *See the Appendix.* □

If $N = 1$, then (3.23) and (3.24) are respectively described as

$$p_i(\delta^*) = \mathbb{E}\Big[\frac{1}{e^{rT}}\Big(\frac{-\mathbb{E}^{P^{\delta^*}}[\epsilon_T^1]}{-\mathbb{E}^{P^{\delta^*}}[\epsilon_T^1] + \frac{\delta_1{}^*}{A}\mathrm{Var}^{P^{\delta^*}}(\epsilon_T^1)^2}\frac{U^{E'}(W_T^E(\delta_1^*; S_T))}{\mathbb{E}[U^{E'}(W_T^E(\delta_1^*; S_T))]} + \frac{\frac{\delta_1{}^*}{A}\mathrm{Var}^{P^{\delta^*}}(\epsilon_T^1)^2}{-\mathbb{E}^{P^{\delta^*}}[\epsilon_T^1] + \frac{\delta_1{}^*}{A}\mathrm{Var}^{P^{\delta^*}}(\epsilon_T^1)^2}M_T(\nu., 0)\Big)g_i(S_T)\Big]$$

and

$$Z_T^{\delta^*} = \frac{1}{e^{rT}}\Big(\frac{-\mathbb{E}^{P^{\delta^*}}[\epsilon_T^1]}{-\mathbb{E}^{P^{\delta^*}}[\epsilon_T^1] + \frac{\delta_1{}^*}{A}\mathrm{Var}^{P^{\delta^*}}(\epsilon_T^1)^2}\frac{U^{E'}(W_T^E(\delta_1^*; S_T))}{\mathbb{E}[U^{E'}(W_T^E(\delta_1^*; S_T))]} + \frac{\frac{\delta_1{}^*}{A}\mathrm{Var}^{P^{\delta^*}}(\epsilon_T^1)^2}{-\mathbb{E}^{P^{\delta^*}}[\epsilon_T^1] + \frac{\delta_1{}^*}{A}\mathrm{Var}^{P^{\delta^*}}(\epsilon_T^1)^2}M_T(\nu., 0)\Big).$$

Fig.3.3 shows the parameter c_1^*,

$$c_1^* \equiv \frac{-\mathbb{E}^{P^{\delta^*}}[\epsilon_T^1]}{-\mathbb{E}^{P^{\delta^*}}[\epsilon_T^1] + \frac{\delta_1{}^*}{A}\mathrm{Var}^{P^{\delta^*}}(\epsilon_T^1)^2}, \tag{3.27}$$

which appears in the above expressions. The left-side figure illustrates the case that the end-user holds a position only in the OTM put option whose strike price is $K = 0.9$, and the right-side figure illustrates the case that the end-user holds a position only in the OTM call option whose strike price is $K = 1.1$. These results shown in this figure are calculated based on the Monte Carlo simulation results obtained in the previous section.

[5] If $-\mathbb{E}^{P^{\delta^*}}\Big[\epsilon_T^i\Big] > 0$ and $\mathrm{Cov}^{P^{\delta^*}}\Big(\epsilon_T^i, \sum_{j=1}^N \frac{\delta_j{}^*}{A}\epsilon_T^j\Big) > 0$, then, $c(\delta^*, A) \equiv \frac{\partial}{\partial A}\log\Big(-\mathbb{E}^{P^{\delta^*}}\Big[\epsilon_T^i\Big]\Big) < 0$ and we can obtain the following relationship,

$$0 < \frac{1}{1 - Ac(\delta^*, A)}, \frac{-Ac(\delta^*, A)}{1 - Ac(\delta^*, A)} < 1$$

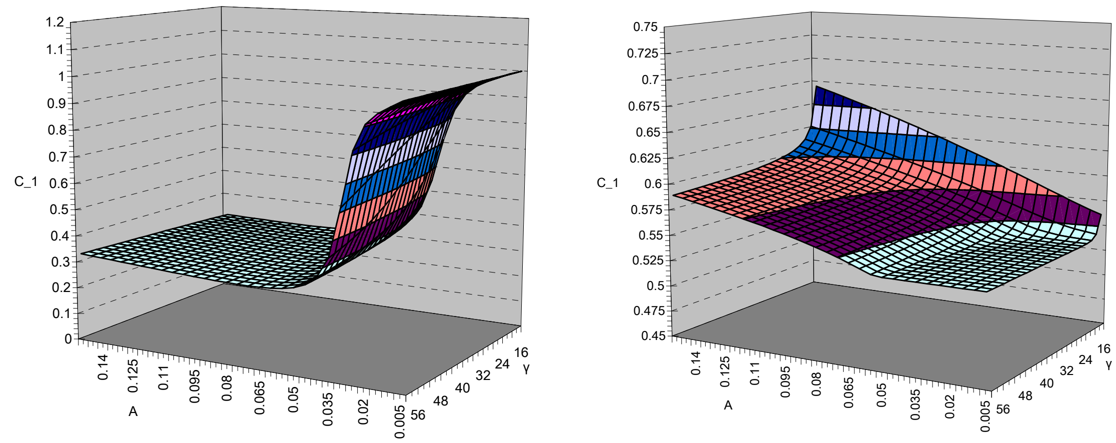

Fig. 3.3: The parameter c_1^* in equilibrium

(1) The left-side figure is for the case that the end-user holds a position only in the OTM put option whose strike price is $K = 0.9$ and the right-side figure is for the case that the end-user holds a position only in the OTM call option whose strike price is $K = 1.1$. These parameters shown in this result are calculated based on the simulation results obtained in the previous section.

It can be observed in Fig.3.3 that c_1^*s have strictly positive values under suitable conditions of preference parameters for the market-maker and the end-user. If c_1^* is equal to zero, it is apparent from the expression provided above that the pricing kernel in equilibrium $Z_T^{\delta^*}$ will be $\frac{1}{e^{rT}} M_T(\nu., 0)$, which is the pricing kernel when the net demand of end-users for options is zero (see Proposition 2). In contrast, the results in Fig.3.3 indicate that the utility function of the end-user and the net demand of end-users for options essentially affect the pricing kernel in equilibrium and, as a result, option prices via the pricing kernel $Z_T^{\delta^*}$ represented by the above expression.

When $N > 1$, the condition of (3.25) is not necessarily acceptable, so that the pricing kernel represented by (3.24) is essentially dependent on each i. Although this marginal pricing kernel is not necessarily the joint pricing kernel that is not dependent on i, by the following idea, we can deduce the joint pricing kernel that prices all of the options traded in the market consistently. Breeden and Litzenberger[1978] provide a risk-neutral probability density function with option prices quoted in actual option market: when $C(t, T; K)$ denotes the price of an option written on S_t, struck at K, and expiring at time T, they show that the risk-neutral probability density function f^* can be expressed as follows:

$$f^*(S_T) = e^{r(T-t)} \frac{\partial^2 C(t, T; K)}{\partial K^2} \mid_{K=S_T} . \tag{3.28}$$

In equilibrium assumed in this chapter, each option price $C(t,T;K)$ appearing in the above representation is substituted by (3.23), and based on the representation of (3.28), we can deduce the joint pricing kernel that is not dependent on i and the prices of all options traded in the market consistently with the end-user net demand for options.

According to the proposition provided above, the (marginal) pricing kernel $Z_T^{\delta^*}$ in equilibrium is a weighted sum of $U^{E'}(W_T^E(\delta^*;S_T))/\mathbb{E}[U^{E'}(W_T^E(\delta^*;S_T))]$ and $M_T(\nu.,0)$ and weights on those of terms are respectively given by $\frac{1}{1-Ac(\delta^*,A)}$ and $\frac{-Ac(\delta^*,A)}{1-Ac(\delta^*,A)}$ (under the condition of (3.25)). In particular, that pricing kernel is affected by the end-user's utility function U^E, the risk tolerance parameter for the option market-maker A, each option hedging error ϵ_T^i, $i=1,2,\cdots,N$, and the end-user net demand for options $\delta^* = (\delta_1^*,\delta_2^*,\cdots,\delta_N^*)^t \in \mathbb{R}^N$ in equilibrium. This $Z_T^{\delta^*}$ induces several interesting implications for empirical evidence provided by the recent studies on the features of financial option markets. For simplicity, we limit the case to $N=1$ and let us demonstrate these implications in the following subsections.

The Net Demand and the Variance Risk Premium

Let us examine a relationship between the net demand of end-users for options and the variance risk premium induced by the stochastic volatility environment assumed in the previous section. In this subsection, we temporarily assume that $\theta_t \equiv -k\sigma_t$ and $\eta_t \equiv v$ ($k,\, v \in \mathbb{R}_{++}$) for the parameters in (3.1), that is, the stochastic volatility model proposed by Heston[1993], and for mathematical tractability, we also assume that the drift term in (3.1) can be replaced with $\mu_t = r + \phi\sigma_t$, $\exists\phi \in \mathbb{R}$.

The variance risk premium is the premium that is associated with the compensation for the time-variation in the conditional return variance. When investing in a security, an investor faces at least two sources of uncertainty, namely, the uncertainty about the return as captured by the return variance and the uncertainty about the return variance itself. It is important to understand how investors deal with the uncertainty in the return variance to effectively manage risk and allocate assets, to accurately price and hedge derivative securities, and to understand the behavior of financial asset prices in general. Under the assumptions of (3.1) and (3.2), the variance risk premium is given by $-\mathrm{Cov}_t\Big(\frac{dM_t}{M_t}, d\sigma_t^2\Big)/dt$ (see Cochrane[2001]). Recent studies, such as those by Bakshi and Kapadia[2003], Low and Zhang[2005] and Carr and Wu[2009], report the negative variance risk premium in the stock and currency option markets. In this subsection, we derive an explicit representation of the variance risk premium with the pricing kernel (3.24) in equilibrium provided in the previous subsection and demonstrate empirical evidence of the variance risk premium reported by recent studies, that is, the existence and the negativity of the variance risk premium.

Let $M_T^{\delta^*}$ denotes $M_T^{\delta^*} \equiv e^{rT} Z_T^{\delta^*}$ ($Z_T^{\delta^*}$ is given by (3.24)). The Radon-Nikodym derivative process for $M_T^{\delta^*}$ is descrived as follows:

$$\begin{aligned} M_t^{\delta^*} &= \mathbb{E}\Big[M_T^{\delta^*} \mid \mathcal{F}_t\Big] = \mathbb{E}\Big[e^{rT} Z_T^{\delta^*} \mid \mathcal{F}_t\Big] \\ &= \mathbb{E}\Big[\Big(c_1^* \frac{U^{E'}(W_T^E(\delta^*; S_T))}{\mathbb{E}[U^{E'}(W_T^E(\delta^*; S_T))]} + (1-c_1^*) M_T(\nu.,0)\Big) \mid \mathcal{F}_t\Big] \\ &= \frac{c_1^*}{\mathbb{E}[U^{E'}(W_T^E(\delta^*; S_T))]} \mathbb{E}\Big[U^{E'}(W_T^E(\delta^*; S_T)) \mid \mathcal{F}_t\Big] + (1-c_1^*)\mathbb{E}\Big[M_T(\nu.,0) \mid \mathcal{F}_t\Big]. \end{aligned} \tag{3.29}$$

In particular, we can derive

$$\begin{aligned} & dU^{E'}(W_t^E(\delta^*; S_t)) \\ &= \frac{\partial}{\partial t} U^{E'}(W_t^E(\delta^*; S_t)) dt + \frac{\partial}{\partial S_t} U^{E'}(W_t^E(\delta^*; S_t)) dS_t + \frac{1}{2}\frac{\partial^2}{\partial {S_t}^2} U^{E'}(W_t^E(\delta^*; S_t))(dS_t)^2 \\ &= \Big(\frac{\partial}{\partial t} U^{E'}(W_t^E(\delta^*; S_t)) + \frac{\partial}{\partial S_t} U^{E'}(W_t^E(\delta^*; S_t)) \mu_t S_t + \frac{1}{2}\frac{\partial^2}{\partial {S_t}^2} U^{E'}(W_t^E(\delta^*; S_t)) \sigma_t^2 {S_t}^2\Big) dt \\ &+ \Big(\frac{\partial}{\partial S_t} U^{E'}(W_t^E(\delta^*; S_t))\Big) \sigma_t S_t \big(\sqrt{1-\rho^2} dB_t^1 + \rho dB_t^2\big). \end{aligned}$$

Thus,

$$\begin{aligned} & U^{E'}(W_T^E(\delta^*; S_T)) = W_0^E + \\ & \int_0^T \Big(\frac{\partial}{\partial t} U^{E'}(W_t^E(\delta^*; S_t)) + \frac{\partial}{\partial S_t} U^{E'}(W_t^E(\delta^*; S_t)) \mu_t S_t + \frac{1}{2}\frac{\partial^2}{\partial {S_t}^2} U^{E'}(W_t^E(\delta^*; S_t)) \sigma_t^2 {S_t}^2\Big) dt \\ & + \int_0^T \Big(\frac{\partial}{\partial S_t} U^{E'}(W_t^E(\delta^*; S_t))\Big) \sigma_t S_t \big(\sqrt{1-\rho^2} dB_t^1 + \rho dB_t^2\big). \end{aligned}$$

When $0 \leq u \leq T$, we obtain the following expression

$$\begin{aligned} & \mathbb{E}\Big[U^{E'}(W_T^E(\delta^*; S_T)) \mid \mathcal{F}_u\Big] = W_0^E + \\ & \int_0^u \Big(\frac{\partial}{\partial t} U^{E'}(W_t^E(\delta^*; S_t)) + \frac{\partial}{\partial S_t} U^{E'}(W_t^E(\delta^*; S_t)) \mu_t S_t + \frac{1}{2}\frac{\partial^2}{\partial {S_t}^2} U^{E'}(W_t^E(\delta^*; S_t)) \sigma_t^2 {S_t}^2\Big) dt \\ & + \int_u^T \mathbb{E}\Big[\Big(\frac{\partial}{\partial t} U^{E'}(W_t^E(\delta^*; S_t)) + \frac{\partial}{\partial S_t} U^{E'}(W_t^E(\delta^*; S_t)) \mu_t S_t \\ & + \frac{1}{2}\frac{\partial^2}{\partial {S_t}^2} U^{E'}(W_t^E(\delta^*; S_t)) \sigma_t^2 {S_t}^2\Big) \mid \mathcal{F}_u\Big] dt \\ & + \int_0^u \Big(\frac{\partial}{\partial S_t} U^{E'}(W_t^E(\delta^*; S_t))\Big) \sigma_t S_t \big(\sqrt{1-\rho^2} dB_t^1 + \rho dB_t^2\big). \end{aligned}$$

Considering the fact of $\mathbb{E}\Big[M_T(\nu.,0) \mid \mathcal{F}_u\Big] = M_u(\nu.,0)$, $0 \le u \le T$, we can show that the Radon-Nikodym derivative process $M_u^{\delta^*}$, $0 \le u \le T$, is

$$\begin{aligned} M_u^{\delta^*} =& \frac{c_1^*}{\mathbb{E}[U^{E'}(W_T^E(\delta^*;S_T))]} \times \\ & \Big\{ W_0^E + \int_0^u \Big(\frac{\partial}{\partial t} U^{E'}(W_t^E(\delta^*;S_t)) + \frac{\partial}{\partial S_t} U^{E'}(W_t^E(\delta^*;S_t))\mu_t S_t \\ & + \frac{1}{2}\frac{\partial^2}{\partial {S_t}^2} U^{E'}(W_t^E(\delta^*;S_t))\sigma_t^2 {S_t}^2 \Big) dt \\ & + \int_u^T \mathbb{E}\Big[\Big(\frac{\partial}{\partial t} U^{E'}(W_t^E(\delta^*;S_t)) + \frac{\partial}{\partial S_t} U^{E'}(W_t^E(\delta^*;S_t))\mu_t S_t \\ & + \frac{1}{2}\frac{\partial^2}{\partial {S_t}^2} U^{E'}(W_t^E(\delta^*;S_t))\sigma_t^2 {S_t}^2 \Big) \mid \mathcal{F}_u \Big] dt \\ & + \int_0^u \Big(\frac{\partial}{\partial S_t} U^{E'}(W_t^E(\delta^*;S_t)) \Big) \sigma_t S_t \big(\sqrt{1-\rho^2} dB_t^1 + \rho dB_t^2 \big) \Big\} \\ & + (1-c_1^*) M_u(\nu.,0) \end{aligned}$$

and its stochastic differential is

$$\begin{aligned} dM_u^{\delta^*} =& \frac{c_1^*}{\mathbb{E}[U^{E'}(W_T^E(\delta^*;S_T))]} \Big\{ \Big(\frac{\partial}{\partial S_u} U^{E'}(W_u^E(\delta^*;S_u)) \Big) \sigma_u S_u \big(\sqrt{1-\rho^2} dB_u^1 + \rho dB_u^2 \big) \Big\} \\ & + (1-c_1^*) dM_u(\nu.,0). \end{aligned}$$

From (3.2),

$$\begin{aligned} M_u(\nu.,0) &= \exp\Big(-\int_0^u \nu_t dB_t^1 - \frac{1}{2}\int_0^u \nu_t^2 dt \Big) \\ &= \exp\Big(-\int_0^u \frac{\phi}{\sqrt{1-\rho^2}} dB_t^1 - \frac{1}{2}\int_0^u \Big(\frac{\phi}{\sqrt{1-\rho^2}} \Big)^2 dt \Big) \\ &= \exp\Big(-\frac{\phi}{\sqrt{1-\rho^2}} B_u^1 - \frac{1}{2}\Big(\frac{\phi^2 u}{1-\rho^2} \Big) \Big), \end{aligned}$$

so its stochastic differential is described as

$$dM_u(\nu.,0) = -\frac{\phi}{\sqrt{1-\rho^2}} M_u(\nu.,0) dB_u^1. \tag{3.30}$$

On the other hand, the following expressions can be derived by solving (3.1):

$${\sigma_u}^2 = e^{-2ku}{\sigma_0}^2 + \frac{v^2}{2k}\Big(1 - e^{-2ku}\Big) + 2v\int_0^u e^{2k(t-u)}\sigma_t dB_t^2$$

and

$$d{\sigma_u}^2 = (v^2 - 2k{\sigma_u}^2)du + 2v\sigma_u dB_u^2. \tag{3.31}$$

According to (3.30) and (3.31), it is clear that two $\mathcal{F}_u$-adapted processes, $M_u(\nu.,0)$ and ${\sigma_u}^2$, are mutually independent because (B_t^1, B_t^2) is a standard 2-dimensional Brownian motion. Thus, $M_u(\nu.,0)$ does not price the stochastic variance ${\sigma_u}^2$. So if we define a $\mathcal{F}_u$-adapted process ξ_u^* as a variance risk premium in equilibrium, that is, $\xi_u^* \equiv -\mathrm{Cov}_u\Big(\frac{dM_u^{\delta^*}}{M_u^{\delta^*}}, d\sigma_u^2\Big)/du$, then we can derive the following representation for ξ_u^*:

$$\begin{aligned}
\xi_u^* &\equiv -\frac{\mathrm{Cov}_u\Big(\dfrac{dM_u^{\delta^*}}{M_u^{\delta^*}}, d\sigma_u^2\Big)}{du} \\
&= \frac{-2c_1^* v\rho}{\mathbb{E}[U^{E'}(W_T^E(\delta^*;S_T))]}\mathrm{Cov}_u\Big(\frac{S_u\dfrac{\partial}{\partial S_u}U^{E'}(W_u^E(\delta^*;S_u))}{M_u^{\delta^*}}\sigma_u dB_u^2, \sigma_u dB_u^2\Big)/du \\
&= \frac{-2c_1^* v\rho}{\mathbb{E}[U^{E'}(W_T^E(\delta^*;S_T))]}\frac{S_u\dfrac{\partial}{\partial S_u}U^{E'}(W_u^E(\delta^*;S_u))}{M_u^{\delta^*}}{\sigma_u}^2 \\
&= \frac{-2c_1^* v\rho}{\mathbb{E}[U^{E'}(W_T^E(\delta^*;S_T))]}\frac{S_u U^{E''}(W_u^E(\delta^*;S_u))\dfrac{\partial}{\partial S_u}W_u^E(\delta^*;S_u)}{M_u^{\delta^*}}{\sigma_u}^2.
\end{aligned}$$

Let us summarize the result obtained above:

Proposition 9 *If we assume that $\theta_t \equiv -k\sigma_t$ and $\eta_t \equiv v$ (k, $v \in \mathbb{R}_{++}$) for the parameters in (3.1), that is, the stochastic volatility model proposed by Heston[1993], and that the drift term in (3.1) is given by $\mu_t = r + \phi\sigma_t$, $\exists\phi \in \mathbb{R}$, then the variance risk premium ξ_u^* in equilibrium is explicitly expressed by the following formula:*

$$\xi_u^* = \psi_u(\delta^*, A, U^E, v, \rho, S_u) \times {\sigma_u}^2 \quad , \tag{3.32}$$

where

$$\begin{aligned}
&\psi_u(\delta^*, A, U^E, v, \rho, S_u) \\
&\equiv \frac{-2c_1^* v\rho}{\mathbb{E}[U^{E'}(W_T^E(\delta^*;S_T))]}\frac{S_u U^{E''}(W_u^E(\delta^*;S_u))\Big(1+\delta^*\dfrac{d}{dS_u}\dfrac{e^{-r(T-u)}}{M_u}\mathbb{E}\Big[g_1(S_T)M_T \mid \mathcal{F}_u\Big]\Big)}{M_u^{\delta^*}}.
\end{aligned}$$

The above proposition provides some interesting implications for empirical evidence of the existence and the negativity of the variance risk premium shown in the recent studies.

In the case that the net demand of end-users for options is zero, that is , $\delta_1^* = 0$, we have the result of $p_1(0) = p_1^0$ from Proposition 2. That is, when the end-user net demand for options δ^* is zero, the variance risk premium in equilibrium $\xi_u^* \equiv -\mathrm{Cov}_u\left(\frac{dM_u^0}{M_u^0}, d\sigma_u^2\right)/du$ also should be zero because of the definition of p_i^0. However, when the end-user net demand for options δ^* is not zero, the sign of the variance risk premium ξ_u is perfectly coincident with the sign of ρ, which is the correlation between volatility changes and changes in the asset price, under the conditions of $c_1^* > 0$, $U^{E'} > 0$, $U^{E''} < 0$, and $1 + \delta^* \frac{dg}{dS_T}(S_T) > 0$. In fact, as we have shown in Fig.3.1, Fig.3.2, and Fig.3.3, the conditions of $c_1^* > 0$ and $1 + \delta^* \frac{dg}{dS_T}(S_T) > 0$ are valid in equilibrium if γ (the risk aversion parameter for the end-user) ranges from approximately 10 to 13, so that our model predicts that the variance risk premium ξ_u^* in equilibrium will have the same sign with ρ under the above condition for γ. In fact, recent studies provide evidence that ρ is generally negative in, for example, stock index markets, so the negativity of the variance risk premium reported by empirical studies such as those by Bakshi and Kapadia[2003], Low and Zhang[2005], and Carr and Wu[2009] might be caused by the negativity of ρ. However, on the other hand, Carr and Wu[2009] also indicate that the negativity of the variance risk premium can not be explained only by the sign of ρ and/or other systematic risk factors such as Fama-French's three factors. On the basis of the discussion explored above, we may say that there is a possibility that (positive) end-user net demand for options of convex payoffs in equilibrium, which is also shown empirically by Gârleanu et al.[2009] with a unique dataset for the S&P500 index options [6] , plays a key role in determining the existence and the sign of the variance risk premium.

The Net Demand and the Risk Aversion Smile

The second implication of (3.24) is relevant to topics on the pricing kernel puzzle or the implied risk aversion smile demonstrated by Aït-Sahalia and Lo[2000], Jackwerth[2000], Rosenberg and Engle[2002], and Ziegler[2007].

In an economy with complete markets and risk-averse investors having common and correct expectations (i.e. all subjective expectations coincide with the objective probability measure), the pricing kernel should be a decreasing function of aggregate resources, which are usually proxied by the returns of the market portfolio. However, empirical studies such as those by Aït-Sahalia and Lo[2000], Jackwerth[2000], and Rosenberg and

[6] They use a unique dataset to identify aggregate daily positions of dealers and end-users. They are the first to document that end-users have a net long position in S&P500 index options with large net positions in out-of-the-money (OTM) puts. Since options are in zero net supply, this implies that dealers are short index options.

Engle[2002] show that the marginal utility of investors is increasing over an important range of wealth levels and is not decreasing in wealth, as economic theory would suggest. Ziegler[2007] explores different potential explanations for the implied risk aversion smile, but for plausible parameter values, none of the potential explanations considered in the article is able to account for the implied risk aversion smile within the standard consumption-based framework. The pricing kernel puzzle (the implied risk aversion smile) is the observation that the pricing kernel (the implied risk aversion) might be increasing in some range of the market returns, and we seek to understand this puzzle in terms of the end-user net demand for options with the pricing kernel represented by (3.24).

From (3.24) and $ARA^*(S_T) = \frac{d}{dS_T}\Big(-\log Z_T^{\delta^*}(S_T)\Big)$ (see Jackwerth[2000] and Ziegler[2007]), the following proposition is derived:

Proposition 10 *Under the notation of $\delta^* \equiv \delta_1^*$ and $a(\delta^*) \equiv \mathbb{E}[U^{E'}(W_T^E(\delta^*; S_T))]$, the implied absolute risk aversion $ARA^*(S_T)$ of the representative investor in equilibrium is expressed as follows:* [7]

$$\begin{aligned} ARA^*(S_T) = &- \frac{c_1^* U^{E''}(W_T^E(\delta^*; S_T))}{c_1^* U^{E'}(W_T^E(\delta^*; S_T)) + a(\delta^*)(1-c_1^*)M_T^{S_T}(\nu.,0)(S_T)}\Big(1+\delta^*\frac{dg_1}{dS_T}(S_T)\Big) \\ &- \frac{a(\delta^*)(1-c_1^*)\dfrac{d}{dS_T}M_T^{S_T}(\nu.,0)(S_T)}{c_1^* U^{E'}(W_T^E(\delta^*; S_T)) + a(\delta^*)(1-c_1^*)M_T^{S_T}(\nu.,0)(S_T)}. \end{aligned} \tag{3.33}$$

A noteworthy point of (3.33) is that the shape of the implied absolute risk aversion in the equilibrium is directly affected by the end-user net demand for options.

For intuitive interpretation and simplicity, let us assume that the price of risk $\nu.$ is zero. In this case, it is clear that $M_T^{S_T}(0,0) \equiv 1$, thus $Z_T^{\delta^*}$ and $ARA^*(S_T)$ derived above are respectively represented as follows:

$$\begin{aligned} Z_T^{\delta^*}(S_T) &= \frac{1}{e^{rT}}\Big(c_1^*\frac{U^{E'}(W_T^E(\delta^*; S_T))}{\mathbb{E}[U^{E'}(W_T^E(\delta^*; S_T))]} + (1-c_1^*)\Big), \\ ARA^*(S_T) &= -\frac{c_1^* U^{E''}(W_T^E(\delta^*; S_T))}{c_1^* U^{E'}(W_T^E(\delta^*; S_T)) + a(\delta^*)(1-c_1^*)}\Big(1+\delta^*\frac{dg_1}{dS_T}(S_T)\Big). \end{aligned} \tag{3.34}$$

Because $1 + \delta^*\frac{dg_1}{dS_T}(S_T) = \frac{\partial}{\partial S_T}W_T^E(\delta^*; S_T)$, from (3.34) it is clear that the shape of the pricing kernel $Z_T^{\delta^*}(S_T)$ and the implied absolute risk aversion $ARA^*(S_T)$ are both affected

[7] When $\nu. > 0$, then $M_T^{S_T}(\nu.,0)(S_T)$ is a decreasing function of S_T.

by the shape of $W_T^E(\delta^*; S_T)$ essentially in equilibrium. In particular, in the case that the utility function of the end-user satisfies the condition of $(U^E)' > 0$ and $(U^E)'' < 0$, we can verify that both the pricing kernel $Z_T^{\delta^*}(S_T)$ and the implied absolute risk aversion $ARA^*(S_T)$ in equilibrium are increasing over the range of S_T on which the terminal wealth function of the end-user $W_T^E(\delta^*; S_T)$ is decreasing.

Fig.3.4 shows the shape of the terminal wealth function of the end-user based on the simulation result in the case that the end-user is permitted to hold a position only in the OTM call option whose strike price is $K = 1.1$, which is examined in the previous section. The end-user net demands in equilibrium δ^*s shown in this figure are those obtained in the case of $A = 0.15$ (δ^* varies according to the value of γ). This result clearly shows that there is a range of S_T on which the terminal wealth function of the end-user $W_T^E(\delta^*; S_T)$ is decreasing. This range is essentially induced by the negative net demand of the end-user for the option, which is determined by the preference parameters of the market-maker (A) and the end-user (γ) in equilibrium.

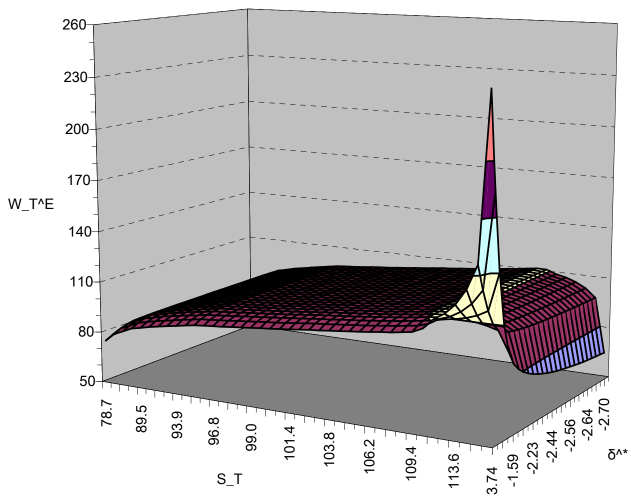

Fig. 3.4: The shape of terminal wealth for the end-user : $W_T^E(\delta^*; S_T)$

This figure is for the case that the end-user holds a position only in the OTM call option whose strike price is $K = 1.1$.

The similar results on the shape of the wealth function of the representative investor are provided by the several recent studies. In particular, Driessen and Maenhout[2007] empirically study the economic benefits of giving investors access to index options in the standard portfolio problem by using data on the S&P500 index options, and they show that the standard expected-utility investors and common studied behavioral investors *never* have a positive demand for straddles and OTM puts given observed prices. They provide evidence empirically that CRRA investors find it always optimal to short OTM puts and ATM straddles and the terminal wealth functions of these investors are partly decreasing functions to the terminal asset price S_T. Thus, according to our results derived in this section, it is also inferred from these recent studies that the pricing kernel puzzle and the implied risk aversion smile found by Aït-Sahalia and Lo[2000], Jackwerth[2000], Rosenberg and Engle[2002], and Ziegler[2007] are essentially induced by the end-user net (negative) demand for options in equilibrium.

3.4 Concluding Remarks

In this chapter, we study financial option prices and their related topics in terms of demand-pressure effects of options contracts. With a self-contained model of equilibrium between the supply and demand for options, each of the portfolio optimization problems for the market-maker and end-user of options contracts is examined. Deriving equilibrium demand pressures for options, we provide an explicit representation of the pricing kernel in equilibrium between the supply and demand for options, which is a function of those of equilibrium demand pressures.

On the basis of the demand-based pricing kernel in equilibrium derived from our model setup, we provide some important implications for empirical evidence which have been provided in recent studies related to option pricing anomalies. In particular, under suitable conditions, we find that the negative variance risk premium, which is shown empirically by, for example, Bakshi and Kapadia[2003], Low and Zhang[2005], and Carr and Wu[2009], can be explained by a positive demand pressure for options of convex payoffs. Moreover, we also demonstrate the pricing kernel puzzle or the implied risk aversion smile in terms of demand-pressure effects of options. We find that these option pricing anomalies can also be explained by a non-zero demand pressure for options.

To the best of our knowledge, this study is the first to provide some implications for the well-known empirical evidence, that is, the negative variance risk premium and the pricing kernel puzzle, in terms of the end-user net demand for financial options. Compared with already existing studies, this study allows to shed light on the role of the end-user net demand for financial options to the features of option market prices. However, it remains some challenges for future research on the field discussed in this chapter. First, it is an important problem open for consideration to investigate some implications of the end-user net demand for options under the setting of more realistic utility functions of the market-maker and the end-user. Second, the liquidity of the financial futures is also important because it affects the performance of the delta-neutral hedging strategy employed by the option market-maker. Since the liquidity of the futures is essentially related to transaction costs for the delta-neutral hedging strategy, there is a possibility that it affects the quotation for option prices from the market-maker, and as a result, the net demand of end-users and the prices of financial options in equilibrium.

Appendix 3.A Expected Delta-Hedged Gain and Loss

Proposition 11 *If $C_t[\lambda.]$ denotes the time-t call option price which is consistent with the underlying asset price process (3.1) and the equivalent martingale measure $\mathbb{Q}(\nu., \lambda.)$, then the following inequation can be derived:*

$$C_t[0] - C_t[\lambda.] \leq (\geq) \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u[\lambda.]}{\partial \tilde{\sigma}_u}\Big] du \leq (\geq) \Big(1 + r(T-t)\Big)\Big(C_t[0] - C_t[\lambda.]\Big),$$

for $\lambda_u \geq (\leq) 0$. If we are able to assume $r \approx 0$, then this inequation can be further simplified to the approximation presented below:

$$C_t[0] - C_t[\lambda.] \approx \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u[\lambda.]}{\partial \tilde{\sigma}_u}\Big] du. \tag{3.35}$$

Proof *(1) First, let us assume $\lambda_u \geq 0$. We have the following inequations on the option premium,*

$$C_u[\lambda.] - C_u[0] \leq 0 \quad (\forall u \in [t, T])$$

and

$$\frac{\partial}{\partial u}\mathbb{E}^{\mathbb{P}}\Big[C_u[\lambda.] - C_u[0]\Big] = \mathbb{E}^{\mathbb{P}}\Big[\frac{\partial}{\partial u}\Big(C_u[\lambda.] - C_u[0]\Big)\Big] \geq 0.$$

Thus,

$$\begin{aligned} C_t[0] - C_t[\lambda.] &= \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u[\lambda.]}{\partial \tilde{\sigma}_u}\Big] du + r\int_t^T \mathbb{E}^{\mathbb{P}}\Big[C_u[\lambda.] - C_u[0]\Big] du \\ &\geq \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u[\lambda.]}{\partial \tilde{\sigma}_u}\Big] du + r(T-t)\Big(C_t[\lambda.] - C_t[0]\Big). \\ \therefore \quad \Big(1 + r(T-t)\Big)\Big(C_t[0] - C_t[\lambda.]\Big) &\geq \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u[\lambda.]}{\partial \tilde{\sigma}_u}\Big] du. \end{aligned}$$

We also have a inequation of $\mathbb{E}^{\mathbb{P}}\Big[C_u[\lambda.] - C_u[0]\Big] \leq 0$, so the following inequation can be obtained,

$$C_t[0] - C_t[\lambda.] \leq \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u[\lambda.]}{\partial \tilde{\sigma}_u}\Big] du.$$

Thus we can derive the following inequation with two inequations derived above,

$$C_t[0] - C_t[\lambda.] \leq \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u[\lambda.]}{\partial \tilde{\sigma}_u}\Big] du \leq \Big(1 + r(T-t)\Big)\Big(C_t[0] - C_t[\lambda.]\Big).$$

(2) In the case of $\lambda_u < 0$, we have the following inequations,

$$C_u[\lambda.] - C_u[0] \geq 0 \quad (\forall u \in [t,T])$$

and

$$\frac{\partial}{\partial u}\mathbb{E}^{\mathbb{P}}\Big[C_u[\lambda.] - C_u[0]\Big] = \mathbb{E}^{\mathbb{P}}\Big[\frac{\partial}{\partial u}\Big(C_u[\lambda.] - C_u[0]\Big)\Big] \leq 0.$$

Thus we can derive the inequation asserted in this Proposition with the similar approach to the discussion of (1). □

Appendix 3.B Proof of Proposition 7

Proof

$$\begin{aligned}
\frac{\partial p_i(\delta)}{\partial \delta_k} &= \frac{\partial}{\partial \delta_k}\Big({p_i}^0 - \mathbb{E}^{\mathbb{P}^\delta}\Big[\epsilon_T^i\Big]\Big) \\
&= \frac{1}{A}\frac{\mathbb{E}\Big[\exp\Big(-\sum_{j=1}^N \frac{\delta_j}{A}\epsilon_T^j\Big)\epsilon_T^i\epsilon_T^k\Big]\mathbb{E}\Big[\exp\Big(-\sum_{j=1}^N \frac{\delta_j}{A}\epsilon_T^j\Big)\Big]}{\Big(\mathbb{E}\Big[\exp\Big(-\sum_{j=1}^N \frac{\delta_j}{A}\epsilon_T^j\Big)\Big]\Big)^2} \\
&\qquad - \frac{\mathbb{E}\Big[\exp\Big(-\sum_{j=1}^N \frac{\delta_j}{A}\epsilon_T^j\Big)\epsilon_T^i\Big]\mathbb{E}\Big[\exp\Big(-\sum_{j=1}^N \frac{\delta_j}{A}\epsilon_T^j\Big)\epsilon_T^k\Big]}{\Big(\mathbb{E}\Big[\exp\Big(-\sum_{j=1}^N \frac{\delta_j}{A}\epsilon_T^j\Big)\Big]\Big)^2} \\
&= \frac{1}{A}\Big(\mathbb{E}^{\mathbb{P}^\delta}\Big[\epsilon_T^i\epsilon_T^k\Big] - \mathbb{E}^{\mathbb{P}^\delta}\Big[\epsilon_T^i\Big]\mathbb{E}^{\mathbb{P}^\delta}\Big[\epsilon_T^k\Big]\Big) \\
&= \frac{1}{A}\mathrm{Cov}^{\mathbb{P}^\delta}\Big(\epsilon_T^i, \epsilon_T^k\Big) \quad , \quad i = 1, 2, \cdots, N.
\end{aligned}$$

The rest of the assertions are trivial.

□

Appendix 3.C Proof of Proposition 8

Proof *(3.23) can be directly derived by (3.13) and (3.21). (3.26) can be also obtained easily by the following fact:*

$$
\begin{aligned}
\frac{\partial}{\partial A}\{-\mathbb{E}^{P^{\delta^*}}\Big[\epsilon_T^i\Big]\} &= -\frac{1}{A^2}\frac{\mathbb{E}\Big[\Big(\sum_{j=1}^N \delta_j\epsilon_T^j\Big)\exp\Big(-\sum_{j=1}^N \frac{\delta_j}{A}\epsilon_T^j\Big)\epsilon_T^i\Big]\mathbb{E}\Big[\exp\Big(-\sum_{j=1}^N \frac{\delta_j}{A}\epsilon_T^j\Big)\Big]}{\Big(\mathbb{E}\Big[\exp\Big(-\sum_{j=1}^N \frac{\delta_j}{A}\epsilon_T^j\Big)\Big]\Big)^2} \\
&\quad + \frac{1}{A^2}\frac{\mathbb{E}\Big[\exp\Big(-\sum_{j=1}^N \frac{\delta_j}{A}\epsilon_T^j\Big)\epsilon_T^i\Big]\mathbb{E}\Big[\Big(\sum_{j=1}^N \delta_j\epsilon_T^j\Big)\exp\Big(-\sum_{j=1}^N \frac{\delta_j}{A}\epsilon_T^j\Big)\Big]}{\Big(\mathbb{E}\Big[\exp\Big(-\sum_{j=1}^N \frac{\delta_j}{A}\epsilon_T^j\Big)\Big]\Big)^2} \\
&= -\frac{1}{A}\mathrm{Cov}^{P^{\delta^*}}\Big(\epsilon_T^i, \sum_{j=1}^N \frac{{\delta_j}^*}{A}\epsilon_T^j\Big).
\end{aligned}
$$

□

Appendix 3.D Monte Carlo Simulation Results

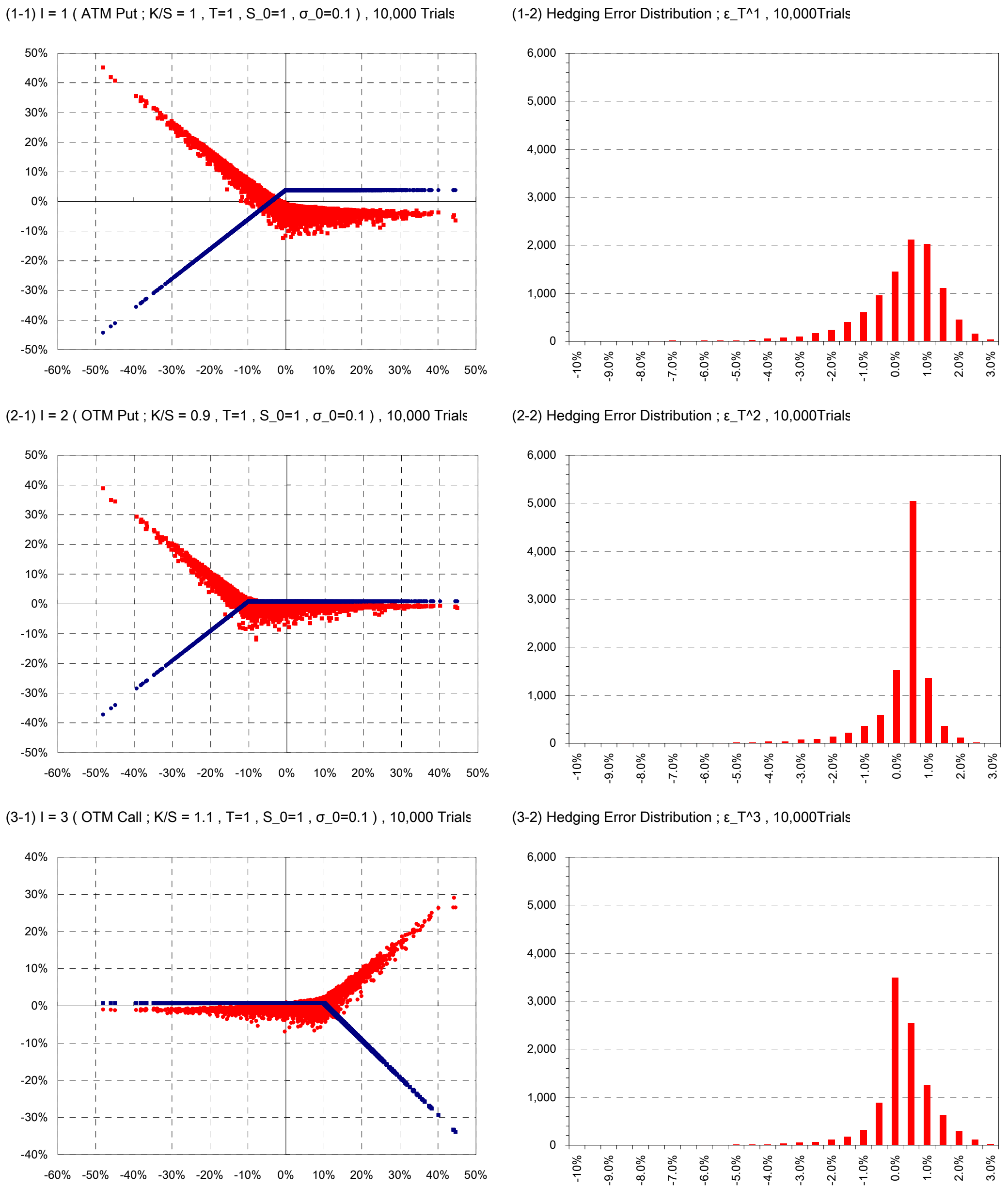

Fig. 3.5: The Distributions of Delta-Neutral Hedging Errors ϵ_T^i, $i = 1, 2, 3$

These figures show the distributions of delta-neutral hedging errors ϵ_T^i, $i = 1, 2, 3$, examined by Monte Carlo simulations of 10,000 trials. The top, middle, and bottom figure are respectively correspondent to $i = 1$, $i = 2$, and $i = 3$. In particular, the left-side figures show delta-neutral hedging simulation results for $i = 1$, $i = 2$, and $i = 3$, respectively, in which the horizontal axis means cumulative rate of returns of the underlying asset at maturity and the vertical line means the profit and loss level of the option contract or the delta-neutral hedging performance. The right-side figures show the distributions of the delta-neutral hedging errors ϵ_T^i in which the horizontal axis means the level of each delta-neutral hedging error at maturity and the vertical line means the frequency of the delta-neutral hedging errors. These simulation results are obtained under the parameter conditions of $T = 1$, $S_0 = 1$, $\kappa = 2.52$, $\gamma = 0.01$, $\eta = 0.23$, and $\rho = -0.3$, and the options of $i = 1$, $i = 2$, and $i = 3$ are respectively correspondent to an ATM put of $K_1 = 1.0$, an OTM put of $K_2 = 0.9$, and an OTM call of $K_3 = 1.1$. The each initial price p_i^0 is calculated by the closed formula for option valuation proposed by Heston[1993] setting the volatility risk premium parameter as zero.

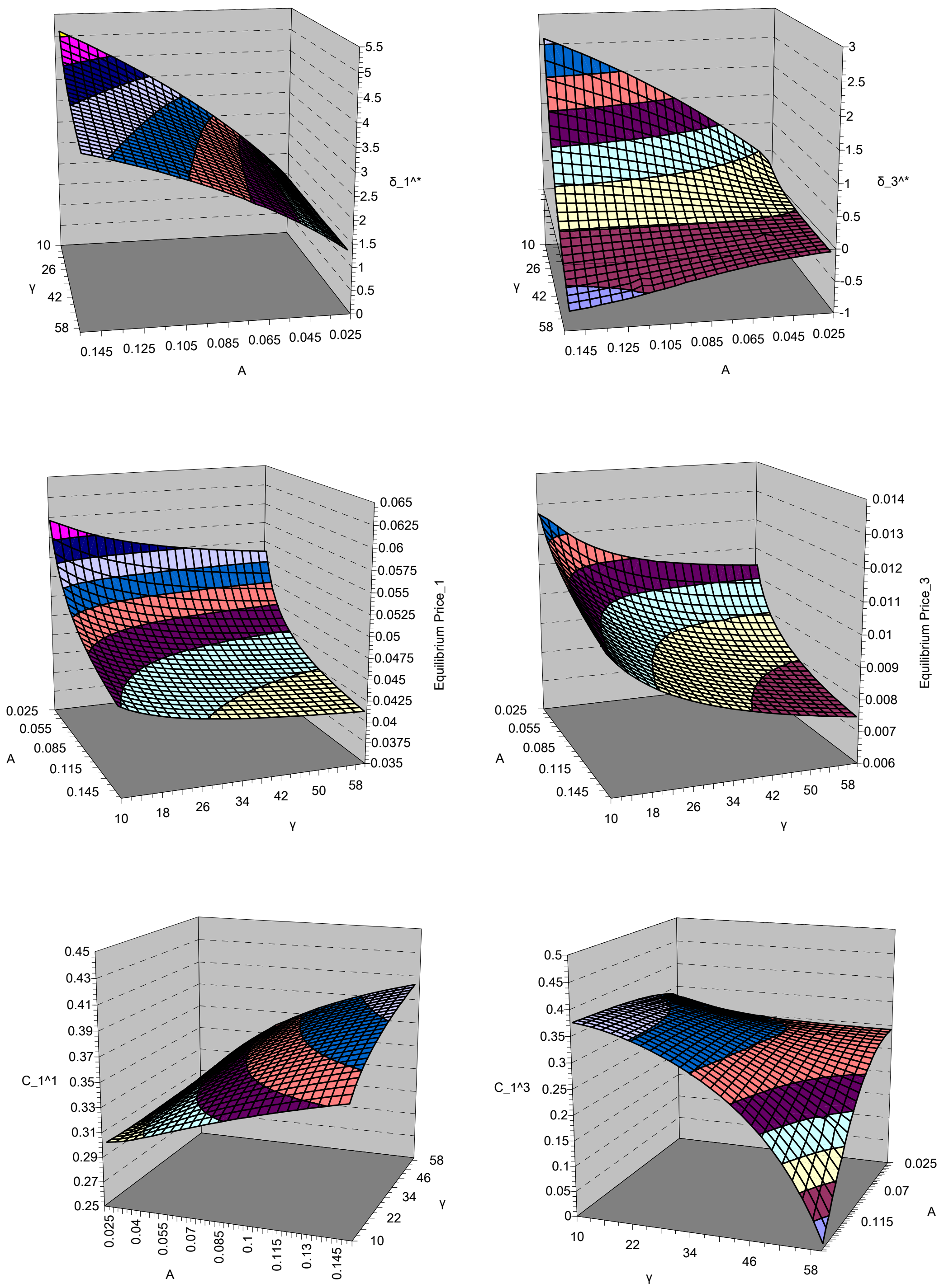

Fig. 3.6: The End-User Net Demand δ_i^* and the Option Prices $p_i(\delta^*)$ in Equilibrium

---

These figures show the end-user net demand δ_i^*, the option prices $p_i(\delta^*)$, and the parameter c_1^i which appears in the representation of (4.19) in equilibrium in the case that the end-user can hold positions in the options of both $i = 1$ (the ATM put of $K_1 = 1.0$) and $i = 3$ (the OTM call of $K_3 = 1.1$). In particular, the left-side figures are the results for the option of $i = 1$ and the right-side figures are the results for the option of $i = 3$. The top, middle, and bottom figures are respectively exhibit the δ_i^*, $p_i(\delta^*)$, and c_1^i for $i = 1$ and $i = 3$.

Chapter 4

Understanding Delta-hedged Option Returns in Stochastic Volatility Environments

4.1 Introduction

In developing risk management strategies for financial option portfolios in incomplete markets, it is necessary to specify the risk factors in the markets and select an option pricing model which is consistent with those of specified risks. In particular, for the practitioners it is essential to consider the matters mentioned above for their risk management processes. In particular, in incomplete markets, option portfolios can not be hedged perfectly due to the unhedgeable risks such as stochastic volatility and jump and are exposed to the risk of significant losses especially in the financial crisis period.

On the basis of these aspects mentioned above, a rich body of studies on empirical option prices and delta-hedged option returns in financial option markets has developed in recent years with some stylized empirical analyses. Coval and Shumway[2001] examine expected option returns in the context of mainstream asset pricing theory and their results strongly suggest that something besides market risk is important in pricing the risk associated with option contracts. They imply that systematic stochastic volatility may be an important factor in pricing assets. Bakshi and Kapadia[2003] and Low and Zhang[2005] study delta-hedged option returns in a stock index option market and currency option markets, respectively, and they provide evidence that expected delta-hedged option returns are not zero because of negative stochastic volatility risk premiums. Goyal and Saretto[2009] study a cross-section of stock option returns by sorting stocks on the difference between historical realized volatility and at-the-money implied volatility. They find that a zero-cost trading strategy that is long (short) in a position with a large positive (negative) difference between these two volatility measures produces an economically and statistically significant return due to some unknown risk factors or mispricing. Broadie, Chernov, and Johannes[2009] conclude that option portfolio returns can be well explained if we consider jump risk premiums or model parameter estimation risk. They assume that investors account for uncertainty in the spot volatility and parameters when pricing options.

Although these studies identify and investigate the sources of financial option prices in terms of some systematic risk factors or mispricing separately, they do not demonstrate any relative contribution to option prices between systematic risk factors and mispricing based on a unified approach.

Jones[2006] presents the most recent research that provides a unified approach to demonstrate the relative contribution of the sources of stock index option prices based on a non-linear factor analysis. He examines the historical performances of equity index option portfolios in the period from January 1986 to September 2000 and shows that priced risk factors such as stochastic volatility and jump contribute to their extraordinary average returns but are insufficient to explain their magnitudes, particularly for short-term out-of-the-money puts. Although this may be the only study that provides a unified approach to demonstrate the relative contribution of the sources of financial option prices based on a stylized model, the author does not reveal any sources besides the priced risk factors such as stochastic volatility and jump that contribute to the option portfolios' extraordinary average returns. The author also does not show the time dependency of the relative contribution between the systematic risk factors and other potential sources such as mispricing to financial option prices especially during the period of the recent financial crisis because his empirical analysis is based on the period from January 1986 to September 2000.

In this chapter, we present the relative contribution analysis between the effect of a systematic risk factor and the effect of *parameter estimation risk* of an option valuation model on financial option prices based on a historical simulation in the pre- and post Lehman crisis period. Theoretical models often assume that the economic agent who makes an optimal financial decision knows the true parameters of the model. But the true parameters are rarely if ever known to the decision maker. In reality, model parameters have to be estimated based on historical information and, hence, the model's usefulness depends partly on how good the estimates are. This gives rise to estimation risk in virtually all option valuation models.

We assume that financial option prices are determined by option market participants basaed on an option valuation model which is consistent with the historical information and the market participants' preferences. The option market participants try to estimate the (unknown) true model parameters "with some error" based on historical information, and they price financial options being consistent with that estimated model. Under this assumption, we provide a novel representation of delta-hedged option returns in a stochastic volatility environment. The representation of delta-hedged option returns provided in this chapter consists of two terms; volatility risk premium and parameter estimation risk. In an empirical analysis, we examine delta-hedged option returns of the USD-JPY currency options based on a historical simulation from October 2003 to June 2010. We find that the delta-hedged option returns for OTM put options are strongly affected by parameter estimation risk as well as the volatility risk premium, especially in the post-Lehman shock period.

To the best of our knowledge, this is the first empirical research on the relative contribution analysis of the effects of systematic risk factors and parameter estimation risk on delta-hedged option returns in a stochastic volatility environment. Of course, there are some prior studies on the effect of parameter estimation risk on pricing financial options (Boyle and Ananthanarayanan[1977], Green and Figlewski[1999], Bunnin, Guo, and Ren[2002], Cont[2006], Broadie, Chernov, and Johannes[2009], etc.) , but there is no empirical evidence that shows the relative contribution between the risk premiums and parameter estimation risk to delta-hedged option returns or the time dependency of that contribution. In our empirical study, approximately 13 % of the value of the OTM currency option premium is generated by the existence of parameter estimation risk in the post-Lehman crisis period, and this effect on option prices is more significant than the effect of the volatility risk premium. One of the most important implications of our study is that the sign and the level of the expected delta-hedged option returns do not necessarily explain the existence of volatility risk premiums. An important point to emphasize is that there may be additional important factors such as parameter estimation risk that make an impact on delta-hedged option returns, rendering standard hedging-based tests on volatility risk premiums explored and examined by, for example, Bakshi and Kapadia[2003] and Low and Zhang[2005], unreliable.

This chapter is organized as follows: Section 2 describes the model structure and provides an explicit representation of delta-hedged option returns. An estimation methodology for the time-varying volatility risk premium in the USD-JPY currency option market is also explored in this section. Section 3 describes the basic methodology used in our empirical analysis, and Section 4 illustrates the nature of the delta-hedged option returns and presents empirical findings on the relative contributions of the effects of the volatility risk premium and parameter estimation risk on delta-hedged option returns. Section 5 summarizes the main results of this chapter.

4.2 The Model and the Methodology

4.2.1 An explicit representation for delta-hedged option returns

We start with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\geq 0})$, $t \in [0, T]$ and consider a two dimensional exchange rate process that allows return volatility to be stochastic under the physical probability measure $\mathbb{P}$:

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu_t dt + \sigma_t \sqrt{1-\rho^2} dW_t^1 + \sigma_t \rho dW_t^2, \\ d\sigma_t &= \theta_t dt + \eta_t dW_t^2, \end{aligned} \tag{4.1}$$

where $\mu_t \equiv \mu(t, S_t, \sigma_t)$, $\theta_t \equiv \theta(t, S_t, \sigma_t)$, and $\eta_t \equiv \eta(t, S_t, \sigma_t)$ are $\{\mathcal{F}_t\}_{t\geq 0}$-adapted stochastic processes with deterministic functions μ, θ, and η, respectively, which allow the above equations to have a strong solution. (W_t^1, W_t^2) denote a standard 2-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we give the information set $\mathcal{F}_t$ as a sigma-algebra of $\sigma\{W_s^1, W_s^2 | s \leq t\} \vee \mathcal{N}$, where $\mathcal{N}$ is the null set. The ρ is a constant which is in $[-1, 1] \subset \mathbb{R}^1$. Moreover, in the following, $r_d \in \mathbb{R}$ and $r_f \in \mathbb{R}$ denote the domestic and the foreign risk-free interest rate, respectively. In this chapter, we assume that the parameters and state variables σ_t are unobserved and cannot be perfectly estimated from historical information.

We limit the model of an exchange rate process to a stochastic volatility model and do not consider other factors such as the jump. Although this model is rather restrictive, but in Andersen, Bollerslev, Diebold and Labys[2000], based on ten years of high-frequency returns for the Deutschemark - U.S. Dollar and Japanese Yen - U.S. Dollar exchange rates, they provide indirect support for the assertion of a jumpless diffusion with a fact that the presence of jumps is likely to result in a violation of the empirical normality of the standardized returns.

It is well known that the absence of arbitrage opportunities is essentially equivalent to the existence of a probability measure $\mathbb{Q}$, equivalent to the physical probability measure $\mathbb{P}$, under which the discounted prices process is an $\mathcal{F}_t$-adapted martingale; such a probability will be called equivalent martingale measure. Any equivalent martingale measure $\mathbb{Q}$ is characterized by a continuous version of its density process with respect to $\mathbb{P}$ which can be written from the integral form of martingale representation:

$$M_t \equiv \frac{d\mathbb{Q}}{d\mathbb{P}}\,|_{\mathcal{F}_t} \equiv \exp\Big(-\int_0^t \nu_u dW_u^1 - \int_0^t \lambda_u dW_u^2 - \frac{1}{2}\int_0^t \nu_u^2 du - \frac{1}{2}\int_0^t \lambda_u^2 du\Big),$$

where (ν_t, λ_t) is adapted to $\mathcal{F}_t$ and satisfies the integrability conditions $\int_0^T \nu_u^2 du < \infty$ and $\int_0^T \lambda_u^2 du < \infty$ a.s.. These two processes of ν_t and λ_t satisfy the following market price of risk equation, $\sigma_t\rho\lambda_t + \sigma_t\sqrt{1-\rho^2}\nu_t = r_f - r_d + \mu_t$, and, in particular, are determined so that the discounted price process $e^{-(r_d-r_f)t}S_t$ can be a martingale under this equivalent martingale measure $\mathbb{Q}$ [1]. Two processes of ν_t and λ_t are interpreted as the price of risk premia relative respectively to the two sources of uncertainty W_t^1 and W_t^2. In particular, if $\Lambda_t \equiv M_t B_t$ denotes the discount factor process where $B_t \equiv \exp\big(-r_d t\big)$, then the price of volatility risk λ_t is defined as $\lambda_t \equiv -\mathrm{Cov}_t(\frac{d\Lambda_t}{\Lambda_t}, d\sigma_t)$ (See, e.g., Cochrane[2001]) and a positive correlation between the discount factor process Λ_t and the volatility process σ_t implies a negative λ_t. To understand clearly, for example, if we could assume the stochastic volatility model proposed by Heston[1993] for (4.1), that is to say that $\theta_t \equiv -\kappa\sigma_t$ and $\eta_t \equiv v$ where κ and v are constants, and the representative agent with a power utility function in the financial market, then we can derive the following equation [2]:

$$\mathrm{Cov}_t\Big(\frac{d\Lambda_t}{\Lambda_t}, d\sigma_t\Big) = -\gamma\rho v\sigma_t,$$

where $\gamma \in \mathbb{R}$ is the risk aversion parameter for the representative agent. This relation suggests that the market price of volatility risk λ_t is proportional to ρ, and if the correlation between volatility changes and changes in the exchange rate is negative, then the market price of volatility risk is also negative. Hereinafter we also describe $\mathbb{Q}$ by $\mathbb{Q}[\lambda_t]$ to emphasize that $\mathbb{Q}$ is depend on the process of λ_t.

$C_t^M \equiv G(t, S_t, \sigma_t)$ denotes the *time-t market price* of an European-type call option [3] on the underlying exchange rate S_t. This C_t^M is struck at K, expiring at time T, and represented by a $C^{1,2,2}$-function $G(t, S_t, \sigma_t) : \mathbb{R}^3 \to \mathbb{R}^1$. In the following, we consider delta-hedged option returns of $C_t^M \equiv G(t, S_t, \sigma_t)$ for the representative option market participant under model parameter estimation risk.

In the actual market, the option market participants could not necessarily price options based on the true equation (4.1) because they essentially could not know the true

[1] See § 9.3 in Shreve[2004] for details.

[2] Details will be described in section 4.1.

[3] In this section, we focus on an European call option, but the discussion and results provided in this section can apply more generally to other options such as put options and straddle options.

parameters of (4.1) due to limited information or sampling errors, which are also pointed out by the previous studies such as those by Boyle and Ananthanarayanan[1977], Green and Figlewski[1999], and Broadie, Chernov, and Johannes[2009]. That is to say that the option market participants might believe the another set of parameters that are not necessarily true and price options based on those of misestimated parameters while they try to estimate the true dynamics (4.1) based on historical market information. So we assume that the option market participants estimate the another set of model parameters $\tilde{\mu}_t \equiv \tilde{\mu}(t, S_t, \sigma_t)$, $\tilde{\theta}_t \equiv \tilde{\theta}(t, S_t, \sigma_t)$, $\tilde{\eta}_t \equiv \tilde{\eta}(t, S_t, \sigma_t)$ and $\tilde{\rho} \in [-1, 1]$ for the drift of the exchange rate price process, the drift of the volatility process, the diffusion coefficient of the volatility process, and the correlation between volatility changes and changes in the exchange rate, respectively. The parameters of $\tilde{\mu}_t$, $\tilde{\theta}_t$, $\tilde{\eta}_t$, and $\tilde{\rho}$ may not be equal to the parameters of μ_t, θ_t, η_t, and ρ in (4.1), respectively, because of estimation errors caused by limited information. Under such condition, it is natural to assume that the option market participants price options based on the following alternative model for an exchange rate process $\tilde{S}_t$ [4] :

$$
\begin{aligned}
\frac{d\tilde{S}_t}{\tilde{S}_t} &= \tilde{\mu}_t dt + \tilde{\sigma}_t \sqrt{1-\tilde{\rho}^2} dW_t^1 + \tilde{\sigma}_t \tilde{\rho} dW_t^2, \\
d\tilde{\sigma}_t &= \tilde{\theta}_t dt + \tilde{\eta}_t dW_t^2.
\end{aligned} \tag{4.2}
$$

The market price of the option at time t, $C_t^M \equiv G(t, S_t, \sigma_t)$, determined by the representative option market participant which is consistent with the model (4.2) satisfies the following pricing equation:

$$
\begin{aligned}
&\frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 G}{\partial S^2}(t, S_t, \sigma_t) + \frac{1}{2}\tilde{\eta}_u^2 \frac{\partial^2 G}{\partial \sigma^2}(t, S_t, \sigma_t) \\
&\qquad + \tilde{\rho}\tilde{\eta}_t \sigma_t S_t \frac{\partial^2 G}{\partial S \partial \sigma}(t, S_t, \sigma_t) + (r_d - r_f) S_t \frac{\partial G}{\partial S}(t, S_t, \sigma_t) \\
&\qquad + (\tilde{\theta}_t - \lambda_t) \frac{\partial G}{\partial \sigma}(t, S_t, \sigma_t) + \frac{\partial G}{\partial t}(t, S_t, \sigma_t) - r_d G(t, S_t, \sigma_t) = 0.
\end{aligned} \tag{4.3}
$$

Due to such parameter estimation risk on the exchange rate dynamics introduced above, the representative option market participant try to hedge according to the misspecified function G, and this aspect leads to the following discussion.

[4] We assume that the SDE (4.2) has a strong solution for $\tilde{S}_t$ under the parameters of $\tilde{\mu}_t$, $\tilde{\theta}_t$, $\tilde{\eta}_t$, and $\tilde{\rho}$.

When $0 \leq \tau \leq T - t$, we can derive a following equation via Ito's lemma:

$$\begin{aligned} C^M_{t+\tau} = C^M_t &+ \int_t^{t+\tau} \frac{\partial G}{\partial S}(u, S_u, \sigma_u) dS_u \\ &+ \int_t^{t+\tau} \frac{\partial G}{\partial \sigma}(u, S_u, \sigma_u) d\sigma_u + \int_t^{t+\tau} \mathcal{D}G(u, S_u, \sigma_u) du, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \mathcal{D}G(t, S_t, \sigma_t) = \frac{\partial G}{\partial t}(t, S_t, \sigma_t) &+ \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 G}{\partial S^2}(t, S_t, \sigma_t) \\ &+ \frac{1}{2}\eta_t^2 \frac{\partial^2 G}{\partial \sigma^2}(t, S_t, \sigma_t) + \rho \eta_t \sigma_t S_t \frac{\partial^2 G}{\partial S \partial \sigma}(t, S_t, \sigma_t). \end{aligned}$$

If we define the delta-hedged gain and loss (hereinafter, DHGL) $\Pi^G_{t,t+\tau}$ for C^M_t in the period of $[t, t+\tau]$ by

$$\begin{aligned} \Pi^G_{t,t+\tau} \equiv C^M_{t+\tau} - C^M_t &- \int_t^{t+\tau} \frac{\partial G}{\partial S}(u, S_u, \sigma_u) dS_u \\ &- \int_t^{t+\tau} \Big(r_d G(u, S_u, \sigma_u) - (r_d - r_f) S_u \frac{\partial G}{\partial S}(u, S_u, \sigma_u) \Big) du, \end{aligned}$$

then, from (4.4) it follows

$$\begin{aligned} \Pi^G_{t,t+\tau} &= \int_t^{t+\tau} \frac{\partial G}{\partial \sigma}(u, S_u, \sigma_u) d\sigma_u + \int_t^{t+\tau} \mathcal{D}G(u, S_u, \sigma_u) du \\ &\quad - \int_t^{t+\tau} \Big(r_d G(u, S_u, \sigma_u) - (r_d - r_f) S_u \frac{\partial G}{\partial S}(u, S_u, \sigma_u) \Big) du \\ &= \int_t^{t+\tau} \mathcal{L}G(u, S_u, \sigma_u) du + \int_t^{t+\tau} \eta_u \frac{\partial G}{\partial \sigma}(u, S_u, \sigma_u) dW^2_u, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} \mathcal{L}G(u, S_u, \sigma_u) = \theta_u &\frac{\partial G}{\partial \sigma}(u, S_u, \sigma_u) \\ &+ \mathcal{D}G(u, S_u, \sigma_u) - \Big[r_d G(u, S_u, \sigma_u) - (r_d - r_f) S_u \frac{\partial G}{\partial S}(u, S_u, \sigma_u) \Big]. \end{aligned} \tag{4.6}$$

Thanks to (4.3),

$$\begin{aligned} &\frac{\partial G}{\partial t}(t, S_t, \sigma_t) + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 G}{\partial S^2}(t, S_t, \sigma_t) - r_d G(t, S_t, \sigma_t) + (r_d - r_f) S_t \frac{\partial G}{\partial S}(t, S_t, \sigma_t) \\ &= -(\tilde{\theta}_t - \lambda_t) \frac{\partial G}{\partial \sigma}(t, S_t, \sigma_t) - \frac{1}{2}\tilde{\eta}_u^2 \frac{\partial^2 G}{\partial \sigma^2}(t, S_t, \sigma_t) - \tilde{\rho}_t \tilde{\eta}_t \sigma_t S_t \frac{\partial^2 G}{\partial S \partial \sigma}(t, S_t, \sigma_t). \end{aligned} \tag{4.7}$$

Substituting (4.7) into (4.5), we have

$$\begin{aligned}\Pi^G_{t,t+\tau} = &\int_t^{t+\tau} \Big[\frac{1}{2}(\eta_u^2 - \tilde{\eta}_u^2)\frac{\partial^2 G}{\partial \sigma^2}(u, S_u, \sigma_u) \\ &+ (\rho_u\eta_u - \tilde{\rho}_u\tilde{\eta}_u)\sigma_u S_u \frac{\partial^2 G}{\partial S \partial \sigma}(u, S_u, \sigma_u) + (\theta_u - \tilde{\theta_u})\frac{\partial G}{\partial \sigma}(u, S_u, \sigma_u)\Big] du \\ &+ \int_t^{t+\tau} \lambda_u \frac{\partial G}{\partial \sigma}(u, S_u, \sigma_u) du + \int_t^{t+\tau} \eta_u \frac{\partial G}{\partial \sigma}(u, S_u, \sigma_u) dW_u^2.\end{aligned} \tag{4.8}$$

Thus, taking expectation to (4.8) under the physical measure $\mathbb{P}$, the expected DHGL for the representative option market participant can be derived in the following equation:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[\Pi^G_{t,t+\tau}\big] = &\int_t^{t+\tau} \mathbb{E}^{\mathbb{P}}\Big[\frac{1}{2}(\eta_u^2 - \tilde{\eta}_u^2)\frac{\partial^2 G}{\partial \sigma^2}(u, S_u, \sigma_u) \\ &+ (\rho_u\eta_u - \tilde{\rho}_u\tilde{\eta}_u)\sigma_u S_u \frac{\partial^2 G}{\partial S \partial \sigma}(u, S_u, \sigma_u) + (\theta_u - \tilde{\theta_u})\frac{\partial G}{\partial \sigma}(u, S_u, \sigma_u)\Big] du \\ &+ \int_t^{t+\tau} \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial G}{\partial \sigma}(u, S_u, \sigma_u)\Big] du.\end{aligned}$$

Thus, we have the following proposition:

Proposition 12 *If the representative option market participant tries to hedge based on the misspecified function G, then the expectation of the delta-hedged gain and loss from time t to $t+\tau$, $\Pi^G_{t,t+\tau}$, for the market participant, that is, $\mathbb{E}^{\mathbb{P}}[\Pi^G_{t,t+\tau}]$, is represented by the following formula:*

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[\Pi^G_{t,t+\tau}\big] = &\int_t^{t+\tau} \mathbb{E}^{\mathbb{P}}\Big[\frac{1}{2}(\eta_u^2 - \tilde{\eta}_u^2)\frac{\partial^2 G}{\partial \sigma^2}(u, S_u, \sigma_u) \\ &+ (\rho\eta_u - \tilde{\rho}\tilde{\eta}_u)\sigma_u S_u \frac{\partial^2 G}{\partial S \partial \sigma}(u, S_u, \sigma_u) + (\theta_u - \tilde{\theta_u})\frac{\partial G}{\partial \sigma}(u, S_u, \sigma_u)\Big] du \\ &+ \int_t^{t+\tau} \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial G}{\partial \sigma}(u, S_u, \sigma_u)\Big] du.\end{aligned} \tag{4.9}$$

An interpretation for the expected DHGL (4.9) is given as follows. If the price process of an underlying exchange rate assumed by the representative option market maker coincide with the true price process, that is to say, $\theta_t = \tilde{\theta}_t$, $\eta_t = \tilde{\eta}_t$ and $\rho = \tilde{\rho}$, then the first term in the right side of (4.9) equals zero and, as a result, the representation of the expected DHGL under the physical measure $\mathbb{P}$ will be

$$\mathbb{E}^{\mathbb{P}}\big[\Pi^G_{t,t+\tau}\big] = \int_t^{t+\tau} \mathbb{E}^{\mathbb{P}}\big[\lambda_u \frac{\partial G}{\partial \sigma}(u, S_u, \sigma_u)\big] du.$$

Thus, the expected DHGL will be perfectly determined only by the level of the volatility risk premium λ_t. In addition to the above assumption, if we also assume a linear form of $\lambda_u \equiv \lambda\sigma_u \ (\lambda \in \mathbb{R}^1)$ for the volatility risk premium as well as the stochastic volatility model proposed by Heston[1993], the sign of the expected DHGL will coincide with the sign of the volatility risk premium parameter λ because $\sigma_u > 0$ and $\frac{\partial C_u}{\partial \tilde{\sigma}_u} > 0$. However, if the parameters for the representative option market maker, $\tilde{\theta}_t$, $\tilde{\eta}_t$ and $\tilde{\rho}_t$, are not equal with the true parameters, that is, θ_t, η_t, and ρ_t, the sign and the magnitude of the expected DHGL will be affected by the parameter estimation risk term represented by the first term in the right side of (4.9), as well as the volatility risk premium term.

Let us provide a proposition related to the second term in the right side of (4.9) for the purpose of obtaining a more testable representation of the expected DHGL.

Proposition 13 *If $C_t^M[\lambda_u]$ denotes the time-t call option price which is consistent with the underlying exchange rate process (4.2) and the equivalent martingale measure $\mathbb{Q}[\lambda_u]$, then the following inequation can be derived:*

$$C_t^M[0] - C_t^M[\lambda_u] \le (\ge) \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \sigma_u}\Big] du \le (\ge) \Big(1 + r_d(T-t)\Big)\Big(C_t^M[0] - C_t^M[\lambda_u]\Big),$$

where $\lambda_u \ge (\le) 0$. Moreover, if we are able to assume $r_d \approx 0$, then this inequation can be further simplified to the approximation presented below:

$$C_t^M[0] - C_t^M[\lambda_u] \approx \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \sigma_u}\Big] du. \tag{4.10}$$

Proof *See the Appendix.* □

Under the zero interest rate financial policy in Japan after the year of 2000, we are allowed to consider r_d to be approximately zero. So we can assume that (4.10) will be well-suited for the exchange rates whose base currency is the Japanese Yen.

Substituting the approximation of (4.10) for (4.9), we find that an explicit representation of the expected DHGL for the representative option market maker can be described as follows:

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\big[\Pi_{t,t+\tau}^G\big] \approx \int_t^{t+\tau} & \mathbb{E}^{\mathbb{P}}\Big[\frac{1}{2}(\eta_u^2 - \tilde{\eta}_u^2)\frac{\partial^2 G}{\partial \sigma^2}(u, S_u, \sigma_u) \\ & + (\rho\eta_u - \tilde{\rho}\tilde{\eta}_u)\sigma_u S_u \frac{\partial^2 G}{\partial S \partial \sigma}(u, S_u, \sigma_u) + (\theta_u - \tilde{\theta}_u)\frac{\partial G}{\partial \sigma}(u, S_u, \sigma_u)\Big] du \\ & + C_t^M[0] - C_t^M[\lambda_u]. \end{aligned} \tag{4.11}$$

In the following section, we will simulate the DHGLs with historical market data and estimate the left side of (4.11) empirically. If we can valuate the second term in the right side of (4.11) in each period, then we will be able to provide a contribution analysis between the effect of parameter uncertainty and the effect of the volatility risk premium on the expected DHGL based on (4.11) empirically. As mentioned above, we assume that the representative option market participant determines option market prices being consistent with (4.2) by estimating the set of parameters in (4.2), that is to say, $\tilde{\theta}_t$, $\tilde{\eta}_t$, and $\tilde{\rho}$, based on historical market price data. Thus, under the assumption stated above, obtaining the set of parameters in (4.2) with historical market price data and calibrating the volatility risk premium implied in option market prices in each period, we will be able to provide a contribution analysis on the expected DHGL explicitly. Let us describe an explicit way of providing the contribution analysis in the following subsection.

An estimation strategy for $\tilde{\theta}_t$, $\tilde{\eta}_t$, and $\tilde{\rho}$ with historical price data will be provided in the next section. In the next subsection, we provide a calibration methodology for the volatility risk premium parameter implied in option market prices in each period with estimated parameters of $\tilde{\theta}_t$, $\tilde{\eta}_t$, and $\tilde{\rho}$.

For simplicity, hereinafter we assume $\theta_t \equiv -k\sigma_t$, $\eta_t \equiv v$, $\tilde{\theta}_t \equiv -\tilde{k}\sigma_t$, and $\tilde{\eta}_t \equiv \tilde{v}$ (k, v, $\tilde{k}$, and $\tilde{v}$ are constants), that is to say, the stochastic volatility model proposed by Heston[1993] and investigate the expected DHGL represented by (4.11) in detail under such assumptions. Heston[1993] assumes linear form of $\lambda_t[\sigma] \equiv \lambda\sigma_t$ for the volatility risk premium and we are also assume that form in line with Heston[1993].

4.2.2 Estimation Strategy for the Volatility Risk Premium

As mentioned above, we assume that the representative option market participant prices options based on the model under the assumptions of $\tilde{\theta}_t \equiv -\tilde{k}\sigma_t$ and $\tilde{\eta}_t \equiv \tilde{v}$ in (4.2), or

$$\begin{aligned} \frac{d\tilde{S}_t}{\tilde{S}_t} &= \tilde{\mu}_t dt + \tilde{\sigma}_t\sqrt{1-\tilde{\rho}^2}dW_t^1 + \tilde{\sigma}_t\tilde{\rho}dW_t^2, \\ d\tilde{\sigma}_t &= -\tilde{k}\tilde{\sigma}_t dt + \tilde{v}dW_t^2. \end{aligned} \tag{4.12}$$

If we also assume the formula of the volatility risk premium as $\lambda_t \equiv \lambda\tilde{\sigma}_t$ ($\lambda \in \mathbb{R}$), that is to say, a linear form on the volatility, (4.12) will be rewrited under the risk neutral measure $\mathbb{Q}[\lambda\tilde{\sigma}_t]$ as follows:

$$\begin{aligned} \frac{d\tilde{S}_t}{\tilde{S}_t} &= (r_d - r_f)dt + \tilde{\sigma}_t\sqrt{1-\tilde{\rho}^2}d\tilde{W}_t^1 + \tilde{\sigma}_t\tilde{\rho}d\tilde{W}_t^2, \\ d\tilde{\sigma}_t &= -(\tilde{k}+\lambda)\tilde{\sigma}_t dt + \tilde{v}d\tilde{W}_t^2, \end{aligned} \tag{4.13}$$

where $\tilde{W}_t \equiv \left(\tilde{W}_t^1, \tilde{W}_t^2\right)^t$ is two-dimensional Brownian motion under $\mathbb{Q}[\lambda\tilde{\sigma}_t]$ whose each component is represented as follows:

$$\tilde{W}_t^1 \equiv W_t^1 + \int_0^t \nu_u du \quad \text{and} \quad \tilde{W}_t^2 \equiv W_t^2 + \lambda \int_0^t \tilde{\sigma}_u du.$$

Under (4.13), we can derive the expectation of instantaneous variance at time t under $\mathbb{Q} \equiv \mathbb{Q}[\lambda\tilde{\sigma}_t]$,

$$\mathbb{E}_t^{\mathbb{Q}}[\tilde{\sigma}_u^2] = \tilde{\sigma}_t^2 \exp(-2(\tilde{k}+\lambda)(u-t)) + \frac{\tilde{v}^2}{2(\tilde{k}+\lambda)}\Big(1 - \exp(-2(\tilde{k}+\lambda)(u-t))\Big), \quad t \le u.$$

Thus the expectation of realized variance $RV_{t,T}$ in the period of $[t,T]$ under $\mathbb{Q}$ will be

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}[RV_{t,T}] &= \mathbb{E}_t^{\mathbb{Q}}[\frac{1}{T-t}\int_t^T \tilde{\sigma}_u^2 du] = \frac{1}{T-t}\int_t^T \mathbb{E}_t^{\mathbb{Q}}[\tilde{\sigma}_u^2]du \\ &= \frac{\tilde{v}^2}{2(\tilde{k}+\lambda)} + \frac{\exp(-2(\tilde{k}+\lambda)T) - \exp(-2(\tilde{k}+\lambda)t)}{2(\tilde{k}+\lambda)(T-t)}\Big(\frac{\tilde{v}^2}{2(\tilde{k}+\lambda)} - \tilde{\sigma}_t^2\Big). \end{aligned} \tag{4.14}$$

On the other hand, Carr and Wu[2009] provide the formula for the risk neutral expected value of return variance which can be well approximated with the value of a particular portfolio of options [5].

Proposition 14 (Carr and Wu(2009)) *Under no arbitrage, the time-t risk-neutral expected value of the return quadratic variation of an asset over horizon $[t,T]$ can be approximated by the continuum of European out-of-the-money option prices across all strikes $K > 0$ and at the same maturity date T*

$$\mathbb{E}_t^{\mathbb{Q}}[RV_{t,T}] = \frac{2}{T-t}\int_0^\infty \frac{\Theta_t(K,T)}{B_t(T)K^2}dK, \tag{4.15}$$

[5] Carr and Wu[2009] assume that the futures price F_t solves the following stochastic differential equation,

$$dF_t = F_{t-}\sigma_{t-}dW_t + \int_{(-\infty,\infty)\backslash 0} F_{t-}\left(e^x - 1\right)\left[\mu(dx,dt) - \nu_t(x)dxdt\right]$$

(see Carr and Wu[2009] for details on a notation). The equation represented above models the futures price change as the summation of the increments of two orthogonal martingales: a purely continuous martingale and a purely discontinuous (jump) martingale. This decomposition is generic for any continuous time martingales. So, in general, Proposition 2 should be stated including the effect of jump component. But, in this chapter, we only assume a *continuous* martingale in order to represent an underlying exchange rate process, so we leave the term induced by the jump component out of (4.15) in Proposition 2.

where $B_t(T)$ denotes the time-t price of a bond paying one dollar at T, $\Theta_t(K,T)$ denotes the time-t value of an out-of-the-money option with strike price $K > 0$ and maturity $T \geq t$ (a call option when $K > F_t$ and a put option when $K \leq F_t$).

Proof *See proof of Proposition1 in Carr and Wu[2009].* □

Using the set of parameters in (4.2) and option prices $\Theta_t(K,T)$ quoted in an option market, we can estimate λ explicitly based on the following equation

$$\begin{aligned} \frac{2}{T-t}\int_0^\infty \frac{\Theta_t(K,T)}{B_t(T)K^2}dK = & \frac{\tilde{v}^2}{2(\tilde{k}+\lambda)} \\ & + \frac{\exp(-2(\tilde{k}+\lambda)T)-\exp(-2(\tilde{k}+\lambda)t)}{2(\tilde{k}+\lambda)(T-t)}\Big(\frac{\tilde{v}^2}{2(\tilde{k}+\lambda)}-\tilde{\sigma}_t^2\Big), \end{aligned} \tag{4.16}$$

which can be derived with (4.14) and (4.15). Thus, using the λ estimated with the equation (4.16), we can calculate $C_t^M[0] - C_t^M[\lambda_u]$, the second term in the right hand side of (4.11), at each time t using the closed formula for the European-type call option proposed by Heston[1993]:

$$C_t^M[\lambda\tilde{\sigma}_t] = S_tP_1 + e^{-r\tau}KP_2,$$

where

$$\begin{aligned} P_j &= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-\sqrt{-1}\phi ln(K)}F_j}{\sqrt{-1}\phi}\right]d\phi, \\ F_j &= e^{C+D\tilde{\sigma}_t^2+\sqrt{-1}\phi ln(S_t)}, \\ C &= (r_d-r_f)\tau\phi\sqrt{-1}+\frac{1}{4}\left[(\beta_j-2\tilde{\rho}\tilde{v}\phi\sqrt{-1}+h)\tau-2ln\Big(\frac{1-ge^{h\tau}}{1-g}\Big)\right], \\ D &= \frac{\beta_j-2\tilde{\rho}\tilde{v}\phi\sqrt{-1}+h}{4\tilde{v}^2}\Big(\frac{1-e^{h\tau}}{1-ge^{h\tau}}\Big), \\ g &= \frac{\beta_j-2\tilde{\rho}\tilde{v}\phi\sqrt{-1}+h}{\beta_j-2\tilde{\rho}\tilde{v}\phi\sqrt{-1}-h}, \\ h &= \sqrt{(2\tilde{\rho}\tilde{v}\phi\sqrt{-1}-\beta_j)^2-4\tilde{v}^2(2u_j\phi\sqrt{-1}-\phi^2)}, \qquad (j=1,2) \end{aligned} \tag{4.17}$$

and

$$\tau = T-t, u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, \beta_1 = 2\tilde{k}+\lambda-2\tilde{\rho}\tilde{v}, \beta_2 = 2\tilde{k}+\lambda.$$

Needless to say, we can also derive the closed formula for the European-type put option by using (4.17) and the put-call parity relation and will use these closed formulas for pricing currency options in the following empirical simulations.

4.2.3 An Analytical Process for a Contribution Analysis

In this subsection, let us summarize an analytical process for a contribution analysis on the expected DHGL of the currency option, which is explored in the next section.

In this chapter, as mentioned in the previous section, the expected DHGL of the currency option consists of the volatility risk premium and parameter estimation risk, and the brief analytical process for a contribution analysis on the expected DHGL is as follows:

Step 1 *Each of the DHGLs for an option portfolio is simulated based on historical data of an exchange rate at each time point.*

Step 2 *The volatility risk premium parameter is estimated on the basis of the approach demonstrated in the previous subsection and the effect of the volatility risk premium,* $C_t^M[0] - C_t^M[\lambda_u]$ *in (4.11), on the DHGL is calculated with estimated volatility risk premium parameter and the option pricing formula proposed by Heston[1993] at each time when the delta-neutral hedging strategy for an option portfolio is started.*

Step 3 *The effect of parameter estimation risk on the DHGL for an option portfolio is calculated by subtracting the calculation result of* $C_t^M[0] - C_t^M[\lambda_u]$ *in Step 2 from the result of the DHGL simulated in Step 1.*

Step 4 *Then, we average each of the time-series results of DHGLs, the effects of the volatility risk premium on the DHGL, and the effects of parameter estimation risk on the DHGL, and decompose the expected DHGL into two effects from the volatility risk premium and parameter estimation risk.*

In the next section, we provide more detailed explanation on the data and methodology for an empirical implementation.

4.3 Data and Methodology for an Empirical Implementation

4.3.1 Description of the OTC Currency Option Market and Data

In our empirical study, we examine the expected DHGL and its contribution analysis with the USD-JPY spot exchange rate and the USD-JPY currency options with maturities of one month traded on the OTC market. The OTC currency option market has some special features and conventions. First, option prices in the OTC market are quoted in terms of deltas and implied volatilities instead of strikes and money prices, as in the organized option exchanges. At the time of settling a given deal, the implied volatility quotes are translated to money prices using the Garman-Kohlhagen formula, which is the equivalent of the Black-Scholes formula for currency options. This arrangement is convenient for option dealers in that they do not have to change their quotes every time the spot exchange rate moves. However, it is important to note that this does not mean that option dealers necessarily believe that the Black-Scholes assumptions are valid. They use the formula only as a one-to-one nonlinear mapping between the volatility delta space (where the quotes are made) and the strike premium space (in which the final specification of the deal is expressed for the settlement). Second, most transactions in the market involve option combinations. The popular combinations are straddles, risk reversals, and strangles. Among these, the most liquid combination is the standard delta-neutral straddle contract, which is a combination of a call and a put with the same strike. The strike price is set, together with the quoted implied volatility space, such that the delta of the straddle computed on the basis of the Garman-Kohlhagen formula is zero.

Because the standard straddle is by design delta neutral on the deal date, its price is not sensitive to the market price of the underlying foreign currency. However, it is sensitive to changes in volatility. Because of its sensitivity to volatility risk, delta-neutral straddles are widely used by participants in the OTC market to hedge and trade volatility risk. If the volatility risk is priced in the OTC market, then delta-neutral straddles are the best instruments through which to observe the risk premium. For this reason, Coval and Shumway[2001] use delta-neutral straddles in their empirical study of expected returns on equity index options and find that a volatility risk premium is priced in the equity index option market.

We use the WM/Reuter closing spot rate for the USD-JPY spot exchange rate data, the LIBOR 1M interest rates for the domestic (Japan) and the foreign (United States of America) interest rates, and quoted implied volatility data from Bloomberg. The implied volatility data is from the European type put and call OTC currency options with maturities of one month and strike prices of 5 delta, 10 delta, 15 delta, 25 delta, 35 delta and ATM, respectively. All the data are daily-based data. In the following empirical simulation, we price the options using bid prices quoted in the actual market at each time point in order to take account of transactions costs when simulating the profit and loss generated by a delta-neutral hedging strategy with a short position of each of the European options. Our data sample starts in October 2003 (because of data availability of the implied volatility in the USD-JPY currency option market) and ends in June 2010.

Fig.4.3, Fig.4.4 and Fig.4.5 show the time series data for the USD-JPY WM/Reuter closing spot rate, the ATM implied volatility of the USD-JPY European option, and the mid-bid price spread of the ATM implied volatility which indicates the level of transactions costs for selling strategies of the European ATM options at each time point, respectively. Table 4.7 provides descriptive statistics for the implied volatilities in the period from October 2003 to June 2010. In this table we can find the feature of "volatility skew", which indicates that the implied volatilities for OTM puts are higher than those for OTM calls during the period under consideration.

4.3.2 Parameter Estimation for the Heston[1993] Stochastic Volatility Model

In this chapter, we estimate a set of parameters for the Heston[1993] stochastic volatility model expressed in equation (4.12) with the maximum likelihood method proposed by Aït-Sahalia[2001] and Aït-Sahalia and Kimmel[2007]. Aït-Sahalia and Kimmel[2007] provide an approximation formula for the likelihood function using the Hermite series expansion of the transition probability density of the Heston[1993] stochastic volatility model and propose a methodology for estimating the parameters of multivariate diffusion processes via the maximum likelihood method with discrete-sampled price data. They derive a closed form likelihood function used explicitly to estimate the parameters of a two-dimensional diffusion process consisting of an underlying asset price and its instantaneous volatility or the option price associated with it. In this study, we use a historical 20-day realized volatility as a proxy for the instantaneous volatility and estimate the model parameters based on the maximum likelihood method proposed by Aït-Sahalia and Kimmel[2007]. We update the model parameters daily using the historical daily data of 1,750 days with a rolling estimation procedure.

4.3.3 Estimation of the Volatility Risk Premium

To estimate the volatility risk premium parameter λ with equation (4.16), we need to calculate the integral term in that equation by a discretization of that integral. As mentioned in the previous subsection, we have only a grid of 11 implied volatility points in terms of the strike price, so that we first interpolate the implied volatilities at different moneyness levels with the polynomial approximation methodology proposed by Brunner and Hafner[2003] to obtain a fine curve of implied volatilities [6]. Then we calculate the value of the integral in (4.16) by applying the numerical integral technique. In this simulation $B_t(T)$ in (4.16) is $\exp(-r_d(T-t))$, which is a zero coupon bond price whose maturity date is T.

4.3.4 Estimation of the Expected DHGL

We study an empirical analysis based on a historical simulation to estimate the magnitude of the expected DHGL for the delta-hedged option strategy with a short position of the one-month ATM-forward straddle [7] or the one-month OTM delta-25 put. In particular, beginning on our simulation on the date of October 31, 2003, we compute the DHGLs for those of strategies at the end date after one month and repeat the same computations on the following day after October 31, 2003. The final simulation starts on May 31, 2010 and ends at June 30, 2010. We finally collect the DHGL results for 1,717 samples for each delta-hedged option strategy through those iterated simulations. We employ the Garman-Kohlhagen model, as an extension of the Black-Scholes model to currency options, to compute the delta of the short option positions for tractability, even though the delta computed from the Garman-Kohlhagen model may differ from the delta computed from the stochastic volatility model assumed in this study. If we use C to denote the European call option price on an exchange rate, P as the European put option price, S_0 as the spot rate level on that exchange rate, r_f as the foreign risk free rate, r_d as the domestic risk free rate, σ as the volatility, T as the maturity, and $N(\cdot)$ as the cumulative standard normal distribution function, Garman-Kohlhagen[1983] provides the closed formula for the prices of European currency options as follows:

$$C = S_0 e^{-r_f T} N(d_1) - K e^{-r_d T} N(d_2), \quad P = K e^{-r_d T} N(-d_2) - S_0 e^{-r_f T} N(-d_1),$$

where

$$d_1 = \frac{ln(\frac{S_0}{K}) + (r_d - r_f + \frac{\sigma^2}{2})}{\sigma\sqrt{T}}, \quad d_2 = \frac{ln(\frac{S_0}{K}) + (r_d - r_f - \frac{\sigma^2}{2})}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}. \qquad (4.18)$$

[6]Brunner and hafner[2003] approximate the implied volatility $\sigma_t^T(K)$, whose the strike price is K and the maturity date is T, as a polynomial as follows; $\sigma_t^T(K) = \beta_0 + \beta_1 M + \beta_2 M^2 + D\beta_3 M^3$, where $D \equiv 0(if M \leq 0), \equiv 1(if M > 0)$ and $M \equiv \log\left(\frac{K}{F_t^T}\right)/\sqrt{T-t}$, under the definition that F_t^T is a forward rate whose the maturity date is T at each time t.

[7]The ATM-forward straddle contract is a combination of a call and a put option with the same strike price of ATM forward rate and the same maturity to the underlying forward contract.

If the prices of European currency options are represented as described above, the delta values of call and put options are computed with the following respective formulas

$$\triangle_{call} = e^{-r_f T} N(d_1), \quad \triangle_{put} = e^{-r_f T}\big(N(d_1) - 1\big).$$

Bakshi and Kapadia[2003] and Low and Zhang[2005] provide a simulation exercise that shows using the Black-Scholes delta-hedge ratio instead of the stochastic volatility counterpart has a negligible effect on the DHGL results, and they insist that the empirical analysis results regarding the existence and the sign of the volatility risk premium would not be affected by the decision regardless of which model is used to compute the delta. So we use the Garman-Kohlhagen model to compute the delta in line with the studies of Bakshi and Kapadia[2003] and Low and Zhang[2005]. In our delta-neutral hedge simulations, the volatility σ in the equation (4.18) is estimated using historical daily return data of 20 days.

We rebalance the delta-hedged option portfolio daily and measure the DHGL $\Pi^G_{t,T}$ in the period from the contract date t to the maturity date T using the following formula:

$$\Pi^G_{t,T} = C^M_t - C^M_T - \sum_{n=0}^{N-1} \Delta_{t_n}(S_{t_n} - S_{t_{n+1}}) + \sum_{n=0}^{N-1}(r_d C^M_t - (r_d - r_f)\Delta_{t_n} S_{t_n})\frac{T-t}{N},$$

where $t_0 = t, t_1, t_2, \cdots, t_N = T$ are time steps during the period $[t, T]$ and $\triangle_{t_n}$ is the delta value of the option portfolio at time t_n. In this study, we do not take transactions costs in the delta-neutral hedge operations with spot contracts into consideration because the effects of those transactions costs to the DHGL results are actually negligible due to the high liquidity and the low level of such costs in the USD-JPY exchange rate market. [8]

4.4 An Empirical Analysis

4.4.1 Estimation Results for the Model Parameters

Fig.4.6, Fig.4.7 and Fig.4.8 in Appendix show the time series results of estimated parameters, $\tilde{\kappa}$, $\tilde{v}$, and $\tilde{\rho}$, respectively, in (4.12), which represents the Heston[1993] stochastic volatility model. These parameters are estimated with the maximum likelihood methodology proposed by Aït-Sahalia and Kimmel[2007]. In particular, Fig.4.8 shows the time series of estimated $\tilde{\rho}$, which is the correlation between volatility changes and changes in

[8] As we mentioned before, we take account of transactions costs only in selling the option contracts in our empirical study.

the exchange rate, and we find that the level of $\tilde{\rho}$ is almost negative during the period under consideration.

Fig.4.1 shows the time series of the volatility risk premium parameter λ estimated with (4.16). The λ is also almost negative during the period under consideration and the result of negative market volatility risk premium is consistent with evidence provided by Low and Zhang[2005]. However, this time series of the λ does not have a time consistency and it moves to negative values significantly after the Sub-prime crisis in 2007 followed by the largest negative period during the Lehman-crisis between September 2008 and October 2008.

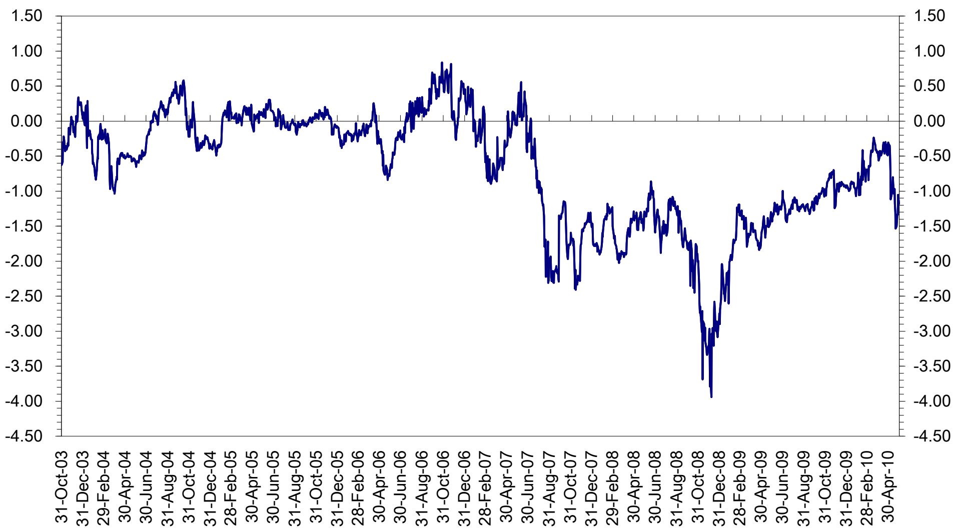

Fig. 4.1: The time series of the volatility risk premium parameter λ

This figure shows a time series result of the λ estimated at each time point based on the equation (4.16) from October 31, 2003 to May 31, 2010. The model parameters in the equation (4.12) are estimated by the maximum liklihood method proposed by Aït-Sahalia and Kimmel[2007], and we update the model parameters daily based on historical 1,750 days daily data with a rolling estimation procedure. We calculate the integral term in the equation (4.16) by a discretizaion and the numerical integral technique. We interpolate implied volatilities at different moneyness levels with a polynomial approximation methodology proposed by Brunner and hafner[2003] to obtain a fine curve of implied volatilities.

We also show the statistical significance on the level of the volatility risk premium parameter λ exhibited in Fig.4.1. Table 4.1 summarizes the statistics for the λ. The top row of Table 4.1 shows the statistics on the λ for overall period from October 31, 2003 to May 31, 2010, and the middle and the bottom rows show the same statistics for the first half period between October 31, 2003 and December 29, 2006 and the following half period between January 2, 2007 and May 31, 2010, respectively. The second column of Table 4.1 shows the number of observations in each period. In the third column, we show the percentage of the λ values that are negative. In the period from January 2, 2007 to May 31, 2010, we can find that almost all the λ have negative values. The percentage of negative values is 94.2 % in that period. The unconditional means and the standard deviations of the λ are listed in the fourth and the fifth columns of Table 4.1, respectively.

The sample mean of the λ is negative for each period, while the high standard deviations of the estimated λ make the means appear to not be significantly different from zero. However, it is misleading to use the unconditional standard deviation to test the mean because serial correlation in the time series of λ can cause the standard deviation to be a biased measure of the actual random error. The next three columns in Table 4.1 show that the first three autocorrelation coefficients are quite large and decay slowly. This result indicates that the time series may follow an autoregressive process. We also show the partial autocorrelation coefficients in Table 4.1. The first-order partial autocorrelation coefficient is large in all cases, while the second- and third-order autocorrelation coefficients become much smaller. The pattern for both autocorrelation coefficients and partial autocorrelation coefficients suggests the fitting of an autoregressive process of order three (AR(3)) to the time series of the λ. The AR(3) process for the volatility risk premium parameter is examined by the following model

$$\lambda_t = \alpha + \beta_1 \lambda_{t-1} + \beta_2 \lambda_{t-2} + \beta_3 \lambda_{t-3} + \epsilon_t,$$

where λ_t is the time-t volatility risk premium parameter and ϵ_t is a white noise process. Its unconditional mean is given by the following formula

$$\mathbb{E}[\lambda_t] = \frac{\alpha}{1 - \beta_1 - \beta_2 - \beta_3},$$

which implies that the null hypothesis of a zero unconditional mean is equivalent to the null hypothesis that the intercept of the AR(3) process is equal to zero.

Table 4.1: Summary statistics on the volatility risk premium parameter λ

Period	No. of obs.	% of $\lambda <$ 0	Spl. Mean	Spl. Std. Dev.	(1) Auto Corr.			(2) Partial Auto Corr.			(3) AR3 Int.	
					*Lag*1	*Lag*2	*Lag*3	*Lag*1	*Lag*2	*Lag*3	α	*pVal*
Total	1,717	75.3 %	-0.686	0.831	0.989	0.983	0.976	0.989	0.171	0.037	-0.006	0.046
(A)	826	55.0 %	-0.068	0.324	0.957	0.925	0.897	0.957	0.099	0.053	-0.001	0.362
(B)	891	94.2 %	-1.260	0.742	0.979	0.965	0.951	0.979	0.169	0.033	-0.025	0.004

The sample period is from October 31, 2003 to May 31, 2010. Period (A) in this table is the first half period from October 31, 2003 to December 29, 2006 and Period (B) is the following half period from January 2, 2007 to May 31, 2010. The second column of this table shows the number of observations in each time series. In the third column, we present the percentage of the λ which has a negative value. The unconditional means and standard deviations of the λ are respectively exhibited in the fourth and the fifth columns in this table.

We estimate the parameters of the AR(3) process introduced above and show estimated intercept and its p-value for the t-statistic in the last two columns of Table 4.1. The intercept is significantly negative at the 5 % level in overall period under consideration and significantly negative at the 1 % level during the second half period from January 2, 2007 to May 31, 2010, whereas it is insignificant, although negative, during the first half period from October 31, 2003 to December 29, 2006. These results, that is to say that the volatility risk premium parameter λ is negative, are consistent with Low and Zhang[2005] which provides an evidence of the negative volatility risk premium in currency option markets, but are also clearly show that the volatility risk premium parameter λ in the USD-JPY currency option market does not have a time consistency but essentially has a stochastic nature. This feature of time-varying volatility risk premium parameter (or time-varying volatility risk premium) in currency option markets is not demonstrated by Low and Zhang[2005] and we should notice that the existence of volatility risk premium in the USD-JPY currency option market is not necessarily significant at any time.

In equilibrium, we can obtain an explicit relation between the risk aversion parameter for the representative agent and the volatility risk premium parameter. According to Heston[1993],

$$-\lambda\tilde{\sigma}_t = \mathrm{Cov}_t\Big(\frac{d\Lambda_t}{\Lambda_t}, d\tilde{\sigma}_t\Big)/dt, \tag{4.19}$$

where Λ_t is the stochastic discount factor process in the dynamic equilibrium setting. In the case that the utility function of the representative agent assumes the following power utility form,

$$U_t = \exp(-\delta t)\frac{S_t^{1-\gamma}}{1-\gamma},$$

where δ is the subjective discount factor and γ is the risk aversion parameter, we can obtain a representation of $\Lambda_t = \exp(-\delta t)S_t^{-\gamma}$ easily. Thus, by (4.12) and Ito's lemma, the following equation can be derived:

$$\mathrm{Cov}_t\Big(\frac{d\Lambda_t}{\Lambda_t}, d\tilde{\sigma}_t\Big)/dt = -\gamma\tilde{\rho}\tilde{v}\tilde{\sigma}_t. \tag{4.20}$$

Equations of (4.19) and (4.20) lead to an explicit relation between the volatility risk premium parameter λ and the risk aversion parameter γ, that is to say, $\gamma = \lambda/(\tilde{\rho}\tilde{v})$. This equation enables us to provide an empirical analysis for the time series of γ in equilibrium. Fig.4.2 shows the estimation result on the time series of γ. This figure shows that it moves to significantly positive values after the sub-prime crisis in 2007 followed by the longest positive period during the Lehman crisis from September 2008 to October 2008. Then, it moves down slowly but seems to rise again during the period of the European financial crisis from April 2010 to May 2010.

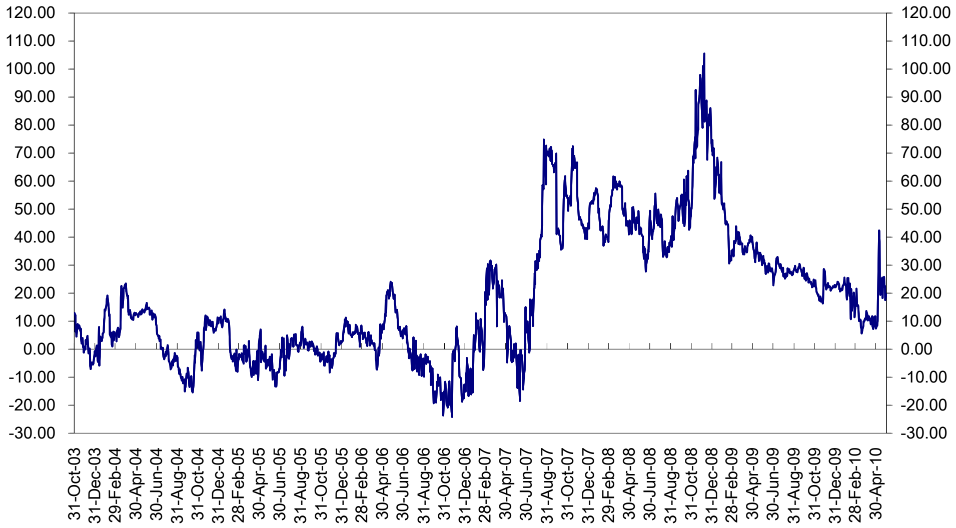

Fig. 4.2: Risk aversion parameter γ

The power utility is assumed to the utility function for the representative agent, and this figure shows the time series of the risk aversion parameter estimated by the relation of $\gamma = \lambda/(\tilde{\rho}\tilde{v})$ during the period from October 2003 to May 2010. The model parameters in the equation (4.12) are estimated by the maximum liklihood method proposed by Aït-Sahalia and Kimmel[2007] and we update the model parameters daily based on historical 1,750 days daily data with a rolling estimation procedure. The value of λ used in this result is from the result exhibited in Fig.4.1.

4.4.2 A Contribution Analysis on the Expected DHGL

To demonstrate a contribution analysis on the expected DHGL represented by (4.11), we first calculate $C_t^M[0] - C_t^M[\lambda]$, which is the second term on the right side of the equation (4.11), using the set of estimated parameters $(\tilde{k}, \tilde{v}, \tilde{\rho})$ and the volatility risk premium parameter λ shown in Fig.4.1. Combining this calculation result for $C_t^M[0] - C_t^M[\lambda]$ with the DHGL estimated with simulated delta-hedged option returns on the basis of (4.11), the effect of parameter estimation risk, the first term on the right side of the equation (4.11), can be estimated.

Table 4.2 reports the contribution of the volatility risk premium ("RP" in Table4.2), which is represented by the second term on the right side of the equation (4.11), and the contribution of parameter estimation risk ("PER" in Table4.2), which is represented by the first term on the right side of the equation (4.11), to the expected DHGL estimated by our historical simulation. To standardize the influence of the difference of the exchange rate level at each time point, we calculate the return defined by $\Pi^G_{t,t+\tau}/S_t$ for each simulation and recognize this return as a proxy for the DHGL at each time point. To investigate the effects of the strike price on the results of the contribution analysis for the expected DHGL, historical simulations of short positions of the OTM delta-25 1M puts as well as the ATM 1M straddles are examined. We segment the results into ten levels in terms of the level of volatility risk premium parameter λ at each time point to ensure the time dependencies of this attribution analysis. In this table, "No. of Obs." means the number of samples and "Mean of λ" shows the average of the λ in each interval. "RP" shows the contribution induced by the volatility risk premium to the total expected DHGL and "PER" shows the contribution induced by parameter estimation risk to the total expected DHGL. These attribution results shown in this table are monthly-based basis point returns.

Table 4.2: Contribution Analysis for the Expected Delta-hedged Gain Loss

(1) Interval of λs	(2) No. of Obs.	(3) Mean of λ	(4) Expected DHGL: ATM Straddle Short			(5) Expected DHGL: OTM delta-25 Put Short		
			(4-1) RP	(4-2) PER	(4-3) Total	(5-1) RP	(5-2) PER	(5-3) Total
Overall	1,717	-0.69	9.7	-3.8	5.9	3.6	2.2	5.8
$\lambda \in (-4, -3.5]$	4	-3.76	89.3	73.2	162.4	32.8	53.1	86.0
$\lambda \in (-3.5, -3]$	18	-3.15	66.6	37.7	104.3	24.3	63.4	87.6
$\lambda \in (-3, -2.5]$	25	-2.78	59.2	62.9	122.1	21.8	57.1	78.9
$\lambda \in (-2.5, -2]$	62	-2.22	34.5	12.7	47.2	12.7	19.5	32.2
$\lambda \in (-2, -1.5]$	186	-1.73	25.5	8.2	33.7	9.5	-0.6	8.9
$\lambda \in (-1.5, -1]$	304	-1.27	15.2	4.9	20.1	5.7	-3.3	2.4
$\lambda \in (-1, -0.5]$	233	-0.75	7.8	-3.1	4.7	2.9	0.9	3.7
$\lambda \in (-0.5, 0]$	461	-0.03	0.6	-26.2	-25.6	0.2	1.6	1.9
$\lambda \in (0, +0.5]$	388	0.17	-1.3	-5.4	-6.7	-0.4	3.6	3.2
$\lambda \in (+0.5, +1]$	36	0.62	-4.2	-9.9	-14.1	-1.5	-0.5	-2.0

Starting on our simulation at the date of October 31, 2003, we compute the DHGL of each strategy at the end date after one month and repeat the same computations on the following day after October 31, 2003. The final simulation starts at May 31, 2010 and ends at June 30, 2010. We finally collect the DHGL results of 1,717 samples for each delta-hedged option strategy through those iterated simulations. We employ the Garman-Kohlhagen model, as an extension of the Black-Scholes model to currency options, to compute the delta of the short option positions for tractability. In this table, "No. of Obs." means the number of samples in each interval for the λ, and "Mean of λ" shows the average of the λ in each interval. "RP" shows the contribution induced by the volatility risk premium to the total expected DHGL and "PER" shows the contribution induced by parameter estimation risk to the total expected DHGL. These attribution results shown in this table are monthly-based basis point returns.

In this table, it is clear that the total level of expected DHGL increases in proportion as the volatility risk premium parameter λ decreases for both the delta-hedged ATM straddle short strategy and the delta-hedged OTM delta-25 put short strategy, and both effects of volatility risk premium and parameter estimation risk on the total expected DHGL make a significant impact on that variation. This result also indicates that, in the case of $\lambda \leq -2.5$, the total expected DHGL is more than 100bp per month for the delta-hedged ATM straddle short strategy and approximately 80bp per month for the delta-hedged OTM delta-25 put short strategy. In particular, the effect of parameter estimation risk on the total expected DHGL seems to be more significant than the effect of volatility risk premium for the OTM delta-25 put strategy. One of the most important implications of this result is that the sign and the level of the expected delta-hedged option returns do not generally explain the existence of the volatility risk premiums. There are additional important factors that make an impact on delta-hedged option returns such as parameter estimation risk, rendering standard hedging-based tests on volatility risk premiums explored by, for example, Bakshi and Kapadia[2003] and Low and Zhang[2005], unreliable. We also find a large negative expected DHGL when the λ is in the interval between -0.5 and 0 for the case of ATM straddle short strategy. This result is entirely generated by the effect of parameter estimation risk and is especially affected by several jumps in the USD-JPY exchange rate market during the period of the European financial crisis from April 2010 to May 2010.

Table 4.3: Summary Statistics of the DHGL for the ATM Straddle Short Strategy

Period	No. of Obs.	% of Pos.	Sample Mean	(1) Auto Corr.			(2) Partial Auto Corr.			(3) AR3 Int.	
				$Lag1$	$Lag2$	$Lag3$	$Lag1$	$Lag2$	$Lag3$	α	$pVal$
Panel 1 : DHGLs (RP+PER)											
Total	1,717	55.3 %	5.9	0.875	0.779	0.698	0.875	0.059	0.023	0.7	0.222
(A)	826	47.5 %	-8.3	0.861	0.775	0.704	0.861	0.133	0.044	-1.0	0.862
(B)	891	62.6 %	19.1	0.874	0.772	0.686	0.874	0.036	0.017	2.4	0.072
Panel 2 : The effects of volatility risk premium (RP)											
Total	1,717	75.3 %	9.7	0.986	0.979	0.971	0.986	0.211	0.046	0.1	0.055
(A)	826	55.0 %	0.8	0.951	0.920	0.894	0.951	0.154	0.082	0.0	0.268
(B)	891	94.2 %	17.9	0.979	0.966	0.954	0.979	0.202	0.042	0.3	0.021
Panel 3 : The effects of parameter estimation risk (PER)											
Total	1,717	50.2 %	-3.8	0.863	0.761	0.675	0.863	0.062	0.020	-0.4	0.678
(A)	826	47.0 %	-9.1	0.862	0.779	0.709	0.862	0.137	0.045	-1.1	0.876
(B)	891	53.2 %	1.2	0.863	0.754	0.662	0.863	0.039	0.015	0.3	0.432

The sample period of these simulation results is from October 31, 2003 to June 30, 2010. Period (A) in this table is the first term from October 31, 2003 to December 29, 2006 and Period (B) is the following term from January 2, 2007 to June 30, 2010. The second column of this table lists the number of observations in each time series. In the third column, we report the percentage of the DHGL which has a positive value. The unconditional means of the DHGL are listed in the fourth column. The sample means and intercepts (α) shown in this table are monthly-based basis point returns.

To understand the statistical significance of the expected DHGL shown in Table 4.2, we employ the same methodology as that used for the analysis in Table 4.1. Table 4.3 and Table 4.4 show the statistical significance of the total expected DHGL, the effect of volatility risk premium on the total expected DHGL, and the effect of parameter estimation risk on the total expected DHGL. In particular, Table 4.3 shows the result for ATM straddle short strategy and Table 4.4 shows the result for OTM delta-25 put short strategy. Period (A) in these tables is the first term from October 31, 2003 to December 29, 2006 and Period (B) is the following term from January 2, 2007 to June 30, 2010. The second column lists the number of observations in each time series. In the third column, we report the percentage of the DHGL which has a positive value. The unconditional means of the DHGL are listed in the fourth column. The sample means and intercepts (α) shown in these tables are monthly-based basis point returns.

Table 4.4: Summary Statistics of the DHGL for the OTM Put Short Strategy

Period	No. of Obs.	% of Pos.	Sample Mean	(1) Auto Corr.			(2) Partial Auto Corr.			(3) AR3 Int.	
				*Lag*1	*Lag*2	*Lag*3	*Lag*1	*Lag*2	*Lag*3	α	*pVal*
Panel 1 : DHGLs (RP+PER)											
Total	1,717	65.8 %	5.8	0.904	0.825	0.757	0.904	0.042	0.024	0.5	0.101
(A)	826	63.3 %	4.2	0.862	0.776	0.707	0.862	0.127	0.053	0.5	0.112
(B)	891	68.0 %	7.3	0.911	0.833	0.765	0.911	0.019	0.019	0.7	0.179
Panel 2 : The effects of volatility risk premium (RP)											
Total	1,717	75.3 %	3.6	0.984	0.976	0.969	0.984	0.235	0.084	0.0	0.052
(A)	826	55.0 %	0.3	0.915	0.874	0.847	0.915	0.221	0.143	0.0	0.254
(B)	891	94.2 %	6.6	0.976	0.964	0.952	0.976	0.219	0.070	0.1	0.020
Panel 3 : The effects of parameter estimation risk (PER)											
Total	1,717	63.2 %	2.2	0.900	0.818	0.746	0.900	0.039	0.022	0.0	0.294
(A)	826	63.2 %	3.9	0.863	0.778	0.709	0.863	0.126	0.054	0.0	0.128
(B)	891	63.2 %	0.7	0.906	0.824	0.752	0.906	0.016	0.017	0.1	0.447

The sample period of these simulation results is from October 31, 2003 to June 30, 2010. Period (A) in this table is the first term from October 31, 2003 to December 29, 2006 and Period (B) is the following term from January 2, 2007 to June 30, 2010. The second column of this table lists the number of observations in each time series. In the third column, we report the percentage of the DHGL which has a positive value. The unconditional means of the DHGL are listed in the fourth column. The sample means and intercepts (α) shown in this table are monthly-based basis point returns.

In Table 4.3, we find that the p-value for the intercept of the AR3-process for the total expected DHGL is 0.222 during the entire period, so that the statistical significance of the total expected DHGL does not seem to be high. However, if we focus on Period (B), the p-value on the intercept of the AR3-process decreases to 0.072 and it is significantly different from zero at the 10 % level. This result seems to be induced by the statistical significance of the volatility risk premium during Period (B) (see the middle panel in

Table 4.3). In contrast, the statistical significance of the effect of parameter estimation risk on the total expected DHGL does not seem to be high during either period, while parameter estimation risk might affect the level of the total expected DHGL.

Table 4.4 shows the same result as Table 4.3 for the delta-hedged OTM delta-25 put short strategy. In this case, the effect of parameter estimation risk on the total expected DHGL seems to be high relative to the case of the delta-hedged ATM straddle short strategy (see the bottom panel in Table 4.4). This result suggests that the effect of parameter estimation risk on the total expected DHGL for the OTM delta-25 put short strategy is more significant when compared with the ATM straddle short strategy.

Finally, to investigate the magnitude of the effect of parameter estimation risk on the total expected DHGL in the pre- and post-financial crisis periods, we report a subperiod contribution analysis for the total expected DHGL in Table 4.5 and Table 4.6 using the same simulation results exhibited in Table 4.2. We divide the overall period into two periods of pre- and post-Lehman shock: the former period is from October 2003 to September 2008 and the latter period is from October 2008 to June 2010. In Table 4.5, "VRP Para." denotes the volatility risk premium parameter in each period. For each of the ATM straddle and the OTM delta-25 put short strategies, "Pre." shows the option premium for each corresponding strategy, and "RP" and "PER" show the contributions of the volatility risk premium and parameter estimation risk on the total expected DHGL, respectively. "Tot." shows the total expected DHGL. Each value introduced above is shown in terms of the average ("Ave.") in each period and also in terms of the relative percentage to the corresponding option premium ("Rel."). The average value of the option premium and each contribution to the total expected DHGL shown in these tables are monthly-based basis point returns.

As mentioned above, we obtain our daily delta-hedged option returns by selling a corresponding option and maintaining a delta-neutral portfolio using the spot contract of the USD-JPY exchange rate until the option matures. In calculating the delta-hedged return of the one-month option sold on a given trading day, say day 0, we use the information for days 1 to 22, assuming that there are 22 trading days before the option maturity date. Then, for the delta-hedged return of the one-month option sold on day 1, we use the information of day 2, day 3, up to day 23. Consequently, the delta-hedged returns of day 0 and day 1 options use information from an overlapping period between days 2 and 22. To address the concern that our evidence on a contribution analysis for delta hedged option returns is driven by the common information in the overlapping periods, we also construct a time-series of non-overlapping delta hedged option returns for each option strategy. Specifically, for each option strategy, we construct a monthly series of delta-hedged returns on the one-month options sold at the first trading day of each month in our sample period. Because the delta-hedged returns of the beginning-of-month option only depend on the information of the trading days in the same month, they are non-overlapping. Table 4.6 reports the result for the non-overlapping returns of each of the two delta-hedged option strategies.

Table 4.5: Relative Contribution Comparison of the Expected DHGL between Pre- and Post Lehman Crisis : Based on a time-series of overlapping results

	VRP	(A) ATM Straddle				(B) OTM delta-25 Put			
	Para.	(A1) Pre.	(A2) RP	(A3) PER	(A4) Tot.	(B1) Pre.	(B2) RP	(B3) PER	(B4) Tot.
Panel 1 : Overall Period [From October 2003 to June 2010]									
(1) Ave.	-1.37	256.6	9.7	-3.8	5.9	51.0	3.6	2.2	5.8
(2) Rel.	-	100.0 %	3.8 %	-1.5 %	2.3 %	100.0 %	7.0 %	4.4 %	11.4 %
Panel 2 : Pre-Lehman Crisis [From October 2003 to September 2008]									
(1) Ave.	-0.89	217.9	5.1	-13.2	-8.1	42.8	1.9	-1.0	0.9
(2) Rel.	-	100.0 %	2.4 %	-6.1 %	-3.7 %	100.0 %	4.4 %	-2.3 %	2.0 %
Panel 3 : Post-Lehman Crisis [From October 2008 to June 2010]									
(1) Ave.	-2.80	370.9	23.1	24.2	47.3	75.3	8.7	11.8	20.5
(2) Rel.	-	100.0 %	6.2 %	6.5 %	12.7 %	100.0 %	11.5 %	15.6 %	27.2 %

Starting on our simulation at the date of October 31, 2003, we compute the DHGL of each strategy at the end date after one month, and repeat the same computations on the following day after October 31, 2003. The final simulation starts at May 31, 2010 and ends at June 30, 2010. We finally collect the DHGL results of 1,717 samples through those iterated simulations. We employ the Garman-Kohlhagen model, as an extension of the Black-Scholes model to currency options, to compute the delta of the short option positions for tractability. In this table, "VRP Para." denotes the volatility risk premium parameter in each period. "Pre." shows the option premium for each corresponding strategy. "RP" and "PER" show the contribution of the volatility risk premium and parameter estimation risk to the total expected DHGL, respectively. "Tot." shows the total expected DHGL. Each value introduced above is shown in terms of the average ("Ave.") in each period and also shown in terms of the relative percentage to corresponding option premium ("Rel."). The average value of option premium and each contribution to the total expected DHGL shown in this table are monthly-based basis point returns.

We find several important pieces of evidence on the delta-hedged option returns in Table 4.5 and 4.6. First, in the case of the ATM straddle short strategy, the effect of parameter estimation risk on the total expected DHGL does not seem to be high in the overall period from October 2003 to June 2010 because the relative value of the effect of parameter estimation risk to the corresponding option premium is less than 2 % in terms of the absolute value. It is clear that the total expected DHGL in that period is well explained by the effect of the volatility risk premium. However, we should notice that the effect of parameter estimation risk becomes more significant in the post-Lehman shock period, that is to say, from October 2008 to June 2010, because the relative value of that effect to the corresponding option premium is more than 6 % and the relative contribution of that effect to the total expected DHGL is much larger than that of the effect of the volatility risk premium.

Second, in the case of the OTM delta-25 put short strategy, the effect of parameter estimation risk on the total expected DHGL is more significant in the overall period from October 2003 to June 2010, which differs from the result for the case of the ATM straddle short strategy. In Table 4.6, we find that the effect of parameter estimation risk is much larger than the effect of the volatility risk premium in the overall period, and in the post-Lehman shock period, that is, from October 2008 to June 2010, the effect of parameter uncertainty on the total expected DHGL in terms of the relative value to the corresponding option premium becomes 13 % or more.

Table 4.6: Relative Contribution Comparison of the Expected DHGL between Pre- and Post Lehman Crisis : Based on a time-series of non-overlapping results

	VRP	(A) ATM Straddle				(B) OTM delta-25 Put			
	Para.	(A1) Pre.	(A2) RP	(A3) PER	(A4) Tot.	(B1) Pre.	(B2) RP	(B3) PER	(B4) Tot.
Panel 1 : Overall Period [From October 2003 to June 2010]									
(1) Ave.	-1.41	258.7	9.6	0.0	9.6	51.8	3.6	3.7	7.3
(2) Rel.	-	100.0 %	3.7 %	0.0 %	3.7 %	100.0 %	7.0 %	7.2 %	14.1 %
Panel 2 : Pre-Lehman Crisis [From October 2003 to September 2008]									
(1) Ave.	-0.92	218.3	5.1	-12.1	-7.0	43.2	1.9	1.5	3.4
(2) Rel.	-	100.0 %	2.3 %	-5.6 %	-3.2 %	100.0 %	4.3 %	3.5 %	7.8 %
Panel 3 : Post-Lehman Crisis [From October 2008 to June 2010]									
(1) Ave.	-2.76	372.3	22.3	34.1	56.4	75.8	8.5	9.9	18.4
(2) Rel.	-	100.0 %	6.0 %	9.2 %	15.2 %	100.0 %	11.2 %	13.0 %	24.3 %

Starting on our simulation at the date of October 31, 2003, we compute the DHGL of each strategy at the end date after one month, and repeat the same computations on the following day after October 31, 2003. The final simulation starts at May 31, 2010 and ends at June 30, 2010. We finally collect the DHGL results of 1,717 samples through those iterated simulations. We employ the Garman-Kohlhagen model, as an extension of the Black-Scholes model to currency options, to compute the delta of the short option positions for tractability. In this table, "VRP Para." denotes the volatility risk premium parameter in each period. "Pre." shows the option premium for each corresponding strategy. "RP" and "PER" show the contribution of the volatility risk premium and parameter estimation risk to the total expected DHGL, respectively. "Tot." shows the total expected DHGL. Each value introduced above is shown in terms of the average ("Ave.") in each period and also shown in terms of the relative percentage to corresponding option premium ("Rel."). The average value of option premium and each contribution to the total expected DHGL shown in this table are monthly-based basis point returns.

From these empirical results, we can recognize that parameter estimation risk makes an significant impact on option premiums, especially in the post-financial crisis period, as well as the volatility risk premium, and the effect of parameter estimation risk seems to cause much higher option premiums. Needless to say, we find that the sign and the level of the expected delta-hedged option returns do not generally explain the existence of volatility risk premiums. Moreover, it needs to be emphasized that there are additional important factors such as parameter estimation risk that make an impact on delta-hedged option returns, rendering standard hedging-based tests on volatility risk premiums explored by previous studies unreliable.

4.5 Concluding Remarks

In this chapter, we provide a novel representation of delta-hedged option returns in a stochastic volatility environment. The representation of delta-hedged option returns provided in this chapter consists of two terms: volatility risk premium and parameter estimation risk. In an empirical analysis, we examine delta-hedged option returns based on the result of a historical simulation with the USD-JPY currency option market data from October 2003 to June 2010. We find that the delta-hedged option returns for OTM put options are strongly affected by parameter estimation risk as well as the volatility risk premium, especially in the post-Lehman shock period.

This study is the first to provide empirical evidence on the effect of parameter estimation risk on dellta-hedged option returns. However, the analysis examined in this chapter constitutes a first step toward a more detailed investigation of the empirical characteristics of delta-hedged option returns. A next step might be a detailed analysis on the impact of jump risk on delta-hedged option returns. A further direction of this study will be to provide a contribution analysis of delta-hedged option returns with the effect of parameter estimation risk, as well as the effects of volatility and jump risk premiums.

Appendix 4.A Proof of Proposition 13

The following equation can be derived by applying Ito's lemma to the market price $C_t^M = G(t, S_t, \sigma_t)$,

$$\begin{aligned} C_{t+\tau}^M = C_t^M &+ \int_t^{t+\tau} \frac{\partial G}{\partial S}(u, S_u, \sigma_u) dS_u \\ &+ \int_t^{t+\tau} \frac{\partial G}{\partial \sigma}(u, S_u, \sigma_u) d\sigma_u + \int_t^{t+\tau} \mathcal{D}G(u, S_u, \sigma_u) du, \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} \mathcal{D}G(t, S_t, \sigma_t) = \frac{\partial G}{\partial t}(t, S_t, \sigma_t) &+ \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 G}{\partial S^2}(t, S_t, \sigma_t) \\ &+ \frac{1}{2}\tilde{\eta}_t^2 \frac{\partial^2 G}{\partial \sigma^2}(t, S_t, \sigma_t) + \tilde{\rho}_t \tilde{\eta}_t \sigma_t S_t \frac{\partial^2 G}{\partial S \partial \sigma}(t, S_t, \sigma_t), \end{aligned}$$

and $C_t^M = G(t, S_t, \sigma_t)$ is also solves the equation (4.7). Under the two equations of (4.7) and (4.21), we can derive the following equation with the market price C_t^M,

$$\begin{aligned} C_{t+\tau}^M = C_t^M &+ \int_t^{t+\tau} \frac{\partial C_u^M}{\partial S_u} dS_u + \int_t^{t+\tau} \Big(r_d C_u^M - (r_d - r_f) S_u \frac{\partial C_u^M}{\partial S_u} \Big) du \\ &+ \int_t^{t+\tau} \lambda_u \frac{\partial C_u^M}{\partial \sigma_u} du + \int_t^{t+\tau} \tilde{\eta}_u \frac{\partial C_u^M}{\partial \sigma_u} dW_u^2. \end{aligned} \tag{4.22}$$

If $C_t^M[\lambda_u]$ denotes the time-t call option price which is consistent with the underlying exchange rate process (4.2) and the equivalent martingale measure $\mathbb{Q}[\lambda_u]$, the following equation can be derived because of the fact that $C_T^M[\lambda_u] = C_T^M[0]$ at the maturity date.

$$\begin{aligned} C_t^M[\lambda_u] &+ \int_t^{t+\tau} \frac{\partial C_u^M[\lambda_u]}{\partial S_u} dS_u + \int_t^{t+\tau} \Big(r_d C_u^M[\lambda_u] - (r_d - r_f) S_u \frac{\partial C_u^M[\lambda_u]}{\partial S_u} \Big) du \\ &+ \int_t^{t+\tau} \lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \sigma_u} du + \int_t^{t+\tau} \tilde{\eta}_u \frac{\partial C_u^M[\lambda_u]}{\partial \sigma_u} dW_u^2 \\ &= C_t^M[0] + \int_t^{t+\tau} \frac{\partial C_u^M[0]}{\partial S_u} dS_u + \int_t^{t+\tau} \Big(r_d C_u^M[0] - (r_d - r_f) S_u \frac{\partial C_u^M[0]}{\partial S_u} \Big) du \\ &+ \int_t^{t+\tau} \tilde{\eta}_u \frac{\partial C_u^M[0]}{\partial \sigma_u} dW_u^2. \end{aligned}$$

Rearranging the above equation and taking expectation to the rearranged equation, we can obtain the following expression.

$$C_t^M[0] - C_t^M[\lambda_u] = \int_t^T \mathbb{E}^{\mathbb{P}} \Big[\lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \sigma_u} \Big] du + r_d \int_t^T \mathbb{E}^{\mathbb{P}} \Big[C_u^M[\lambda_u] - C_u^M[0] \Big] du. \tag{4.23}$$

First, let us assume $\lambda_u \geq 0$. We have the following inequations on the option premium,

$$C_u^M[\lambda_u] - C_u^M[0] \leq 0 \quad (\forall u \in [t, T])$$

and

$$\frac{\partial}{\partial u} \mathbb{E}^{\mathbb{P}} \Big[C_u^M[\lambda_u] - C_u^M[0] \Big] = \mathbb{E}^{\mathbb{P}} \Big[\frac{\partial}{\partial u} \Big(C_u^M[\lambda_u] - C_u^M[0] \Big) \Big] \geq 0.$$

Thus,

$$\begin{aligned} C_t^M[0] - C_t^M[\lambda_u] &= \int_t^T \mathbb{E}^{\mathbb{P}} \Big[\lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \sigma_u} \Big] du + r_d \int_t^T \mathbb{E}^{\mathbb{P}} \Big[C_u^M[\lambda_u] - C_u^M[0] \Big] du \\ &\geq \int_t^T \mathbb{E}^{\mathbb{P}} \Big[\lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \sigma_u} \Big] du + r_d (T - t) \Big(C_t^M[\lambda_u] - C_t^M[0] \Big), \\ \therefore \quad \Big(1 + r_d (T - t) \Big) \Big(C_t^M[0] - C_t^M[\lambda_u] \Big) &\geq \int_t^T \mathbb{E}^{\mathbb{P}} \Big[\lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \sigma_u} \Big] du. \end{aligned}$$

We also have an inequation of $\mathbb{E}^{\mathbb{P}}\Big[C_u^M[\lambda_u] - C_u^M[0]\Big] \leq 0$, so the following inequation can be obtained,

$$C_t^M[0] - C_t^M[\lambda_u] \leq \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \sigma_u}\Big] du.$$

Thus we can derive the following inequation with two inequations derived above,

$$C_t^M[0] - C_t^M[\lambda_u] \leq \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \sigma_u}\Big] du \leq \Big(1 + r_d(T-t)\Big)\Big(C_t^M[0] - C_t^M[\lambda_u]\Big).$$

In the case of $\lambda_u < 0$, we have the following inequations,

$$C_u^M[\lambda_u] - C_u^M[0] \geq 0 \quad (\forall u \in [t, T])$$

and

$$\frac{\partial}{\partial u}\mathbb{E}^{\mathbb{P}}\Big[C_u^M[\lambda_u] - C_u^M[0]\Big] = \mathbb{E}^{\mathbb{P}}\Big[\frac{\partial}{\partial u}\Big(C_u^M[\lambda_u] - C_u^M[0]\Big)\Big] \leq 0.$$

Thus we can derive the inequation asserted in this Proposition with the similar approach to the discussion explored above. □

Appendix 4.B Time Series Data

Table 4.7: Summary Statistics for Implied Volatilities

Statistics	D5Put	D10Put	D15Put	D25Put	D35Put	ATM	D35Call	D25Call	D15Call	D10Call	D5Call
Mean	16.2	15.2	14.6	13.4	12.7	12.0	11.5	11.1	11.0	11.2	11.3
St. Dev.	7.7	6.7	6.4	5.5	5.1	4.6	4.3	3.9	3.9	3.9	4.0
Min	7.4	6.4	6.9	6.5	6.0	5.8	5.5	4.6	4.8	3.9	5.7
1'st q.	10.5	10.2	9.8	9.3	9.1	8.8	8.5	8.4	8.3	8.5	8.4
Median	15.0	14.1	13.5	12.4	11.8	11.3	10.9	10.7	10.5	10.6	10.7
3'rd q.	19.0	17.7	16.9	15.6	14.7	14.0	13.5	13.3	13.3	13.3	13.5
Max	59.7	54.0	53.7	48.3	46.0	43.0	40.4	38.1	37.0	36.1	35.9
Skew	1.9	1.8	1.9	1.8	1.9	1.8	1.7	1.7	1.5	1.5	1.4
Kurt	5.1	4.5	5.4	5.7	6.2	6.3	6.2	6.4	5.3	4.8	4.3

The implied volatility data is from the European type put and call OTC currency options with maturities of one month and strike prices of 5 delta, 10 delta, 15 delta, 25 delta, 35 delta and ATM. The statistics listed in this table are estimated in the period from October 2003 to June 2010.

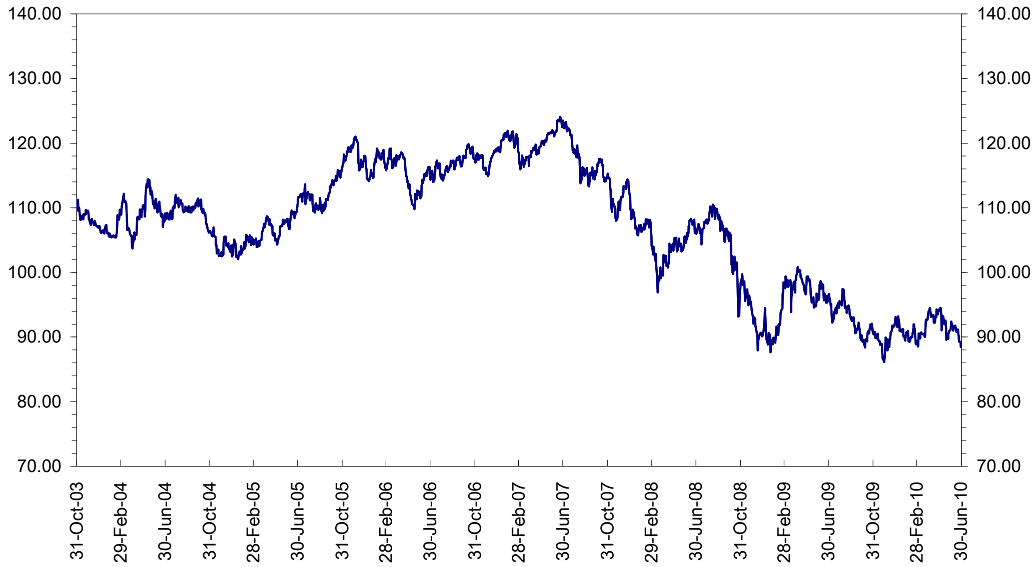

Fig. 4.3: USD-JPY WM/Reuter Closing Spot Rate

This figure shows the time-series data in the period from October 31, 2003 to June 30, 2010.

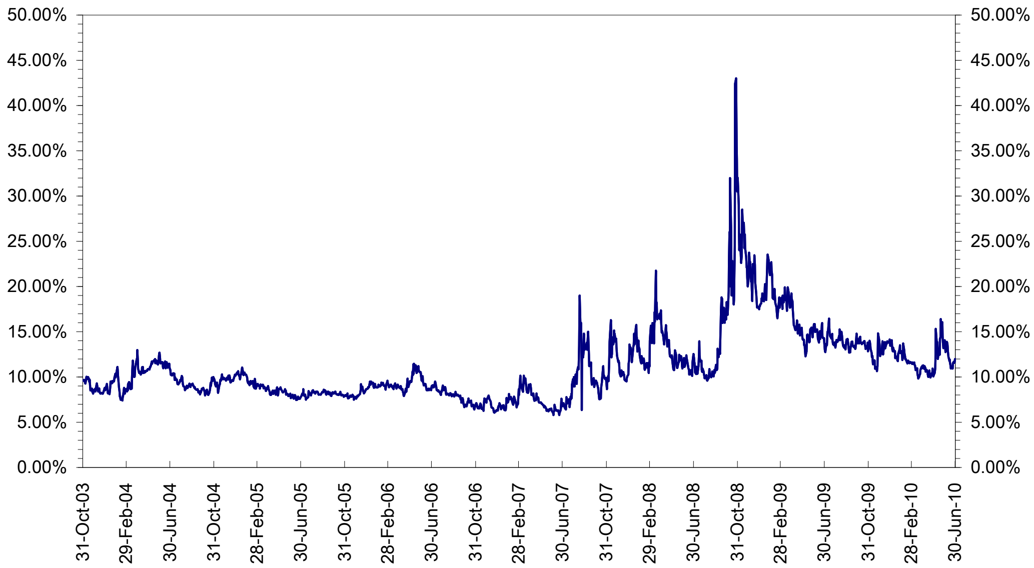

Fig. 4.4: USD-JPY 1M ATM Implied Volatility (Mid Price)

This figure shows the time-series data in the period from October 31, 2003 to June 30, 2010.

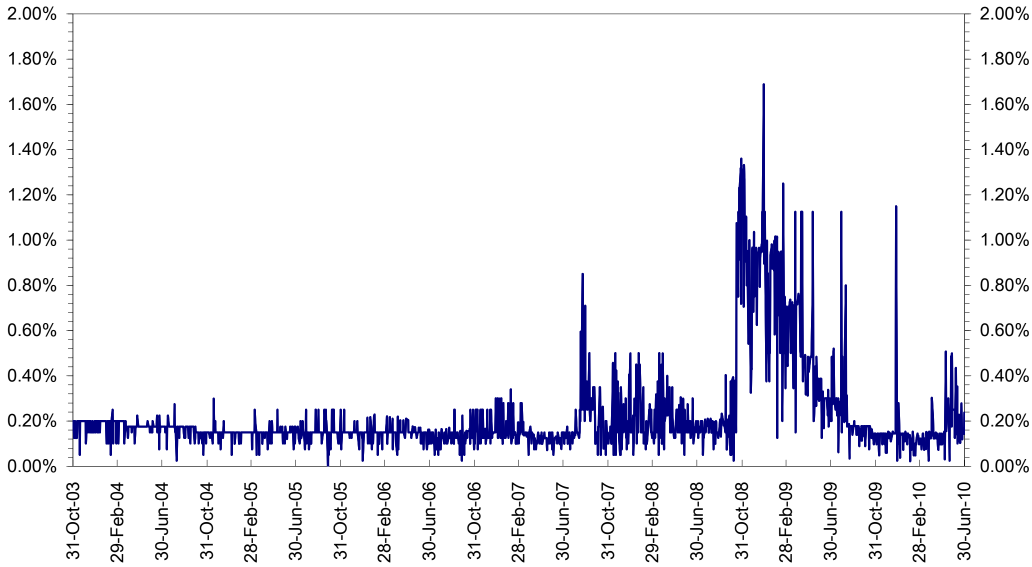

Fig. 4.5: USD-JPY 1M ATM Mid-Bid Spread

This figure shows the time-series data in the period from October 31, 2003 to June 30, 2010.

Appendix 4.C Estimation Results of the Heston Model

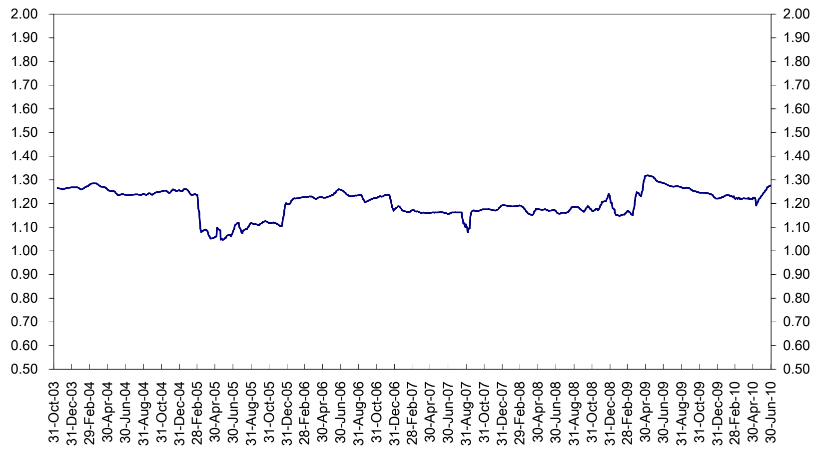

Fig. 4.6: The time-series of the estimated parameter : $\tilde{k}$

This figure shows estimation results of $\tilde{\kappa}$ at each time point in the period from October 31, 2003 to June 30, 2010. These parameters are estimated with the maximum liklihood method proposed by Aït-Sahalia and Kimmel[2007] and we update the parameters daily based on historical 1,750 days daily data with a rolling estimation procedure.

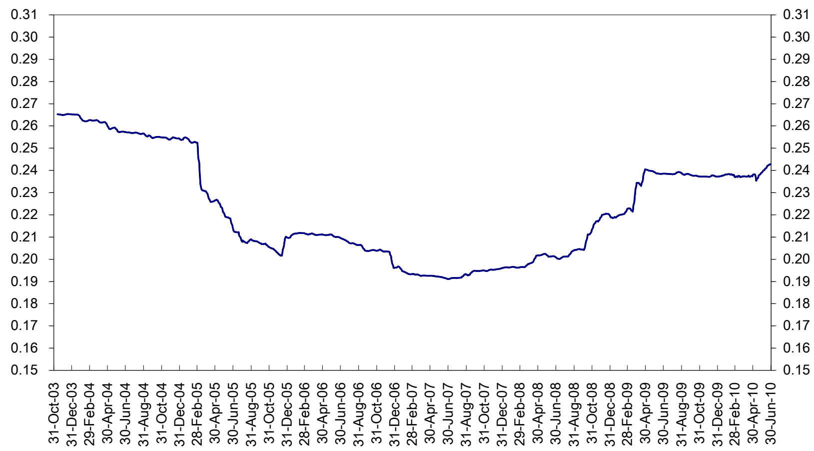

Fig. 4.7: The time-series of the estimated parameter : $\tilde{v}$

This figure shows estimation results of $\tilde{v}$ at each time point in the period from October 31, 2003 to June 30, 2010. These parameters are estimated with the maximum liklihood method proposed by Aït-Sahalia and Kimmel[2007] and we update the parameters daily based on historical 1,750 days daily data with a rolling estimation procedure.

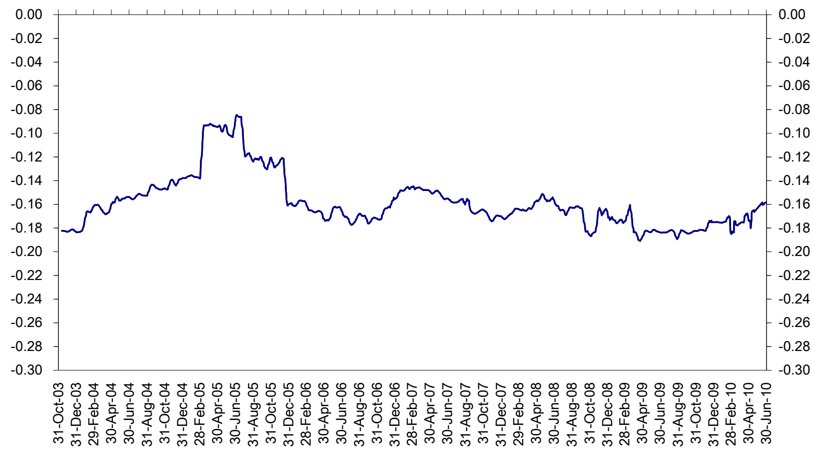

Fig. 4.8: The time-series of the estimated parameter : $\tilde{\rho}$

This figure shows estimation results of $\tilde{\rho}$ at each time point in the period from October 31, 2003 to June 30, 2010. These parameters are estimated with the maximum liklihood method proposed by Aït-Sahalia and Kimmel[2007] and we update the parameters daily based on historical 1,750 days daily data with a rolling estimation procedure.

Chapter 5

Concluding Remarks

In this study, we investigate and identify risk premiums in higher order moments of financial asset returns under various economic settings.

In chapter 2, we investigate the skewness risk premium in the financial market under a general equilibrium setting. Extending the long-run risks (LRR) model proposed by Bansal and Yaron[2004] by introducing a stochastic jump intensity for jumps in the LRR factor and the variance of consumption growth rate, we provide an explicit representation for the skewness risk premium, as well as the volatility risk premium, in equilibrium. On the basis of the representation for the skewness risk premium, we propose a possible reason for the empirical fact of time-varying and negative risk-neutral skewness. Moreover, we also provide an equity risk premium representation of a linear factor pricing model with the variance and skewness risk premiums. The empirical results prove that the skewness risk premium, as well as the variance risk premium, has superior predictive power for future aggregate stock market index returns. Compared with the variance risk premium, the results show that the skewness risk premium plays an independent and essential role for predicting the market index returns.

In chapter 3, we study financial option prices in terms of demand pressure effects based on the preferences of the representative market-maker and the representative end-user. Assuming an incomplete market governed by a stochastic volatility factor in underlying asset price processes, we demonstrate that the demand pressure for an option contract directly impacts traded option prices due to the covariance of the unhedgeable parts of a demanded option and the other traded options. Moreover, considering each of optimization problems for the representative market-maker and the representative end-user independently, we derive the equilibrium demand pressures for traded option contracts and provide an explicit representation for the pricing kernel in equilibrium as a function of those of the equilibrium demand pressures. Finally, we provide some implications in the existence of the variance risk premium and the shape of the implied risk aversion function with the pricing kernel derived above.

Moreover, in chapter 4, we provide a novel representation of delta-hedged option returns in a stochastic volatility environment. The representation of delta-hedged option returns provided in this chapter consists of two terms: volatility risk premium and parameter estimation risk. In an empirical analysis, we examine the delta-hedged option returns based on a historical simulation of a currency option market from October 2003 to June 2010. We find that the delta-hedged option returns for OTM put options are strongly affected by parameter estimation risk as well as the volatility risk premium, especially in the post-Lehman shock period.

In conclusion, it is found from these results that the uncertainty of the higher order moments of financial asset returns, such as the variance and skewness, are consistently priced in equilibrium and we can provide the way of a deeper understanding on the existence of the risk premiums in the higher order moments examined by recent academic studies through both theoretical and empirical approaches.

Owing to their outstanding empirical properties, the risk premiums in higher order moments of financial asset returns have been focused in many fields in the financial economics, such as the estimation of the preference of the representative agent, the estimation of the market sentiment, and the application to investment strategies in financial markets. In particular, Bollerslev, Gibson, and Zhou[2011] propose a method for estimating the representative investor's risk aversion with an intimate link between the stochastic volatility risk premium and the coefficient of risk aversion for the representative investor within the standard intertemporal asset pricing framework. Han[2008] examines whether investor sentiment about the stock market affects prices of the S & P 500 options. The author suggests that one channel for investor sentiment to affect option prices is through the demand pressure effect, which is investigated in Chapter 3 in this study in terms of a relationship with the variance risk premium. Kostakis, Panigirtzoglou, and Skiadopoulos[2011] address the empirical implementation of the static asset allocation problem by developing a forward-looking approach that uses information from market option prices. They find that the use of risk-adjusted implied distributions times the market and makes the investor better off compared with the case where she uses historical returns' distributions to calculate her optimal strategy. The results provided in their study can be also related directly to evidence of the information content of the risk premiums in higher order moments. For a deeper understanding on risk premiums in higher order moments of financial asset returns, further insight into these aspects, in particular, the relationship between the results in this study and these findings in the previous studies, is left for future work.

Bibliography

[1] Aït-Sahalia, Y., 2001, Closed-form likelihood expansions for multivariate diffusions, Working Paper, Princeton University.

[2] Aït-Sahalia, Y., and Kimmel, R., 2007, Maximum likelihood estimation of stochastic volatility models, *Journal of Financial Economics* **83**, 413-452.

[3] Aït-Sahalia, Y., and Lo, A., 1998, Nonparametric estimation of state-price-densities implicit in financial asset prices, *The Journal of Finance* **53**, 499-547.

[4] Aït-Sahalia, Y., and Lo, A., 2000, Nonparametric Risk Management and Implied Risk Aversion, *Journal of Econometrics*, **94**, 9-51.

[5] Aït-Sahalia, Y., Wang, W., and Yared, F., 2001, Do option markets correctly price the probabilities of movement of the underlying asset？, *Journal of Econometrics* **102**, 67-110.

[6] Andersen, T., G., Bollerslev, T., Diebold, F., X., and Labys, P., 2000, Exchange Rate Returns Standardized by Realized Volatility are (Nearly) Gaussian, *Multinational Finance Journal*, **4**, 159-179.

[7] Bakshi, G., and Kapadia, N., 2003, Delta-hedged gains and Negative Market Volatility Risk Premium, *The Review of Financial Studies*, **16**, 527-566.

[8] Bakshi, G., N., Kapadia, and D., Madan, 2003, Stock Return Characteristics, Skew Laws, and the Differential Pricing of Individual Equity Options, *Review of Financial Studies* **16**, 101-143.

[9] Bali, T., G., and A., Hovakimian, 2009, Volatility Spread and Expected Stock Returns, *Management Science* **55**, 1797-1812.

[10] Bansal, R., and A. Yaron, 2004, Risks for the Long Run: A Potential Resolution of Asset-pricing Puzzles, *Journal of Finance* **59**, 1481-1509.

[11] Bollen, N., and Whaley, R., 2004, Does Net Buying Pressure Affect the Shape of Implied Volatility Functions?, *The Journal of Finance*, **59**, 711-753.

[12] Bollerslev, T., Gibson, N., and H. Zhou, 2011, Dynamic estimation of volatility risk premia and investor risk aversion from option-implied and realized volatilities, *Journal of Econometrics*, **160**, 235-245.

[13] Bollerslev, T., N. Sizova, and G. Tauchen, 2012, Volatility in Equilibrium: Asymmetries and Dynamic Dependencies, *Review of Finance* **16**, 31-80.

[14] Bollerslev, T., G. Tauchen, and H. Zhou, 2010, Expected Stock Returns and Variance Risk Premia, *Review of Financial Studies* **22**, 4463-4492.

[15] Boyle, P., and Ananthanarayanan, A., 1977, The Impact of Variance Estimation in Option Valuation Models, *Journal of Financial Economics* **5**, 375-387.

[16] Branger, N., and C. Völkert, 2010, What is the Equilibrium Price of Variance Risk? A Long-Run Risk Model with Two Volatility Factors, Working Paper.

[17] Breeden, D., T., and Litzenberger, R., H., 1978, Prices of State-contingent Claims Implicite on Option Prices, *Journal of Business*, **51**, 621-651.

[18] Broadie, M., Chernov, M., and Johannes, M., 2009, Understanding Index Option Returns, *The Review of Financial Studies* **22**, 4493-4529.

[19] Brunner, B., and Hafner, R., 2003, Arbitrage-Free Estimation of the Risk-Neutral Density from the Implied Volatility Smile, *Journal of Computational Finance* **7**, 75-106.

[20] Bunnin, F., O., Guo, Y., and Ren, Y., 2002, Option pricing under model and parameter uncertainty using predictive densities, *Statistics and Computing* **12**, 37-44.

[21] Campbell, J., Y., and R., J., Shiller, 1988, The dividend-price ratio and expectations of future dividends and discount factors, *Review of Financial Studies* **1**, 195-227.

[22] Carr, P., and Wu, L., 2009, Variance Risk Premiums, *The Review of Financial Studies*, **22**, 1311-1341.

[23] Chan, W., H., and J., M., Maheu, 2002, Conditional Jump Dynamics in Stock Market Returns, *Journal of Business & Economic Statistics* **20**, 377-389.

[24] Chang, B., Y., P. Christoffersen, and K., Jacobs, 2013, Market skewness risk and the cross section of stock returns, *Journal of Financial Economics* **107**, 46-68.

[25] Christoffersen, P., K., Jacobs, and C., Ornthanalai, 2012, Dynamic jump intensities and risk premiums: Evidence from S & P500 returns and options, *Journal of Financial Economics* **106**, 447-472.

[26] Cochrane, J., H., 2001. Asset Pricing. Princeton University Press, Princeton.

[27] Cochrane, J., H., and Hansen, L., P., 1992, Asset pricing explorations for macroeconomics, in 1992 NBER Macroeconomics Annual (NBER, Cambridge, MA).

[28] Cont, R., 2006, Model Uncertanity and its Impact on the Pricing of Derivative Instruments, *Mathmatical Finance* **16**, 519-547.

[29] Cont, R., and P. Tankov. 2004, Financial Modeling with Jump Processes. London: Chapman & Hall.

[30] Coval, J., and Shumway, T., 2001, Expected Option Returns, *The Journal of Finance* **56**, 983-1009.

[31] Demeterfi, K., E., Derman, M., Kamal, and J., Zou, 1999, A Guide to Volatility and Variance Swaps, *The Journal of Derivatives* **6**, 9-32.

[32] Drechsler, I. and A. Yaron, 2011, What's vol got to do with it?, *Review of Financial Studies* **24**, 1-45.

[33] Driessen, J., T., Lin, and X., Lu, 2012, Why do option prices predict stock returns?, Working Paper.

[34] Driessen, J., and Maenhout, P., 2007, An Empirical Portfolio Perspective on Option Pricing Anomalies, *Review of Finance*, **11**, 561-603.

[35] Epstein, L., and S. Zin, 1989, Substitution, risk aversion and the temporal behavior of consumption and asset returns: A theoretical framework, *Econometrica* **57**, 937-969.

[36] Eraker, B., 2008, The volatility premium, Working Paper, Department of Finance, University of Wisconsin.

[37] Gârleanu, N., Pedersen, L., and Poteshman, A., 2009, Demand-Based Option Pricing, *The Review of Financial Studies*, **22**, 4259-4299.

[38] Galai, D., 1983, The Components of the Return from Hedging Options Against Stocks, *Journal of Business* **56**, 45-54.

[39] Garman, M., and Kohlhagen, S., 1983, Foreign currency option values, *Journal of International Money and Finance* **2**, 231-237.

[40] Goyal, A., and Saretto, A., 2009, Cross-section of option returns and volatility, *Journal of Financial Economics* **94**, 310-326.

[41] Green, T., C., and Figlewski, S., 1999, Market Risk and Model Risk for a Financial Institution Writing Options, *The Journal of Finance*, **54**, 1465-1499.

[42] Han, Bing, 2008, Investor Sentiment and Option Prices, *The Review of Financial Studies*, **21**, 387-414.

[43] Heston, S., 1993, A closed-form solution for options with stochastic volatility with applications to bonds and currency options, *The Review of Financial Studies*, **6**, 327-343.

[44] Jackwerth, J., C., 2000, Recovering Risk Aversion from Option Prices and Realized Returns, *The Review of Financial Studies*, **13**, 433-451.

[45] Jones, C., S., 2006, A Nonlinear Factor Analysis of S & P 500 Index Option Returns, *The Journal of Finance* **61**, 2325-2363.

[46] Kang, J., and Park, H., 2008, The information content of net buying pressure: Evidence from the KOSPI 200 index option market, *Journal of Financial Markets*, **11**, 36-56.

[47] Kostakis, A., Panigirtzoglou, N., and Skiadopoulos, G., 2011, Market Timing with Option-Implied Distributions: A Forward-Looking Approach, *Management Science*, **57**, 1231-1249.

[48] Lettau, M., and S. Ludvigson, 2001, Consumption, Aggregate Wealth, and Expected Stock Returns, *Journal of Finance* **56**, 815-849.

[49] Low, B., S., and Zhang, S., 2005, The Volatility Risk Premium Embedded in Currency Options, *Journal of Financial and Quantitative Analysis*, **40**, 803-832.

[50] Mehra, R., and Prescott, E., C., 1985, The equity premium: A puzzle, *Journal of Monetary Economics*, **15**, 145-161.

[51] Rehman, Z., and G., Vilkov, 2012, Risk-Neutral Skewness: Return Predictability and Its Sources, Working Paper.

[52] Rosenberg, J. V., and R. F. Engle, 2002, Empirical Pricing Kernel, *Journal of Financial Economics*, **64**, 341-372.

[53] Shiu, Y., Pan, G., Lin, S., and Wu, T., 2010, Impact of Net Buying Pressure on Change in Implied Volatility: Before and After the Onset of the Subprime Crisis, *The Journal of Derivatives*, **17**, 54-66.

[54] Shreve, S., E., 2004, Stochastic Calculus for Finance II: Continuous-Time Models, Springer.

[55] Yan, S., 2011, Jump risk, stock returns, and slope of implied volatility smile, *Journal of Financial Economics* **99**, 216-233.

[56] Ziegler, A., 2007, Why Does Implied Risk Aversion Smile?, *The Review of Financial Studies*, **20**, 859-904.