HIAS-E-17

Liquidity Shocks and Asset Prices

Pablo A. Guerron-Quintana
Federal Reserve Bank of Philadelphia
Ryo Jinnai
Hitotsubashi University

December 1, 2015

Hitotsubashi Institute for Advanced Study, Hitotsubashi University
2-1, Naka, Kunitachi, Tokyo 186-8601, Japan
tel:+81 42 580 8604 http://hias.ad.hit-u.ac.jp/

HIAS discussion papers can be downloaded without charge from:
http://hdl.handle.net/10086/27202
https://ideas.repec.org/s/hit/hiasdp.html

Liquidity Shocks and Asset Prices

Pablo A. Guerron-Quintana Ryo Jinnai*

December 1, 2015

Abstract

In models of liquidity, stock market booms tend to follow adverse liquidity shocks. This result is clearly at odds with the data. We demonstrate that allowing for endogenous productivity corrects this puzzling price dynamics. Negative growth prospects decrease equity prices because of a long-run predictable component in dividend growth.

1 Introduction

The severity and widespread impact of the Great Recession has propelled substantial research to better understand the interconnection between the financial sector and the aggregate economy. One strand of this literature emphasizes the role of shocks affecting the liquidity of assets, i.e., how easily assets can be pledged as collateral. Yet these models of liquidity (Jermann and Quadrini (2012) and Kiyotaki and Moore (2012)) have often been criticized because of their counterfactual stock market dynamics (Shi (2015)). More precisely, these models tend to predict a stock market boom following an adverse liquidity shock to the underlying asset. Furthermore, the counterfactual asset price response is robust to a wide range of specifications such as nominal rigidities, habit formation in consumption, and adjustment costs in capital or investment. This prediction is troublesome not only because it is counterfactual but also because it arises at precisely the heart of the transmission mechanism they feature. It is therefore crucial to correct these models before we use them, for instance, as a laboratory to evaluate policies.

In this paper, we argue that this stock market anomaly is a consequence of the disconnect between productivity and liquidity. To support our claim, we construct an endogenous growth

*Guerron-Quintana: Federal Reserve Bank of Philadelphia, email: pablo.guerron@phil.frb.org. Jinnai: Hitotsubashi University, email: rjinnai@ier.hit-u.ac.jp. We thank Thorsten Drazberg and Daniel Sanches for useful comments/discussions and Yang Liu for research assistance. Ryo Jinnai gratefully acknowledges the financial support from the Ministry of Education, Culture, Sports, Science and Technology of the Japanese Government through JSPS KAKENHI Grant and the Hitotsubashi Institute for Advanced Study. The views expressed here are those of the authors and do not necessarily reflect those of the Federal Reserve Bank of Philadelphia or the Federal Reserve System.

model with cash strapped investors. Specifically, we augment Shi (2015)'s model with a stylized learning-by-doing mechanism.[1] In this framework, an adverse liquidity disturbance has a persistent, in fact permanent, negative effect on the level of macroeconomic variables. This prediction is consistent with recent empirical work documenting that recessions associated with financial crises tend to be long and painful (Cerra and Saxena (2008), Jorda, Schularick, and Taylor (2014), and Reinhart and Rogoff (2009)). Because of a long-run predictable component in dividend growth associated with an adverse liquidity disturbance, the stock market plunges following the unfavorable shock if the intertemporal elasticity of substitution is large enough. This is essentially the same insight as the one used in the long-run risk literature in finance (Bansal and Yaron (2004)).

2 Model

The economy is populated by a continuum of households, with measure one. Each household has a unit measure of members. At the beginning of the period, all members of a household are identical and share the household's assets. During the period, members are separated from each other, and each member receives a shock that determines the role of the member in the period. A member will be an entrepreneur with probability $\pi \in [0, 1]$ and will be a worker with probability $1 - \pi$. These shocks are i.i.d. among the members and across time.

A period is divided into four stages: household's decisions, production, investment, and consumption. In the household's decision stage, all members of a household are together to pool their assets: s_t units of equity claims. An equity is the ownership of a unit of capital. Aggregate shocks to exogenous state variables are realized. Because all the members of the household are identical in this stage, the household evenly divides the assets among the members. The head of the household also gives contingency plans to each member as follows. If one becomes an entrepreneur, he or she spends i_t units of consumption goods to invest, sets aside x_t^e units of consumption goods for the consumption stage, and makes necessary trades in the stock market so that he or she returns to the household with s_{t+1}^e units of equity claims. In contrast, if the member becomes a worker, he or she supplies l_t units of labor, sets aside x_t^w units of consumption goods for the consumption stage, and makes necessary trades in the stock market so that he or she returns to the household with s_{t+1}^w units of equity claims. After receiving these instructions, the members go to the market and will remain separated from each other until the consumption stage.

At the beginning of the production stage, each member receives the shock whose realization determines whether the individual is an entrepreneur or a worker. Competitive firms produce final

[1] We introduce two modifications to Shi (2015). First, we abstract away from the government sector. Second, we allow perfect risk sharing within a household at the consumption stage. These two changes allow us to obtain a clean asset pricing equation which we will use extensively in the discussion of the result.

consumption goods y_t using labor service l_t^D and capital service k_t^D with the production technology

$$y_t = A_t \left(k_t^D\right)^{\alpha} \left(l_t^D\right)^{1-\alpha}.$$

Here, α is the capital share, the superscript D indicates demand, and A_t is the technology level which both the households and the firms take as given. After production, a worker receives wage income, and an individual receives compensation for capital. A fraction δ of capital depreciates.

The third stage in the period is the investment stage where entrepreneurs seek finance and undertake investment projects. We assume that an entrepreneur can transform any amount i_t units of consumption goods into i_t units of new capital. Individuals trade assets to finance investment and to achieve the asset holdings instructed earlier by their households. Markets close at the end of this stage.

The individuals return to their households in the consumption stage. The head of the household lets an entrepreneur consume c_t^e units of consumption goods and lets a worker consume c_t^w units of consumption goods. The feasibility constraint in this stage is

$$\pi x_t^e + (1-\pi)\, x_t^w = \pi c_t^e + (1-\pi)\, c_t^w. \tag{1}$$

The instructions have to satisfy the intra-temporal budget constraints:

$$x_t^e + i_t = r_t s_t + q_t \left((1-\delta)\, s_t + i_t - s_{t+1}^e\right) \tag{2}$$

for an entrepreneur where r_t and q_t are the rental price and the equity price respectively, and

$$x_t^w = r_t s_t + q_t \left((1-\delta)\, s_t - s_{t+1}^w\right) + w_t l_t \tag{3}$$

for a worker where w_t is the wage rate. We introduce the following equity-market frictions. An entrepreneur can issue new equity on at most a fraction θ of investment. In addition, she can sell at most a fraction ϕ_t of existing capital in the market. Effectively, these constraints introduce a lower bound to the capital holdings of an entrepreneur at the end of the period:

$$s_{t+1}^e \geq (1-\theta)\, i_t + (1-\phi_t)\,(1-\delta)\, s_t. \tag{4}$$

ϕ_t is an exogenous, random variable we call a liquidity shock. A similar constraint applies to workers, i.e., $s_{t+1}^w \geq (1-\phi_t)\,(1-\delta)\, s_t$, but we omit it because it does not bind in equilibrium for workers are net buyers of equities under our calibration. There are non-negativity constraints for i_t, l_t, x_t^w, c_t^e, c_t^w, and s_{t+1}^w, but we omit them too because they do not bind in equilibrium either. An exception is the entrepreneur's intra-temporal resource transfer, whose non-negativity

constraint is given by

$$x_t^e \geq 0. \tag{5}$$

The head of the household chooses instructions to its members to maximize the value function defined as

$$v\left(s_t; K_t, \phi_t\right) = \max\left\{\pi \frac{\left(c_t^e\right)^{1-\rho}}{1-\rho} + (1-\pi)\frac{\left[c_t^w\left(1-l_t\right)^{\eta}\right]^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t\left[v\left(s_{t+1}; K_{t+1}, \phi_{t+1}\right)\right]\right\}$$

subject to (1), (2), (3), (4), (5), and the accounting identity

$$s_{t+1} = \pi s_{t+1}^e + (1-\pi)\, s_{t+1}^w.$$

As in Shi (2015), we will restrict our attention to the case in which $1 < q_t < 1/\theta$ always hold. The first inequality $1 < q_t$ implies that the marginal costs of investment are smaller than the marginal revenues. The second inequality $q_t < 1/\theta$ implies that the entrepreneur needs extra liquidity for down payment whenever she expands the scale of the investment project. These conditions jointly imply that both the entrepreneur's liquidity constraint (4) and her non-negativity constraint for intra-temporal resource transfer (5) must be binding at the optimum. The appendix includes the proof of this claim as well as the derivations of the optimality conditions and other important results.

Optimal labor supply condition requires that the marginal utility of leisure is equated to the marginal utility of receiving wage income,

$$\eta \frac{c_t^w}{1-l_t} = w_t.$$

Optimal resource allocation in the consumption stage requires that the marginal utility of entrepreneur's consumption is equated to the marginal utility of worker's consumption,

$$\left(c_t^e\right)^{-\rho} = \left(c_t^w\right)^{-\rho}\left(1-l_t\right)^{\eta(1-\rho)}.$$

Finally, the following equation determines the price of equity,

$$q_t = \mathbb{E}_t\left[\beta\left(\frac{c_{t+1}^e}{c_t^e}\right)^{-\rho}\left(r_{t+1} + (1-\delta)\, q_{t+1} + \pi\lambda_{t+1}^e\left(r_{t+1} + q_{t+1}\phi_{t+1}\left(1-\delta\right)\right)\right)\right]$$

where λ_t^e is the variable Shi (2015) calls liquidity services

$$\lambda_t^e = \frac{q_t - 1}{1-\theta q_t}.$$

The numerator is the marginal benefit of investment, and the denominator is the downpayment

required to increase investment marginally. As such, λ_t^e measures the marginal benefit of providing liquidity to an entrepreneur.

The competitive equilibrium is defined in a standard way. Market clearing conditions are

$$\pi c_t^e + (1-\pi)\, c_t^w + \pi i_t = A_t \left(k_t^D\right)^{\alpha} \left(l_t^D\right)^{1-\alpha},$$

$$l_t^D = (1-\pi)\, l_t,$$

$$k_t^D = K_t = s_t,$$

and

$$\pi s_{t+1}^e + (1-\pi)\, s_{t+1}^w = (1-\delta)\, s_t + \pi i_t.$$

Capital stock evolves according to

$$K_{t+1} = (1-\delta)\, K_t + \pi i_t.$$

The aggregate stock market value $Stock_t$ is measured by

$$Stock_t = q_t K_{t+1}.$$

We assume that asset resalability ϕ_t obeys

$$\log\left(\frac{\phi_t}{\phi}\right) = \rho_\phi \log\left(\frac{\phi_{t-1}}{\phi}\right) + \varepsilon_t. \tag{6}$$

ε_t is an i.i.d. shock.

We compare results under two alternative assumptions regarding the source of growth. In one, we assume that the technology level A_t grows at a constant rate:

$$A_t = A_0 \left(\gamma^{1-\alpha}\right)^t,$$

where A_0 is a scale parameter. This is a standard assumption in the business cycle research including Shi (2015). In the second case, we assume that the level of A_t is endogenous;

$$A_t = A_0 \left(K_t\right)^{1-\alpha}.$$

We interpret this assumption following Arrow (1962), Sheshinski (1967), and Romer (1986); namely, knowledge is not only a by-product of investment but also a public good that anyone

Table 1: Parameters and Calibration Targets

	Value		
Parameter	EXO	END	Calibration Target
β; discount factor	0.994	0.994	Exogenously chosen
ρ; relative risk aversion	1/1.85	1/1.85	Exogenously chosen
π; fraction of entrepreneurs	0.06	0.06	Annual fraction of investing firms = 0.24
η; curvature in leisure utility	2.66	2.66	Hours of work = 0.25
α; capital share	0.36	0.36	Labor income share $(1-\alpha) = 0.64$
δ; capital depreciation rate	0.015	0.015	Annual investment/capital = 0.076
A_0; scale parameter	1	0.183	Normalization; steady-state growth = 1.004
γ; steady-state growth	1.004	-	Steady-state growth = 1.004
ϕ; steady-state resalability	0.12	0.12	Capital stock/annual output = 3.32
ρ_ϕ; persistence in resalability	0.9	0.9	Exogenously chosen
θ; fraction of new equity	0.12	0.12	Set equal to ϕ

can access at zero cost.[2] Under this assumption, the production function is written as

$$Y_t = A_0 K_t \left[(1-\pi)\, l_t\right]^{1-\alpha}.$$

The long-run tendency for capital to experience diminishing returns is eliminated. As such, the economy can grow in the long run.

3 Calibration

Table 1 reports our calibration, which is similar to Shi (2015) both in values and in calibration strategies. We choose the discount factor β, the fraction of investors π, the capital share α, and the persistence in resalability ρ_ϕ exogenously. The relative risk aversion parameter is set at $\rho = 1/1.85$ so that the implied inter-temporal elasticity of substitution (the inverse of ρ) is consistent with the value used in the finance literature (Kung and Schmid (2015)) and within the credible set estimated by Schorfheide, Song, and Yaron (2014). We set the steady state growth rate at $\gamma = 1.004$ and normalize the scale parameter by $A_0 = 1$ in the exogenous growth model. We calibrate the scale parameter A_0 in the endogenous growth model so that the model's non-stochastic steady state growth rate is 1.004. Following Shi (2015), we assume that the fraction of new equity θ is equal to the steady state resalability ϕ. We calibrate ϕ as well as the curvature in leisure in the utility function η and the capital depreciation rate δ using the following targets (all of which are used by Shi (2015)): the aggregate hours of work in the steady state (0.25), the ratio of capital to annual output in the steady state (3.32), and the ratio of annual investment to capital in the steady state (0.076).

[2] We want to emphasize that this assumption is made for tractability. Similar results would follow from more elaborated versions of endogenous growth.

4 Results

We suppose that all the detrended variables are in the non-stochastic steady state at time $t = 0$. At the beginning of $t = 1$, there is an unanticipated 10% drop in liquidity from the steady state. Thereafter ϕ_t follows the process in (6) with $\varepsilon_t = 0$ for all $t \geq 2$. Figure 1 plots percentage deviations of selected variables from their realizations in an alternative scenario in which no shock hits the economy for all $t \geq 1$. As we can see, a negative liquidity shock is contractionary in general with the exception of consumption defined as $\pi c_t^e + (1-\pi) c_t^w$, which rises in the short run. A few ways to overturn this counterfactual prediction are known in the literature such as nominal wage rigidities (Ajello (2014)), insight from the news shock literature (Guerron-Quintana and Jinnai (2015)), and investment adjustment costs (Shi (2015)). Our exercise also reveals that the effect of the shock is more persistent in the endogenous growth model than in the exogenous growth model. In fact, it is permanent in the former while it is transitory in the latter.

Our main interest is in asset prices. In the exogenous growth model, the equity price q_t counterfactually rises in response to a negative liquidity shock while it drops in the endogenous growth model. The same pattern is true for the aggregate stock market value $Stock_t = q_t K_{t+1}$. The following equation is useful to understand the factors behind these results:

$$Stock_t = \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}^e}{c_t^e} \right)^{-\rho} (\alpha y_{t+1} - \pi i_{t+1} + Stock_{t+1}) \right]. \tag{7}$$

This equation holds in both the endogenous and exogenous growth models. Taking a first-order log approximation in the endogenous growth model, we find

$$\begin{aligned} \underbrace{\log\left(\frac{Stock_t}{\gamma^t Stock_0}\right)}_{\text{IRF}} = & \underbrace{\log\left(\frac{K_t/K_0}{\gamma^t}\right)}_{\text{realized change in trend}} \\ & \underbrace{+\left(1-\beta\gamma^{1-\rho}\right)\sum_{j=1}^{\infty}\left(\beta\gamma^{1-\rho}\right)^{j-1}\mathbb{E}_t\left[(1-\rho)\log\left(\frac{K_{t+j}/K_t}{\gamma^j}\right)\right]}_{\text{expected change in trend}} \\ & \underbrace{\begin{aligned} &+\left(1-\beta\gamma^{1-\rho}\right)\sum_{j=1}^{\infty}\left(\beta\gamma^{1-\rho}\right)^{j-1} \\ &\times\mathbb{E}_t\left[-\rho\log\left(\frac{\hat{c}_{t+j}^e}{\hat{c}_t^e}\right)+\frac{\alpha\hat{y}}{\alpha\hat{y}-\pi\hat{\imath}}\log\left(\frac{\hat{y}_{t+j}}{\hat{y}}\right)-\frac{\pi\hat{\imath}}{\alpha\hat{y}-\pi\hat{\imath}}\log\left(\frac{\hat{\imath}_{t+j}}{\hat{\imath}}\right)\right], \end{aligned}}_{\text{cycle}} \end{aligned} \tag{8}$$

where variables with a hat indicate that they are divided by the endogenous trend term; for example, $\hat{y}_t$ is defined as $\hat{y}_t = y_t/K_t$. Variables without time subscript denote non-stochastic steady state values of the corresponding variables. γ denotes the (gross) growth rate of the

economy in the non-stochastic steady state. The left-hand side represents the impulse response function of the stock market in period t. The right-hand side breaks it into three factors, of which two are contributions of the endogenous trend term, while the remaining one is the contribution of fluctuations around the trend.

The top panel of Figure 2 plots the contributions of each factor to the dynamic response of the stock market. Contributions of the cyclical terms are positive throughout the simulation. However, the endogenous trend terms more than offset the cyclical contributions in our framework. Not surprisingly, the contribution of the realized change in the endogenous trend term is important only in the medium run because the trend term K_t is a stock variable. It is therefore the expectation about the future trend term that is crucial to generate a negative stock market response in the short run. The mechanism is intuitive and simple. Because the endogenous trend term will be low in the future, so are future cash flows (remember that they are cointegrated). This expectation depresses the current value of the stock.

Yet this prediction is sensitive to some parameter values. To show this point, we plot the impulse response function under the alternative calibration with $\rho = 1.0$, i.e., log utility, in the middle panel. Contributions of the expected change in the future trend disappear, and as a result, the aggregate stock market value positively responds to a negative liquidity shock in the short run. To understand the mechanism, it is important to realize that a liquidity shock affects not only cash flows but also the entrepreneur's consumption profile and hence the stochastic discount factor. Specifically, the impulse response function of the entrepreneur's consumption (not depicted) monotonically decreases for $t \geq 1$, namely,

$$\underbrace{\log\left(\frac{c_t^e}{\gamma^t c_0^e}\right)}_{\text{IRF in period } t} > \underbrace{\log\left(\frac{c_{t+1}^e}{\gamma^{t+1} c_0^e}\right)}_{\text{IRF in period } t+1}$$

for $t \geq 1$. Rearranging terms, we find

$$\frac{c_{t+1}^e}{c_t^e} < \gamma.$$

This inequality implies that the entrepreneur's consumption growth after a negative liquidity shock (c_{t+1}^e/c_t^e for $t \geq 1$) is lower than the growth rate in an alternative scenario in which no liquidity shock hits the economy (γ). Its effect on the stock market value is clear from equation (7); the stochastic discount factor $\beta\left(c_{t+1}^e/c_t^e\right)^{-\rho}$ becomes larger (the discount rate $1-\beta\left(c_{t+1}^e/c_t^e\right)^{-\rho}$ becomes smaller) after a negative liquidity shock than before, and hence, the current stock market value would increase even if the future cash flows are unchanged. Put it differently, the marginal rate of substitution between consumption goods in the future and consumption goods in current period rises after the shock, making a claim of a unit of consumption goods in the future more valuable in terms of consumption goods in the current period. This effect is stronger when the relative risk aversion is larger (the inter-temporal elasticity of substitution is smaller). $\rho = 1$ is the threshold

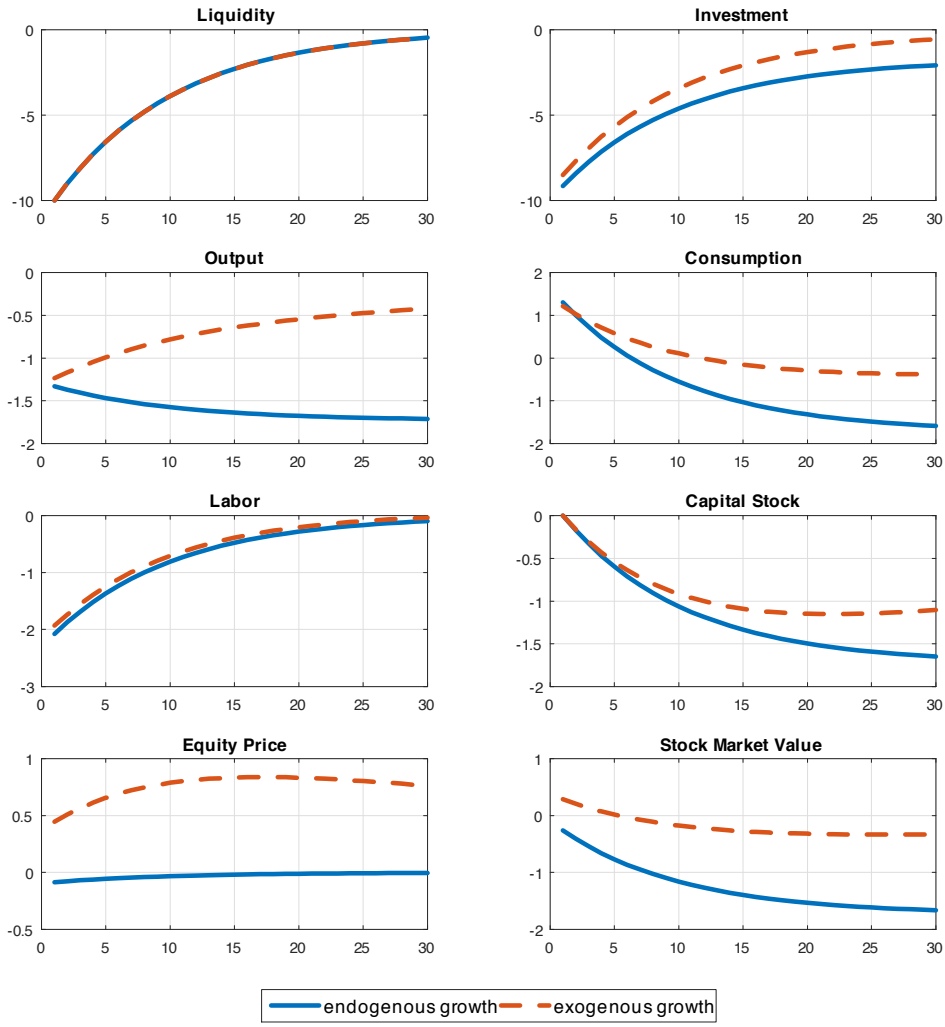

Figure 1: Impulse Response Functions to a Liquidity Shock

value at which this effect through the relative price (i.e., the marginal rate of substitution) perfectly offsets the aforementioned effect through cash flows. The essentially same insight is well known in the long-run risk literature in finance (Bansal and Yaron (2004)), emphasizing the importance of a large inter-temporal elasticity of substitution to obtain a reasonable stock market response to a change in growth prospects. The long-run risk is an endogenous outcome in our model caused by financial shocks. This implication might be of interest in the literature searching for a structural origin of the long-run risk.[3]

The decomposition also sheds a light on why in the exogenous growth model stock market responds positively to a negative liquidity shock. This is because the aggregate stock market value is driven by the cyclical terms alone. We illustrate this point in the bottom panel of Figure 2. Clearly, the cyclical terms are positive contributors to the aggregate stock market value, and this prediction is robust as Shi (2015) discusses in detail.

5 Conclusion

Models of liquidity have become fashionable since the onset of the Great Recession. This is so because they provide a natural connection between financial frictions and the depth of the crisis. These models, however, tend to have the unappealing feature of predicting counterfactual dynamics of asset prices. In this paper, we show that this anomaly is readily fixed once we allow for endogenous productivity.

References

Ajello, A. (2014): "Financial Intermediation, Investment Dynamics and Business Cycle Fluctuations," Board of Governors Working Paper.

Arrow, K. J. (1962): "The Economic Implications of Learning by Doing," *The Review of Economic Studies*, 29(3), pp. 155–173.

Bansal, R., and A. Yaron (2004): "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *The Journal of Finance*, 59(4), 1481–1509.

Cerra, V., and S. C. Saxena (2008): "Growth Dynamics: The Myth of Economic Recovery," *American Economic Review*, 98(1), 439–57.

Croce, M. M. (2014): "Long-Run Productivity Risk: A New Hope for Production-Based Asset Pricing?," *Journal of Monetary Economics*, 66, 13–31.

[3]Previous work includes Croce (2014), Jinnai (2015), Kaltenbrunner and Lochstoer (2010), and Kung and Schmid (2015).

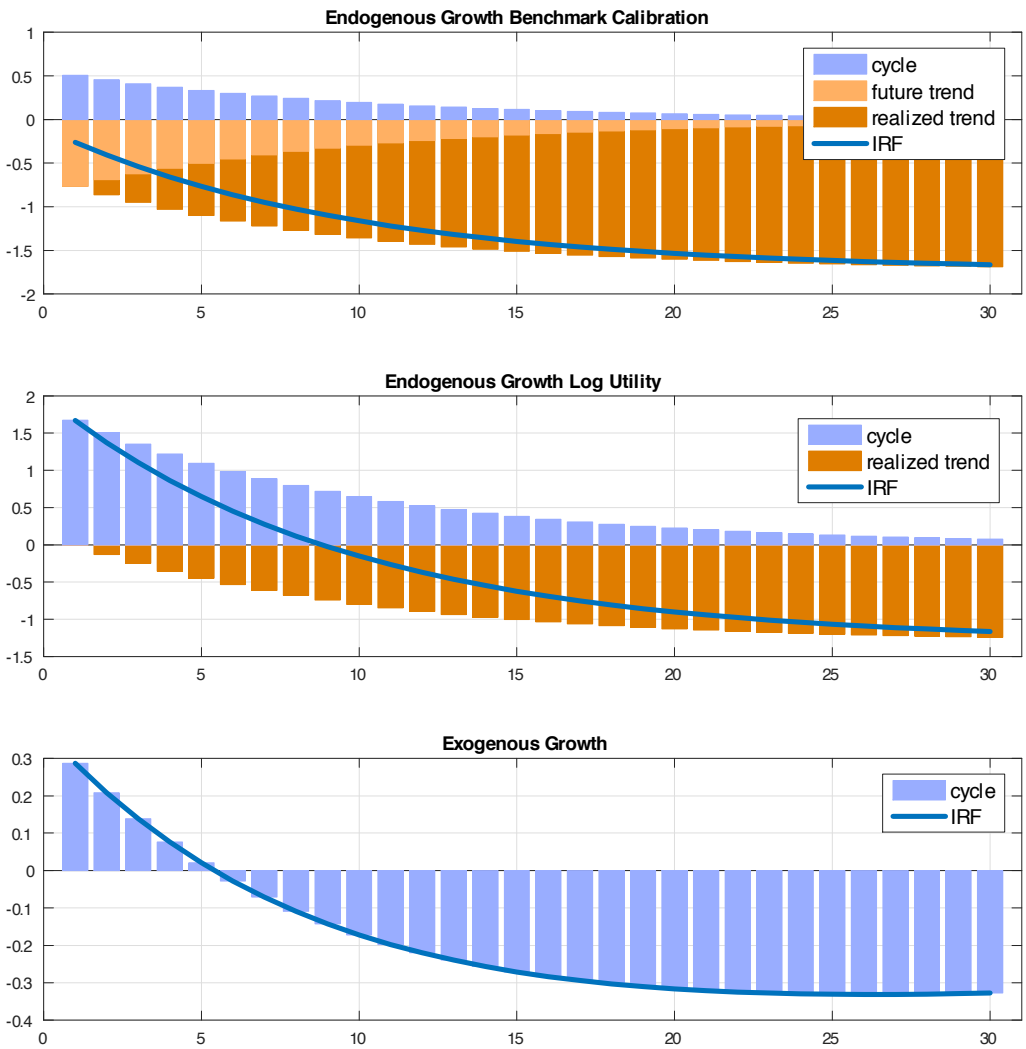

Figure 2: Factor Decompositions of Stock Market Responses

Guerron-Quintana, P. A., and R. Jinnai (2015): "Financial Frictions, Trends, and the Great Recession," Unpublished Manuscript.

Jermann, U., and V. Quadrini (2012): "Macroeconomic Effects of Financial Shocks," *American Economic Review*, 102(1), 238–71.

Jinnai, R. (2015): "Innovation, product cycle, and asset prices," *Review of Economic Dynamics*, 18(3), 484 – 504.

Jorda, O., M. Schularick, and A. M. Taylor (2014): "The Great Mortgaging: Housing Finance, Crises, and Business Cycles," Working Paper 20501, National Bureau of Economic Research.

Kaltenbrunner, G., and L. A. Lochstoer (2010): "Long-Run Risk through Consumption Smoothing," *Review of Financial Studies*, 23, 3190–3224.

Kiyotaki, N., and J. Moore (2012): "Liquidity, Business Cycles, and Monetary Policy," Working Paper 17934, National Bureau of Economic Research.

Kung, H., and L. Schmid (2015): "Innovation, Growth, and Asset Prices," *The Journal of Finance*, 70, 1001–1037.

Reinhart, C., and K. Rogoff (2009): *This Time Is Different: Eight Centuries of Financial Folly*, vol. 1. Princeton University Press, 1 edn.

Romer, P. M. (1986): "Increasing Returns and Long-Run Growth," *Journal of Political Economy*, 94(5), pp. 1002–1037.

Schorfheide, F., D. Song, and A. Yaron (2014): "Identifying Long-Run Risks: A Bayesian Mixed-Frequency Approach," NBER Working Papers 20303, National Bureau of Economic Research, Inc.

Sheshinski, E. (1967): "Optimal accumulation with learning by doing," in *Essays on the theory of optimal economic growth*, ed. by K. Shell, pp. 31–52. MIT Press, Cambridge, MA.

Shi, S. (2015): "Liquidity, assets and business cycles," *Journal of Monetary Economics*, 70(0), 116 – 132.

6 Appendix

6.1 Solving Household's Problem

We assume that $1 < q_t < 1/\theta$ always hold. We argue that the liquidity constraint (4) must be binding at the optimum. Suppose otherwise. Then the head of the household can increase i_t by $\Delta > 0$ and increase x_t^e by $(q_t - 1)\Delta$ without violating any constraints as long as Δ is sufficiently small. These changes do not alter s_{t+1} but increase the amount of resource available in the consumption stage $\pi x_t^e + (1-\pi) x_t^w$, implying that the initial choice is not optimal. Next, we argue that the non-negativity constraint (5) must be binding at the optimum. Suppose otherwise. Then the head of the household can decrease x_t^e by $\Delta > 0$, increase s_{t+1}^e by $(1/q_t)\Delta$, increase s_{t+1}^w by $(\pi/(1-\pi))(1/q_t)\Delta$, and decrease x_t^w by $(\pi/(1-\pi))\Delta$ without violating any constraints as long as $\Delta \leq x_t^e$. These changes alter neither s_{t+1} nor the amount of resource available in the consumption stage, but make the liquidity constraint (4) slack. Because the household can increase utility if (4) is not binding, the initial choice is not optimal.

With the binding liquidity and non-negativity constraints, the household's problem is rewritten as follows.

$$v(s_t; K_t, Z_t) = \max \left\{ \pi \frac{(c_t^e)^{1-\rho}}{1-\rho} + (1-\pi) \frac{[c_t^w (1-l_t)^\eta]^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[v(s_{t+1}; K_{t+1}, Z_{t+1}) \right] \right\}$$

subject to

$$r_t s_t + w_t l_t + q_t \left((1-\delta) s_t - s_{t+1}^w \right) = \frac{1}{1-\pi} \left[\pi c_t^e + (1-\pi) c_t^w \right]$$

and

$$s_{t+1} = \pi \left[\frac{1}{q_t} (1+\lambda_t^e)(r_t + q_t (1-\delta)\phi_t) + (1-\phi_t)(1-\delta) \right] s_t + (1-\pi) s_{t+1}^w$$

where

$$\lambda_t^e = \frac{q_t - 1}{1 - q_t \theta}.$$

First order conditions are

$$(c_t^e)^{-\rho} = \mu_t, \tag{9}$$

$$(c_t^w)^{-\rho} (1-l_t)^{\eta(1-\rho)} = \mu_t, \tag{10}$$

$$\eta (c_t^w)^{-\rho} (1-l_t)^{\eta(1-\rho)} \frac{c_t^w}{1-l_t} = \mu_t w_t, \tag{11}$$

and

$$\beta E_t \left[v'(s_{t+1}; K_{t+1}, Z_{t+1}) \right] = \mu_t q_t. \tag{12}$$

Envelope condition is

$$\begin{aligned} & v'\left(s_t; K_t, Z_t\right) \\ = \; & \beta E_t\left[v'\left(s_{t+1}; K_{t+1}, Z_{t+1}\right)\right]\pi\left[\frac{1}{q_t}\left(1+\lambda_t^e\right)\left(r_t+q_t\left(1-\delta\right)\phi_t\right)+\left(1-\phi_t\right)\left(1-\delta\right)\right] \\ & +\left(1-\pi\right)\mu_t\left(r_t+q_t\left(1-\delta\right)\right). \end{aligned} \tag{13}$$

Combining (9) and (10), we find

$$\left(c_t^e\right)^{-\rho} = \left(c_t^w\right)^{-\rho}\left(1-l_t\right)^{\eta(1-\rho)}.$$

Combining (10) and (11), we find

$$\eta\frac{c_t^w}{1-l_t} = w_t.$$

Combining (12) and (13), we find

$$v'\left(s_t; K_t, Z_t\right) = \mu_t\left(r_t+\left(1-\delta\right)q_t+\pi\lambda_t^e\left(r_t+q_t\phi_t\left(1-\delta\right)\right)\right). \tag{14}$$

Combining (9), (12), and (14), we find

$$q_t = E_t\left[\beta\left(\frac{c_{t+1}^e}{c_t^e}\right)^{-\rho}\left(r_{t+1}+\left(1-\delta\right)q_{t+1}+\pi\lambda_{t+1}^e\left(r_{t+1}+q_{t+1}\phi_{t+1}\left(1-\delta\right)\right)\right)\right].$$

6.2 Model summary

6.2.1 Summary before Detrending

$$\eta\frac{c_t^w}{1-l_t} = w_t,$$

$$\left(c_t^e\right)^{-\rho} = \left(c_t^w\right)^{-\rho}\left(1-l_t\right)^{\eta(1-\rho)},$$

$$\lambda_t^e = \frac{q_t-1}{1-q_t\theta},$$

$$q_t = E_t\left[\beta\left(\frac{c_{t+1}^e}{c_t^e}\right)^{-\rho}\left(r_{t+1}+\left(1-\delta\right)q_{t+1}+\pi\lambda_{t+1}^e\left(r_{t+1}+q_{t+1}\phi_{t+1}\left(1-\delta\right)\right)\right)\right],$$

$$i_t = \frac{r_t+q_t\left(1-\delta\right)\phi_t}{1-q_t\theta}K_t,$$

$$\alpha\frac{y_t}{K_t} = r_t$$

$$\left(1-\alpha\right)\frac{y_t}{\left(1-\pi\right)l_t} = w_t,$$

$$\pi c_t^e+\left(1-\pi\right)c_t^w+\pi i_t = y_t,$$

$$y_t = A_t (K_t)^{\alpha} [(1-\pi) l_t]^{1-\alpha},$$

$$K_{t+1} = (1-\delta) K_t + \pi i_t.$$

6.2.2 Exogenous Growth Model

If $A_t = A_0 (\gamma^{1-\alpha})^t$, the variables are detrended as follows.

$$\eta \frac{\hat{c}_t^w}{1-l_t} = \hat{w}_t,$$

$$(\hat{c}_t^e)^{-\rho} = (\hat{c}_t^w)^{-\rho} (1-l_t)^{\eta(1-\rho)},$$

$$\lambda_t^e = \frac{q_t - 1}{1 - q_t \theta},$$

$$q_t = E_t \left[\beta \left(\gamma \frac{\hat{c}_{t+1}^e}{\hat{c}_t^e} \right)^{-\rho} \left(r_{t+1} + (1-\delta) q_{t+1} + \pi \lambda_{t+1}^e \left(r_{t+1} + q_{t+1} \phi_{t+1} (1-\delta) \right) \right) \right],$$

$$\hat{\imath}_t = \frac{r_t + q_t (1-\delta) \phi_t}{1 - q_t \theta} \hat{K}_t,$$

$$\alpha \frac{\hat{y}_t}{\hat{K}_t} = r_t$$

$$(1-\alpha) \frac{\hat{y}_t}{(1-\pi) l_t} = \hat{w}_t,$$

$$\pi \hat{c}_t^e + (1-\pi) \hat{c}_t^w + \pi \hat{\imath}_t = \hat{y}_t,$$

$$\hat{y}_t = A_0 \left(\hat{K}_t \right)^{\alpha} [(1-\pi) l_t]^{1-\alpha},$$

$$\gamma \hat{K}_{t+1} = (1-\delta) \hat{K}_t + \pi \hat{\imath}_t$$

where a hat variable denotes the original variable divided by γ^t, for example, $\hat{c}_t^e = c_t^e / \gamma^t$.

The following parameters are exogenously chosen. $\beta = 0.994$, $\pi = 0.06$, $\alpha = 0.36$, $\gamma = 1.004$, and $A_0 = 1$. The ratio of annual investment to capital in the steady state is set at $4\pi \hat{\imath} / \hat{K} = 0.076$. Survival rate of capital $(1-\delta)$ is backed out from

$$1 - \delta = \underbrace{\gamma - \frac{1}{4} \frac{4\pi i}{K}}_{\text{known}}.$$

Aggregate hours of work in the steady state is set at $(1-\pi) l = 0.25$. Hours of work per worker in the steady state is backed out from

$$l = \underbrace{\frac{1}{1-\pi} (1-\pi) l}_{\text{known}}.$$

The ratio of capital to annual output is set at $\hat{K}/(4\hat{y}) = 3.32$. Steady state output is backed out from

$$\hat{y} = \underbrace{\left(\frac{\hat{K}}{\hat{y}}\right)^{\frac{\alpha}{1-\alpha}} (1-\pi)\, l}_{\text{known}}.$$

Other steady state values are backed out as follows.

$$\hat{K} = \underbrace{(4\hat{y})\,\frac{\hat{K}}{4\hat{y}}}_{\text{known}}.$$

$$\hat{\imath} = \underbrace{\frac{\hat{K}}{4\pi}\frac{4\pi\hat{\imath}}{\hat{K}}}_{\text{known}}.$$

$$\hat{w} = \underbrace{(1-\alpha)\,\frac{\hat{y}}{(1-\pi)\,l}}_{\text{known}}.$$

$$r = \underbrace{\alpha\frac{\hat{y}}{\hat{K}}}_{\text{known}}.$$

The following equations are solved for λ^e, q, and ϕ.

$$\lambda^e = \frac{q-1}{1-q\phi},$$

$$q = \beta\gamma^{-\rho}\left(r + (1-\delta)\,q + \pi\lambda^e\left(r + q\phi\,(1-\delta)\right)\right),$$

$$\hat{\imath} = \frac{r + q\,(1-\delta)\,\phi}{1-q\phi}\hat{K}.$$

The following equations are solved for $\hat{c}^e$, $\hat{c}^w$, and η.

$$\eta\frac{\hat{c}^w}{1-l} = \hat{w},$$

$$(\hat{c}^e)^{-\rho} = (\hat{c}^w)^{-\rho}\,(1-l)^{\eta(1-\rho)},$$

$$\pi\hat{c}^e + (1-\pi)\,\hat{c}^w + \pi\hat{\imath} = \hat{y}.$$

6.2.3 Endogenous growth model

If $A_t = A_0\,(K_t)^{1-\alpha}$, variables are detrended as follows.

$$\eta\frac{\hat{c}_t^w}{1-l_t} = \hat{w}_t,$$

$$(\hat{c}_t^e)^{-\rho} = (\hat{c}_t^w)^{-\rho} (1 - l_t)^{\eta(1-\rho)},$$

$$\lambda_t^e = \frac{q_t - 1}{1 - q_t\theta},$$

$$q_t = E_t \left[\beta \left(\gamma_t \frac{\hat{c}_{t+1}^e}{\hat{c}_t^e} \right)^{-\rho} \left(r_{t+1} + (1-\delta) q_{t+1} + \pi \lambda_{t+1}^e \left(r_{t+1} + q_{t+1}\phi_{t+1} (1-\delta) \right) \right) \right],$$

$$\hat{\imath}_t = \frac{r_t + q_t (1-\delta) \phi_t}{1 - q_t\theta},$$

$$\alpha \hat{y}_t = r_t$$

$$(1-\alpha) \frac{\hat{y}_t}{(1-\pi) l_t} = \hat{w}_t,$$

$$\pi \hat{c}_t^e + (1-\pi) \hat{c}_t^w + \pi \hat{\imath}_t = \hat{y}_t,$$

$$\hat{y}_t = A_0 \left[(1-\pi) l_t \right]^{1-\alpha},$$

$$\gamma_t = (1-\delta) + \pi \hat{\imath}_t.$$

where $\gamma_t = K_{t+1}/K_t$ and a hat variable denotes the original variable divided by K_t, for example, $\hat{c}_t^w = c_t^w / K_t$.

The following parameters are exogenously chosen. $\beta = 0.994$, $\pi = 0.06$, $\alpha = 0.36$, and $\gamma = 1.004$. The ratio of annual investment to capital in the steady state is set at $4\pi i_t / K_t|_{s.s.} = 4\pi \hat{\imath} = 0.076$. Hence

$$\hat{\imath} = \underbrace{\frac{0.076}{4\pi}}_{\text{known}}.$$

$(1-\delta)$ is found as

$$1 - \delta = \underbrace{\gamma - \pi \hat{\imath}}_{\text{known}}.$$

Aggregate hours of work in the steady state is set at $(1-\pi) l = 0.25$. Steady state individual hours of work in the steady state is backed out from

$$l = \underbrace{\frac{1}{1-\pi} (1-\pi) l}_{\text{known}}.$$

The ratio of capital to annual output is set at $K_t / (4y_t)|_{s.s.} = 1/(4\hat{y}_t)|_{s.s.} = 3.32$. Hence

$$\hat{y} = \left(\frac{1}{4} \right) \left(\frac{1}{3.32} \right).$$

$$A_0 = \underbrace{\hat{y} \left[(1-\pi) l \right]^{\alpha - 1}}_{\text{known}}.$$

$$\hat{w} = \underbrace{(1-\alpha)\frac{\hat{y}}{(1-\pi)\,l}}_{\text{known}}.$$

$$r = \underbrace{\alpha\hat{y}}_{\text{known}}.$$

The following equations are solved for λ^e, q, and ϕ.

$$\lambda^e = \frac{q-1}{1-q\phi},$$

$$q = \beta\gamma^{-\rho}\left(r + (1-\delta)\,q + \pi\lambda^e\left(r + q\phi\,(1-\delta)\right)\right),$$

$$\hat{\imath} = \frac{r + q\,(1-\delta)\,\phi}{1-q\phi}.$$

The following equations are solved for $\hat{c}^e$, $\hat{c}^w$, and η.

$$\eta\frac{\hat{c}^w}{1-l} = \hat{w},$$

$$(\hat{c}^e)^{-\rho} = (\hat{c}^w)^{-\rho}\,(1-l)^{\eta(1-\rho)},$$

$$\pi\hat{c}^e + (1-\pi)\,\hat{c}^w + \pi\hat{\imath} = \hat{y}.$$

6.3 Asset pricing equation

Pricing equation for equity is given by

$$q_t = E_t\left[\beta\left(\frac{c^e_{t+1}}{c^e_t}\right)^{-\rho}\left(r_{t+1} + (1-\delta)\,q_{t+1} + \pi\lambda^e_{t+1}\left(r_{t+1} + q_{t+1}\phi_{t+1}\,(1-\delta)\right)\right)\right].$$

Multiply K_{t+1} to both sides,

$$Stock_t = E_t\left[\beta\left(\frac{c^e_{t+1}}{c^e_t}\right)^{-\rho}\left(\alpha y_{t+1} + (1-\delta)\,q_{t+1}K_{t+1} + \pi\lambda^e_{t+1}\left(r_{t+1} + q_{t+1}\phi_{t+1}\,(1-\delta)\right)K_{t+1}\right)\right].$$

Because $i_t/K_t = (r_t + q_t\phi_t\,(1-\delta))\,/\,(1-q_t\theta)$ in our model economy,

$$\begin{aligned}
Stock_t &= E_t\left[\beta\left(\frac{c^e_{t+1}}{c^e_t}\right)^{-\rho}\left(\alpha y_{t+1} + (1-\delta)\,q_{t+1}K_{t+1} + \pi\lambda^e_{t+1}\,(1-q_{t+1}\theta)\,i_{t+1}\right)\right] \\
&= E_t\left[\beta\left(\frac{c^e_{t+1}}{c^e_t}\right)^{-\rho}\left(\alpha y_{t+1} + (1-\delta)\,q_{t+1}K_{t+1} + \pi\,(q_{t+1}-1)\,i_{t+1}\right)\right] \\
&= E_t\left[\beta\left(\frac{c^e_{t+1}}{c^e_t}\right)^{-\rho}\left(\alpha y_{t+1} - \pi i_{t+1} + Stock_{t+1}\right)\right].
\end{aligned}$$