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Asset Bubbles, Endogenous Growth, and Financial Frictions*

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Abstract

This paper analyzes the existence and the effects of bubbles in an endogenous growth model with financial frictions and heterogeneous investments. Bubbles are likely to emerge when the degree of pledgeability is in the middle range, implying that improving the financial market might increase the potential for asset bubbles. Moreover, when the degree of pledgeability is relatively low, bubbles boost long-run growth; when it is relatively high, bubbles lower growth. Furthermore, we examine the effects of a bubble burst, and show that the effects depend on the degree of pledgeability, i.e., the quality of the financial system. Finally, we conduct a full welfare analysis of asset bubbles.

Key words: Asset Bubbles, Endogenous Growth, and Financial Frictions

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1 Introduction

Many countries have experienced large movements in asset prices, called asset bubbles, which are associated with significant fluctuations in real economic activity. A notable example is the recent global economic upturn and downturn before and after the financial crisis of 2007. Many economists and policy makers want to understand why bubbles emerge and how they affect real economies.[1] However, it is still not clear how financial market conditions affect the existence condition of bubbles. In this study, we first examine the relationship between the emergence of asset bubbles and financial conditions, in other words, whether bubbles are more likely to occur in financially developed or less-developed economies.

Empirically, there is a complicated relationship between financial market conditions and asset bubbles. Emerging market economies, such as those in South East Asia, often experience bubble-like dynamics. Caballero (2006) and Caballero and Krishnamurthy (2006) found that financial imperfection is a key element in bubbles in emerging market economies. However, not all countries with less developed financial markets experience bubble-like dynamics. For example, the financial systems in some African countries are less developed than in Asia (UNECA, 2006), yet they have not experienced bubble-like macro dynamics. This may suggest that financial quality below a certain threshold cannot sustain asset bubbles. In fact, countries in the South East Asia began to develop their financial markets in the 1980s, and this was one of the reasons for their high growth rates (World Bank, 1993). On the other hand, improving financial conditions might promote asset bubbles. For example, Allen (2001) pointed out that the financial liberalization resulting from financial system development in these countries was a factor in the emergence of bubbles in the 1990s.[2] Additionally, advanced economies like the U.S. experienced information frictions problems in financial markets such as subprime problems, suggesting that advanced economies may also face financial imperfections (see Campello et al, 2010; Brunnermeier and Sannikov, 2014).

From these observations, it seems that financial market conditions and the emergence of bubbles may have a non-linear relationship. In other words, bubbles may

[1]See, for example, Akerlof and Shiller (2009).

[2]The Japanese economy experienced asset bubbles in the 1980s, but the structural reforms of the Japanese financial system and subsequent financial liberalization materialized before the rise in asset prices. See Shigemi (1995) for a more detailed discussion.

not occur in financially underdeveloped or in well-developed economies. They tend to occur in countries with an intermediate level of financial development. The first purpose of this study is to formulate this non-linear relationship theoretically. For this purpose, we use an endogenous growth model with heterogeneous investments and financial market imperfections. In our model, entrepreneurs switch between productive and unproductive states. In the productive state, entrepreneurs' investments yield high returns, while they yield low returns in the unproductive state. In addition, entrepreneurs can pledge only a fraction of the returns from their investments.

The endogenous growth model with heterogeneous investments is crucial to formulating an intuitive understanding of the non-linear relationship. For example, Farhi and Tirole (2012) recently examined the existence of bubbles and found that they can exist when the pledgeability level is low, although their main focus was on the effects of outside liquidity. However, they assumed homogeneous investment opportunities. Hence, if pledgeability is very low, the interest rate becomes very low and the growth rate, which equals zero in the steady-state in their study, becomes relatively high compared to the interest rate. Thus, based on these assumptions, bubbles can exist despite very poor financial market conditions.[3] On the other hand, if there are heterogeneous investments, the market interest rate may not decrease much, even if the financial market is very poor. Because the return from low-yield investments becomes the lower bound for the interest rate. Thus, the growth rate becomes very low compared to the interest rate, and bubbles cannot exist in very poor financial market conditions. This result suggests that improving financial market conditions might increase the emergence of bubbles if the financial market starts from a state of underdevelopment.[4]

Based on the existence condition of bubbles, we can also examine the relationship between technological progress and the conditions leading to asset bubbles. Scheinkman (2014) recently pointed out the importance of this relationship. Since technological progress is a factor for promoting economic growth rates, it seems to increase the existence of bubbles. However, if it also increases the interest rate,

[3]Caballero (2006) and Caballero and Krishnamurthy (2006) both assume a high exogenously given growth rate in emerging countries. Thus, even their model cannot capture the non-linear relationship.

[4]In this sense, our model is related to Matsuyama's (2007, 2008) model showing that a better credit market may be more prone to financing what he calls bad investments that do not have positive spillover effects on future generations.

technological progress may not lead to bubbles. Moreover, bubbles may in turn affect investment financing with technological progress, suggesting that there is a two-way feedback relationship between technological progress and bubbles. We will show that the type of technological progress affects this relationship, and will derive results consistent with Scheinkman's (2014) stylied facts.

Moreover, it is not yet obvious how bubbles affect economic growth. The second purpose of this study is to investigate the macroeconomic effects of bubbles. Here we examine whether bubbles enhance or impair growth, as well as the relationship between these macroeconomic effects and financial conditions. In the process, we analyze how financial conditions determine the effects of bubbles' collapse on the economic growth rate.

We will show that the effect of bubbles on economic growth depends on financial market conditions. Bubbles have both crowd-out and crowd-in effects on investment and growth rates. Since bubbles crowd savings away from investments, bubbles decrease the economic growth rate. On the other hand, bubbles increase the rate of return on savings and improve borrowers' networth, which in turn crowds in their future investments. That is, bubbles endogenously generate "the balance sheet effect" emphasized by Bernanke and Gertler (1989). Our main finding is that the relative impact of these effects depends on the degree of pledgeability. When the pledgeability level is relatively low, the crowd-in effect dominates the crowd-out effect and bubbles enhance the economic growth rate. On the other hand, if the pledgeability level is relatively high, the crowd-out effect dominates, and bubbles decrease the economic growth rate.

This examination also has important implications for the effects of bubbles after they burst, which our results suggest is not uniform. A country's financial condition has a significant effect on the growth path after the collapse of bubbles. If the imperfection of the financial market is relatively high (i.e., if pledgeability is relatively low), the bubble burst decreases the growth rate permanently. This implies that in economies with low pledgeability, bubbles can temporarily mask low economic growth rates due to the poor financial market conditions. On the other hand, economies with high pledgeability will experience a decline in the economic growth rate immediately after the bubble bursts, but will recover and achieve a high growth rate. That is, the burst may enhance the long-run growth rate if the financial markets are in relatively good condition.

Moreover, this result implies that if the temporary negative productivity shock is sufficiently large, the level of total output becomes permanently lower than the pre-bubble trend level, despite recovery in the economic growth. This result is consistent with empirical evidence on the effects on growth of various types of financial crises. For example, Cerra and Saxena (2008) show that most financial crises are associated with a decline in growth that leaves output permanently below its pre-crisis trend.

Finally, we conduct a rigorous full welfare analysis of asset bubbles in an infinitely-lived agent model with heterogeneous investments and financial market imperfections. In our framework, we assume that bubbles will collapse with positive probability and that entrepreneurs are risk-averse. Entrepreneurs care about increased volatility in consumption arising from the collapse of a bubble. We consider the welfare effects of this increased volatility from the bubble's burst. We find analytically that bubbles increase welfare, regardless of whether they increase or decrease the long-run economic growth rate and even if these are expected to collapse. The economic intuition for this result lies in the consumption-smoothing effects of bubbles. In this economy, entrepreneurs face borrowing constraints and cannot consume smoothly against idiosyncratic shocks to the productivity of investment. In this situation, the circulation of bubble assets serves as an insurance device against idiosyncratic productivity shocks, thereby increasing welfare.

The rest of this paper is organized as follows. Subsection 1.1 provides a literature review. In section 2, we present our basic model, both with and without bubbles. In section 3, we present the dynamics of bubbles. In section 4, we examine the existence condition of bubbles, and in section 5, we examine the effects of bubbles on economic growth rates. In section 6, we show how the effects of the bubbles' burst are related to financial market conditions. In section 7, we conduct a full welfare analysis of bubbles, and section 8 concludes the paper.

1.1 Related Work in the Literature

Our study considers the existence of bubbles in an infinitely lived agents model. With regard to the existence of bubbles in infinite horizon economies, it is commonly thought that bubbles cannot arise in deterministic sequential market economies with a finite number of infinitely lived agents (Tirole, 1982). The Tirole model assumes a perfect financial market, that is, agents can borrow and lend freely. Tirole showed that in such an environment, no equilibrium with bubbles exists. Our result is consistent with the Tirole result. That is, when the financial market is perfect in the point that pledgeability is equal to one, bubbles cannot arise even in our setting. We show that bubbles can arise even in an infinitely lived agents model if the financial market is imperfect. Of course, the possibility of bubbles in infinite horizon economies with borrowing constraints has been recongnized in seminal papers on deterministic fiat money (deterministic bubbles) (Bewley, 1980; Townsend, 1980; and Scheinkman and Weiss, 1986). These seminal papers proved the existence of a monetary equilibrium in an endowment economy where no borrowing and lending are allowed.[5] Given these studies, important studies by Kocherlakota (1992) and Santos and Woodford (1997) more explicity examined the (necessary) conditions for the existence of deterministic bubbles. Additionally, the recent important paper by Hellwig and Lorenzoni (2009) proved that the resulting set of equilibrium allocations with self-enforcing private debt is equivalent to the allocations sustained with rational bubbles. All of these studies are, however, based on an endowment economy. Our paper is in line with research examining bubbles in an infinitely lived agents model. Our paper's contribution is that we develop a full-blown macroeconomic model with heterogeneous investments and financial frictions, and provide a full characterization on the relationship between the existence of bubbles and financial frictions in a production economy.

There are many papers examining the relationship between bubbles and investment. However, in the literature, the crowd-out and crowd-in effects are examined separately. The conventional wisdom (Samuelson, 1958; Tirole, 1985) suggests that bubbles crowd investment out and lower output. According to the traditional view, the financial market is perfect and all savings in the economy flow to investment. In this situation, bubbles crowd savings away from investment once they appear in the economy. Saint-Paul (1992), Grossman and Yanagawa (1993), and King and Ferguson (1993) extend the Samuelson-Tirole model to economies with endogenous growth, and show that bubbles reduce investment and lower long run economic growth.[6][7] Recently, however, some studies such as Woodford (1990), Caballero and

[5] As Kocherlakota (1992) points out, although Scheinkman and Weiss (1986) implicitly provide examples of bubbles in an infinitely lived agents model, they do not explicitly give the necessary conditions for the existence of bubbles. Kocherlakota provided the conditions.

[6] This crowd-out effect of bubbles has been criticized because it seems inconsistent with the historical evidence that investment and economic growth rates tend to surge when bubbles arise, and then stagnate when they burst.

[7] Olivier (2000) shows that the conclusions reached by Saint-Paul (1992), Grossman and Yana-

Krishnamurthy (2006), Kiyotaki and Moore (2008), Kocherlakota (2009) developed a model with financial frictions, and showed that bubbles crowd investment in and increase output.[8] These studies demonstrate that financial market imperfections prevent the transfer of enough resources to those with investments from those without investments, resulting in underinvestment. Bubbles help to transfer resources between them.

One novel point of our study is that we have combined these two effects and shown the degree of financial imperfection, i.e., the degree of pledgeability, is crucial for understanding which of these effects is dominant. Martin and Ventura (2012) also investigated whether bubbles are expansionary. There are some significant differences. First, Martin and Ventura (2012) assume that no agent can borrow or lend through financial markets because none of the returns from investment can be pledgeable. That is, they consider a situation where financial markets are completely shut down.[9] On the other hand, in our model, entrepreneurs are allowed to borrow as long as they offer pledgeable assets (collateral) to secure debts. Our main focus is to investigate the relationship between the degree of pledgeability and bubbles. We show that both the emergence and the effects of bubbles are significanlty dependent on the degree of pledgeability, that is, the degree of financial imperfection. Second, Martin and Ventura (2012) use a two-period overlapping generations model assuming that young agents with investment opportunities cannot borrow at all because financial markets are completely shut down, but they can create new bubble assets in every period. This assumption of a new bubble creation in every period directly produces wealth effects for the young and is crucial for crowd-in effects of bubbles. That is, there are no crowd-in effects and only Tirole's (1985) crowd-out effects without this assumption. They investigated the conditions of new bubble creations for the existence of bubbles. On the other hand, our model abstracts from such new bubble creation, and instead assumes that agents live infinitely, and their type changes stochastically in each period. Entrepreneurs buy bubbles for specula-

gawa (1993), and King and Ferguson (1993) crucially depend on the type of asset being speculated on. Bubbles in equity markets can be growth-enhancing while bubbles in unproductive assets are growth-impairing.

[8]Hirano and Yanagawa (2010), Martin and Ventura (2011), Miao and Wang (2011), Wang and Wen (2012), and Aoki and Nikolov (2013) also show the crowd-in effect of bubbles.

[9]In Woodford (1990), no returns from investment can be pledgeable. In Kocherlakota (2009), agents can borrow against bubbles in land prices. However, without such bubbles, there is no borrowing or lending.

tive purposes when they have low productivity, and sell them when they are high productivity. Since bubbles increase the rate of return on savings, this speculative activity endogenously improves borrowers' net worth and generates crowd-in effects. Third, financial frictions are crucial for the existence of bubbles in our model with infinitely-lived agents, while in OLG models, as Tirole (1985) shows, bubbles can arise even in a perfect finanical market if an economy is dynamically inefficient. Additionally, our paper uses an infinitely lived agents model, while Farhi and Tirole (2012) and Martin and Ventura (2012) are based on overlapping generations models. As Farhi and Tirole (2012) point out, the potential benefit of using an infinitely lived agents model would be that it is in principle more suitable for realistic quantitative explorations which the recent macroeconomic literature emphasizes.

Caballero and Krishnamurthy (2006) developed a theory of stochastic bubbles in emerging markets using an overlapping generations model, though with exogenously given growth rates and international interest rates. They implicitly assume a low pledgeability level, and that without bubbles, the domestic interest rate was lower than the international interest rate. Hence, our argument is a generalization of their argument. Kiyotaki and Moore (2008) is also releted to our study. In their theory, since deterministic fiat money facilitates exchange for its high liquidity, people hold money despite its low rate of return, emphasizing the role of money as a medium of exchange. In our model, however, we emphasize the role of bubbles as a store of value. Entrepreneurs buy and sell bubble assets for speculative purposes because they have a high return.

Our paper is also related to the growth literature. As Levine (1997) and Beck et al. (2000) show empirically, it is widely accepted that improving financial market conditions enhances long-run economic growth. However, the effect on growth volatility is not yet clear. In our study, stochastic bubbles tend to occur when financial markets have an intermediate level of financial development. This suggests that growth volatility tends to be high in the middle range of financial development, which can offer an explanation for empirical findings from Easterly *et al.* (2000) and Kunieda (2008) that growth volatility is high when financial development is an intermediated level.

In terms of welfare effects of bubbles, our result that bubbles enhance the consumption-smoothing effect shares similarity with Bewley (1980) who examined deterministic fiat money as a means of self-insurance against idiosyncratic income risk. There are some significant differences. First, our model is based on a production economy with investment opportunities and focuses on idiosyncratic shocks to productivity of investment, while the Bewley's model is based on an endowment economy and focuses on income shocks. Second, we consider an economy where borrowing and lending are allowed (i.e., we consider the whole range of pledgeability of collateral), while Bewley considered an economy where financial markets are completely shut down. Third, in our model, entrepreneurs can employ other means to save besides bubble assets, i.e., through lending or by investing in their own investment projects, while in Bewley's model, fiat money is the only means of saving. Fourth, we examine the welfare effects of stochastic bubbles, while Bewley's model deals with determinisitic fiat money.

2 The Model

Consider a discrete-time economy with one homogeneous good and a continuum of entrepreneurs. A typical entrepreneur has the following expected discounted utility:

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t \log c_t^i \right], \tag{1}$$

where i is the index for each entrepreneur, and c_t^i is the entrepreneur's consumption at date t. $\beta \in (0, 1)$ is the subjective discount factor and E_0 is the expectation operator conditional on date 0 information.

At each date, each entrepreneur meets high-productivity investment projects (hereinafter H-projects) with probability p, and low-productivity ones (L-projects) with probability $1 - p$.[10] The investment technologies are as follows:

$$y_{t+1}^i = \alpha_t^i z_t^i, \tag{2}$$

where $z_t^i (\geq 0)$ is the investment level at date t, and y_{t+1}^i is the output at date $t + 1$. α_t^i is the marginal productivity of the investment at date t. $\alpha_t^i = \alpha^H$ if the entrepreneur has H-projects, and $\alpha_t^i = \alpha^L$ if he/she has L-projects. We assume

[10] Gertler and Kiyotaki (2010), Kiyotaki and Moore (2008), and Kocherlakota (2009) use a similar setting. In Woodford (1990), the entrepreneurs have investment opportunities in alternating periods.

$\alpha^H > \alpha^L$.[11] The probability p is exogenous, and independent across entrepreneurs and over time. At the beginning of each date t, entrepreneurs know whether they have H-projects or L-projects. We call entrepreneurs with H-projects (L-projects) "H-types" ("L-types").

In this economy, we assume that because of frictions in a financial market, the entrepreneur can pledge at most a fraction θ of the future return from investment to creditors (See Hart and Moore (1994) and Tirole (2006) for the foundations of this setting.). Thus, in order for debt contracts to be credible, debt repayment cannot exceed the pledgeable value. That is, the borrowing constraint becomes:

$$r_t b_t^i \leq \theta \alpha_t^i z_t^i, \tag{3}$$

where r_t and b_t^i are the gross interest rate, and the amount of borrowing at date t, respectively. The parameter $\theta \in [0,1]$, which is assumed to be exogenous, can be naturally taken to be the degree of imperfection of the financial market.

In this paper, we consider an economy with asset bubbles, called a "bubble economy". We define bubble assets as those producing no real return, that is, the asset's fundamental value is zero. Aggregate supply of bubble assets is assumed to be constant over time X. Here, following Weil (1987), we consider stochastic bubbles, in the sense that they may collapse. In each period, bubble prices become zero (i.e., bubbles burst) at a probability of $1-\pi$ conditional on survival in the previous period. A lower π means riskier bubbles because they have a higher probability of collapsing. In line with the literature, once bubbles collapse, they do not arise again (their reappearance is not expected ex-ante.). This implies that bubbles persist with a probability $\pi(<1)$ and that their prices are positive until they revert to zero. Let P_t^x be the per unit price of bubble assets at date t. $P_t^x = P_t > 0$ if bubbles survive at date t with probability π, and $P_t^x = 0$ if they collapse at date t with probability $1-\pi$. As we will show, P_t is endogenously determined in equilibrium. Let x_t^i be the level of bubble assets purchased by type i entrepreneur at date t. Each entrepreneur has the following three constraints: flow of funds constraint, the borrowing constraint,

[11] We can also consider a model where capital goods are produced through the investment technology. For example, let $k_{t+1}^i = \alpha_t^i z_t^i$ be the investment technology, where k is capital goods. Capital fully depreciates in one period. Consumption goods are produced by the following aggregate production function: $Y_t = K_t^\sigma N_t^{1-\sigma} \bar{k}_t^{1-\sigma}$, where K and N are the aggregate capital and labor input, and $\bar{k}$ is the economy's per-labor capital, capturing the externality to generate endogenous growth. In this type of model, we can obtain the same results as this study.

(3), and the short-sale constraint:

$$c_t^{*i} + z_t^{*i} + P_t^x x_t^i = y_t^{*i} - r_{t-1}^* b_{t-1}^{*i} + b_t^{*i} + P_t^x x_{t-1}^i, \tag{4}$$

$$x_t^i \geq 0, \tag{5}$$

where * represents the bubble economy. Both sides of (4) include bubbles. $P_t^x x_{t-1}^i$ on the right hand side is the sales of bubble assets, and $P_t^x x_t^i$ on the left hand side is the new purchase of them. We define the net worth of the entrepreneur in the bubble economy as $e_t^{*i} \equiv y_t^{*i} - r_{t-1}^* b_{t-1}^{*i} + P_t^x x_{t-1}^i$. We assume that (3) is the borrowing constraint, that is, bubbles do not contribute to pledgeable value or collateral. Even so, bubbles can lead to increased investments by improving the borrowers' net worth, as we will explain in detail in section 2.4.[12]

We should add a few remarks about the short-sale constraint (5). As Kocherlakota (1992) showed, the short-sale constraint is important for the existence of bubbles in deterministic economies with a finite number of infinitely lived agents. Without the constraint, bubbles always represent an arbitrage opportunity for an infinitely lived agent, who can gain by permanently reducing holdings of the asset. However, it is well known that in such economies, equilibria can only exist if agents are constrained not to engage in Ponzi schemes. Kocherlakota (1992) demonstrated that the short-sale constraint is one of no-Ponzi-game conditions and hence, it can support bubbles by eliminating the agent's ability to permanently reduce his holdings of the asset (see Kocherlakota (1992) for details.). In our model, without the short sale constraint, entrepreneurs can obtain funds infinitely by short-selling bubble assets. As a result, the interest rate rises sufficiently in the credit market and bubbles grow faster than the growth rate of the economy. Therefore, bubbles cannot be sustained. In other words, without the short-sale constraint, bubbles cannot

[12]We can relax the assumption concerning the borrowing constraint. For example, we can consider a case where the entrepreneur can use both a fraction θ of the return from investment and a fraction θ^x of the expected return from bubble assets as collateral. In this case, the borrowing constraint can be written as:

$$r_t^* b_t^{*i} \leq \theta \alpha_t^i z_t^i + \theta^x \pi P_{t+1} x_t^i.$$

It is shown that if θ^x is sufficiently small, H-types do not purchase bubble assets in equilibrium. For determinisitc bubbles, i.e., $\pi = 1$, unless $\theta^x = 1$, H-types do not purchase bubble assets in equilibrium. Kocherlakota (2009) analyzes this special case with $\theta^x = 1$ under contingent debt contracts where even H-types buy bubble assets. In our model, we focus on the case where θ^x is sufficiently small so that H-types do not purchase bubble assets in equilibrium. We explore this point in greater detail in the Technical Appendix.

arise in equilibrium.

2.1 Optimal Behavior of Entrepreneurs

We now characterize the equilibrium behavior of entrepreneurs in the bubble economy. We consider the equilibrium where $\alpha^L \leq r_t^* < \alpha^H$. In equilibrium, the interest rate must be at least as high as α^L, since no agent lends to projects if $r_t^* < \alpha^L$.

For H-types at date t, the borrowing constraint (3) binds since $r_t^* < \alpha^H$ and the investment in bubbles is not attractive, that is, (5) also binds. We will verify this result in the Technical Appendix. Since the utility function is log-linear, each entrepreneur consumes a fraction $1-\beta$ of the net worth in every period, that is, $c_t^{*i} = (1-\beta)(y_t^{*i} - r_{t-1}^* b_{t-1}^{*i} + P_t^x x_{t-1}^i)$.[13] Then, by using (3), (4), and (5), the investment function of H-types at date t can be written as:

$$z_t^{*i} = \frac{\beta(y_t^{*i} - r_{t-1}^* b_{t-1}^{*i} + P_t^x x_{t-1}^i)}{1 - \dfrac{\theta\alpha^H}{r_t^*}}. \tag{6}$$

This is a popular investment function in financial constraint problems.[14] We see that the investment equals the leverage, $1/\left[1-(\theta\alpha^H/r_t)\right]$, times savings, $\beta(y_t^{*i} - r_{t-1}^* b_{t-1}^{*i} + P_t^x x_{t-1}^i)$. Leverage increases with θ and is greater than one in equilibrium, implying that when θ is larger, H-types can finance more investment, z_t^{*i}. We also learn that the presence of bubble assets increases entrepreneurs' net worth. In our model, entrepreneurs buy bubble assets for speculative purposes when they have L-projects, and sell those assets when they have opportunities to invest in H-projects.

For L-types at date t, since $c_t^{*i} = (1-\beta)e_t^{*i}$, the budget constraint (4) becomes

$$z_t^{*i} + P_t^x x_t^i - b_t^{*i} = \beta e_t^{*i}.$$

Each L-type allocates savings, βe_t^{*i}, to three assets, i.e., z_t^{*i}, x_t^i, and $(-b_t^{*i})$. Each L-type chooses optimal amounts for b_t^{*i}, x_t^i, and z_t^{*i} such that the expected marginal utility from investing in these three assets is equalized. By solving the utility maximization problem explained in the Technical Appendix, we can derive the L-type's

[13] See, for example, chapter 1.7 of Sargent (1988).

[14] See, for example, Bernanke and Gertler (1989), Bernanke et al. (1999), Holmstrom and Tirole (1998), Kiyotaki and Moore (1997), and Matsuyama (2007, 2008).

demand function for bubble assets:

$$P_t x_t^i = \frac{\pi \frac{P_{t+1}}{P_t} - r_t^*}{\frac{P_{t+1}}{P_t} - r_t^*} \beta e_t^{*i}, \tag{7}$$

From (7), we learn that an entrepreneur's portfolio decision depends on the survival probability of bubbles, π. When π is high, the bursting probability is low, and the demand for bubble assets increases.

The remaining fraction of savings is split across z_t^{*i} and $(-b_t^{*i})$:

$$z_t^{*i} + (-b_t^{*i}) = \frac{(1-\pi)\frac{P_{t+1}}{P_t}}{\frac{P_{t+1}}{P_t} - r_t^*} \beta e_t^{*i}.$$

Since investing in L-projects (z_t^{*i}) and secured lending to other entrepreneurs ($-b_t^{*i}$) are both safe assets, $z_t^{*i} \geq 0$ if $r_t^* = \alpha^L$, and $z_t^{*i} = 0$ if $r_t^* > \alpha^L$. That is, the following conditions must be satisfied: $(r_t^* - \alpha^L) z_t^{*i} = 0$, $z_t^{*i} \geq 0$, and $r_t^* - \alpha^L \geq 0$. Moreover, when $r_t^* = \alpha^L$, investing in L-projects and secured lending to other entrepreneurs are indifferent for L-types.

2.2 Equilibrium

We denote the aggregate consumption of H-and L-types at date t as C_t^{*H} and C_t^{*L}, respectively. Similarly, let Z_t^{*H}, Z_t^{*L}, B_t^{*H}, and B_t^{*L} be the aggregate investment and the aggregate borrowing of each type, respectively, and X_t be the aggregate investment in bubbles. Then, the market clearing conditions for goods, credit, and bubbles are:

$$C_t^{*H} + C_t^{*L} + Z_t^{*H} + Z_t^{*L} = Y_t^*, \tag{8}$$

$$B_t^{*H} + B_t^{*L} = 0, \tag{9}$$

$$X_t = X, \tag{10}$$

where Y_t^* is the aggregate output at date t.

The competitive equilibrium is defined as a set of prices $\{r_t^*, P_t^x\}_{t=0}^{\infty}$ and quantities $\left\{c_t^{*i}, b_t^{*i}, z_t^{*i}, y_{t+1}^{*i}, C_t^{*H}, C_t^{*L}, B_t^{*H}, B_t^{*L}, Z_t^{*H}, Z_t^{*L}, X_t, Y_{t+1}^*\right\}_{t=0}^{\infty}$, such that (i) the market clearing conditions, (8), (9), and (10) are satisfied, and (ii) each entrepreneur chooses consumption, borrowing, bubble assets, and investments to maximize the expected discounted utility (1) under the constraints (2), (3), (4), and (5).

2.3 Bubbleless Economy:

To examine the effects of bubbles, we first examine an economy without bubbles as a benchmark case. Our model without bubbles is based on Kiyotaki (1998). Let the economy without * represent the bubbleless economy, in which $P_t^x = 0$ for any t. The entrepreneur's net worth in the bubbleless economy is defined as $e_t^i \equiv y_t^i - r_{t-1}b_{t-1}^i$. Obviously, if θ is sufficiently high, all total savings are used only for H-projects and $r_t = \alpha^H$. Hence, we focus on the case where the interest rate is strictly lower than α^H and the borrowing constraint binds for H-types, $\alpha^L \leq r_t < \alpha^H$.

Since there are no bubbles, the investment function for H-types at date t can be written as:

$$z_t^i = \frac{\beta(y_t^i - r_{t-1}b_{t-1}^i)}{1 - \dfrac{\theta\alpha^H}{r_t}}. \tag{11}$$

By aggregating (11), we have:

$$Z_t^H = \frac{\beta E_t^H}{1 - \dfrac{\theta\alpha^H}{r_t}} = \frac{\beta p Y_t}{1 - \dfrac{\theta\alpha^H}{r_t}}, \tag{12}$$

where E_t^H is the aggregate net worth of H-types at date t. Since every entrepreneur has the same opportunity to invest in H-projects with probability p in each period, the aggregate net worth of H-types at date t is a fraction p of the aggregate output at date t, i.e., $E_t^H = pY_t$.

For L-types, if $r_t = \alpha^L$, lending and borrowing to invest are indifferent. Thus, how much they invest in their own projects is indeterminate at an individual level. However, their aggregate investment level is determined by the goods market clearing condition, (8):

$$Z_t^H + Z_t^L = \beta Y_t. \tag{13}$$

βY_t is the total savings. If $r_t > \alpha^L$, Z_t^L must be zero. Thus, the following conditions must be satisfied: $Z_t^L(r_t - \alpha^L) = 0$, $Z_t^L \geq 0$, $r_t - \alpha^L \geq 0$.

The aggregate output is

$$Y_{t+1} = \alpha^H Z_t^H + \alpha^L Z_t^L.$$

In the bubbleless economy, Y_t equals the aggregate wealth of entrepreneurs, A_t, i.e., $Y_t = A_t$. The growth rate of $Y_t = A_t$ becomes:

$$g_t \equiv \frac{Y_{t+1}}{Y_t} = \beta\alpha^H - \beta(\alpha^H - \alpha^L)l_t, \tag{14}$$

where $l_t \equiv Z_t^L/\beta Y_t$ is the ratio of low-productivity investments to total investments. As long as the amount of L-projects, l_t, is zero, total savings are allocated only to H-projects, and the economic growth rate becomes $\beta\alpha^H$, which is the same as the growth rate under $\theta = 1$. If $l_t > 0$, however, the difference in productivity between H-projects and L-projects, $\alpha^H - \alpha^L$, decreases the growth rate and g_t becomes $\beta\alpha^H - \beta(\alpha^H - \alpha^L)l_t$.

Next, we examine the equilibrium level of l_t and r_t. The key point is the size of Z_t^H relative to total savings βY_t. Since Z_t^H is an increasing function of θ, $Z_t^H > \beta Y_t$ at $r_t = \alpha^L$ if θ is sufficiently high. That is, if the possible borrowing level of H-projects is sufficiently high, r_t becomes greater than α^L in equilibrium due to the tightness of the credit market. Thus, L-types have no incentives to invest in their L-projects, and l_t becomes zero in equilibrium. g_t becomes $\beta\alpha^H$ and r_t should satisfy $Z_t^H = \beta Y_t$. On the other hand, if θ is low and $Z_t^H < \beta Y_t$ at $r_t = \alpha^L$, then r_t equals α^L and l_t becomes $1 - p/(1 - \frac{\theta\alpha^H}{\alpha^L}) > 0$ in equilibrium. In summary, we can derive the following Proposition.

Proposition 1 *When bubbles do not exist, the equilibrium interest rate, r_t, and the equilibrium growth rate, g_t, are the following increasing functions of θ:*

$$r_t = r(\theta) = \begin{cases} \alpha^L, & \text{if } 0 \leq \theta < (1-p)\dfrac{\alpha^L}{\alpha^H}, \\[2ex] \dfrac{\theta\alpha^H}{1-p}, & \text{if } (1-p)\dfrac{\alpha^L}{\alpha^H} \leq \theta < 1-p, \\[2ex] \alpha^H, & \text{if } 1-p \leq \theta \leq 1. \end{cases}$$

$$g_t = g(\theta) = \beta\alpha^H - \beta(\alpha^H - \alpha^L)L(\theta), \tag{15}$$

where $L(\theta) = Max[1 - \frac{p}{1-\frac{\theta\alpha^H}{\alpha^L}}, 0]$.

In the bubbleless economy, once the initial output, Y_0, is given, then the economy achieves the balanced growth path immediately, i.e., there are no transitionary dynamics. Figure 1 depicts Proposition 1. We take θ on the horizontal axis, and g and r on the vertical axis. As we will show later, the necessary condition for the existence of stochastic bubbles is $g > r$ under the bubbleless economy. Hence, the relationship between g and r is important for our results. Figure 1 shows that both the relation between g and θ and the relation between r and θ are non-linear. Hence, it is shown that under some parameter conditions, only in the middle range of θ is g greater than r. The intuitive reason for this result is as follows.

The growth rate generated by L-projects, $\beta\alpha^L$, is lower than the rate of return of L-projects α^L. When θ is sufficiently low, H-types cannot gather sufficient funds and most are invested in L-projects. Consequently, the growth rate becomes sufficiently low and close to (but higher than) $\beta\alpha^L$, and the growth rate is lower than the interest rate, α^L, i.e., $g(\theta = 0) < r(\theta = 0)$. In the middle range of θ, the interest rate is still α^L since H-projects are not enough to absorb all total savings, but the growth rate can be higher than α^L since most of the savings are invested in H-projects, leading to high economic growth, i.e., $g(\theta) > r(\theta)$ for the middle range of θ. If θ becomes sufficiently high, however, all total savings are invested in H-projects and the growth rate becomes $\beta\alpha^H$, but the interest rate becomes high and equal to α^H if θ is close to 1, i.e., $g(\theta) < r(\theta)$ for sufficiently high θ. Hence, only in the middle range of θ, $g(\theta) > r(\theta)$.

From this intuitive explanation, we can understand that heterogeneous investment opportunities are crucial for this result. In the middle range of θ, most resources are allocated to H-projects, which leads to high economic growth but the interest rate remains low at α^L. In the Appendix, we provide more discussion about the theoretical characteristics wherein g tends to be greater than r only when θ is in the middle range in the bubbleless economy.

Moreover, we can see that to derive this result, two types of technologies are not crucial and we can extend this argument to a more general environment with a continuum of productivity. Let us suppose, for example, that there are continuously distributed investment opportunities with different productivities from $\underline{\alpha}$ to $\overline{\alpha}$ and the population share of entrepreneurs who have lower technology is sufficiently high. In this case, when θ is almost 0, most resources are allocated to the lowest technology and the growth rate becomes close to $\beta\underline{\alpha}$, which is lower than $\underline{\alpha}$, i.e., $g(\theta) < r(\theta)$ around $\theta = 0$. When θ goes up to the middle range, the interest rate becomes higher than $\underline{\alpha}$ because there are a continuum of technologies. However, the interest rate is determined by the rate of return of the marginal type of technology. On the other hand, resources can be allocated to the technologies with higher productivity than the marginal technology. Hence, the growth rate can be higher than the rate of return of the marginal type, that is, $g(\theta) > r(\theta)$ in the middle range of θ. On the other hand, if θ becomes close to 1, almost all resources are allocated to the project with $\overline{\alpha}$, and the interest rate becomes close to $\overline{\alpha}$, which is higher than the economy's growth rate $\beta\overline{\alpha}$ under $\theta = 1$, i.e., $g(\theta) < r(\theta)$ for sufficiently high θ. In both the Appendix and the Technical Appendix, we explain this continuum case more rigorously, and show that under some conditions, g is greater than r only in the middle range of θ even in the continuum case.

2.4 Economy with Bubbles

We are now in a position to derive the dynamics of the bubble economy. Since we assume that rational bubbles are stochastic, that is, bubbles persist with probability $\pi < 1$, we focus on the dynamics until bubbles collapse, i.e., $P_t^x = P_t > 0$.

(8) can be rewritten as

$$Z_t^{*H} + Z_t^{*L} + P_t X = \beta A_t^*, \tag{16}$$

or

$$Z_t^{*H} + Z_t^{*L} = \beta Y_t^* - (1-\beta) P_t X,$$

where $A_t^* \equiv Y_t^* + P_t X$ is the entrepreneur's aggregate wealth in the bubble economy at date t. Compared to (13), we can see that the resources allocated to the real investments $Z_t^{*H} + Z_t^{*L}$ becomes from βY_t^* to $\beta Y_t^* - (1-\beta) P_t X$ by the existence of bubbles, $P_t X > 0$. This reduction in resources is the crowd-out effect of bubbles, which is similar to the effect in the traditional literature such as Tirole (1985). Since a part of the total savings is invested in the bubble assets, the resources allocated to real investments should be crowded out.

On the other hand, bubbles have another effect because the investment level of H-projects is determined as:

$$Z_t^{*H} = \frac{\beta p A_t^*}{1 - \dfrac{\theta \alpha^H}{r_t^*}} = \frac{\beta p Y_t^*}{1 - \dfrac{\theta \alpha^H}{r_t^*}} + \frac{\beta p P_t X}{1 - \dfrac{\theta \alpha^H}{r_t^*}}, \tag{17}$$

where pA_t^* is the aggregate wealth of H-types at date t. (More details about the aggregation of each variable will be explained in the Technical Appendix). This second term is the crowd-in effect of bubbles on investment. Intuitively, since the possible borrowing level is an increasing function of A_t^*, H-types can gather more funds by the existence of bubbles and thus increase their investments. Moreover, the investments expand more than the direct increase in the net worth because of the leverage effect. Bubbles endogenously generate "balance sheet effects". In other words, bubbles work to reallocate the resource toward productive investments.

It is worth noting why A_t^* can be higher than Y_t^* and the positive "balance sheet effects" work though we exclude the possibility of bubble creations in every period. As we will describe in more detail below, the equilibrium rate of the return on bubble assets becomes higher than the rate of return on low-productivity investments, α^L. Hence, the existence of bubbles can improve net worth, A_t^*, by increasing the rate of return on savings at $t-1$. This is why the crowd-in effect works even without bubble creations in every period, and the "balance sheet effects" are generated endogenously.[15]

In summary, although the bubbles crowd out the resource allocated to real investments, they reallocate resources toward high-productivity investments through the crowd-in effect. Moreover, when θ is low, a high share of resources are allocated to low-productivity investments if there are no bubbles. Hence, the crowd-in effect may dominate the crowd-out effect when θ is low. We will rigorously prove this intuition in the later section.

Next, we examine the equilibrium interest rate. When

$$P_t X > Max\left[\beta A_t^* - \frac{\beta p A_t^*}{1-\dfrac{\theta\alpha^H}{\alpha^L}}, 0\right] \Leftrightarrow \phi_t > L(\theta),$$

only H-types invest and the equilibrium interest rate $r_t^* (> \alpha^L)$ is determined to satisfy

$$\phi_t = 1 - \frac{p}{1-\frac{\theta\alpha^H}{r_t^*}} \Leftrightarrow r_t^* = \frac{\theta\alpha^H(1-\phi_t)}{1-p-\phi_t},$$

where $\phi_t \equiv P_t X/\beta A_t^*$ is the size of the bubbles (the share of the value of the bubble

[15] We provide an analysis about the effects of bubble creations within our framework in the Technical Appendix.

assets). It follows then that r_t^* is an increasing function of ϕ_t because of the tightness of the credit market. On the other hand, if $\phi_t \leq L(\theta)$, the interest rate becomes $r_t^* = \alpha^L$ and both L-types and H-types invest in equilibrium. Thus, in the bubble economy, the equilibrium interest rate is:

$$r_t^* = Max\left[\alpha^L, \frac{\theta\alpha^H(1-\phi_t)}{1-p-\phi_t}\right]. \tag{18}$$

This means that as long as the size of the bubbles is small, the interest rate stays low at α^L, but when the bubbles become large enough, then the interest rate starts to rise.

In this paper, we will examine the relationship between the economic growth rate and asset bubbles with three types of examination: (i) the relationship between the economic growth rate and ϕ_t in the bubble economy, (ii) the relationship between the economic growth rate and θ in the bubble economy, and (iii) a comparison between the economic growth rate in the bubble economy and that in the bubbleless economy for each θ. We will first examine (i).

Together with (17), we have the evolution of aggregate output:

$$Y_{t+1}^* = \begin{cases} \alpha^H \dfrac{\beta p A_t^*}{1-\frac{\theta\alpha^H}{\alpha^L}} + \alpha^L\left(\beta Y_t - (1-\beta)P_tX - \dfrac{\beta p A_t^*}{1-\frac{\theta\alpha^H}{\alpha^L}}\right) & \text{if } \phi_t \leq L(\theta), \\ \alpha^H \frac{\beta p A_t^*}{1-\frac{\theta\alpha^H}{r_t^*}} = \alpha^H(\beta Y_t - (1-\beta)P_tX) & \text{if } \phi_t \geq L(\theta). \end{cases} \tag{19}$$

When the bubbles are small, both L-types and H-types invest in equilibrium. The first and second terms in the first line represent aggregate output at date $t+1$ produced by H-and L-types, respectively. When the bubbles are large, then only H-types invest.

By rearranging (19), we can derive the economic growth rate:

$$\frac{Y_{t+1}^*}{Y_t^*} = \begin{cases} \beta\alpha^H - \beta(\alpha^H-\alpha^L)L(\theta) + \left((\alpha^H-\alpha^L)\beta(1-L(\theta)) - (1-\beta)\alpha^L\right)\dfrac{P_tX}{Y_t^*} & \text{if } \phi_t \leq L(\theta), \\ \beta\alpha^H - (1-\beta)\alpha^H\frac{P_tX}{Y_t^*} & \text{if } \phi_t \geq L(\theta), \end{cases} \tag{20}$$

where $\frac{P_tX}{Y_t^*} = \frac{\beta\phi_t}{1-\beta\phi_t}$ and $\frac{\beta\phi_t}{1-\beta\phi_t}$ is an increasing function of ϕ_t. The dynamic system of this economy is mainly characterized by (20), although we have not yet derived the equilibrium ϕ_t. By the existence of bubbles P_tX, the amount of $\frac{p}{1-\frac{\theta\alpha^H}{\alpha^L}}\beta P_tX = \beta(1-L(\theta))P_tX$ shifts from L-projects to H-projects by the crowd-in effect and the net contribution to Y^*_{t+1} of this effect is $(\alpha^H-\alpha^L)\beta(1-L(\theta))P_tX$. Conversely, P_tX prevents $(1-\beta)P_tX$ resources from allocation to real investments by the crowd-out effect of bubbles, and the negative impact on Y^*_{t+1} is $(1-\beta)\alpha^L P_tX$. Hence, $(\alpha^H - \alpha^L)\beta(1-L(\theta))-(1-\beta)\alpha^L\frac{P_tX}{Y^*_t}$ shows the crowd-in and crowd-out effects of bubbles, and we will derive in a later section that $\left((\alpha^H-\alpha^L)\beta(1-L(\theta))-(1-\beta)\alpha^L\right)$ is positive as long as bubbles satisfy the existence condition. In other words, the crowd-in effect dominates the crowd-out effect, and the growth rate $\frac{Y^*_{t+1}}{Y^*_t}$ in the bubble economy is an increasing function of the size of the bubbles ϕ_t as long as $\phi_t \leq L(\theta)$. On the other hand, if $\phi_t \geq L(\theta)$, only H-types are producing, and the growth rate $\frac{Y^*_{t+1}}{Y^*_t}$ in the bubble economy is a decreasing function of the size of the bubbles ϕ_t. In other words, the relationship between the economic growth rate and bubble size is inverted U-shaped and $L(\theta) = Max[1-\frac{p}{1-\frac{\theta\alpha^H}{\alpha^L}}, 0]$ is the size of the bubbles that maximizes the economic growth rate.

3 Dynamics of Rational Bubbles

Next, we examine the dynamics of rational bubbles and derive the equilibrium ϕ_t.

From the definition of $\phi_t \equiv P_tX/\beta A^*_t$, ϕ_t evolves over time as

$$\phi_{t+1} = \frac{\frac{P_{t+1}}{P_t}}{\frac{A^*_{t+1}}{A^*_t}}\phi_t. \tag{21}$$

The evolution of the size of the bubbles depends on the relationship between the growth rate of aggregate wealth and that of the bubbles.

When we aggregate (7), and solve for P_{t+1}/P_t, we obtain the required rate of return on bubble assets:

$$\frac{P_{t+1}}{P_t} = \frac{r^*_t(1-p-\phi_t)}{\pi(1-p)-\phi_t} > r^*_t \geq \alpha^L,\ if\ \pi < 1. \tag{22}$$

$(1-p-\phi_t)/[\pi(1-p)-\phi_t]$ captures the risk premium on bubble assets, which is greater than one as long as $\pi < 1$; the required rate of return is strictly greater than the interest rate. From this and the relationship $r_t^* \geq r_t$, we learn that bubbles increase the rate of return on savings compared to the bubbleless economy as long as bubbles persist. This high rate of return on bubble assets increases entrepreneurs' net worth.[16]

Using (19) and $A_{t+1}^* = Y_{t+1}^* + P_{t+1}X = Y_{t+1}^* + (P_{t+1}/P_t)\,\beta\phi_t A_t^*$, the growth rate of the aggregate wealth in the bubble economy can be written as:

$$\frac{A_{t+1}^*}{A_t^*} = \begin{cases} \beta\{\alpha^H(1-L(\theta)) + \alpha^L(L(\theta)-\phi_t) + \dfrac{P_{t+1}}{P_t}\phi_t\} & \text{if } \phi_t \leq L(\theta), \\ \\ \beta\{\alpha^H(1-\phi_t) + \dfrac{P_{t+1}}{P_t}\phi_t\} & \text{if } \phi_t \geq L(\theta). \end{cases} \tag{23}$$

From (18), (22), and the definition of entrepreneurs' aggregate wealth, (21) can be rewritten as:

$$\phi_{t+1} = \begin{cases} \dfrac{\dfrac{(1-p-\phi_t)}{\pi(1-p)-\phi_t}}{\left(1+\dfrac{\alpha^H-\alpha^L}{\alpha^L-\theta\alpha^H}p\right)\beta + \dfrac{(1-\pi)(1-p)}{\pi(1-p)-\phi_t}\beta\phi_t}\phi_t & \text{if } \phi_t \leq L(\theta), \\ \\ \dfrac{\theta}{\beta}\dfrac{1}{\pi(1-p)-(1-\theta)\phi_t}\phi_t & \text{if } \phi_t \geq L(\theta). \end{cases} \tag{24}$$

Using (24), we examine the sustainable dynamics of ϕ_t. For stochastic bubbles to be sustainable, the following condition must be satisfied for any t: $0 < \phi_t < 1$. If this condition is violated, the bubbles explode, i.e., bubbles do not exist.

As examined in the previous studies (Tirole 1985; Farhi and Tirole 2012), there is a continuum of starting values for the share of bubbles in total savings that are consistent with equilibrium. The dynamics of bubbles take three patterns. The first is that bubbles become too large and explode to $\phi_t \geq 1$. The economy cannot sustain this dynamic path, and thus, bubbles cannot exist in this pattern. The second pattern is that ϕ_t becomes smaller over time and converges to zero as long as bubbles persist. This path is referred to as asymptotically bubbleless, where as

[16] For deterministic bubbles, i.e., $\pi = 1$, we have $P_{t+1}/P_t = r_t^* \geq \alpha^L$ in equilibrium. Even in this case, bubbles affect the long-run economic growth rate on the balanced growth path if and only if $P_{t+1}/P_t = r_t^* > \alpha^L$. See our working paper version (the first submitted version to REStud, Hirano and Yanagawa. 2010 July, CARF Working Paper) for details.

long as bubbles persist, their effects decrease, eventually becoming small. The third pattern is that ϕ_t converges to a positive value for as long as bubbles survive.

From (24), we can derive that ϕ_t must be constant over time, unless ϕ_t is asymptotically bubbleless. Following Weil (1987), we refer to this equilibrium with constant ϕ^* as the "stochastic steady-state", where entrepreneurs' wealth, the bubbles, and the output grow at the same constant rate as long as the bubbles persist, $A^*_{t+1}/A^*_t = P_{t+1}/P_t = Y^*_{t+1}/Y^*_t$.

4 Existence Condition of Stochastic Bubbles

In this section, we examine the existence condition of stochastic bubbles. In other words, we investigate whether a dynamic path with bubbles does not explode. Mathematically, we will check whether the dynamic system (24) has a non-negative steady-state, $\phi_t = \phi^*$. As we show below, the financial market condition, θ, is crucial to the existence of bubbles. (Hereafter, proofs of all Propositions are given in Appendix).

Proposition 2 *Stochastic bubbles with survival probability π can exist if and only if θ satisfies the following condition,*

$$\underline{\theta} \equiv Max\left[\frac{\alpha^L[1-\pi\beta(1-p)] - p\beta\pi\alpha^H}{\alpha^H(1-\pi\beta)}, 0\right] < \theta < \overline{\theta} \equiv \pi\beta(1-p).$$

Moreover, we can use the structure of the bubbleless economy to characterize the existence condition. The existence condition for bubbles (both stochastic and deterministic bubbles) is that the growth rate is not lower than the interest rate under the bubbleless economy. This condition is consistent with the existence condition stated in Tirole's (1985) study, although the Tirole model is based on an exogenous growth model with overlapping generations, while ours is based on an endogenous growth model with infinitely-lived agents.

Proposition 3 *The necessary condition for the existence of a bubble is that the equilibrium growth rate is not lower than the equilibrium interest rate under the bubbleless economy.*

From Proposition 2, we can see that bubbles tend to exist when the degree of financial imperfection, θ, is in the middle range. In other words, improving conditions in the financial markets might increase the existence of bubbles when the initial condition of θ is low.[17] This result is in sharp contrast to the results from previous studies, such as Farhi and Tirole (2012), in which bubbles are more likely to emerge when the financial market is more imperfect (i.e., when pledgeability is more limited).[18]

The intuition for this result is as follows. If θ is low, H-types cannot borrow sufficiently, and the economic growth rate must be low, even with bubbles. On the other hand, the interest rate cannot be lower than α^L, since there is an opportunity to invest in L-projects, even if θ is low. Hence, under a very low level of θ, bubbles grow at a higher rate than the economy, and hence cannot exist. Since we assume heterogeneous investment opportunities, the interest rate has the lower bound, and we thus obtain a different result from that of Farhi and Tirole (2012).

Figure 1 is a typical case representing the relationship between θ and bubble regions.[19] The Figure shows that with sufficiently high or low financial friction, bubbles cannot exist, suggesting that in financially underdeveloped or well-developed economies, bubbles are not likely to arise. They are likely to emerge in countries in an intermediate stage of financial development.[20] As we explained in the Introduction, based on the experiences in advanced economies like the U.S., θ in the real world may be away from $\theta = 1$, i.e., perfect pledgeability.

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[17] Researchers such as Kaminsky and Reinhart (1999) and Allen and Gale (1999) point out that financial liberalization causes bubbles. Based on our model, an interpretation of this effect is as follows. For instance, before financial liberalization, the economy is in non-bubble regions. After liberalization, θ increases, and the borrowing constraint is relaxed, causing the economy to enter bubble regions.

[18] Note that if $\frac{\alpha^L[1-\pi\beta(1-p)]-p\beta\pi\alpha^H}{\alpha^H(1-\pi\beta)} < 0$, then stochastic bubbles can arise even at $\theta = 0$.

[19] Though the growth rate is strictly greater than the interest rate, bubbles cannot arise in the economy unless agents expect to be able to pass bubbles on to other agents. This expectation is a sufficient condition for bubbles to exist. Here, we assume that the condition is satisfied when bubbles appear.

[20] Readers may wonder why phenomena that look like bubbles occur repeatedly in the real world where the financial system is continually developing over time, though our model suggests that bubbles do not appear in high θ regions. We propose one interpretation from our model. In our study, we assume a common θ for both high and low investments. However, we can use different values of θ for those projects. In such a case, the important factor for the existence of bubbles is θ^H, which is placed only on high-profit investments. Taking this into account, consider the situation in which existing projects with α^L disappear. Then, new investment opportunities appear in the economy that are more profitable than the existing α^H. In such a situation, the θ for these new projects is important for the existence of a bubble. If the θ is low, the economy will again enter bubble regions, even if it was previously in non-bubble regions with a high θ. In the real world, this process might repeat itself.

As we also explained in the introduction, this theoretical result is consistent with empirical observations, and implies that growth volatility tends to be high when θ is in the middle range because stochastic bubbles tend to occur. Hence, this result can be an explanation about the recent empirical results by Easterly *et al.* (2000) or Kunieda (2008), who show that growth volatility is high when financial development is at an intermediate level.[21]

4.1 Discussion:

4.1.1 Technological Progress and Asset Bubbles

Based on the existence condition of bubbles and the impacts of asset bubbles on the long-run economic growth rate, we can clarify the relationship between technological progress and asset bubbles. Scheinkman (2014) highlighted the importance of this relationship.[22]. Given $\theta < \pi\beta(1-p)$, when we solve the existence condition $\underline{\theta}$ for α^H/α^L, we get:

$$\frac{\alpha^H}{\alpha^L} > \frac{[1-\pi(1-p)\beta]}{(1-\pi\beta)\theta + p\beta\pi}.$$

That is, bubbles are more likely to arise as inequality in productivity, α^H/α^L, rises. This result suggests that we need to clarify the types of technological progress to understand the relationship between the existence condition and technological progress.

Proposition 4 *Effects of Technological Innovation on Asset Bubbles: (i) Suppose a technological innovation that increases α^H, termed "high-tech specific progress". This "high-tech specific progress" increases the existence of bubbles. (ii) Suppose technological innovation that increases both α^H and α^L by Δ simultaneously, termed "general progress". This technological progress is neutral in terms of the existence of bubbles.*

From these results, we can draw at least two points. First, it is possible that technological progress, such as high-tech specific progress, increases the existence of

[21] Aghion et al. (1999) and Matsuyama (2007, 2008) show that macroeconomic volatility is high when financial market development is at an intermediate level, though we have a rather different source of high volatility. In our study, volatility occurs because bubbles appear. In these other studies, volatility comes from the interest rate or the quality of investments.

[22] We thank an anonymous referee who pointed out this implication. We also thank Jose Scheinkman for thoughtful comments on this point.

bubbles. Second, general technological progress does not contribute to the existence of bubbles. High-tech specific innovation increases the economy's growth rate, and thus the economy enters a bubble region. Moreover, the appearance of bubbles makes it easier to finance investments with technological progress (i.e., crowd-in high-tech investments) and enhance long-run economic growth (we will show the effects of bubbles on long-run economic growth on the balanced growth path in Proposition 6.). This implies that there is a two-way feedback relationship between high-tech innovation and asset bubbles, consistent with the stylized fact explored by Scheinkman (2014), who showed that asset bubbles tend to appear in periods when a new technology arrives, and bubbles make it easier to finance innovative investments. Scheinkman also pointed out that bubbles may play a positive role in economic growth. The arrival of new technologies roughly corresponds to high-tech specific progress in our setting.

4.1.2 Maximum Bursting Probability of Bubbles and Financial Conditions

We can also see the relationship between the maximum bursting probability and financial conditions from the existence condition.[23] We define $1-\underline{\pi}(\theta)$ as the maximum bursting probability consistent with a given level of θ. As shown in Proposition 3, the necessary condition for bubbles to exist is $g > r$ under the bubbleless economy. Since $g-r$ is hump-shaped with θ, this means that $1-\underline{\pi}(\theta)$ is hump-shaped with θ. The following Proposition summarizes this result.

Proposition 5 *Let $1-\underline{\pi}(\theta)$ be the maximum bursting probability consistent with a given level of θ. Then, $1-\underline{\pi}(\theta)$ is hump-shaped with θ. That is, $1-\underline{\pi}(\theta)$ is an increasing function of θ in $\underline{\theta}(\pi=1) < \theta < \alpha^L(1-p)/\alpha^H$, and is a decreasing function of θ in $\alpha^L(1-p)/\alpha^H \leq \theta < \bar{\theta}(\pi=1)$.*

Proposition 5 suggests that in the early stage of financial development, as the economy develops financially, riskier bubbles with high bursting probability can arise. Once the level of financial development passes a certain level, then riskier bubbles are less likely to arise, and once the level of finanicial development reaches a sufficiently high state, bubbles cannot arise from Proposition 2. The intuion is that if the bursting probability of bubbles is high, to compensate entrepreneurs for

[23]We thank an anonymous referee who pointed out this implication.

their risk, bubbles would need to grow so fast that they would become too large to be feasible in equilibrium. Financial development increases the economy's growth rate and therefore makes it possible for riskier bubbles to arise in equilibrium. However, after a certain level of financial development, the interest rate rises with financial development, which in turn makes it difficult for riskier bubbles to occur in equilibrium.

5 Asset Bubbles and Economic Growth

In this section, we examine the relationship between the growth rate and θ in the stochastic steady-state (i.e., the balanced growth path) under the bubble eonomy. We will show here that the effect of bubbles on the economic growth rate depends on the financial market condition, θ, even if the existence condition for bubbles is satisfied.

First, we formally derive $\phi^*(\theta)$, which is the relative bubble size under the stochastic steady-state, $\phi_t = \phi^*$ for any t, as long as bubbles persist. From (24), this ϕ^* becomes a function of θ, and can be written as:

$$\phi^*(\theta) = \begin{cases} \dfrac{\pi - \dfrac{1-\pi\beta(1-p)}{\left[1+(\frac{\alpha^H-\alpha^L}{\alpha^L-\theta\alpha^H})p\right]\beta - \beta(1-p)}}{1 - \dfrac{1-\pi\beta(1-p)}{\left[1+(\frac{\alpha^H-\alpha^L}{\alpha^L-\theta\alpha^H})p\right]\beta - \beta(1-p)}}(1-p) & \text{if } \underline{\theta} < \theta \leq \theta^m, \\ \\ \dfrac{\pi\beta(1-p)-\theta}{\beta(1-\theta)} & \text{if } \theta^m \leq \theta < \overline{\theta}. \end{cases} \tag{25}$$

where θ^m (provided in the Appendix) is the degree of the financial market condition that achieves the bubble size, $\phi^*(\theta^m) = L(\theta^m)$. Hence, we have $\phi^*(\theta) < L(\theta^m)$ for any θ in $\underline{\theta} < \theta < \theta^m$, and $\phi^*(\theta) > L(\theta^m)$ for any θ in $\theta^m < \theta < \overline{\theta}$. In this section, we do not consider the asymptotically bubbleless path, since it does not affect the long-run economic growth rate. We examine the dynamics on the asymptotically bubbleless path in the Appendix. From $t = 0$, the bubble size follows (25). Moreover, from (25), we can easily see that $\phi^*(\theta)$ is an increasing function of θ in $\underline{\theta} < \theta \leq \theta^m$, and a decreasing function of θ in $\theta^m \leq \theta < \overline{\theta}$. That is, $\phi^*(\theta)$ is at the maximum value at θ^m.

Combining (20) and (25), we can rewrite the growth rate of Y_t^* as:

$$g_t^* = \frac{Y_{t+1}^*}{Y_t^*} = \begin{cases} \begin{array}{c} \beta\alpha^H - \beta(\alpha^H - \alpha^L)L(\theta) \\ + \left(\beta(\alpha^H - \alpha^L)(1 - L(\theta)) - (1-\beta)\alpha^L\right) \frac{\beta\phi^*(\theta)}{1-\beta\phi^*(\theta)} \end{array} & \text{if } \underline{\theta} < \theta \leq \theta^m, \\ \\ \alpha^H(\beta - (1-\beta)\frac{\beta\phi^*(\theta)}{1-\beta\phi^*(\theta)}) & \text{if } \theta^m \leq \theta < \overline{\theta}. \end{cases} \tag{26}$$

The condition of $\beta(\alpha^H - \alpha^L)(1 - L(\theta)) - (1-\beta)\alpha^L > 0$ is equivalent to $\theta > \frac{\alpha^L[1-\pi\beta(1-p)]-p\beta\pi\alpha^H}{\alpha^H(1-\pi\beta)}$, which is the existence condition of stochastic bubbles in Proposition 2. Hence, in the region of $\underline{\theta} < \theta \leq \theta^m$, g_t^* is an increasing function of θ because $\frac{\beta\phi^*(\theta)}{1-\beta\phi^*(\theta)}$ is an increasing function of θ in this region. Conversely, g_t^* is a decreasing function of $\phi^*(\theta)$ in the region of $\theta^m \leq \theta < \overline{\theta}$. However, $\phi^*(\theta)$ is a decreasing function of θ in this region, meaning that g_t^* is an increasing function of θ, even in the region of $\theta^m \leq \theta < \overline{\theta}$.

Moreover, the interest rate, $r^*(\theta)$, and bubbles' growth rate, $\frac{P_{t+1}}{P_t}(\theta)$, under the balanced growth path become, respectively:

$$r^*(\theta) = \begin{cases} \alpha^L & \text{if } \underline{\theta} < \theta \leq \theta^m, \\ \\ \dfrac{\theta\alpha^H[1-\phi^*(\theta)]}{1-p-\phi^*(\theta)} & \text{if } \theta^m \leq \theta < \overline{\theta}. \end{cases}$$

and

$$\frac{P_{t+1}}{P_t}(\theta) = \begin{cases} \dfrac{\alpha^L(1-p-\phi^*(\theta))}{\pi(1-p)-\phi^*(\theta)} > \alpha^L & \text{if } \underline{\theta} < \theta \leq \theta^m, \\ \\ \dfrac{\theta\alpha^H[1-\phi^*(\theta)]}{\pi(1-p)-\phi^*(\theta)} > r^*(\theta) & \text{if } \theta^m \leq \theta < \overline{\theta}. \end{cases}$$

We should mention that on the balanced growth path, $\frac{P_{t+1}}{P_t}(\theta) > r^*(\theta) \geq r(\theta) \geq \alpha^L$. That is, bubbles increase the rate of return on savings compared to that under the bubbleless economy.

Next, we compare the growth rate in the bubble economy to that in the bubbleless economy for each θ. We know from (15) that $\beta\alpha^H - \beta(\alpha^H - \alpha^L)L(\theta)$ corresponds to the economic growth rate under the bubbleless economy. Hence, we can easily see that in the region of $\underline{\theta} < \theta \leq \theta^m$, $g_t^* - g_t > 0$ for any θ; that is, bubbles enhance the long-run economic growth rate. Though the existence of bubbles generates the crowd-out effect on investments, the effect that reallocates resources from L-projects toward H-projects is dominant. Hence, bubbles increase the economic growth rate.

On the other hand, in the region of $\theta^m \leq \theta < \overline{\theta}$:

$$g_t^* - g_t = \beta(\alpha^H - \alpha^L)L(\theta) - \alpha^H(1-\beta)\frac{\beta\phi^*(\theta)}{1-\beta\phi^*(\theta)}.$$

The first term captures the crowd-in effect. If there are no bubbles, $L(\theta)$ is invested in L-projects. However, bubbles reallocate resources to H-projects that generate a high rate of return, α^H. Conversely, we have the crowd-out effect: bubbles reduce the resources allocated to real investments. The second term represents this crowd-out effect. Since $L(\theta)$ is large when θ is low, the crowd-in effect dominates the crowd-out effect. Therefore, $g_t^* > g_t$ only when θ is low. In summary, we can derive the following proposition[24]

Proposition 6 *There is a threshold level of θ, $\theta^*(> \theta^m)$. If $\underline{\theta} < \theta \leq \theta^*$, the economic growth rate in the bubble economy is higher than that in the bubbleless economy in each period, as long as the bubbles persist. If $\theta^* < \theta < \overline{\theta}$, then the economic growth rate in the bubble economy is lower than that in the bubbleless economy in each period. If bubbles are deterministic, that is $\pi = 1$, then $\theta^* = \beta(1-p)\alpha^L/\alpha^H$.*

Proposition 6 implies that bubbles enhance economic growth in economies within bubble regions and with relatively low values of θ, but hinder economic growth in economies with relatively high values of θ. Figure 2 illustrates this relationship, and the Appendix describes θ^* in more detail.[25]

We can reinterpret Proposition 6 by using of the dynamics of the wealth. Given the relationship of $A_{t+1}^*/A_t^* = Y_{t+1}^*/Y_t^*$ on the balanced growth path, from (23), we

[24]Our definition of the "crowd in" and "crowd out" effects of bubbles is closely related to the "liquidity" and "leverage" effects described in Farhi and Tirole (2012). Although these two models are different, the crowd-in effect and the liquidity effect capture the situation in which savers can use bubbles to increase the rate of return they earn on their savings. In addition, the crowd-out effect and the leverage effect capture the situation in which higher interest rates reduce the leverage of investors and constrains their investment choices. We appreciate an anonymous referee who pointed out this relation.

[25]The recent macroeconomic literature emphasizes the role of aggregate total factor productivity (TFP) to account for long-run economic growth rate and business fluctuations. In our model, the appearance of a bubble and its collapse endogenously affects TFP. We define TFP in the bubble economy as:

have:

$$g_t^* - g_t = \frac{A_{t+1}^*}{A_t^*} - \frac{A_{t+1}}{A_t} = \begin{cases} \left[\frac{P_{t+1}}{P_t}(\theta) - \alpha^L\right]\beta\phi^*(\theta) > 0 & \text{if } \underline{\theta} < \theta \le \theta^m, \\ \left[\frac{P_{t+1}}{P_t}(\theta) - \alpha^L\right]\beta L(\theta) + \left[\frac{P_{t+1}}{P_t}(\theta) - \alpha^H\right]\beta\left[\phi^*(\theta) - L(\theta)\right] & \text{if } \theta^m \le \theta < \overline{\theta}. \end{cases} \tag{27}$$

Consider the region of $\underline{\theta} < \theta \le \theta^m$. When bubbles appear in the economy, L-types purchase bubble assets by reducing their own L-projects. As long as the bubbles survive at $t+1$, the realized rate of return from bubbles, P_{t+1}/P_t is higher than α^L. Hence, bubbles increase the growth rate of wealth, unless the bubbles burst. Next, we consider the region of $\theta^m \le \theta < \overline{\theta}$. Since $\frac{P_{t+1}}{P_t}(\theta) > \alpha^L$, the first term is positive, and shows that a high rate of return from bubbles increases entrepreneurs' wealth. When there are no bubbles, $L(\theta)$ is allocated to L-projects and earns $\alpha^L L(\theta)$. On the other hand, if there are bubbles, this amount is allocated to the bubbles and they earn $\frac{P_{t+1}}{P_t}(\theta)L(\theta) > \alpha^L L(\theta)$ unless they burst at $t+1$. This is the effect of the first term. However, not only $L(\theta)$, but $\phi^*(\theta) > L(\theta)$ too is allocated to the bubbles, which is captured by the second term. $\phi^*(\theta) - L(\theta)$ earns only $\frac{P_{t+1}}{P_t}(\theta)(\phi^*(\theta) - L(\theta))$ when bubbles exist, but can earn $\alpha^H(\phi^*(\theta) - L(\theta))$ when there are no bubbles. Since $\frac{P_{t+1}}{P_t}(\theta) < \alpha^H$, this second term is negative. Whether the bubbles increase the growth rate is deteremined by the relative size of the first and second terms. Since $L(\theta)$ is a decreasing function of θ, the first term is larger than the second term when θ is low. Thus, in economies with a relatively low θ,

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$$\begin{aligned} TFP_t^* &\equiv Y_{t+1}^*/(Z_t^{*H} + Z_t^{*L}) \\ &= \begin{cases} \frac{\left(1+\frac{\alpha^H-\alpha^L}{\alpha^L-\theta\alpha^H}p\right)\alpha^L - \alpha^L\phi^*(\theta)}{1-\beta\phi^*(\theta)} & in\ \underline{\theta} < \theta < \theta^m, \\ \alpha^H & in\ \theta^m \le \theta < \overline{\theta}. \end{cases} \end{aligned}$$

In the bubbleless economy,

$$\begin{aligned} TFP_t &\equiv Y_{t+1}/(Z_t^H + Z_t^L) \\ &= \begin{cases} \left(1 + \frac{\alpha^H-\alpha^L}{\alpha^L-\theta\alpha^H}p\right)\alpha^L & in\ 0 \le \theta < \frac{\alpha^L}{\alpha^H}(1-p), \\ \alpha^H & in\ \frac{\alpha^L}{\alpha^H}(1-p) \le \theta \le 1. \end{cases} \end{aligned}$$

It is shown that in $\underline{\theta} < \theta < \theta^*$ where bubbles increase the long-run economic growth rate, $TFP_t^* > TFP_t$. Thus, aggregate TFP and the economic growth rates move pro-cyclically, implying that the collapse of bubbles results in production inefficiency and lowers TFP.

bubbles increase the economy's growth rate.

Moreover, Proposition 6 has an implication for regulations related to leverage.[26] If θ is sufficiently high and greater than $\pi\beta(1-p)$, bubbles cannot occur because the economy cannot support growing bubbles in these regions. However, regulations on leverage that force tighter borrowing constraints than θ lowers the interest rate and decreases the bubble's growth rate. Consequently, the economy may enter bubble regions. Moreover, once bubbles occur, they impair long-run economic growth, suggesting that tight leverage regulation might lead to bubbles that retard long-run economic growth. We summarize this point in the following Proposition.

Proposition 7 *Consider an economy with $\theta > \pi\beta(1-p)$, in which bubbles cannot arise. In this economy, consider a leverage regulation κ on the borrowing constraint, $r_t b_t^i \leq \kappa\alpha_t^i z_t^i$, where $\theta^* < \kappa < \theta$. This tight leverage regulation increases the existence of bubbles that lower the long-run economic growth rate.*

6 The Effects of the Collapse of Bubbles

In this section, we examine the effects of the bubble burst. In the main text, we present one experiment and show that the growth path after the bubbles' burst crucially depends on the financial market conditions θ. In the Appendix, we examine the effects of the bubble burst in other experiments.

In our model, an unexpected shock to productivity may cause bubbles to burst as well as bubbles collapse stochastically. Suppose at date $t = s - 1$, the economy is in the stochastic steady-state. There is then an unexpected shock at $t = s$ that decreases productivity from α^H to $\alpha^S < \alpha^H$. We assume that this shock is temporary. Thus, after $t = s + 1$, α^H is expected to recover to its original level. Even if the productivity shock is temporary, bubbles may burst if agents revise their expectations as a result of the shock and expect the value of the bubble asset to be zero. We examine how the collapse of bubbles affects the economic growth rate in this situation. Since the bubble bursts at $t = s$, the growth rate follows (15) from $t = s+1$. This implies that the difference between the growth rates of the bubble and the bubbleless economies can characterize the long-run impact of the bubble burst. Countries whose θ is $\underline{\theta} < \theta \leq \theta^*$ experience high economic growth rates during

[26] We thank Hitoshi Matsushima for posing this question.

the bubble period, but a decreased economic growth rate after the bubble's burst. There are two reasons for this decreased growth rate: the decrease in productivity and the collapse of the bubble itself. Immediately after the bubble's collapse, the economy's growth rate decreases significantly. Moreover, in low θ countries, even after productivity recovers to its original level, the growth rate becomes permanently low and stagnates.[27] This implies that in low θ countries, bubbles can temporarily mask low economic growth rates due to low θ. On the other hand, countries whose θ is $\theta^* < \theta < \pi\beta(1-p)$ also suffer a decreased growth rate due to the decrease in productivity immediately after the burst, but experience recovery and a high economic growth rate after productivity returns to its original level. The burst enhances the long-run growth rate in high θ countries. This result suggests that the effects of the bubble burst are not uniform, but significantly depend on the country's financial conditions. Figure 3 illustrates this point.[28] In other words, the collapse of bubbles exposes the "true" economic condition of each country. In summary, the growth path after the bubbles' collapse significantly depends on θ : The low (high) θ countries experience relatively lower (higher) growth rates; thus the variance in countries' growth rates increases after the bubble bursts, even though the average growth rate is lower than it was before the collapse.

Moreover, even if α^H recovers to its original level after $t = s+1$, if a temporary negative productivity shock is sufficiently large, the level of Y_t after $t = s+1$ becomes permanently lower than the pre-bubble trend level, even when the economic growth rate recovers to g_t after $t = s+1$. This result is consistent with other empirical studies on the effects of various types of financial crises on growth. For example, Cerra and Saxena (2008) show that most financial crises are associated with a decline in growth that leaves output permanently below its pre-crisis trend.[29]

[27]In standard real business cycle models, a temporary productivity shock has only temporal effects on output. However, in our model, even a small temporary shock on the productivity of entrepreneurs' investments has permanent effects on aggregate productivity and the long-run growth rate.

[28]In Figure 3, although the economic growth rate decreases temporarily for one period after the bubble bursts, the economic growth rate immediately adjusts to its new steady-state without bubbles because, in our model, since an idiosyncratic shock to each entrepreneur's productivity is i.i.d, i.e., productivity is not persistent over time for individual agents.

[29]We thank an anonymous referee who pointed out this implication. The same logic applies to a decline in θ. Tighter leverage regulations may cause bubbles to collapse. For example, during what we call "bubble periods" in Japan from the late 1980s to the beginning of the 1990s, the Ministry of Finance in Japan tightened leverage. This policy may have caused bubbles to collapse and thus the long-run stagnation after the crash.

7 Welfare Analysis

In this section, we conduct a full welfare analysis of asset bubbles, and compare the ex-ante welfare of entrepreneurs in the bubbleless economy to the ex-ante welfare in the bubble economy, depending on the financial market condition θ.

7.1 Welfare in the Bubbleless Economy

Let $V_t^{BL}(e_t)$ be the value function of an entrepreneur in the bubbleless economy with a net worth, e_t, at the beginning of period t. Note that because we have already derived decision rules, we compute the entrepreneur's expected indirect utility function evaluated at the time before the entrepreneur knows her type of period t. When we compute the value function, we need to consider the probability that the entrepreneur becomes an H- or L-type.

Solving the value function yields:

$$V_t^{BL}(e_t) = W(\theta) + \frac{1}{1-\beta}\log(e_t), \tag{28}$$

$W(\theta)$ and the full derivations are given in the Technical Appendix. By setting $t = 0$, we can see how improved financial system quality affects an entrepreneur's welfare in the initial period of the bubbleless economy. We obtain the following Proposition.

Proposition 8 $V_0^{BL}(e_0)$ *is an increasing function of* θ *in* $0 \leq \theta < 1-p$*, and is independent of* θ *in* $1-p \leq \theta \leq 1$*. In other words, as long as the borrowing constraint binds, the entrepreneur's welfare in the bubbleless economy monotonically increases as the quality of the financial markets improves.*

Intuitively, in the region of $0 \leq \theta < \alpha^L(1-p)/\alpha^H$, together with an increase in θ, the aggregate output increases. At an individual level, the rate of return on saving in a high-productivity state, that is, the leveraged rate of return on H-projects, $\frac{\alpha^H(1-\theta)}{1-\theta\alpha^H/r}$, rises. Conversely, the rate of return on saving in a low-productivity state, i.e., the interest rate, remains unchanged at α^L. This implies that the increase in aggregate output is consumed by those entrepreneurs who invested with maximum leverage. Therefore, the leveraged rate of return for H-types increases, which improves welfare.

In the region of $\alpha^L(1-p)/\alpha^H \leq \theta < 1-p$, even if θ increases, the aggregate variables (i.e., output level, consumption level, and investment level) remain unchagend because the economic growth rate remains the same, but welfare increases.

The economic intuitions are that in this region, the interest rate rises together with the rise in θ, which generates two competing effects. One is that the leveraged rate of return on H-projects decreases; the other is that the rate of return on savings for L-types increases. This means that the difference in the returns between a high-productivity state and a low-productivity state decreases, which enhances consumption smoothing. In other words, improved financial market conditions enhances consumption smoothing for entrepreneurs. Hence, welfare improves with a rise in θ, even though aggregate output remains unchanged. In summary, in the bubbleless economy, welfare is Pareto-ranked; that is, welfare is at its lowest when $\theta = 0$, but improves as θ increases, and reaches its highest level when $1 - p \leq \theta \leq 1$.

7.2 Welfare in the Bubble Economy

Next, we compute the ex-ante welfare in the bubble economy. Here, we focus on an entrepreneur's welfare in the stochastic steady-state, i.e., the balanced growth path. Since bubbles are expected to collapse with positive probability and entrepreneurs are risk-averse agents, entrepreneurs care about increased volatility in consumption due to the collapse of a bubble. We consider the welfare effects of this increased volatility from the bubbles' collapse. Let $V_t^{BB}(e_t^*)$ be the value function of the entrepreneur in the bubble economy with a net worth of e_t^* at the beginning of period t. We compute the entrepreneur's expected indirect utility function evaluated at the time before the entrepreneur knows her type in period t. We thus need to consider the probability that the bubbles will burst as well as the probability that the entrepreneur becomes an H- or L-type.

Solving the value function yields:

$$V_t^{BB}(e_t^*) = q(\phi^*(\theta), \theta) + \frac{1}{1-\beta} \log(e_t^*), \tag{29}$$

where $q(\phi^*(\theta), \theta)$ and the full derivation are given in the Technical Appendix. By setting $t = 0$, we can see how an improvement in the quality of the financial system affects an entrepreneur's welfare in the initial period of the bubble economy. If the entrepreneur holds bubble assets in period 0, the period 0 net worth, e_0^*, increases as the bubbles emerge, which increases consumption in the initial period. Moreover, this increase in initial net worth has a persistent increase on the future net worth after period 1. Therefore, this increases future consumption throughout the entrepreneur's lifetime, and thus welfare. The second term of (29) captures the accumulated effects, which we call the initial wealth effects of bubbles.

7.3 Comparison of Welfare

To compare welfare, we make the following assumptions: i) without a loss of generality, the population measure of entrepreneurs is set to one; ii) in the initial period, each entrepreneur is equally endowed with output, i.e., $y_0^i = y_0 = Y_0$; iii) the aggregate supply of bubble assets is one, i.e., $X = 1$; and iv) each entrepreneur is equally endowed with one unit of bubble assets in the initial period. Under these assumptions, the entrepreneur's net worth in the initial period can be written as $e_0 = Y_0$ in the bubbleless economy, and $e_0^* = Y_0 + P_0 = Y_0/[1-\beta\phi(\theta)]$ in the bubble economy. Using this relationship, the value function in the bubbleless economy, (28), can be rewritten as:

$$V_0^{BL}(Y_0) = W(\theta) + \frac{1}{1-\beta}\log(Y_0). \tag{30}$$

The value function in the bubble economy, (29), can be rewritten as:

$$V_0^{BB}(Y_0) = q(\phi^*(\theta),\theta) + \frac{1}{1-\beta}\log\left[\frac{1}{1-\beta\phi(\theta)}\right] + \frac{1}{1-\beta}\log(Y_0). \tag{31}$$

The second term in (31) captures the bubbles' initial wealth effects. By comparing (30) to (31), we can find whether bubbles improve or harm welfare. Obviously, $e_0^* > e_0$ because of the bubbles' initial wealth effects. Moreover, we can show $q(\phi^*(\theta),\theta) > W(\theta)$ for any θ in the bubble regions (See the proof of Proposition 9 in the Appendix.). These lead to the following Proposition.

Proposition 9 $V_0^{BB} > V_0^{BL}$ *for any* θ *in the bubble regions, i.e., in* $\underline{\theta} < \theta < \overline{\theta}$. *That is, stochastic bubbles increase an entrepreneur's welfare, regardless of whether bubbles increase or decrease the long-run economic growth rate and even if bubbles are expected to collapse with positive probability.*

Proposition 9 states that stochastic bubbles improve an entrepreneur's welfare, even if they are expected to collapse and even if they reduce long-run economic growth. This result is in sharp contrast with Grossman and Yanagawa's (1993) finding that bubbles reduce welfare by retarding the long-run economic growth rate.[30] Figure 4 shows numerical examples of Proposition 9. The parameter values are set as follows: $\beta = 0.99$, $\alpha^H = 1.1$, $\alpha^L = 1.0$, $p = 0.1$, $Y_0 = 1$. The only difference between the four cases lies in the probability of the bubble's collapse. Lower π indicates a greater risk that the bubbles will burst. We can see that in all four cases, welfare in the bubble economy is greater than welfare in the bubbleless economy.

Intuitively the key lies in the consumption-smoothing effects of bubbles. In this economy, becasue of the borrowing constraint, entrepreneurs cannot consume smoothly against idiosyncratic shocks to the productivity of investment. In this situation, the circulation of bubble assets serves as an insurance device against those idiosyncratic productivity shocks, thereby increasing welfare. This mechanism has similarity with that of Bewley (1980), as described in section 1.1.

More specifically, in the region of $\underline{\theta} < \theta \leq \theta^m$, bubbles increase aggregate output. At an individual level, bubbles increase the rate of return on savings in a low-productivity state, because they are high return savings vehicles. In contrast, the rate of return on savings in a high-productivity state, i.e., the leveraged rate of return on H-projects, $\frac{\alpha^H(1-\theta)}{1-\theta\alpha^H/r^*}$, remains unchanged, because the interest rate remains the same.[31] This means that bubbles decrease the difference in rates of return between the high- and low-productivity states, which enhances consumption-smoothing for entrepreneurs, and thus welfare. Moreover, entrepreneurs holding one unit of bubble assets in the initial period can increase net worth by selling the asset and can consume more. This also increases welfare. On the other hand, there is a negative effect of bubbles on welfare arising from their collapse. When bubbles collapse, all wealth invested in bubble assets is lost, which lowers the entrepreneur's net worth and welfare. Proposition 9 shows that the bubble's consumption-smoothing effect dominates the negative effect. Therefore, bubbles increase entrepreneurs' welfare.

In the region of $\theta^m < \theta < \pi\beta(1-p)$, in addition to the bubbles' initial wealth effects and the adverse effect of the bubble's collapse, the interest rate rises when the bubbles appear. This increases the rate of return on savings for L-types, while

[30] Grossman and Yanagawa (1993) analyze the welfare effects of deterministic bubbles in an overlapping generations framework with endogenous growth.

[31] Entrepreneurs who invested in the bubbles can consume the increase in aggregate output from bubbles and use it for H-and L-projects.

that for H-types decreases because the leveraged rate of return is lower. That is, the difference in rates of return between the high- and low-productivity states decreases compared to that in the bubbleless economies. This enhances consumption smoothing for entrepreneurs, improving welfare.[32] In summary, bubbles enhance the consumption-smoothing effect for entrepreneurs. In this regard, the circulation of bubble assets functions as a substitute for financial development. This is why bubbles improve welfare, even if they are expected to collapse and even if they reduce economic growth.[33][34]

8 Conclusion

In this paper, we developed an infinitely-lived agents model with heterogeneous investments and financial frictions. We examined the effects of bubbles and explored how the existence condition of stochastic bubbles is related to the condition of the financial market. We find that the middle range of pledgeability allows for the existence of stochastic bubbles, which suggests that improving financial market conditions might increase the possibility of bubbles if the financial market is initially underdeveloped.

From the existence condition of bubbles, we explored the relationship between

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[32]Together with the rise in the interest rate, both interest income and the return on bubble assets increase when the entrepreneur is in a low–productivity state, which increases consumption and welfare. Conversely, the entrepreneur in a high-productivity state loses, because the leveraged rate of return per unit of saving decreases, which lowers consumption and welfare. This means that in the region of $\theta^m < \theta \leq \theta^*$ where bubbles increase economic growth, the increase in aggregate output can be consumed by entrepreneurs who lent to H-types and who invested in bubbles, and can also be used for H-projects. In other words, bubbles provide benefits in a low-productivity state.

[33]In $\underline{\theta} < \theta \leq \theta^*$, the aggregate consumption level in the bubble economy is always higher than in the bubbleless economy. In $\theta^* < \theta < \overline{\theta}$, though bubbles decrease the economic growth rate, the aggregate consumption level is increased in the bubble economy for some time because of the wealth effects of consumption. If the bubbles persist for long periods, then the aggregate consumption level in the bubbleless economy will be higher at some point in time in the future.

[34]Regarding the welfare effects of asset bubbles, as in footnote 11, suppose that we introduce workers into our model with the same time preference rate as entrepreneurs, but without investment projects. In this setting, workers do not save and consume all wage income in each period in equilibrium. If bubbles reduce long-run economic growth, workers' welfare shows a greater decrease in a bubble economy than in a bubbleless economy because bubbles lower the wage rate. This suggests that as long as bubbles increase the economic growth rate, both entrepreneurs (bubble-holders) and workers (non-bubble holders) gain, but when bubbles reduce the economic growth rate, non-bubble holders lose, while bubble holders still gain. Hence, the welfare impact of bubbles differs for entrepreneurs and workers.

technological progress and asset bubbles. We showed that there is a two-way feedback relationship between high-tech innovation and asset bubbles. Technological progress such as high-tech specific innovation increases the existence of bubbles, and the existence of bubbles in turn makes it easier to finance high-tech investments. We also examined the relationship between the maximum bursting probability and financial conditions from the existence condition. We find that the relationship is hump-shaped. That is, in the early stage of financial development, as the economy develops financially, riskier bubbles with high bursting probability can arise. Once the level of financial development passes a certain level, then riskier bubbles are less likely to arise.

Moreover, we examined the effects of bubbles on the long-run economic growth rate. We find that the effects are also related to the financial market's condition. If pledgeability is relatively low, bubbles increase the growth rate, but bubbles decrease the economic growth rate when pledgeability is relatively high. This result has an important implication for the effects of a collapsed bubble. A collapsed bubble decreases the economic growth rate when financial market is not so developed, but may enhance the growth rate when the financial market's condition is relatively developed. The finding that the growth path after the bubbles' collapse depends significantly on the financial market conditions implies that the collapse exposes the country's "true" economic conditions.

Finally, we conducted a full welfare analysis of stochastic bubbles in all bubble regions where bubbles enhance and impair growth. We find that bubbles are welfare-improving, even if they are expected to collapse and even if they reduce the economic growth rate, because bubbles enhance consumption-smoothing.

Our model could be extended in several directions. One direction could analyze policy-related issues such as government intervention before and after the collapse of bubbles.[35] Another direction could extend our model into a two-countries model with different pledgeability levels, and investigate how globalization factors, such as capital account liberalization, affects the emergence of bubbles in each country. These would be promising areas for future research.

[35] Lorenzoni (2008) presents an interesting framework to study policies in the presence of pecuniary externalities from amplified asset prices. Analyzing bubbles within Lorenzoni's framework could clarify how regulations might prevent bubbles or the effect of government intervention after a collapse.

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9 Appendices

9.1 Proof of Proposition 2

By defining $\Omega(\phi_t,\theta) \equiv \frac{\frac{(1-p-\phi_t)}{\pi(1-p)-\phi_t}}{\left(1+\frac{\alpha^H-\alpha^L}{\alpha^L-\theta\alpha^H}p\right)\beta+\frac{(1-\pi)(1-p)}{\pi(1-p)-\phi_t}\beta\phi_t}\phi_t$, $\Gamma(\phi_t,\theta) \equiv \frac{\theta}{\beta}\frac{1}{\pi(1-p)-(1-\theta)\phi_t}\phi_t$, (24) can be rewritten as

$$\phi_{t+1} = \begin{cases} \Omega(\phi_t,\theta) & \text{if } \phi_t \leq L(\theta), \\ \\ \Gamma(\phi_t,\theta) & \text{if } \phi_t > L(\theta). \end{cases}$$

Since $L(\theta) = \frac{\alpha^L(1-p)-\theta\alpha^H}{\alpha^L-\theta\alpha^H}$, $L(\theta) > 0$ if and only if $\theta < \frac{\alpha^L}{\alpha^H}(1-p)$. Moreover, $\Omega(L(\theta),\theta) = \Gamma(L(\theta),\theta)$. Bubbles can exist as long as the above dynamics converge to a positive value or 0. If they converge to 0, the dynamics are asymptotically bubbleless. We can easily derive that $\partial\Gamma/\partial\phi_t > 0$ and $\partial^2\Gamma/\partial\phi_t^2 > 0$. Hence, if and only if $\partial\Gamma(0,\theta)/\partial\phi_t < 1 \Leftrightarrow \theta < \pi\beta(1-p)$, $\Gamma(\phi,\theta) = \phi$ has a unique strictly positive solution, $\phi^\Gamma(\theta) = \frac{\pi\beta(1-p)-\theta}{\beta(1-\theta)} > 0$. $\Omega(\phi_t,\theta)$ function is rather complicated, but, by solving $\Omega(\phi,\theta) = \phi$ explicitly, we can find that this equation has only two solutions, 0 and $\phi^\Omega(\theta) = \frac{\pi-\frac{1-\pi\beta(1-p)}{\left[1+(\frac{\alpha^H-\alpha^L}{\alpha^L-\theta\alpha^H})p\right]\beta-\beta(1-p)}}{1-\frac{1-\pi\beta(1-p)}{\left[1+(\frac{\alpha^H-\alpha^L}{\alpha^L-\theta\alpha^H})p\right]\beta-\beta(1-p)}}(1-p)$. Furthermore $\phi^\Omega(\theta) > 0$ if and only if $\partial\Omega(0,\theta)/\partial\phi_t < 1 \Leftrightarrow \theta > \frac{\alpha^L[1-\pi\beta(1-p)]-p\beta\pi\alpha^H}{\alpha^H(1-\pi\beta)}$.

(i-1) Obviously, if $\phi^\Omega(\theta) \leq 0$ and $\phi^\Gamma(\theta) \leq 0$, any bubble path cannot converge to a positive value. Moreover bubbles cannot converge to even 0 since $\phi^\Omega \leq 0 \Leftrightarrow \partial\Omega(0,\theta)/\partial\phi_t \geq 1$. Hence, bubbles cannot exist in this case.

(i-2) Next, we examine the case where $\phi^\Omega(\theta) \leq 0$ and $\phi^\Gamma(\theta) > 0$. In this case, $\phi^\Gamma(\theta)$ is a candidate to achieve a steady state, $\phi^*(\theta)$. However, $\phi^\Omega(\theta) \leq 0$ means $\Omega(\phi,\theta) > \phi$ for any positive ϕ and thus $\Omega(L(\theta),\theta) = \Gamma(L(\theta),\theta) > L(\theta)$. This implies that $\phi^\Gamma(\theta) < L(\theta)$ but this is a contradiction and $\phi^\Gamma(\theta)$ cannot be $\phi^*(\theta)$. Another candidate for ϕ^* is 0. However, $\Omega'(0) \geq 1$ since $\phi^\Omega \leq 0$ and thus bubbles cannot converge even to 0. Hence, bubbles cannot exist in this case.

(i-3) When $\phi^\Omega(\theta) > 0$ and $\phi^\Gamma(\theta) \leq 0$, $\phi^\Omega(\theta)$ is a candidate of $\phi^*(\theta)$. $\phi^\Gamma(\theta) \leq 0$ and $L(\theta) > 0$ imply that $\pi\beta(1-p) < \theta < \frac{\alpha^L}{\alpha^H}(1-p)$. This means $\pi\beta < \frac{\alpha^L}{\alpha^H}$ must be satisfied but it follows $\theta < \frac{\alpha^L}{\alpha^H}(1-p) < \frac{\alpha^L[1-\pi\beta(1-p)]-p\beta\pi\alpha^H}{\alpha^H(1-\pi\beta)}$. Hence, $\partial\Omega(0,\theta)/\partial\phi_t > 1$ and $\phi^\Omega(\theta)$ cannot be strictly positive. Moreover bubbles cannot converge even to 0.

Hence, bubbles cannot exist even in this case.

(i-4) Lastly, we examine the case where $\phi^{\Omega}(\theta) > 0$ and $\phi^{\Gamma}(\theta) > 0$. In this case, $\phi^{\Omega}(\theta)$ or $\phi^{\Gamma}(\theta)$ can realize $\phi^{*}(\theta)$. Hence, bubbles can exist when $\theta > \frac{\alpha^L[1-\pi\beta(1-p)]-p\beta\pi\alpha^H}{\alpha^H(1-\pi\beta)}$ and $\theta < \pi\beta(1-p)$.

9.2 Derivation of (25) and θ^m

By defining θ^m as the value of θ which satisfies $\phi^{\Omega}(\theta^m) = L(\theta^m)$, we can derive that $L(\theta^m) = \phi^{\Omega}(\theta^m) = \Omega(\phi^{\Omega}(\theta^m), \theta^m) = \Omega(L(\theta^m), \theta^m) = \Gamma(L(\theta^m), \theta^m)$. Since ϕ^{Ω} is an increasing function of θ and L is a decreasing function of θ, this θ^m is uniquely determined. From this relation, $L(\theta^m) = \Gamma(L(\theta^m), \theta^m)$ and this means $\phi^{\Omega}(\theta^m) = \phi^{\Gamma}(\theta^m) = L(\theta^m)$. Hence, $\phi^{\Omega}(\theta) < L(\theta)$ if $\theta < \theta^m$ and $\phi^{\Omega}(\theta) > L(\theta)$ if $\theta > \theta^m$. Since ϕ^{Γ} is a decreasing function of θ, comparison of $\phi^{\Gamma}(\theta)$ and $L(\theta)$ is rather complicated. However, by directly solving $\phi^{\Gamma}(\theta) - L(\theta)$, we can show that $\phi^{\Gamma}(\theta) < L(\theta)$ if $\theta < \theta^m$ and $\phi^{\Gamma}(\theta) > L(\theta)$ if $\theta > \theta^m$. Hence, we can derive

$$\phi^{*}(\theta) = \begin{cases} \phi^{\Omega}(\theta) & \text{if } \underline{\theta} < \theta \leq \theta^m, \\ \\ \phi^{\Gamma}(\theta) & \text{if } \theta^m \leq \theta < \overline{\theta}. \end{cases}$$

Next, we derive θ^m explicitly. θ^m is the positive value of θ that makes the following quadratic function zero.

$$\phi^{\Gamma}(\theta) - L(\theta) = \alpha^H(1-\beta)\theta^2 - \left\{\alpha^L(1-\beta+p\beta) - \alpha^H\beta\left[1-\pi(1-p)\right]\right\}\theta - \alpha^L\beta(1-p)(1-\pi).$$

By direct calculation, we have:

$$\begin{aligned} \theta^m \;=\; & \frac{\left\{\alpha^L(1-\beta+p\beta) - \alpha^H\beta\left[1-\pi(1-p)\right]\right\}}{2\alpha^H(1-\beta)} \\ & + \frac{\sqrt{\left\{\alpha^L(1-\beta+p\beta) - \alpha^H\beta\left[1-\pi(1-p)\right]\right\}^2 + 4\alpha^L\alpha^H\beta(1-\beta)(1-p)(1-\pi)}}{2\alpha^H(1-\beta)}. \end{aligned}$$

9.3 Proof of Proposition 3

Since $\underline{\theta}$ ($\bar{\theta}$) is a decreasing (increasing) function of π, the existence condition for deterministic bubbles is the least severe. Hence, we only check the condition. From the Proof of Proposition 2 and (15), if $\alpha^H\beta < \alpha^L$, bubbles cannot exist and the growth rate under the bubbleless economy is strictly lower than the interest rate under the bubbleless economy for any θ. Next, we check the case where $\alpha^H\beta \geq \alpha^L$. By deriving θ which achieves $g(\theta) = r(\theta)$ under the bubbleless economy, we have two solutions, $\theta = \frac{\alpha^L[1-\beta(1-p)]-p\beta\alpha^H}{\alpha^H(1-\beta)}$, $\beta(1-p)$. Moreover, $g(\theta) > r(\theta)$ under the bubbleless economy when $Max\left[\frac{\alpha^L[1-\beta(1-p)]-p\beta\alpha^H}{\alpha^H(1-\beta)}, 0\right] < \theta < \beta(1-p)$ and $g(\theta) < r(\theta)$ under the bubbleless economy when $\theta < \frac{\alpha^L[1-\beta(1-p)]-p\beta\alpha^H}{\alpha^H(1-\beta)}$ or $\beta(1-p) < \theta$. Since $Max\left[\frac{\alpha^L[1-\beta(1-p)]-p\beta\alpha^H}{\alpha^H(1-\beta)}, 0\right] = \underline{\theta}$ and $\beta(1-p) = \overline{\theta}$ for deterministic bubbles, $g(\theta) \geq r(\theta)$ if and only if $\underline{\theta} \leq \theta \leq \overline{\theta}$.

9.4 Proof of Proposition 4

When we solve the existence condition $\underline{\theta}$ for π, we obtain:

$$\underline{\pi} = \frac{r}{g} = \frac{\alpha^L}{\left(1 + \frac{\alpha^H-\alpha^L}{\alpha^H-\theta\alpha^L}p\right)\beta\alpha^L} \text{ in } \underline{\theta}(\pi = 1) < \theta < \alpha^L(1-p)/\alpha^H.$$

From this relationship, we learn that $\underline{\pi}$ is a decreasing function of θ in $\underline{\theta}(\pi = 1) < \theta < \alpha^L(1-p)/\alpha^H$. Hence, $1 - \underline{\pi}$ is an increasing function of θ in this region. That is, a rise in θ increases the economy's growth rate, which makes it possible for even riskier bubbles to arise in equilibrium.

Likewise, when we solve the existence condition $\bar{\theta}$ for π, we obtain:

$$\underline{\pi} = \frac{r}{g} = \frac{\frac{\theta\alpha^H}{1-p}}{\alpha^H\beta} = \frac{\theta}{\beta(1-p)} \text{ in } \alpha^L(1-p)/\alpha^H \leq \theta < \bar{\theta}(\pi = 1).$$

From this relationship, we learn that $\underline{\pi}$ is an increasing function of θ in $\alpha^L(1-p)/\alpha^H \leq \theta < \bar{\theta}(\pi = 1)$. Hence, $1 - \underline{\pi}$ is an decreasing function of θ in this region. That is, a rise in θ increases the interest rate, which makes it difficult for riskier bubbles to arise in equilibrium.

9.5 Explicit derivation of θ^*

At $\theta = \theta^*$, economic growth rate in the bubble economy equals that in the bubbleless economy, i.e.,

$$g^* = \frac{\alpha^H \left\{\beta\left[1-\pi(1-p)\right] + (1-\beta)\theta\right\}}{1-\pi\beta(1-p)} = \left(1 + \frac{\alpha^H - \alpha^L}{\alpha^L - \theta\alpha^H}p\right)\beta\alpha^L.$$

θ^* is the greater value of θ which satisfies this quadratic equation. Hence, we have:

$$\begin{aligned} \theta^* \quad &= \quad \frac{\alpha^L\alpha^H\left[1-\pi\beta^2(1-p)\right] - (\alpha^H)^2\beta\left[1-\pi(1-p)\right]}{2(\alpha^H)^2(1-\beta)} \\ &\quad + \frac{\sqrt{\left\{\alpha^L\alpha^H\left[1-\pi\beta^2(1-p)\right] - (\alpha^H)^2\beta\left[1-\pi(1-p)\right]\right\}^2 + \rho}}{2(\alpha^H)^2(1-\beta)}, \end{aligned}$$

where

$$\rho = 4\alpha^L(\alpha^H)^2\beta(1-\beta)\left\{\alpha^H\left[1-\pi(1-p)\right] - \left[\alpha^L + (\alpha^H - \alpha^L)p\right]\left[1-\pi\beta(1-p)\right]\right\}.$$

9.6 Proof of Proposition 6

By inserting (25) into (26), we can derive the economic growth rate in the stochastic steady-state:

$$g_t^* = \begin{cases} \dfrac{\left[1 + (\dfrac{\alpha^H - \alpha^L}{\alpha^L - \theta\alpha^H})p\right]\beta\alpha^L - \beta(1-p)\alpha^L}{1-\pi\beta(1-p)} & \text{if } \underline{\theta} < \theta \leq \theta^m, \\ \alpha^H\dfrac{\beta\left[1-\pi(1-p)\right] + (1-\beta)\theta}{1-\pi\beta(1-p)} & \text{if } \theta^m \leq \theta < \pi\beta(1-p). \end{cases} \tag{32}$$

First, we prove $g_t^* > g_t$ in $\underline{\theta} < \theta \leq \theta^m$. Comparing (32) to (15) yields $g_t^* - g_t = \frac{\alpha^L\beta(1-p)\left\{\pi\left[1+(\frac{\alpha^H-\alpha^L}{\alpha^L-\theta\alpha^H})p\right]-1\right\}}{1-\pi\beta(1-p)}$. Thus, if $\pi[1+\frac{\alpha^H-\alpha^L}{\alpha^L-\theta\alpha^H}p] > 1$, we can say $g_t^* > g_t$. Since $\pi[1+\frac{\alpha^H-\alpha^L}{\alpha^L-\theta\alpha^H}p] > 1$ is equivalent to the existence condition of stochastic bubbles, therefore $g_t^* > g_t$.

Next, we prove $g_t^*(\theta) \geq g_t(\theta)$ in $\theta^m < \theta \leq \theta^*$, and $g_t^*(\theta) < g_t(\theta)$ in $\theta^* < \theta < \pi\beta(1-p)$. Since g_t^* is a linear function of θ in $\theta^m \leq \theta < \pi\beta(1-p)$, and g_t is a convex function of θ in $\theta^m \leq \theta < \alpha^L(1-p)/\alpha^H$ and a linear function of

θ in $\alpha^L(1-p)/\alpha^H \leq \theta \leq 1$. Moreover, at $\theta = \theta^m$, $\frac{\left[1+(\frac{\alpha^H-\alpha^L}{\alpha^L-\theta\alpha^H})p\right]\beta\alpha^L-\beta(1-p)\alpha^L}{1-\pi\beta(1-p)} = \alpha^H\frac{\beta[1-\pi(1-p)]+(1-\beta)\theta}{1-\pi\beta(1-p)}$ and $g_t^* > g_t$, and at $\theta = \alpha^L(1-p)/\alpha^H$, $g_t^* < g_t$. Therefore, there exists θ^* at which $g_t^*(\theta) = g_t(\theta)$ and $g_t^*(\theta) > g_t(\theta)$ in $\theta^m < \theta < \theta^*$ and $g_t^*(\theta) < g_t(\theta)$ in $\theta^* < \theta < \pi\beta(1-p)$. θ^* is given in the "Explicit derivation of θ^*" in the Appendix.

9.7 Proof of Proposition 8

By differentiating $V_0^{BL}(e_0)$ with respect to θ, we have:

$$\frac{dV_0^{BL}(e_0)}{d\theta} = \begin{cases} \frac{p(\alpha^H-\alpha^L)}{(1-\theta)(\alpha^L-\theta\alpha^H)} > 0 & \text{in } 0 \leq \theta < \frac{\alpha^L}{\alpha^H}(1-p), \\ \\ \frac{1-p-\theta}{(1-\theta)\theta} > 0 & \text{in } \frac{\alpha^L}{\alpha^H}(1-p) \leq \theta < 1-p. \end{cases}$$

9.8 Dynamics on asymptotically bubbleless path

In the main text, we focused on the stochastic steady-state. In this appendix, we will analyze dynamics on asymptotically bubbleless path.

Rewriting (19) by using ϕ_t, output growth rate in the bubble economy can be written as:

$$\frac{Y_{t+1}^*}{Y_t^*} \equiv g_t^* = \begin{cases} \dfrac{\left[(1+\frac{\alpha^H-\alpha^L}{\alpha^L-\theta\alpha^H}p)\beta\alpha^L - \alpha^L\beta\phi_t\right]}{1-\beta\phi_t} & \text{if } \phi_t \leq L(\theta), \\ \\ \dfrac{\alpha^H\beta\left[1-\phi_t\right]}{1-\beta\phi_t} & \text{if } \phi_t > L(\theta). \end{cases} \tag{33}$$

As we learn from (33), as long as bubbles can exist, g_t^* is an increasing function of ϕ_t in $0 < \phi_t < L(\theta)$, and a decreasing function of ϕ_t in $L(\theta) < \phi_t$, and ϕ^* is the maximum value of the bubble size that can be sustained.

There are four different patterns depending on θ. The first pattern is the dynamics in the region of $\underline{\theta} < \theta \leq \theta^m$. In this case, we have $0 < \phi^* \leq L(\theta)$. If the initial ϕ_0 starts from $\phi_0 < \phi^*$, together with the decline in the size of bubbles, high-productivity investments are crowded out, while low-productivity investments are crowded in. As a result, both aggregate productivity and economic growth rate decrease monotonically as long as bubbles survive. When bubbles collapse, then the economic growth rate decreases to g_t from g_t^* discontinuously. Moreover, the adverse impacts of the collapse of bubbles on the economic growth rate becomes larger, the larger the size of bubbles.

The second pattern is the dynamics in the region of $\theta^m < \theta \leq \theta^*$. In this case, we have $0 < L(\theta) < \phi^*$. In this pattern, although $g_t^* > g_t$ in the stochastic steady-state, overinvestment in bubbles occur if ϕ_t is sufficiently large, i.e., $L(\theta) < \phi_t < \phi^*$. Thus, if the initial ϕ_0 starts from $L(\theta) < \phi_0 < \phi^*$, the economic growth rate initially increases because overinvestment in bubbles is reduced. Once ϕ_t becomes smaller than $L(\theta)$, then the economic growth rate starts to decrease because even L-types start to invest, while H-types have to cut back on their investments. Moreover, the bursting of bubbles leads to a discontinuous drop in the economic growth, and the effects of the bubbles' collapse become the largest if ϕ_t takes an intermediate value, i.e., $\phi_t = L(\theta)$.

The third one is the dynamics in the region of $\theta^* < \theta < \alpha^L(1-p)/\alpha^H$. In this case, we have $0 < L(\theta) < \phi^*$. Since $g_t^* < g_t$ in the stochastic steady-state, if the initial ϕ_0 is close to ϕ^*, the economic growth rate initially increases as the size of bubbles becomes smaller, and then decreases.

The fourth pattern is the dynamics in the region of $\alpha^L(1-p)/\alpha^H \leq \theta < \pi\beta(1-p)$. In this case, we have $L(\theta) = 0 < \phi^*$. In this region, bubbles result in crowding H-projects out and lowering the economic growth. Thus, if the initial ϕ_0 starts from $\phi_0 < \phi^*$, together with the decline in the size of bubbles, the crowd-out effect is reduced and the economic growth rate increases monotonically.

9.9 Proof of Proposition 9

The value function of the bubble economy and the bubbleless economy can be written as:

$$V_0^{BL}(e_0) = W(\theta) + \frac{1}{1-\beta}\log(e_0),$$

and

$$V_0^{BB}(e_0^*) = q(\phi^*(\theta), \theta) + \frac{1}{1-\beta}\log(e_0^*).$$

In this Appendix, we prove $V_0^{BB}(e_0^*) > V_0^{BL}(e_0)$ for any θ in the bubble regions.

To do so, let us define $H_0(\kappa,\theta)$ and $Q_0(\kappa,\theta)$ as the followings:

$$H_0(\kappa,\theta) \equiv \frac{1}{1-\beta}\log(1-\beta) + \frac{\beta}{(1-\beta)^2}\log(\beta) + \frac{\beta^2(1-\pi)}{1-\beta\pi}\frac{1}{(1-\beta)^2}J_0 + \frac{1}{1-\beta\pi}\frac{\beta}{1-\beta}\left[\pi H_1(\kappa,\theta) + (1-\pi)H_2(\kappa,\theta)\right],$$

and

$$Q_0(\kappa,\theta) \equiv \frac{1}{1-\beta}\log(1-\beta) + \frac{\beta}{(1-\beta)^2}\log(\beta) + \frac{\beta^2(1-\pi)}{1-\beta\pi}\frac{1}{(1-\beta)^2}J_0 + \frac{1}{1-\beta\pi}\frac{\beta}{1-\beta}\left[\pi Q_1(\kappa,\theta) + (1-\pi)Q_2(\kappa,\theta)\right],$$

where

$$H_1(\kappa,\theta) = p\log\left[\frac{\alpha^H(1-\theta)}{1-\frac{\theta\alpha^H}{\alpha^L}}\right] + (1-p)\log\left[\frac{\pi\alpha^L[1-p-\kappa]}{\pi(1-p)-\kappa}\right],$$

and

$$H_2(\kappa,\theta) = p\log\left[\frac{\alpha^H(1-\theta)}{1-\frac{\theta\alpha^H}{\alpha^L}}\right] + (1-p)\log\left[\frac{\alpha^L[1-p-\kappa]}{1-p}\right],$$

and

$$Q_1(\kappa,\theta) = p\log\left[\frac{\alpha^H(1-\theta)[1-\kappa]}{p}\right] + (1-p)\log\left[\frac{\pi\theta\alpha^H[1-\kappa]}{\pi(1-p)-\kappa}\right],$$

and

$$Q_2(\kappa,\theta) = p\log\left[\frac{\alpha^H(1-\theta)[1-\kappa]}{p}\right] + (1-p)\log\left[\frac{\theta\alpha^H[1-\kappa]}{1-p}\right].$$

and J_0 is given in the welfare derivation in the Technical Appendix.

First, let us prove $V_0^{BB}(e_0^*) > V_0^{BL}(e_0)$ in the case of $\underline{\theta} < \theta < \theta^m$. In this case, $H_0(\kappa = 0,\theta) = W(\theta)$ holds. We also learn that:

$$\frac{\partial H_0(\kappa,\theta)}{\partial\kappa} = \frac{1}{1-\beta\pi}\frac{\beta}{1-\beta}\frac{1-\pi(1-p)}{1-p-\kappa}\frac{\kappa}{\pi(1-p)-\kappa} > 0 \text{ for any } \kappa \text{ in } 0<\kappa<\pi(1-p).$$

I.e., $H_0(\kappa,\theta)$ is an increasing function of κ for any κ in $0 < \kappa < \pi(1-p)$. Moreover, in $\underline{\theta} < \theta < \theta^m$, we know $0 < \phi^*(\theta) < \pi(1-p)$. This means that $H_0(\kappa,\theta)$ is an increasing function of κ for any κ in $0 < \kappa \le \phi^*(\theta) < \pi(1-p)$. Therefore, we have:

$$W(\theta) = H_0(\kappa = 0,\theta) < H_0(\kappa = \phi^*(\theta),\theta) = q(\phi^*(\theta),\theta).$$

Together with $e_0^* > e_0^*$, we can obtain $V_0^{BB}(e_0^*) > V_0^{BL}(e_0)$. We should mention that as we note in the footnote 19, stochastic bubbles can arise even at $\theta = 0$ if $\frac{\alpha^L[1-\pi\beta(1-p)]-p\beta\pi\alpha^H}{\alpha^H(1-\pi\beta)} < 0$. In this case, we know $0 < \phi^*(\theta = 0) < \pi(1-p)$. Hence, in exactly the same proof above mentioned, we can prove $V_0^{BB}(e_0^*) > V_0^{BL}(e_0)$ by putting $\theta = 0$.

Next, we prove $V_0^{BB}(e_0^*) > V_0^{BL}(e_0)$ in the case of $\theta^m \leq \theta < \alpha^L(1-p)/\alpha^H$. In this case, $H_0(\kappa = 0, \theta) = W(\theta)$ and $H_0(\kappa = L(\theta), \theta) = Q_0(\kappa = L(\theta), \theta)$ hold. Moreover, we have:

$$\frac{\partial Q_0(\kappa,\theta)}{\partial \kappa} = \frac{1}{1-\beta\pi}\frac{\beta}{1-\beta}\frac{1-\pi(1-p)}{1-\kappa}\frac{\kappa}{\pi(1-p)-\kappa} > 0 \text{ for any } \kappa \text{ in } 0 < \kappa < \pi(1-p).$$

I.e., $Q_0(\kappa, \theta)$ is an increasing function of κ for any κ in $0 < \kappa < \pi(1-p)$. Also, in $\theta^m \leq \theta < \alpha^L(1-p)/\alpha^H$, we know $0 < L(\theta) \leq \phi^*(\theta) < \pi(1-p)$. This means that $Q_0(\kappa, \theta)$ is an increasing function of κ for any κ in $L(\theta) \leq \kappa \leq \phi^*(\theta) < \pi(1-p)$. Therefoer, together with $H_0(\kappa = 0, \theta) = W(\theta)$ and $H_0(\kappa = L(\theta), \theta) = Q_0(\kappa = L(\theta), \theta)$, we obtain:

$$W(\theta) = H_0(\kappa = 0, \theta) < H_0(\kappa = L(\theta), \theta) = Q_0(\kappa = L(\theta), \theta) \leq Q_0(\kappa = \phi^*(\theta), \theta) = q(\phi^*(\theta), \theta).$$

Since $e_0^* > e_0^*$, we can obtain $V_0^{BB}(e_0^*) > V_0^{BL}(e_0)$.

Finally, we prove $V_0^{BB}(e_0^*) > V_0^{BL}(e_0)$ in the case of $\alpha^L(1-p)/\alpha^H \leq \theta < \bar{\theta}$. In this case, $Q_0(\kappa = 0, \theta) = W(\theta)$ holds. Moreover, $\frac{\partial Q_0(\kappa,\theta)}{\partial \kappa} > 0$ for any κ in $0 < \kappa < \pi(1-p)$ and $0 < \phi^*(\theta) < \pi(1-p)$. This means that $Q_0(\kappa, \theta)$ is an increasing function of κ for any κ in $0 < \kappa \leq \phi^*(\theta) < \pi(1-p)$. Therefore, we have:

$$W(\theta) = Q_0(\kappa = 0, \theta) < Q_0(\kappa = \phi^*(\theta), \theta) = q(\phi^*(\theta), \theta).$$

Together with $e_0^* > e_0^*$, we can obtain $V_0^{BB}(e_0^*) > V_0^{BL}(e_0)$.

9.10 Other implications of the effects of the bubbles' collapse

In this Appendix, we examine the effects of a permanent shock to productivity (or at least, this shock is expected to be permanent at $t = s$.). We will show that even a small unexpected-negative shock to productivity can cause bubbles to burst in economies with relatively low θ, while in economies with relatively high θ, bubbles can be sustained.

Suppose at date $t = s-1$, the economy is in the stochastic steady-state. There is then an unexpected shock at $t = s$ that decreases productivity from α^H to $\alpha^S < \alpha^H$. Here we assume that this shock is permanent . Since $\underline{\theta}$ is a decreasing function of α^H, bubbles must collapse in countries where pledgeability is lower than $\underline{\theta}(\alpha^S)$. This result shows that, even if the shock is common, the effect of the shock differs from country to country. In relatively low θ countries, even a small unexpected-negative shock to productivity can cause bubbles to burst, while in relatively high θ countries, bubbles can be sustained through an adjustment (a decline) in bubble prices (i.e., the decrease in asset prices does not directly mean that the bubbles will burst. It might just be an adjustment process of the bubbles).

Moreover, the above mentioned results suggest that in high θ countries, the effect of the shock can be non-linear. As long as the shock is small, bubbles can be sustained and thus they may not collapse. However, if the shock is sufficiently large, they collapse, which causes a discontinuous drop in the economic growth rate. This non-linear effect of the shock on the economic growth rate shares similarity with the systemic crises discussed in Brunnermeier and Sannikov (2014) and Gertler and Kiyotaki (2015).

9.11 Discussions on under what conditions g tends to be greater than r only in the middle range of θ in the bubbleless economy

We can derive theoretical characteristics regarding under what conditions g tends to be greater than r only when θ is in the middle range in the bubbleless economy. The following characteristics are important. First, p must be small, that is, a fraction of low-productivity entrepreneurs must be relatively large. Second, the productivity difference, α^H/α^L, or time preference rate, β, must not be too large. The intuitions are that when p is low, even if θ rises, the interest rate does not readily rise because a population share of low-productivity entrepreneurs is large, while the growth rate of the economy increases. Moreover, when p is low, the growth rate of the economy becomes low, and it can be lower than the interest rate when θ is sufficiently small. That is, we obtain $g(\theta = 0) < r(\theta = 0)$. In this situation, when θ increases, $g(\theta)$

increases while $r(\theta)$ does not readily rise. As a consequence, $g(\theta)$ becomes greater than $r(\theta)$ when θ reaches the middle range, $\underline{\theta} > 0$.

Likewise, when the productivity difference, α^H/α^L, is relatively small (but must not be too small because bubbles cannot occur), the value of collateral of high-productivity investments is low and so is leverage. As a consequence, economic growth rate is lowered. Also, when time preference rate, β, is relatively small, saving rate is small. This also lowers economic growth rate. In both cases, the economy's growth rate is sufficiently low compared to the interest rate when θ is sufficiently small, and thus we obtain $g(\theta = 0) < r(\theta = 0)$. In this situation, when θ rises, $g(\theta)$ increases while $r(\theta)$ does not readily rise, and as a result, $g(\theta)$ becomes greater than $r(\theta)$ when θ reaches $\underline{\theta} > 0$.

These properties hold, even when we consider the case with continuous productivities. We explore the case with continuous productivities in more detail in the next Appendix and in the Technical Appendix.

9.12 Case with a continuum of productivity

In this Appendix, we present numerical results showing that even in the continuum case, there exist sets of parameter values for which g is greater than r only in the middle range of θ under the bubbleless economy.[36] In the Technical Appendix, we explore the continuum case with analytical examination in greater details, and derive the conditions under which g is greater than r only in the middle range of θ under the bubbleless economy.

Let us suppose that there are continuously distributed investment opportunities with different productivities from $\underline{\alpha}$ to $\overline{\alpha}$, and the density function of productivity is given by $f(\alpha)$.

The savings/investment market clears if and only if

$$\int_r^{\bar{\alpha}} \frac{1}{1 - \frac{\theta\alpha}{r}} f(\alpha) d\alpha = 1. \tag{34}$$

and the growth rate of the economy is given by

$$g = \int_r^{\bar{\alpha}} \frac{\beta\alpha}{1 - \frac{\theta\alpha}{r}} f(\alpha) d\alpha. \tag{35}$$

[36]We thank an anonymous referee who pointed out this implication.

Both g and r are increasing functions of θ, as shown in the Technical Appendix. Note that since the model is an endogenous growth model, g as well as r is affected by a change in θ. Given this property, we solve the model numerically by specifying the density function. We examine whether there exist sets of parameter values under which g is greater than r only in the middle range of θ under the bubbleless economy.

We examine the following five different density functions, i.e., power law distribution, exponential distribution, double power law distribution, bimodal distribution, and u-shape bimodal distribution. Functional forms in each case are shown in the Technical Appendix. An important common property in the five distributions is that the distributions can capture the population share of entrepreneurs with low productivity by exogenous parameter values, which is important for our result. In the Technical Appendix, we explain which exogenous parameter values capture the share. Also note that in international trade literature with firm heterogeneity, it is widely used that productivity of firm follows power law distribution (see, for example, Helpman et al. 2004; Chaney 2008, etc.).

We summarize numerical results in Figure 5. Our numerical solutions show that in all five density functions, there exist sets of parameter values for which g is greater than r only in the middle range of θ under the bubbleless economy. These numerical results verify that our result in the discrete case holds even in the continuum case.

We should mention that from our numerical solutions and analytical examination provided in the Technical Appendix, we can draw important common properties to get the result. (ii-1) The pupolation share of entrepreneurs with low productivity is relatively large, but not too large. (ii-2) $\underline{\alpha}$ is positive (see the condition (i-1) in the Technical Appendix.). (ii-3) $\bar{\alpha}$ is relatively high (see the condition (i-2) in the Technical Appendix.). (ii-4) The population share of entrepreneurs with high productivity is not too small.

The economic intuitions for (ii-1)-(ii-4) are as follows. When θ increases under $g < r$ at $\theta = 0$, the interest rate also rises. However, the interest rate is determined by the marginal type of technology. When the population share of entrepreneurs with low productivity is relatively large, the interest rate does not rise sharply. On the other hand, the resource can be allocated to the technologies with higher productivity than the marginal type of technology. When $\bar{\alpha}$ is relatively high, or the population share of entrepreneurs with high productivity is not too small, this makes the increase in the economy's growth rate higher. As a consequence, the increase in the growth rate is higher than the increase in the interest rate. Hence, $g(\theta) > r(\theta)$ in the middle range of θ. Moreover, $\underline{\alpha}$ must be positive. This is because, if $\underline{\alpha} > 0$, the interest rate has the positive lower bound. On the other hand, as in the discrete case, if $\underline{\alpha} = 0$, $r = 0$ when $\theta = 0$. If this is the case, entrepreneurs have incentives to buy bubbles even when $\theta = 0$, that is, bubbles can arise even when $\theta = 0$.

Technical Appendices

9.13 A continuum of productivity: analytical examination

In this Technical Appendix, we explore the continuum case analytically.

We first derive the necessary and sufficient conditions analytically under which $g > r$ only in the middle range of θ under the bubbleless economy. Second, we derive weaker conditions, because the weaker conditions can be solved explicitly. Then, we solve the model numerically. The numerical solutions are shown in the Appendix.

For this purpose, we first examine the relation between r and θ, and g and θ. We define $h \equiv \frac{\theta}{r}$ $(r = \frac{\theta}{h})$. From (34), we can derive that

$$\begin{aligned}
&\frac{dh}{d\theta}\int_{r}^{\bar{\alpha}} \frac{\alpha}{(1-h\alpha)^2} f(\alpha)d\alpha = \frac{f(r)}{1-\theta}\frac{dr}{d\theta} \\
\Leftrightarrow\ &\frac{dr}{d\theta} = \frac{1-\theta}{f(r)}\left[\int_{r}^{\bar{\alpha}} \frac{\alpha}{(1-h\alpha)^2} f(\alpha)d\alpha\right]\frac{dh}{d\theta}.
\end{aligned} \tag{36}$$

From $\frac{dh}{d\theta} = \frac{1}{r} - \frac{\theta}{r^2}\frac{dr}{d\theta}$, the above equation can be written as:

$$\begin{aligned}
&\left(\frac{1}{r} - \frac{\theta}{r^2}\frac{dr}{d\theta}\right)\int_{r}^{\bar{\alpha}} \frac{\alpha}{(1-h\alpha)^2} f(\alpha)d\alpha = \frac{f(r)}{1-\theta}\frac{dr}{d\theta} \\
\Leftrightarrow\ &\frac{dr}{d\theta} = \frac{\frac{1}{r}\displaystyle\int_{r}^{\bar{\alpha}} \frac{\alpha}{(1-h\alpha)^2} f(\alpha)d\alpha}{\frac{\theta}{r^2}\displaystyle\int_{r}^{\bar{\alpha}} \frac{\alpha}{(1-h\alpha)^2} f(\alpha)d\alpha + \frac{f(r)}{1-\theta}} > 0.
\end{aligned} \tag{37}$$

That is, the interest rate is a monotonically increasing function of θ. Since $\frac{dr}{d\theta} > 0$, from (36), we learn $\frac{dh}{d\theta} > 0$.

By using (36), we also have:

$$\begin{aligned}\frac{dg}{d\theta} &= \frac{dh}{d\theta}\left[\int_r^{\bar{\alpha}} \frac{\beta\alpha^2}{(1-h\alpha)^2} f(\alpha)d\alpha\right] - \frac{dr(\theta)}{d\theta}\frac{\beta r}{1-\theta} f(r(\theta)) \\ &= \frac{dh}{d\theta}\left[\int_r^{\bar{\alpha}} \frac{\beta\alpha(\alpha-r)}{(1-h\alpha)^2} f(\alpha)d\alpha\right] > 0. \qquad (38)\end{aligned}$$

That is, g is also a monotonically increasing function of θ. Moreover, it is obvious that when $\theta = 1$, $r = \bar{\alpha}$ and $g = \beta\bar{\alpha}$. That is, $g > r$ at $\theta = 1$. Note that since the model is an endogeous growth model, g is affected by a change in θ.

Given these results, we derive the necessary and sufficient conditions under which $g > r$ only in the middle range of θ under the bubbleless economy.

First, g must be smaller than r at $\theta = 0$, i.e., $g(\theta = 0) < r(\theta = 0)$. Since $r = \underline{\alpha}$ when $\theta = 0$, the condition of $g(\theta = 0) < r(\theta = 0)$ can be written as:

$$\int_{\underline{\alpha}}^{\bar{\alpha}} \beta\alpha f(\alpha)d\alpha < \underline{\alpha}. \qquad (39)$$

For this condition to hold, $\underline{\alpha}$ must be positive.

Moreover, by using the condition of $\int_r^{\bar{\alpha}} \frac{1}{1-\frac{\theta\alpha}{r}} f(\alpha)d\alpha = 1$, we have:

$$g - r = \int_r^{\bar{\alpha}} \frac{\beta\alpha}{1-h\alpha} f(\alpha)d\alpha - r = \int_r^{\bar{\alpha}} \frac{\beta\alpha - r}{1-h\alpha} f(\alpha)d\alpha.$$

Hence, by defining that

$$r(\theta^{**}) = \beta\bar{\alpha},$$

we can say

$$\text{Under } \theta^{**} < \theta \leq 1,\ g < r. \qquad (40)$$

Therefore, given (40), the necessary and sufficient conditions for $g > r$ only in the middle range of θ are (39) and

$$g > r \text{ for some } 0 < \theta < \theta^{**}. \qquad (41)$$

Theoretically speaking, since both g and r can be solved by (34) and (35), they can be written as functions of exogenous parameter values. But, it is hard to solve them explicitly under general density functions. Hence, we solve them numerically by specifying the density functions.

Before solving numerically, we want to proceed analytically as much as possible. We derive weaker conditions than (41), because the weaker conditions can be derived explicity. We derive the conditions for $\frac{d(g-r)}{d\theta} > 0$ at $\theta = 0$ because if this condition is satisfied together with (39), then g becomes likely to be greater than r only in the middle range of θ.

For this purpose, we examine the dynamics of $g - r = \int_r^{\bar{\alpha}} \frac{\beta\alpha}{1-\frac{\theta\alpha}{r}} f(\alpha)d\alpha - r$, under the constraint of $\int_r^{\bar{\alpha}} \frac{1}{1-\frac{\theta\alpha}{r}} f(\alpha)d\alpha = 1$.

From (36) and (38), we have:

$$
\begin{aligned}
\frac{d(g-r)}{d\theta} &= \frac{dh}{d\theta}\left[\int_r^{\bar{\alpha}} \frac{\beta\alpha^2}{(1-h\alpha)^2} f(\alpha)d\alpha\right] - \frac{dr(\theta)}{d\theta}\left[\frac{\beta r}{1-\theta} f(r(\theta)) + 1\right] \\
&= \frac{dh}{d\theta}\left[\int_r^{\bar{\alpha}} \frac{\alpha}{(1-h\alpha)^2}\left(\beta(\alpha - r(\theta)) - \frac{1-\theta}{f(r(\theta))}\right) f(\alpha)d\alpha\right].
\end{aligned}
$$

Since $\frac{dh}{d\theta} > 0$, if

$$
\int_r^{\bar{\alpha}} \frac{\alpha}{(1-h\alpha)^2}\left(\beta(\alpha - r(\theta)) - \frac{1-\theta}{f(r(\theta))}\right) f(\alpha)d\alpha > 0 \text{ for some } 0 \le \theta < \theta^{**}, \tag{42}
$$

then we have $\frac{d(g-r)}{d\theta} > 0$ for some $0 \le \theta < \theta^{**}$. Given (40), together with (39), (42) is a necessary condition for $g > r$ only in the middle of θ. Here we derive the condition for $\frac{d(g-r)}{d\theta} > 0$ at $\theta = 0$ in (42).

When $\theta = 0$, $r = \underline{\alpha}$. Then, the condition (42) at $\theta = 0$ can be written as:

$$
\begin{aligned}
&\int_{\underline{\alpha}}^{\bar{\alpha}} \alpha f(\alpha)\left[\beta(\alpha - \underline{\alpha}) - \frac{1}{f(\underline{\alpha})}\right] d\alpha > 0 \\
\Leftrightarrow\ & \frac{\int_{\underline{\alpha}}^{\bar{\alpha}} \alpha^2 f(\alpha)d\alpha}{\int_{\underline{\alpha}}^{\bar{\alpha}} \alpha f(\alpha)d\alpha} > \underline{\alpha} + \frac{1}{\beta f(\underline{\alpha})}.
\end{aligned}
$$

Therefore, if the following conditions are satisfied, $g < r$ at $\theta = 0$ and $\frac{d(g-r)}{d\theta} > 0$ at

$\theta = 0$.

$$\frac{\int_{\underline{\alpha}}^{\bar{\alpha}} \alpha^2 f(\alpha) d\alpha}{\int_{\underline{\alpha}}^{\bar{\alpha}} \alpha f(\alpha) d\alpha} > \underline{\alpha} + \frac{1}{\beta f(\underline{\alpha})} > \underline{\alpha} > \int_{\underline{\alpha}}^{\bar{\alpha}} \beta \alpha f(\alpha) d\alpha.$$

In summary, we have derived the following conditions. (i-1) If $\underline{\alpha} > \int_{\underline{\alpha}}^{\bar{\alpha}} \beta \alpha f(\alpha) d\alpha$, $g < r$ at $\theta = 0$. For this condition to be satisfied, $\underline{\alpha}$ must be positive. (i-2) If $\frac{\int_{\underline{\alpha}}^{\bar{\alpha}} \alpha^2 f(\alpha) d\alpha}{\int_{\underline{\alpha}}^{\bar{\alpha}} \alpha f(\alpha) d\alpha} > \underline{\alpha} + \frac{1}{\beta f(\underline{\alpha})}$, $\frac{d(g-r)}{d\theta} > 0$ at $\theta = 0$. This condition is likely to hold if $\bar{\alpha}$ is high. (i-3) There exists θ^{**} such that $g < r$ under $\theta^{**} < \theta \leq 1$. Therefore, under (i-1) and (i-2), g can be greater than r only in the middle range of θ under the bubbleless economy.

Given these analytical results, we solve the model numerically by specifying the density function. The functional forms are provided below. Figure 5 summarizes our numerical solutions, showing that there exist sets of parameter conditions for which the conditions (i-1) and (i-2) are both satisfied, or (39) and (41) are both satisfied.

Moreover, as the numerical results in Figure 5 show, in the cases of the exponential distribution, and the double power law distribution, and in some cases in the bimodal distribution, there exist sets of parameter values for which even if $\frac{d(g-r)}{d\theta} < 0$ at $\theta = 0$, g is greater than r only in the middle range of θ. These results suggest that even if $\frac{d(g-r)}{d\theta} < 0$ at $\theta = 0$, there exist sets of parameter values for which (39) and (41) are both satisfied.

In the following sections, we explain functional forms of each distribution function.

9.13.1 Power law distribution

Consider

$$f(\alpha) = \chi \alpha^{-\eta-1}.$$

χ must satisfy

$$\int_{\underline{\alpha}}^{\bar{\alpha}} \chi \alpha^{-\eta-1} d\alpha = 1.$$

Thus,

$$\chi = \frac{\eta}{\frac{1}{\underline{\alpha}^{\eta}} - \frac{1}{\bar{\alpha}^{\eta}}}.$$

Therefore, the general form of the density function is

$$f(\alpha) = \frac{\eta}{\frac{1}{\underline{\alpha}^{\eta}} - \frac{1}{\bar{\alpha}^{\eta}}} \alpha^{-\eta-1},$$

where η is the shape parameter of the power law distribution. When η is higher, the population share of low productivity is larger.

9.13.2 Exponential distribution

We define a truncated exponential distribution as the following:

$$f(\alpha) = \xi \exp(-\lambda\alpha).$$

ξ must satisfy

$$\int_{\underline{\alpha}}^{\bar{\alpha}} \xi \exp(-\lambda\alpha) d\alpha = 1.$$

Thus,

$$\xi = \frac{\lambda}{\exp(-\lambda\underline{\alpha}) - \exp(-\lambda\bar{\alpha})}.$$

Therefore, the general form of the density function is

$$f(\alpha) = \frac{\lambda}{\exp(-\lambda\underline{\alpha}) - \exp(-\lambda\bar{\alpha})} \exp(-\lambda\alpha),$$

where λ is the shape parameter of the truncated exponential distribution. When λ is higher, the population share of entrepreneurs with low productivity is larger. Also, compared to the power law distribution, the share of entrepreneurs with high productivity is lower in the exponential distribution under the assumption that $f(\underline{\alpha})$ has the same value.

9.13.3 Double power law

Consider a type of double power law distributions. For $\alpha > \alpha^*$, α follows a power law distribution, the density over $\alpha^* < \alpha \leq \bar{\alpha}$ integrates $weightR$. Similarly, for

$\alpha < \alpha^*$, α follows a power law distribution, and the density over $\underline{\alpha} \le \alpha < \alpha^*$ integrates $1 - weightR$. Namely,

$$f(\alpha) = \begin{cases} \chi_R \alpha^{-\eta-1} & for\ \alpha > \alpha^*, \\ \chi_L \alpha^{\eta-1} & for\ \alpha < \alpha^*. \end{cases}$$

χ_R and χ_L satisfy

$$\int_{\alpha^*}^{\bar{\alpha}} \chi_R \alpha^{-\eta-1} d\alpha = weightR,$$

$$\int_{\underline{\alpha}}^{\alpha^*} \chi_L \alpha^{\eta-1} d\alpha = 1 - weightR.$$

Thus,

$$\chi_R = \frac{\eta weightR}{\frac{1}{\alpha^{*\eta}} - \frac{1}{\bar{\alpha}^\eta}}.$$

$$\chi_L = \frac{\eta(1 - weightR)}{\alpha^{*\eta} - \underline{\alpha}^\eta}.$$

Therefore,

$$f(\alpha) = \begin{cases} \frac{\eta weightR}{\frac{1}{\alpha^{*\eta}} - \frac{1}{\bar{\alpha}^\eta}} \alpha^{-\eta-1} & for\ \alpha > \alpha^*, \\ \frac{\eta(1-weightR)}{\alpha^{*\eta} - \underline{\alpha}^\eta} \alpha^{\eta-1} & for\ \alpha < \alpha^*. \end{cases}$$

Moreover, we impose a condition that at $\alpha = \alpha^*$, $f(\cdot)$ is continuous. That is,

$$\frac{\eta weightR}{\frac{1}{\alpha^{*\eta}} - \frac{1}{\bar{\alpha}^\eta}} (\alpha^*)^{-\eta-1} = \frac{\eta(1 - weightR)}{\alpha^{*\eta} - \underline{\alpha}^\eta} (\alpha^*)^{\eta-1}$$

$$\Longleftrightarrow weightR = \frac{1 - (\frac{\alpha^*}{\bar{\alpha}})^\eta}{2 - (\frac{\underline{\alpha}}{\alpha^*})^\eta - (\frac{\alpha^*}{\bar{\alpha}})^\eta}$$

I.e., $weightR$ is written as functions of the exogenous parameter values. Note that when $\alpha^* = \bar{\alpha}$, $weightR = 0$ and when $\alpha^* = \underline{\alpha}$, $weightR = 1$. Also, note that when α^* is lower, the population share of entrepreneurs with low productivity is larger.

9.13.4 Bimodal distribution

First, the density function of a truncated normal distribution defined over $\underline{\alpha} \le \alpha \le \bar{\alpha}$ is given by

$$f(\alpha) = \frac{f_1(\frac{\alpha-\mu}{\sigma})}{\sigma \left(F_1(\frac{\bar{\alpha}-\mu}{\sigma}) - F_1(\frac{\underline{\alpha}-\mu}{\sigma}) \right)},$$

where f_1 and F_1 are the density function and the cumulative distribution function of the standard normal distribution. The parameter μ and σ are the mean and the standard deviation of the corresponding normal distribution.

Next, let us create a bimodal distribution by combining two truncated normal distributions. Let μ_R and σ_R be the parameters of the right distribution, which is defined over $\underline{\alpha} \leq \alpha \leq \bar{\alpha}$. Similarly, let μ_L and σ_L be the parameters of the left distribution, which is also defined over $\underline{\alpha} \leq \alpha \leq \bar{\alpha}$.

We examine a combination of the two truncated normal distributions, whose standard deviations are both unity, i.e., $\sigma_R = 1$ and $\sigma_L = 1$. Truncation is not necessarily symmetric. Consequently, we can determine two mean parameters, μ_R and μ_L, arbitrarily, but $\mu_L < \mu_R$ must hold. Also, we choose the weight on the left distribution, *weightL*. When *weightL* is higher, the population share of entrepreneurs with low productivity is larger. Then the bimodal distribution in this case is given by

$$f(\alpha) = weightL \frac{f_1(\alpha - \mu_L)}{F_1(\bar{\alpha} - \mu_L) - F_1(\underline{\alpha} - \mu_L)} + (1 - weightL) \frac{f_1(\alpha - \mu_R)}{F_1(\bar{\alpha} - \mu_R) - F_1(\underline{\alpha} - \mu_R)}.$$

This is indeed a density function because

$$\begin{aligned} & \int_{\underline{\alpha}}^{\bar{\alpha}} weightL \frac{f_1(\alpha - \mu_L)}{F_1(\bar{\alpha} - \mu_L) - F_1(\underline{\alpha} - \mu_L)} d\alpha + \int_{\underline{\alpha}}^{\bar{\alpha}} (1 - weightL) \frac{f_1(\alpha - \mu_R)}{F_1(\bar{\alpha} - \mu_R) - F_1(\underline{\alpha} - \mu_R)} d\alpha \\ = \; & weightL + (1 - weightL) = 1. \end{aligned}$$

9.13.5 U-shape bimodal

Consider the following U-shape bimodal distribution funtion:

$$f(\alpha) = a_1(\alpha - a_0)^2,$$

where $a_0 = \frac{\underline{\alpha} + \bar{\alpha}}{2}$ and $a_1 = \frac{12}{(\bar{\alpha} - \underline{\alpha})^3}$.

9.14 Derive the demand function for bubble assets of an L-entrepreneur

Each L-type chooses optimal amounts of b_t^{*i}, x_t^i, and z_t^{*i} so that the expected marginal utility from investing in three assets is equalized. The first order conditions with respect to x_t^i and b_t^{*i} are

$$(x_t^i): \ \frac{P_t}{c_t^{*i,\pi}} = \pi\beta\frac{P_{t+1}}{c_{t+1}^{*i,\pi}}, \tag{43}$$

$$(b_t^i): \ \frac{1}{c_t^{*i,\pi}} = \pi\beta\frac{r_t^*}{c_{t+1}^{*i,\pi}} + (1-\pi)\beta\frac{r_t^*}{c_{t+1}^{*i,(1-\pi)}}, \tag{44}$$

where $c_{t+1}^{*i,\pi} = (1-\beta)(\alpha^L z_t^{*i} - r_t^* b_t^{*i} + P_{t+1}x_t^i)$ is the date $t+1$ consumption level of entrepreneur when bubbles survive at date $t+1$, and $c_{t+1}^{*i,(1-\pi)} = (1-\beta)(\alpha^L z_t^{*i} - r_t^* b_t^{*i})$ is the date $t+1$ consumption level when bubbles collapse at date $t+1$.[37] The RHS of (43) is the gain in expected discounted utility from holding one additional unit of bubble assets at date $t+1$. With probability π bubbles survive, in which case the entrepreneur can sell the additional unit at P_{t+1}, but with probability $1-\pi$ bubbles collapse, in which case he/she receives nothing. The denominators reflect the respective marginal utilities of consumption. The RHS of (44) is the gain in expected discounted utility from lending one additional unit. It is similar to the RHS of (43), except for the fact that lending yields r_t^* at date $t+1$, irrespective of whether or not bubbles collapse.

From (4), (43), and (44), we can derive the demand function for bubble assets of an L-type, (7).

9.15 Aggregation in the bubble economy

The great merit of the expressions for each entrepreneur's investment and demand for bubble assets, z_t^{*i} and x_t^i, is that they are linear in period-t net worth, e_t^{*i}. Hence aggregation is easy: we do not need to keep track of the distributions.

From (6), we learn the aggregate H-investments in the bubble economy:

$$Z_t^{*H} = \frac{\beta p A_t^*}{1 - \dfrac{\theta\alpha^H}{r_t^*}},$$

where $A_t^* \equiv Y_t^* + P_t X$ is the aggregate wealth of entrepreneurs at date t in the bubble economy, and pA_t^* is the aggregate wealth of H-types at date t in the bubble economy.

[37]Since the entrepreneur consumes a fraction $1-\beta$ of the current net worth in each period, the optimal consumption level at date $t+1$ is independent of the entrepreneur's type at date $t+1$. It only depends on whether bubbles collapse.

From this investment function, we see that the aggregate H-investments are both history-dependent and forward-looking, because they depend on asset prices, P_t, as well as cash flows from the investment projects in the previous period, Y_t^*. In this respect, this investment function is similar to the investment function in Kiyotaki and Moore (1997). There is a significant difference. In the Kiyotaki-Moore model, the investment function depends on land prices which reflect fundamentals (cash flows from land), while in our model, it depends on bubble prices.

Aggregate L-investments depend on the level of the interest rate:

$$Z_t^{*L} = \begin{cases} \beta A_t^* - \frac{\beta p A_t^*}{1-\frac{\theta\alpha^H}{\alpha^L}} - P_t X & \text{if } \phi_t \leq L(\theta), \\ \\ 0 & \text{if } \phi_t > L(\theta). \end{cases}$$

When the bubble size is small, L-types may invest positive amount. In this case, we know from (16) that aggregate L-investments are equal to aggregate savings of the economy minus aggregate H-investments minus aggregate value of bubbles. When the bubble size becomes large, L-types do not invest.

The aggregate counterpart to (7) is

$$P_t X_t = \frac{\pi \frac{P_{t+1}}{P_t} - r_t^*}{\frac{P_{t+1}}{P_t} - r_t^*} \beta(1-p) A_t^*, \tag{45}$$

where $(1-p)A_t^*$ is the aggregate net worth of L-types at date t in the bubble economy. (45) is the aggregate demand function for bubble assets at date t.

9.16 Behavior of H-types

We verify that H-types do not buy bubble assets in equilibrium, not just on the asymptotically bubbleless path, but also in the stochastic steady-state (i.e., the balanced growth path). When the short sale constraint binds, H-types do not buy bubble assets. In order that the short sale constraint binds, the following condition must hold:

$$\frac{1}{c_t^{*i,\pi}} > \frac{\pi\beta}{c_{t+1}^{*i,\pi}} \frac{P_{t+1}}{P_t}. \tag{46}$$

Since the borrowing constraint is binding for H-types in the bubble economy, we have:

$$\frac{1}{c_t^{*i,\pi}} = \beta E_t \left[\frac{r_t^*}{c_{t+1}^{*i}} \frac{\alpha^H (1-\theta)}{r_t^* - \theta\alpha^H} \right]. \tag{47}$$

We also know that $c_{t+1}^{*i} = (1-\beta)\left[\frac{r_t^* \alpha^H (1-\theta)}{r_t^* - \theta\alpha^H}\right]$ if (46) is true. Inserting (47) into (46) yields

$$\frac{\beta}{c_{t+1}^{*i}} \left[\frac{\alpha^H (1-\theta)}{1 - \frac{\theta\alpha^H}{r_t^*}} - \pi \frac{P_{t+1}}{P_t} \right] > 0. \tag{48}$$

The first term in the bracket of (48) is the rate of return on H-projects with maximum leverage, and the second term in (48) is the expected return on bubbles. As long as the first term is strictly greater than the second term, then H-types do not buy bubbles. That is, if

$$\frac{\alpha^H (1-\theta)}{1 - \frac{\theta\alpha^H}{r_t^*}} > \pi \frac{P_{t+1}}{P_t} = \pi \frac{\phi_{t+1}}{\phi_t} \frac{1 - \beta\phi_t}{1 - \beta\phi_{t+1}} \frac{Y_{t+1}^*}{Y_t^*}, \tag{49}$$

then (48) holds.

In $\underline{\theta} < \theta \leq \theta^m$, by using (24) and (20), (49) can be written as:

$$\frac{\alpha^L \alpha^H (1-\theta)}{\alpha^L - \theta\alpha^H} > \frac{\pi\alpha^L (1 - p - \phi_t)}{\pi(1-p) - \phi_t}. \tag{50}$$

Since the right hand side is an increasing function of ϕ_t and ϕ_t in the stochastic steady-state is the maximum value of equilibrium ϕ_t, (49) holds in all bubble equilibria if (50) holds in the stochastic steady-state. In the stochastic steady-state, ϕ_t is constant, and thus (50) becomes

$$\frac{\alpha^L \alpha^H (1-\theta)}{\alpha^L - \theta\alpha^H} \frac{1 - \pi\beta}{1 - \pi\beta(1-p)} > 0. \tag{51}$$

Therefore, H-types do not buy bubbles in all bubble equilibria in $\underline{\theta} < \theta \leq \theta^m$.

Likewise, in $\theta^m \leq \theta < \pi\beta(1-p)$, by using (24) and (20), (49) can be written as:

$$\frac{\alpha^H (1-\theta)(1-\phi_t)}{p} > \frac{\pi\theta\alpha^H (1-\phi_t)}{\pi(1-p) - \phi_t},$$

which is equivalent to

$$\alpha^H(1-\phi_t)\left[\frac{(1-\theta)}{p}-\frac{\pi\theta}{\pi(1-p)-\phi_t}\right]>0. \tag{52}$$

Since the second term in the bracket is an increasing function of ϕ_t and ϕ_t in the stochastic steady-state is the maximum value of equilibrium ϕ_t, (49) holds in all bubble equilibria if (52) holds in the stochastic steady-state. In the stochastic steady-state, ϕ_t is constant, and thus (52) becomes

$$\frac{\alpha^H(1-\theta)(1-\phi)(1-\pi\beta)}{p[1-\pi\beta(1-p)]}>0. \tag{53}$$

Therefore, H-types do not buy bubbles in all bubble equilibria in $\theta^m \leq \theta < \pi\beta(1-p)$.

9.17 Derivation of Entrepreneur's Value Function

9.17.1 A Bubbleless Economy

Given the optimal decision rules, the Bellman equation can be written as:

$$V_t^{BL}(e_t)=\log c_t+\beta\left[pV_{t+1}^{BL}(R_t^H\beta e_t)+(1-p)V_{t+1}^{BL}(R_t^L\beta e_t)\right], \tag{54}$$

where $R_t^H\beta e_t$ and $R_t^L\beta e_t$ are the net worth of the entrepreneur at date $t+1$. The net worth evolves as $e_{t+1}=R_t^j\beta e_t$, where $j=H$ or L. R_t^H and R_t^L, which are given below, are the realized rate of return per unit of saving from date t to date $t+1$ in the bubbleless economy. R_t^H represents the rate of return per unit of saving in the productive state. R_t^L represents the rate of return per unit of saving in the unproductive state. Note that, irrespective of the entrepreneur's type at date t, the entrepreneur's consumption at date t, c_t, is the same, if the entrepreneur has the same amount of net worth, e_t. This is because the entrepreneur consumes a fraction $1-\beta$ of his/her date t's net worth, e_t.

We guess that the value function, $V^{BL}(e)$, is a linear function of $\log e$:

$$V_t^{BL}(e_t)=W+\vartheta\log e_t. \tag{55}$$

From the above Bellman equation and (55), applying the method of undetermined coefficients yields

$$W = \frac{1}{1-\beta}\log(1-\beta) + \frac{\beta}{(1-\beta)^2}\log(\beta) + \frac{\beta}{(1-\beta)^2}\left[p\log R_t^H + (1-p)\log R_t^L\right] \equiv W(\theta),$$

$$\vartheta = \frac{1}{1-\beta},$$

with

$$R_t^H = \begin{cases} \frac{\alpha^H(1-\theta)}{1-\frac{\theta\alpha^H}{\alpha^L}} & \text{in } 0 \leq \theta \leq \frac{\alpha^L}{\alpha^H}(1-p), \\ \frac{\alpha^H(1-\theta)}{p} & \frac{\alpha^L}{\alpha^H}(1-p) \leq \theta \leq 1-p, \\ \alpha^H & 1-p \leq \theta \leq 1. \end{cases}$$

and

$$R_t^L = \begin{cases} \alpha^L & \text{in } 0 \leq \theta \leq \frac{\alpha^L}{\alpha^H}(1-p), \\ \frac{\theta\alpha^H}{1-p} & \frac{\alpha^L}{\alpha^H}(1-p) \leq \theta \leq 1-p, \\ \alpha^H & 1-p \leq \theta \leq 1. \end{cases}$$

9.17.2 A Bubble Economy

Given the optimal decision rules, the Bellman equation can be written as:

$$\begin{aligned} V_t^{BB}(e_t^*) = \log c_t^* &+ \beta\pi\left[pV_{t+1}^{BB}(R_t^{*H}\beta e_t^*) + (1-p)V_{t+1}^{BB}(R_t^{*L}\beta e_t^*)\right] \\ &+ \beta(1-\pi)\left[pV_{t+1}^{BL}(R_t^{*H}\beta e_t^*) + (1-p)V_{t+1}^{BL}(R_t^{LL}\beta e_t^*)\right], \end{aligned}$$

where $R_t^{*H}\beta e_t^*$, $R_t^{*L}\beta e_t^*$, and $R_t^{LL}\beta e_t^*$ are the net worth of the entrepreneur at date $t+1$ in each state. R_t^{*H}, R_t^{*L}, and R_t^{LL}, which are given in the Technical Appendix, are the realized rate of return per unit of saving in each state from date t to date $t+1$. R_t^{*H} corresponds to the leveraged rate of return on H-projects per unit of saving. R_t^{*L} and R_t^{LL} are the realized rates of return per unit of saving for L-types in period t when the bubbles survive in period $t+1$ and when the bubbles collapse in period $t+1$, respectively. When the bubbles burst in period $t+1$, the realized rate of return per unit of saving decreases for entrepreneurs who were L-types in period t, because they lose all wealth they had invested in bubble assets. Thus, we have $R_t^{LL} < R_t^{*L}$. In contrast, H-types in period t would not have purchased bubble assets in period t, so the realized rate of return per unit of their saving is not affected, even if the bubble bursts.

We guess that the value function, $V^{BL}(e^*)$, is a linear function of $\log e^*$:

$$V_t^{BB}(e_t^*) = q + u \log e_t^*, \tag{56}$$

From the above Bellman equation and (56) together with

$$R_t^{*H} = \begin{cases} \frac{\alpha^H(1-\theta)}{1-\frac{\theta\alpha^H}{\alpha^L}} & \text{in } \underline{\theta} < \theta \leq \theta^m, \\ \frac{\alpha^H(1-\theta)[1-\phi(\theta)]}{p} & \text{in } \theta^m \leq \theta < \pi\beta(1-p), \end{cases}$$

and

$$R_t^{*L} = \begin{cases} \frac{\pi\alpha^L[1-p-\phi(\theta)]}{\pi(1-p)-\phi(\theta)} & \text{in } \underline{\theta} < \theta \leq \theta^m, \\ \frac{\pi\theta\alpha^H[1-\phi(\theta)]}{\pi(1-p)-\phi(\theta)} & \text{in } \theta^m \leq \theta < \pi\beta(1-p), \end{cases}$$

and

$$R_t^{LL} = \begin{cases} \frac{\alpha^L[1-p-\phi(\theta)]}{1-p} & \text{in } \underline{\theta} < \theta \leq \theta^m, \\ \frac{\theta\alpha^H[1-\phi(\theta)]}{1-p} & \text{in } \theta^m \leq \theta < \pi\beta(1-p). \end{cases}$$

Applying the method of undetermined coefficients yields

$$\begin{aligned} q = {} & \frac{\beta(1-\pi)}{1-\beta\pi}W(\theta) + \frac{1}{1-\beta\pi}\log(1-\beta) + \frac{1}{1-\beta\pi}\frac{\beta}{1-\beta}\log\beta \\ & + \frac{1}{1-\beta\pi}\frac{\beta}{1-\beta}\left[\pi J_1 + (1-\pi)J_2\right] \equiv q(\theta), \end{aligned}$$

$$u = \frac{1}{1-\beta},$$

where in $\underline{\theta} < \theta \leq \theta^m$,

$$J_1 = p\log\left[\frac{\alpha^H(1-\theta)}{1-\frac{\theta\alpha^H}{\alpha^L}}\right] + (1-p)\log\left[\frac{\pi\alpha^L[1-p-\phi(\theta)]}{\pi(1-p)-\phi(\theta)}\right],$$

$$J_2 = p\log\left[\frac{\alpha^H(1-\theta)}{1-\frac{\theta\alpha^H}{\alpha^L}}\right] + (1-p)\log\left[\frac{\alpha^L[1-p-\phi(\theta)]}{1-p}\right],$$

and in $\theta^m \leq \theta < \pi\beta(1-p)$,

$$J_1 = p\log\left[\frac{\alpha^H(1-\theta)[1-\phi(\theta)]}{p}\right] + (1-p)\log\left[\frac{\pi\theta\alpha^H[1-\phi(\theta)]}{\pi(1-p)-\phi(\theta)}\right],$$

$$J_2 = p \log \left[\frac{\alpha^H (1-\theta)[1-\phi(\theta)]}{p} \right] + (1-p) \log \left[\frac{\theta \alpha^H [1-\phi(\theta)]}{1-p} \right],$$

and $\phi(\theta)$ is given by (25).

9.18 Effects of bubble creation

We examine the effects of bubble creations in our framework. If H-types can create new bubble assets in every period, and this creation is expected by H-types, this new bubble creation would increase their wealth and investments like Martin and Ventura (2012), but this case is hard to solve analytically in our infinitely lived agents model. In this Appendix, we consider a case which can be solved analytically in our infinitely lived agents model. By the following thought experiment, we learn that in our framework, who can create new bubble assets is important for whether new bubble creations produce additional real effects. We show that if entrepreneurs holding bubble assets can create new bubble assets in every period, the rate of return on bubble assets is affected by the creation, which cancels out the creation's positive effect on entrepreneurs' wealth. Consequently, there are no additional effects of this bubble creation on the existence conditions of bubbles and the long-run economic growth rate.

Suppose that entreprenuers who purchased x_t^i units of bubble assets at date t can create new bubble assets by μx_t^i at date $t+1$. This new bubble creation is assumed to be proportional of holding bubble assets so that the model can be solved analytically. This is the only different point from the model analyzed in the main text. A fraction p of the entrepreneurs who can create new bubble assets at date $t+1$ becomes productive agents, and they can directly increase their net worth by the creation of new bubbles.

Let us derive the demand equation of bubble assets in this case. As in the main text, each L-type chooses optimal amounts of b_t^{*i}, x_t^i, and z_t^{*i} so that the expected marginal utility from investing in three assets is equalized. The first order conditions with respect to x_t^i and b_t^{*i} are

$$(x_t^i): \ \frac{P_t}{c_t^{*i,\pi}} = \pi\beta \frac{(1+\mu)P_{t+1}}{c_{t+1}^{*i,\pi}}, \tag{57}$$

$$(b_t^i): \ \frac{1}{c_t^{*i,\pi}} = \pi\beta \frac{r_t^*}{c_{t+1}^{*i,\pi}} + (1-\pi)\beta \frac{r_t^*}{c_{t+1}^{*i,(1-\pi)}}, \tag{58}$$

where $c_{t+1}^{*i,\pi} = (1-\beta)(\alpha^L z_t^{*i} - r_t^* b_t^{*i} + (1+\mu)P_{t+1}x_t^i)$ is the date $t+1$ consumption level of the entrepreneur when bubbles survive at date $t+1$, and $c_{t+1}^{*i,(1-\pi)} = (1-\beta)(\alpha^L z_t^{*i} - r_t^* b_t^{*i})$ is the date $t+1$ consumption level when bubbles collapse at date $t+1$. The RHS of (57) is the gain in expected discounted utility from holding one additional unit of bubble assets at date $t+1$. With probability π bubbles survive, in which case the entrepreneur can sell the additional unit and the new bubble assets at P_{t+1}, but with probability $1-\pi$ bubbles collapse, in which case he/she receives nothing. The denominators reflect the respective marginal utilities of consumption. The RHS of (58) is the gain in expected discounted utility from lending one additional unit. It is similar to the RHS of (57), except for the fact that lending yields r_t^* at date $t+1$, irrespective of whether or not bubbles collapse.

From (4), (57), and (58), we can derive the demand function for bubble assets of an L-type:

$$P_t x_t^i = \frac{\pi(1+\mu)\frac{P_{t+1}}{P_t} - r_t^*}{(1+\mu)\frac{P_{t+1}}{P_t} - r_t^*}\beta e_t^{*i} \tag{59}$$

When we aggregate (59), and solve for P_{t+1}/P_t, we obtain the required rate of return on bubble assets:

$$\frac{P_{t+1}}{P_t} = \frac{r_t^*}{1+\mu}\frac{(1-p-\phi_t)}{\pi(1-p)-\phi_t}, \tag{60}$$

where $r_t^* = \frac{\theta\alpha^H(1-\phi_t)}{1-p-\phi_t}$. From (60), we learn that the required rate of return on a unit of bubble assets, $\frac{P_{t+1}}{P_t}$, is a decreasing function of the new bubble creation, μ. This means that, the more entrepreneurs can create new bubbles, the more the required rate of return on bubbles decreases.

Because of the new bubble creation, the aggregate supply of bubble assets grows at the rate of μ, i.e., $X_{t+1} = (1+\mu)X_t$. By considering this relation, the evolution of ϕ given by (21) changes into

$$\phi_{t+1} = \frac{(1+\mu)\frac{P_{t+1}}{P_t}}{\frac{A_{t+1}}{A_t}}\phi_t. \tag{61}$$

The growth rate of the aggregate wealth in the bubble economy given by (23) changes into

$$\frac{A_{t+1}^*}{A_t^*} = \begin{cases} \beta\{\alpha^H(1-L(\theta)) + \alpha^L(L(\theta)-\phi_t) + \dfrac{P_{t+1}}{P_t}(1+\mu)\phi_t\} & \text{if } \phi_t \leq L(\theta), \\ \\ \beta\{\alpha^H(1-\phi_t) + \dfrac{P_{t+1}}{P_t}(1+\mu)\phi_t\} & \text{if } \phi_t \geq L(\theta). \end{cases} \tag{62}$$

From (62), we learn that there are two competing effects of the new bubble creation on the growth rate of the aggregate wealth. One effect is that the new bubble creation captured by μ increases the growth rate of the aggregate wealth because, if other things being equal, the new bubble creation itself directly increases the entrepreneurs' wealth. The other effect is that since entrepreneurs anticipate that they can create new bubble assets, the required rate of return captured by $\frac{P_{t+1}}{P_t}$ decreases, which completely cancels out the positive effects of the bubble creation on the aggregate wealth. Moreover, since the required rate of return on bubbles decreases, $(1+\mu)\frac{P_{t+1}}{P_t}$ in (61) remains the same as the analyses in the main text. As a consequence, ϕ on the balanced growth path is not affected by this type of the bubble creation. This means that the aggregate value of bubbles remains unchanged by this type of the bubble creation. Therefore, there are no additional effects of this bubble creation on the existence conditions of bubbles and the long-run economic growth rate.

9.19 Bubbles as Collateral

In this technical appendix, we consider a case where a fraction θ^x of the expected return from bubble assets can be used as collateral as well as a fraction θ of the return from investment. We will show below that even in this case, if θ^x is sufficiently small, H-types do not purchase bubble assets in equilibrium.

In this case, the borrowing constraint can be written as:

$$r_t b_t^i \leq \theta \alpha_t^i z_t^i + \theta^x \pi P_{t+1},$$

As we showed in the "Behavior of H-types" in the Technical Appendix, when the short sale constraint binds, H-types do not buy bubble assets. In order that the short sale constraint binds, the following condition must hold:

$$\frac{1}{c_t^{*i,\pi}} > E_t\left[\frac{r_t^*}{c_{t+1}^{*i}}\frac{\alpha^H - r_t^*}{r_t^* - \theta\alpha^H}\right] + \frac{\pi\beta}{c_{t+1}^{*i,\pi}}\frac{P_{t+1}}{P_t}. \tag{63}$$

Since the borrowing constraint is binding for H-types, we have (47). We also know that $c_{t+1}^{*i} = (1-\beta)\left[\frac{r_t^*\alpha^H(1-\theta)}{r_t^* - \theta\alpha^H}\right]$ if (63) is true. Inserting (47) into (63) yields

$$\frac{\beta}{c_{t+1}^{*i}}\left[\frac{\alpha^H(1-\theta)}{1-\frac{\theta\alpha^H}{r_t^*}} - \pi\frac{P_{t+1}}{P_t} - \frac{\left(\frac{\alpha^H}{r_t} - 1\right)}{1-\frac{\theta\alpha^H}{r_t^*}}\theta^x\pi\frac{P_{t+1}}{P_t}\right] > 0. \tag{64}$$

The first term in the bracket of (64) is the rate of return on H-projects with maximum leverage, and the second term in (64) is the expected return on bubbles, and the third term is the effects of bubbles that relaxes the borrowing constraint. As long as the first term is strictly greater than the second-and- third terms, the short sale constraint binds. We will investigate (64) below in the case of deterministic and stochastic bubbles, respectively.

9.19.1 Deterministic Bubbles ($\pi = 1$)

First, let us consider deterministic bubbles ($\pi = 1$). In this case, whether (64) holds depends on

$$\frac{\alpha^H(1-\theta)}{1-\frac{\theta\alpha^H}{r_t^*}} - \frac{P_{t+1}}{P_t} - \frac{\left(\frac{\alpha^H}{r_t} - 1\right)}{1-\frac{\theta\alpha^H}{r_t^*}}\theta^x\frac{P_{t+1}}{P_t} > 0. \tag{65}$$

In the case of deterministic bubbles, $r_t^* = \frac{P_{t+1}}{P_t}$ holds in equilibrium, i.e., the rate of return on bubbles must equal the interest rate. Then, (65) can be written as:

$$\frac{r_t^*(\alpha^H - r_t^*)(1-\theta^x)}{r_t^* - \theta\alpha^H} > 0.$$

Therefore, as long as $\theta^x < 1$, the short sale constraint binds for H-types, and therefore H-types do not buy bubbles.

9.19.2 Stochastic Bubbles ($\pi < 1$)

Next, we consider stochastic bubbles ($\pi < 1$). In this case, whether (64) holds depends on

$$\frac{\alpha^H(1-\theta)}{1-\frac{\theta\alpha^H}{r_t^*}} - \pi\frac{P_{t+1}}{P_t} > \frac{\left(\frac{\alpha^H}{r_t} - 1\right)}{1-\frac{\theta\alpha^H}{r_t^*}}\theta^x\pi\frac{P_{t+1}}{P_t}. \tag{66}$$

By using (51) and (53), on the balanced growth path, (66) can be written as:

$$\begin{aligned} \frac{\alpha^L\alpha^H(1-\theta)}{\alpha^L-\theta\alpha^H}\frac{1-\pi\beta}{1-\pi\beta(1-p)} &> \frac{\left(\alpha^H-\alpha^L\right)}{\alpha^L-\theta\alpha^H}\theta^x\pi g_t^*, \text{ in } \underline{\theta} < \theta \leq \theta^m, \\ \frac{\alpha^H(1-\theta)(1-\phi)(1-\pi\beta)}{p[1-\pi\beta(1-p)]} &> \frac{\left(\alpha^H-r^*\right)}{r^*-\theta\alpha^H}\theta^x\pi g_t^*, \text{ in } \theta^m \leq \theta < \pi\beta(1-p), \end{aligned} \tag{67}$$

where g_t^* is given by (26). From (67), in both regions of $\underline{\theta} < \theta \leq \theta^m$ and $\theta^m \leq \theta < \pi\beta(1-p)$, we learn that as long as θ^x is small enough so that (67) is satisfied, then the short sale constraint binds for H-types, and therefore H-types do not buy bubbles.

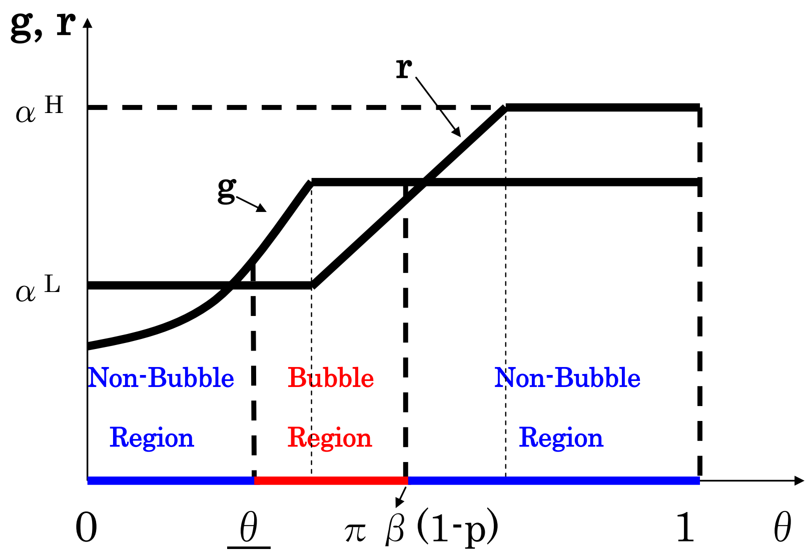


Figure 1: Bubble Region and θ

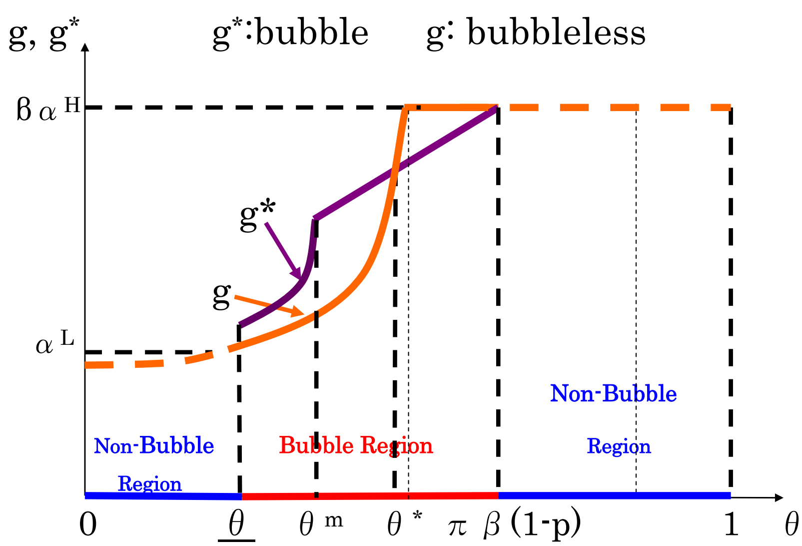


Figure 2: Bubbles and Economic Growth

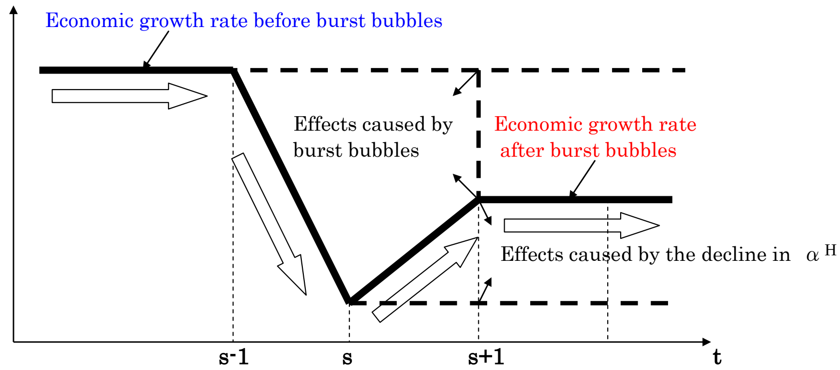


Figure 3-1: Effects of bubble bursts in relatively low θ

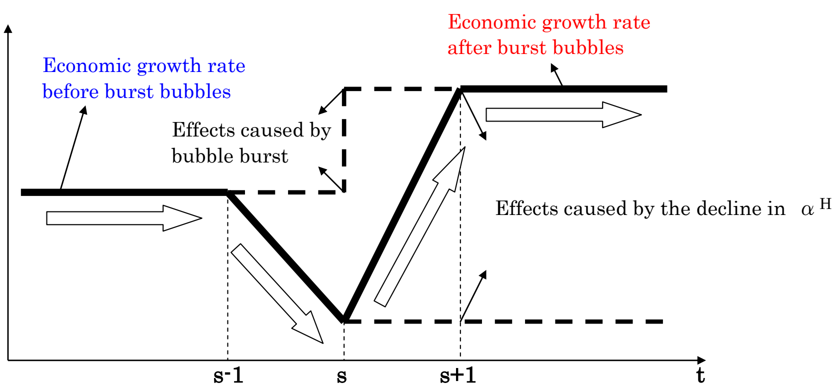


Figure 3-2: Effects of bubble bursts in relatively high θ

Figure 4-1

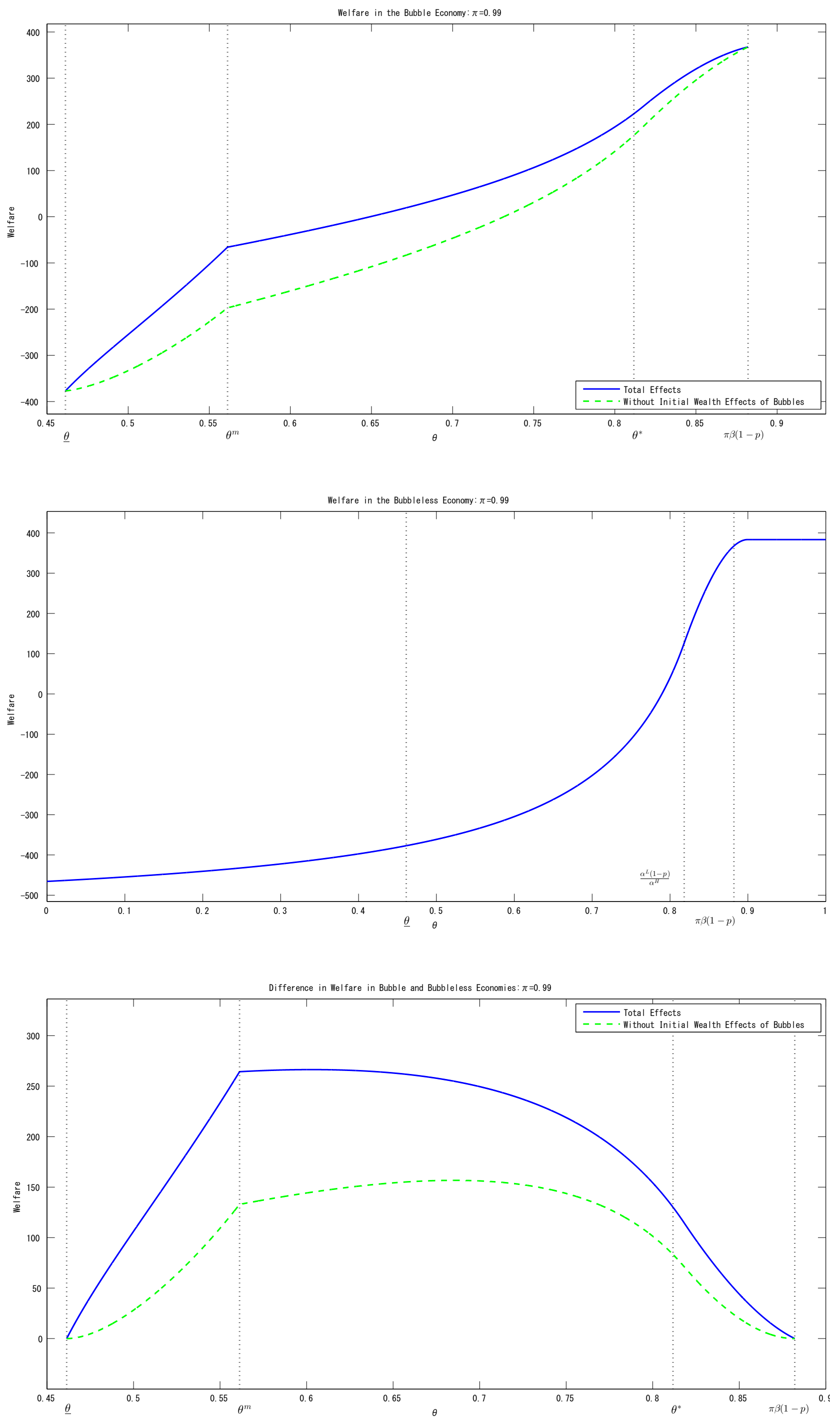

Welfare in the Bubble Economy: π=0.99
Welfare
θ
$\underline{\theta}$
θ^m
θ^*
$\pi\beta(1-p)$
Total Effects
Without Initial Wealth Effects of Bubbles
Welfare in the Bubbleless Economy: π=0.99
Welfare
θ
$\underline{\theta}$
$\frac{\alpha^L(1-p)}{\alpha^H}$
$\pi\beta(1-p)$
Difference in Welfare in Bubble and Bubbleless Economies: π=0.99
Welfare
θ
$\underline{\theta}$
θ^m
θ^*
$\pi\beta(1-p)$
Total Effects
Without Initial Wealth Effects of Bubbles

Figure 4-2

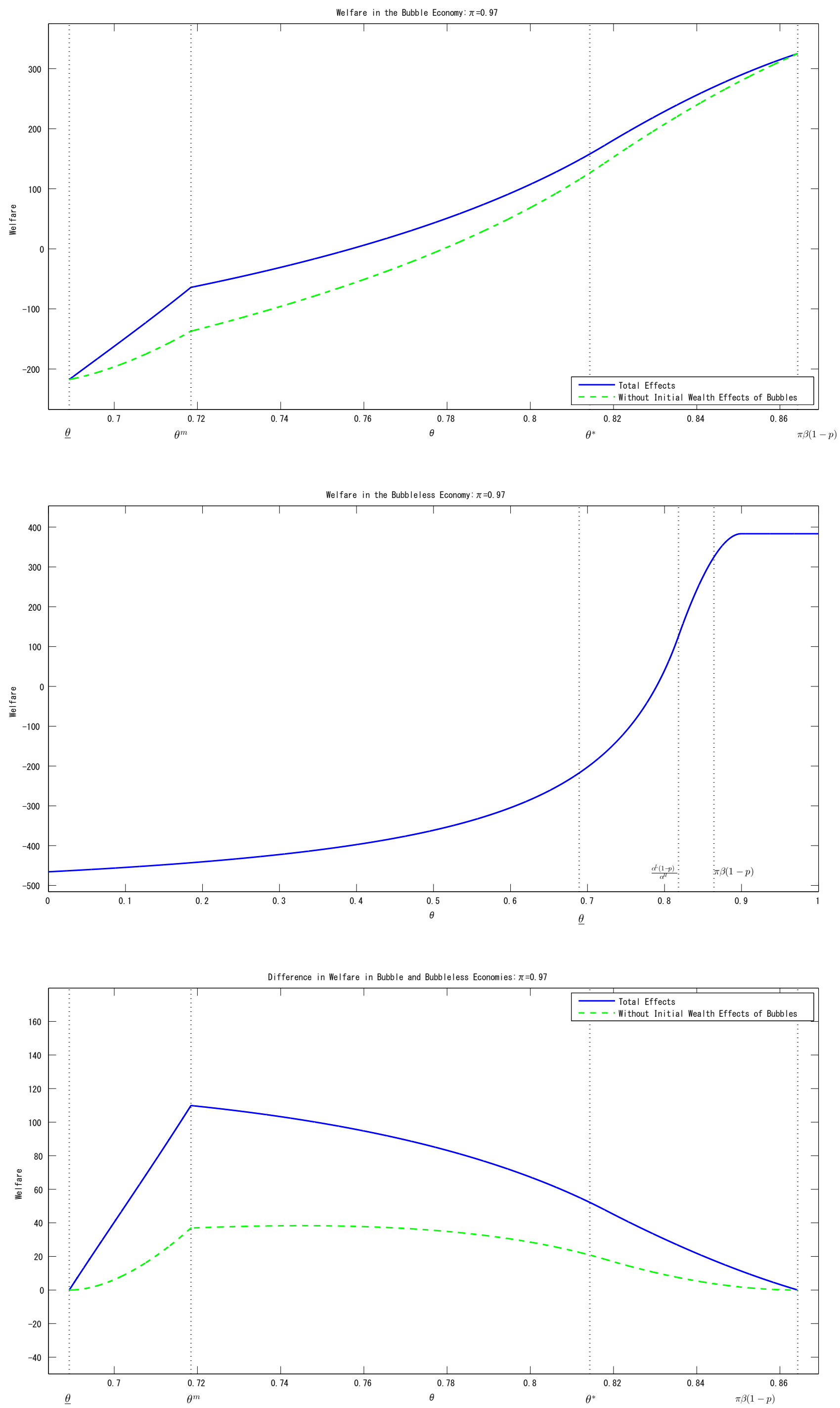
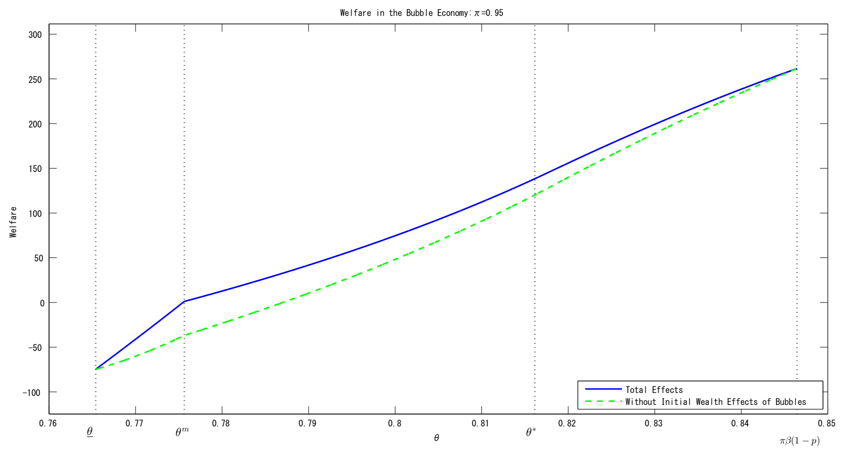
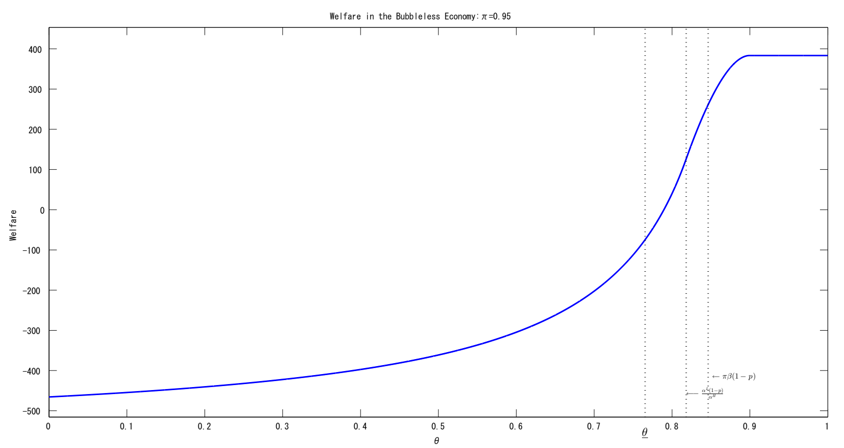

Welfare in the Bubble Economy: π=0.97
Welfare
300
200
100
0
−100
−200
0.7
0.72
0.74
0.76
0.78
0.8
0.82
0.84
0.86
$\underline{\theta}$
θ^m
θ
θ^*
$\pi\beta(1-p)$
Total Effects
Without Initial Wealth Effects of Bubbles
Welfare in the Bubbleless Economy: π=0.97
Welfare
400
300
200
100
0
−100
−200
−300
−400
−500
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
θ
$\underline{\theta}$
$\pi\beta(1-p)$
Difference in Welfare in Bubble and Bubbleless Economies: π=0.97
Welfare
160
140
120
100
80
60
40
20
0
−20
−40
0.7
0.72
0.74
0.76
0.78
0.8
0.82
0.84
0.86
$\underline{\theta}$
θ^m
θ
θ^*
$\pi\beta(1-p)$
Total Effects
Without Initial Wealth Effects of Bubbles

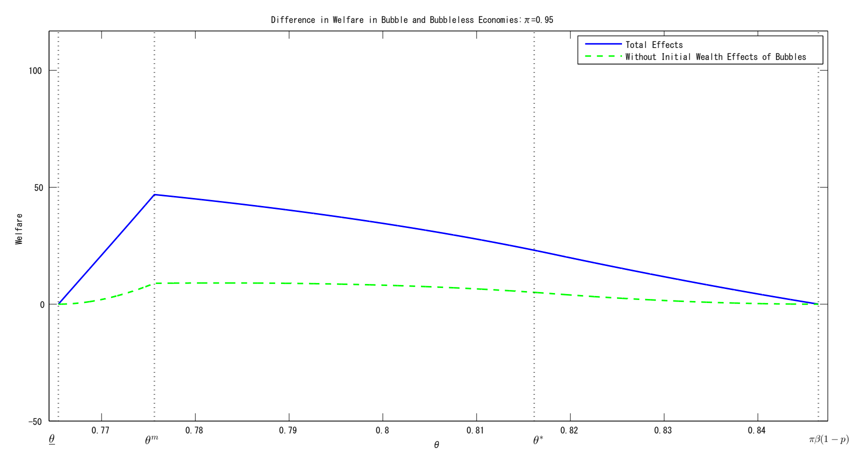
Figure 4-3



Welfare in the Bubble Economy: π=0.95
Welfare
θ
$\underline{\theta}$
θ^m
θ^*
$\pi\beta(1-p)$
Total Effects
Without Initial Wealth Effects of Bubbles

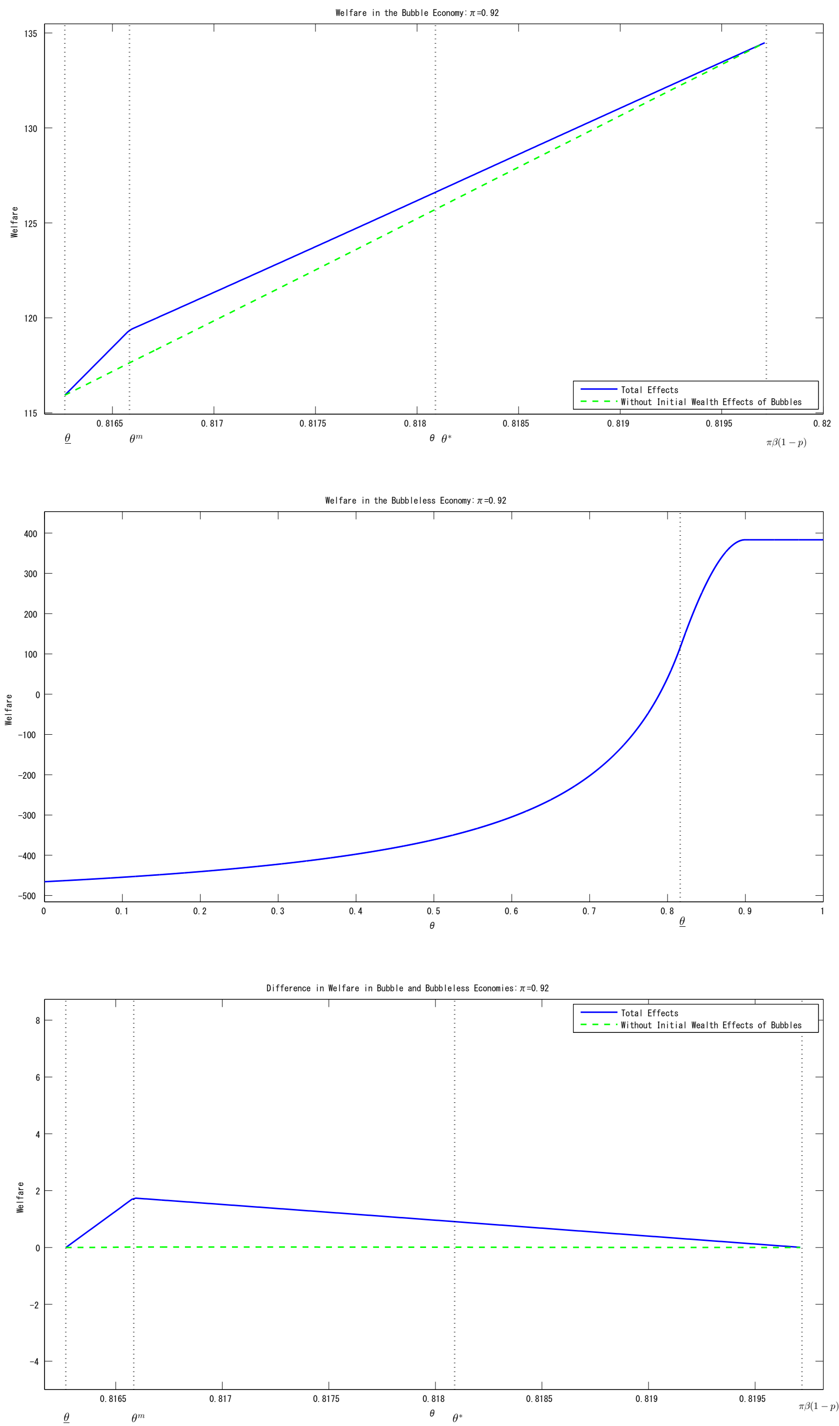


Welfare in the Bubbleless Economy: π=0.95
Welfare
θ
$\underline{\theta}$
$\leftarrow \pi\beta(1-p)$



Difference in Welfare in Bubble and Bubbleless Economies: π=0.95
Welfare
θ
$\underline{\theta}$
θ^m
θ^*
$\pi\beta(1-p)$
Total Effects
Without Initial Wealth Effects of Bubbles

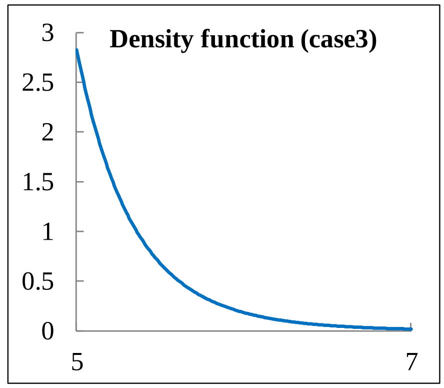
Figure 4-4



Note: Since $\frac{\alpha^L}{\alpha^H}(1-p)$ is close to $\pi\beta(1-p)$, $\frac{\alpha^L}{\alpha^H}(1-p)$ does not appear in the middle graph.

Power law distribution

	Parameter values						
	case1	case2	case3	case4	case5	case6	case7
β	0.97	0.95	0.93	0.9	0.85	0.82	0.8
$\underline{\alpha}$	2.3	3	5	9.7	14	14	14.5
$\overline{\alpha}$	2.7	3.7	7	15	26	28	31
η	35	27	14	10	6	5	4.5
$\underline{\theta}$	0.852	0.902	0.5	0.602	0.178	0.428	0.41
$\overline{\theta}$	0.958	0.93	0.902	0.854	0.78	0.702	0.686



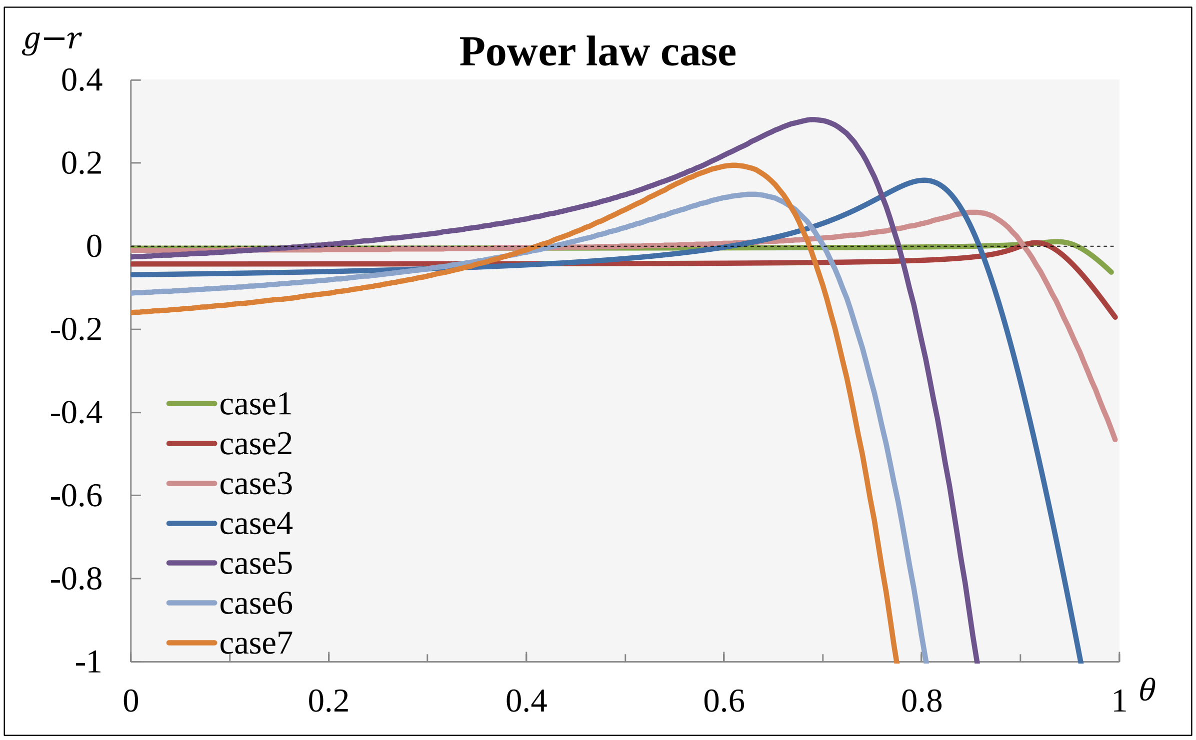
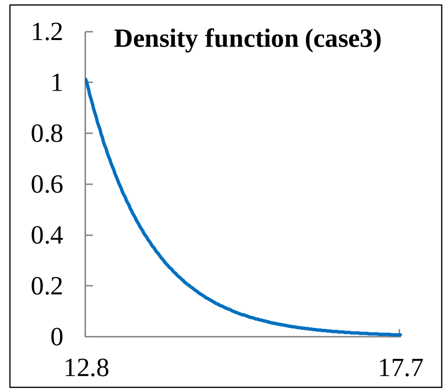


Figure 5-1

Exponential distribution

Parameter values							
	case1	case2	case3	case4	case5	case6	case7
β	0.97	0.95	0.93	0.9	0.85	0.82	0.8
$\underline{\alpha}$	15.25	14	12.8	11.8	10.5	9.2	9.38
$\overline{\alpha}$	17.7	17.7	17.7	17.5	18	18	20.4
λ	2.45	1.49	1.005	0.92	0.7	0.55	0.42
$\underline{\theta}$	0.926	0.874	0.79	0.818	0.748	0.694	0.614
$\overline{\theta}$	0.958	0.93	0.89	0.858	0.794	0.746	0.726



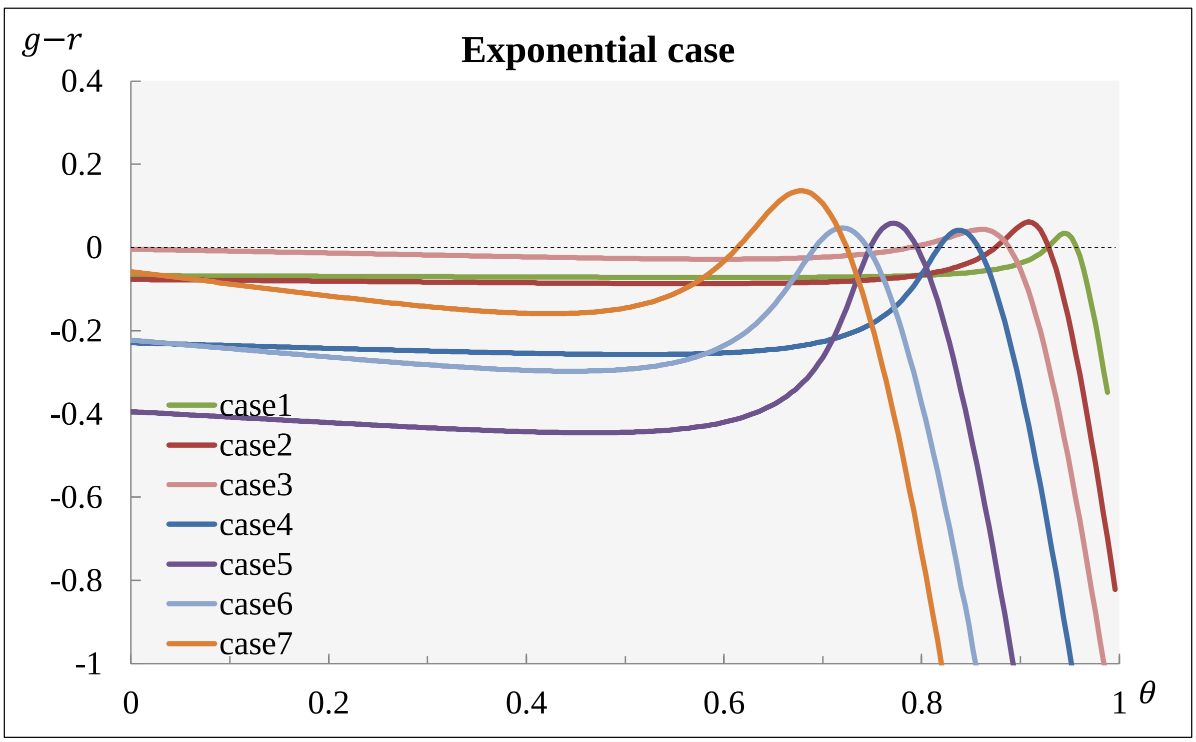
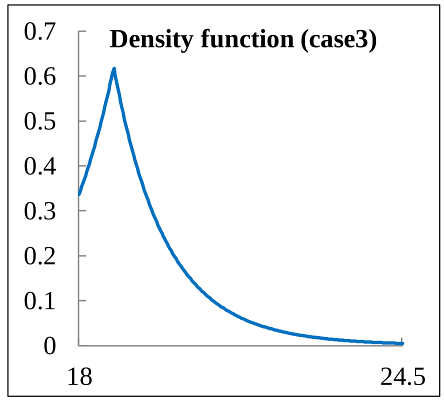


Figure 5-2

Double power law distribution

	Parameter values						
	case1	case2	case3	case4	case5	case6	case7
β	0.97	0.95	0.93	0.9	0.85	0.82	0.8
$\underline{\alpha}$	24.5	20	18	15	10.1	8.5	7.25
$\overline{\alpha}$	28.5	24.85	24.5	22.5	17.9	16.5	15.2
η	36.5	26.9	17	13	8.7	7.8	6.9
α^*	24.62	20.4	18.7	16	11.2	9.75	8.5
$\underline{\theta}$	0.93	0.898	0.854	0.822	0.75	0.71	0.666
$\overline{\theta}$	0.942	0.93	0.894	0.854	0.782	0.754	0.734



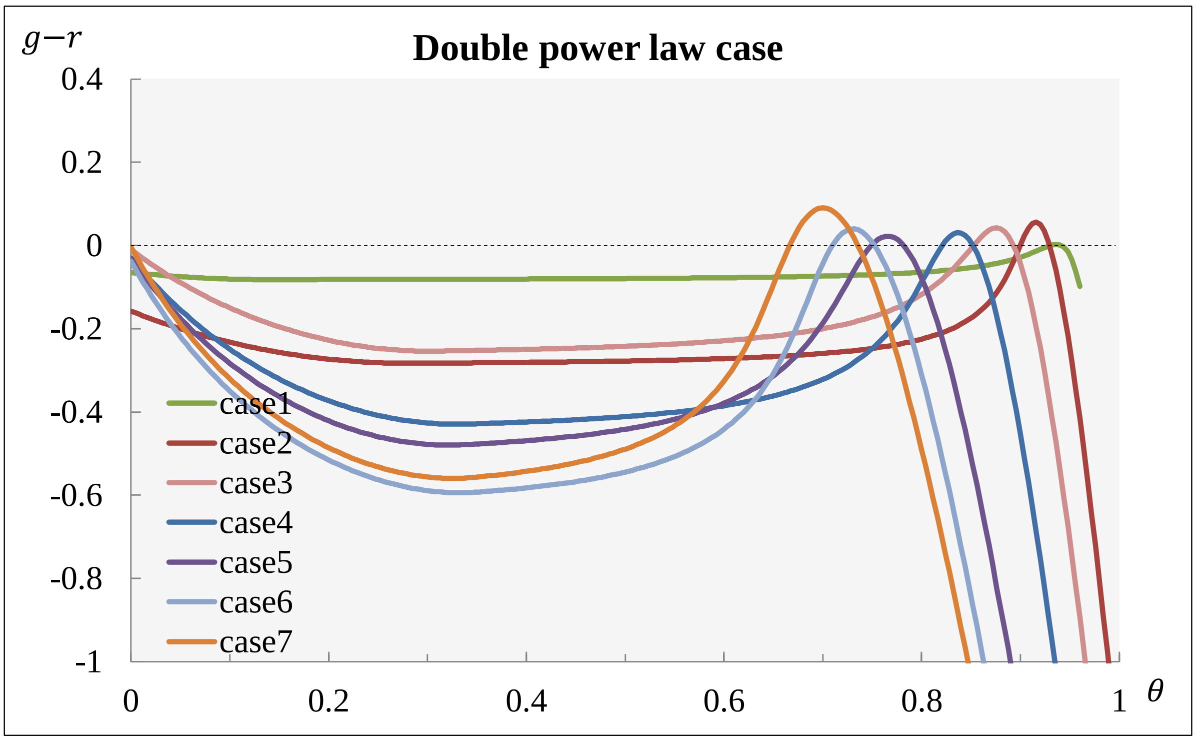
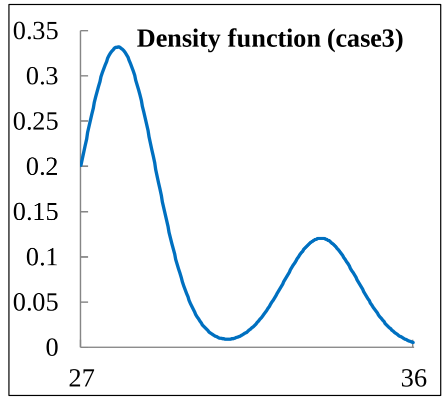


Figure 5-3

Bimodal distribution

	Parameter values						
	case1	case2	case3	case4	case5	case6	case7
β	0.96	0.93	0.9	0.9	0.88	0.88	0.8
$\underline{\alpha}$	51	42	27	34	22	25	12
$\overline{\alpha}$	58	51	36	44	31	34	21
μ_L	51	43	28	34	23	25	12
μ_R	56.5	48.5	33.5	41	28.5	31.5	18.5
$weightL$	0.8	0.75	0.7	0.7	0.7	0.6	0.7
$\underline{\theta}$	0.462	0.602	0.422	0.438	0.418	0.194	0.238
$\overline{\theta}$	0.77	0.674	0.614	0.626	0.598	0.526	0.558



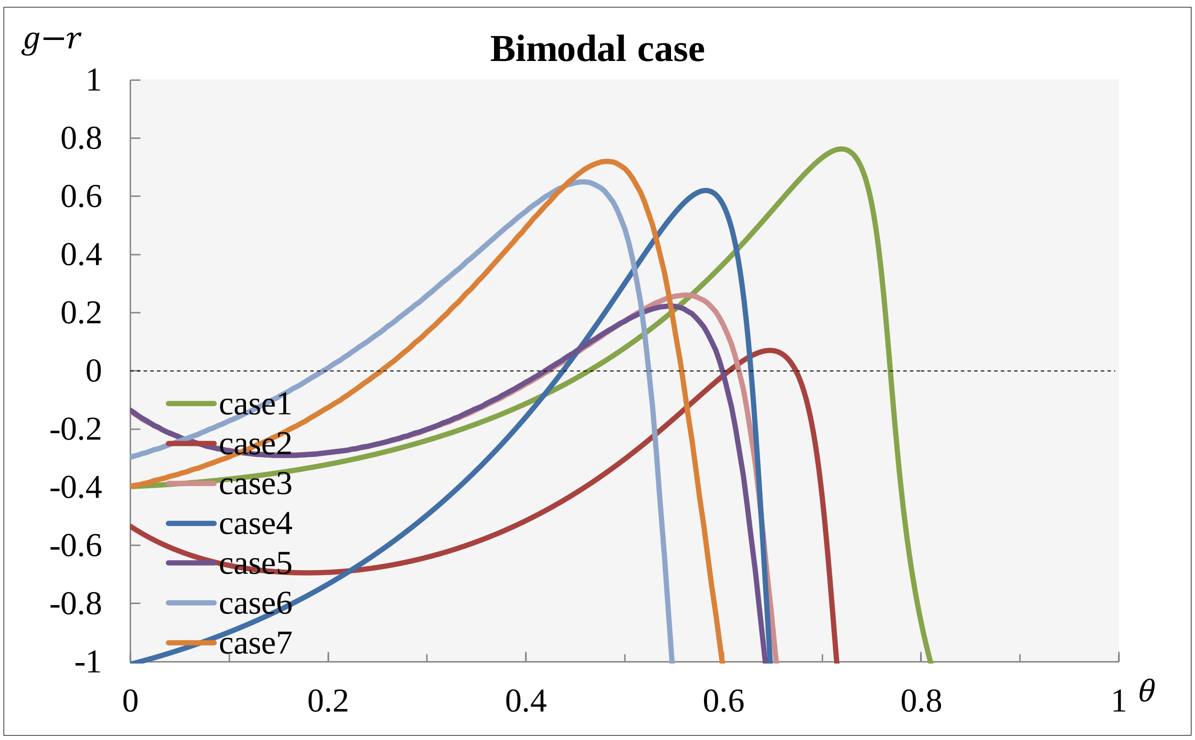
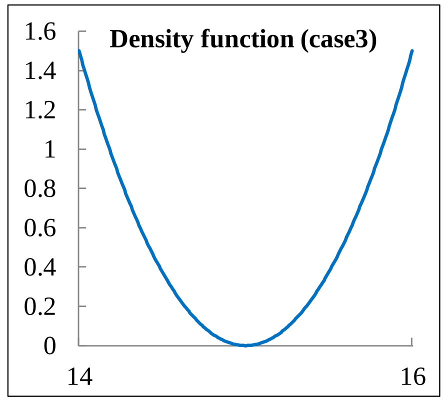


Figure 5-4

U-shape distribution

	Parameter values						
	case1	case2	case3	case4	case5	case6	case7
β	0.96	0.95	0.93	0.9	0.85	0.82	0.8
$\underline{\alpha}$	14.8	14.5	14	13.3	12	11.3	10.8
$\overline{\alpha}$	16	16	16	16	16	16	16
$\underline{\theta}$	0.078	0.05	0.142	0.254	0.158	0.15	0.106
$\overline{\theta}$	0.462	0.458	0.434	0.378	0.366	0.342	0.338



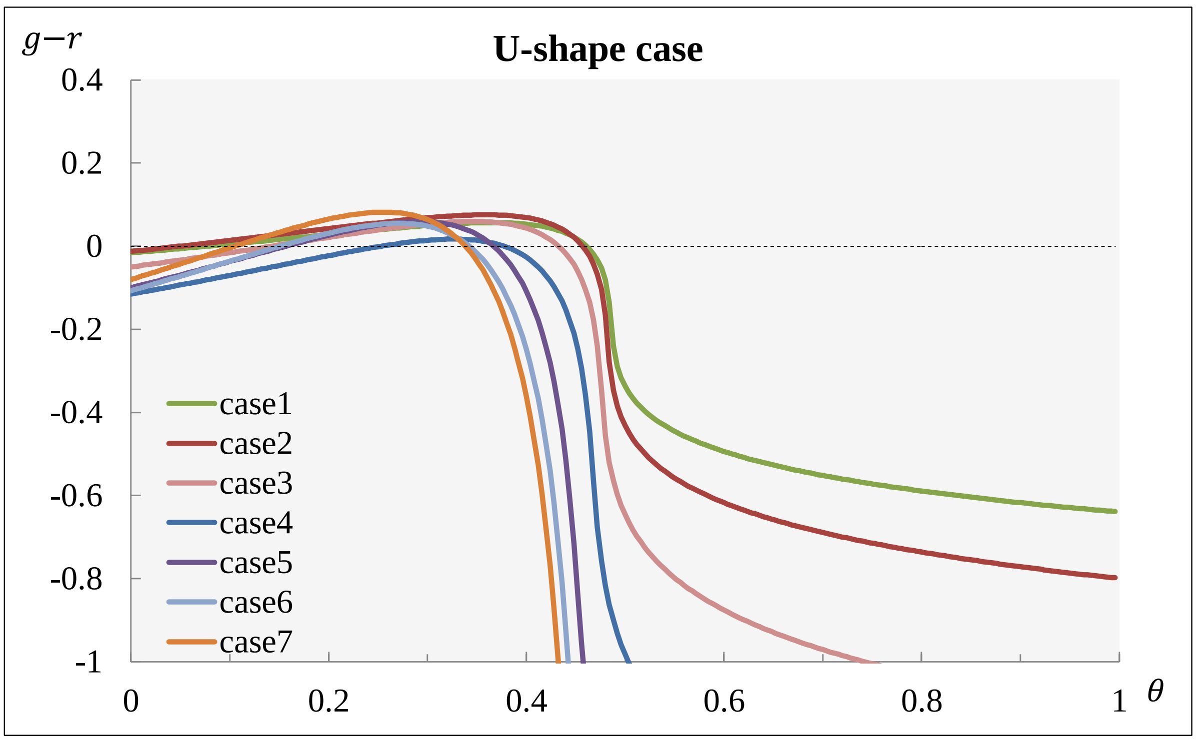


Figure 5-5