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Exchange Rates and Fundamentals: A General Equilibrium Exploration

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Abstract
Engel and West (2005) claim that the observed near random-walk behavior of nominal exchange rates is an equilibrium outcome of a present-value model of a partial equilibrium asset approach when economic fundamentals follow exogenous first-order integrated processes and the discount factor approaches one. Subsequent empirical studies further confirm this proposition by estimating discount factors close to one under distinct identification schemes. In this paper, I argue that the unit market discount factor creates a theoretical trade-off within a neoclassical, two-country, incomplete-market monetary model; on the one hand, the unit discount factor generates near random-walk nominal exchange rates, while, on the other hand, it counterfactually implies perfect consumption risk sharing as well as flat money demand. Bayesian posterior simulation exercises based on post-Bretton Woods data from Canada and the United States reveal difficulties in reconciling the equilibrium random-walk proposition within the canonical model; in particular, the market discount factor is identified as being much smaller than one.

Key Words: Exchange rate; Present-value model; Economic fundamental; Random walk; Two-country model; Incomplete market; Cointegrated TFPs; Perfect risk sharing.

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1. Introduction

Few equilibrium models for nominal exchange rates systematically beat a naïve random-walk counterpart in terms of out-of-sample forecast performance. Since the study of Meese and Rogoff (1983), this robust empirical property of nominal exchange rate fluctuations has stubbornly resisted theoretical challenges to understand the behavior of nominal exchange rates as an equilibrium outcome. Recently developed open-economy dynamic stochastic general equilibrium (DSGE) models also suffer from this problem. Infamous as the disconnect puzzle, open-economy DSGE models fail to generate random-walk nominal exchange rates along an equilibrium path because their exchange rate forecasts are closely related to other macroeconomic fundamentals.

Can we understand near random-walk movements of nominal exchange rates through the lens of solid macroeconomics? To tackle this question, Engel and West (2005, hereafter EW) establish the near random-walk behavior of nominal exchange rates within a partial equilibrium asset approach.\(^1\) Their model implies that equilibrium nominal exchange rates are given as the present discounted values of the expected future economic fundamentals. If economic fundamentals are integrated of order one (hereafter I(1)) and the discount factor approaches one, a nominal exchange rate then follows a near random-walk process in equilibrium.\(^2\)

Because the assumed non-stationarity of economic fundamentals seems to hold without question, subsequent studies within the literature have focused on the empirical validity of the assumption that the discount factor is close to one. Examining data on different currencies and spanning distinct sample periods, Sarno and Sojli (2009) and Balke et al. (2013, hereafter BMW) identify a discount factor based on partial equilibrium asset models similar to that of EW and infer that the estimated discount factor is indeed distributed near to one. In particular, the Bayesian unobserv-

\(^1\)Engel (2014) provides the most recent survey on past studies on nominal exchange rates.

\(^2\)This equilibrium random-walk property is attributable to the fact that only the Beveridge-Nelson trend components in the I(1) economic fundamentals are reflected in present-value calculation at the limit of the unit discount factor. Because the Beveridge-Nelson permanent component is a random walk, the current economic fundamentals lack the power to forecast future depreciation rates even along an equilibrium path. Nominal exchange rates, therefore, need to Granger-cause future economic fundamentals, not vice versa. The empirical exercises of EW based on vector autoregressions (VARs) provide solid evidence for this implication of Granger-causality across different currencies. The cross-sectional and panel regressions by Sarno and Schmeling (2014) also confirm the hypothesis that nominal exchange rates have predictive power for nominal economic fundamentals.
able component (UC) model of BMW estimates money demand shocks as a dominant underlying driver of a long sample of the British pound/U.S. dollar rate. This empirical fact supports the conjecture of EW that persistent unobservable economic fundamentals play a significant role in near random-walk nominal exchange rates.

Nason and Rogers (2008, hereafter NR) attempt to generalize EW’s proposition more rigorously and preserve the random-walk property of nominal exchange rates within a neoclassical two-country monetary dynamic stochastic general equilibrium (DSGE) model that includes incomplete international financial markets. NR rely only on a subset of the first-order necessary conditions (FONCs) of the proposed two-country model to construct the present value model (DSGE-PVM) of nominal exchange rates with the steady state market rate as the corresponding discount factor. In their DSGE-PVM, an equilibrium nominal exchange rate is given as the present discounted values of the expected future values of fundamentals that consist of cross-country consumption and money supply differentials. As claimed in EW, if these fundamentals are I(1), the nominal exchange rate behaves like a near random-walk at the limit of the unit market discount factor.

Utilizing the cross-equation restrictions (CERs) of the DSGE-PVM and specifying the exogenous I(1) processes of the economic fundamentals, NR estimate a restricted UC model for the bilateral exchange rate between Canada and the United States. Their Bayesian posterior inferences using post-Bretton Woods data confirm EW’s proposition, finding that the market discount factor is close to one. Moreover, they observe that permanent shocks to the consumption and money supply differentials dominate the historical movements of the bilateral exchange rate.³

In this paper, I go beyond the theoretical and empirical achievements of NR. My challenge of reconciling random-walk exchange rates within a two-country general equilibrium model begins with three arguments towards NR’s empirical exercise based on the DSGE-PVM. First, NR construct their DSGE-PVM by taking the log-linear approximations of the stochastically de-trended FONCs around the stable, deterministic, steady state of the model. The incompleteness of the international

³This empirical result is consistent with the argument known as the PPP puzzle (Rogoff 1996) because, by incorporating price stickiness, many open-economy DSGE models emphasize the role of mean-reverting monetary policy shocks as the main force driving nominal exchange rates.
financial market in their two-country model, in which only state non-contingent bonds are traded by representative households across the two countries, might lead endogenous variables to exhibit permanent unit-root dynamics. In this case, there is no guarantee that a stable, deterministic, steady state will exist.\footnote{See the detailed discussions of Ghironi (2006) and Boileau and Normandin (2008) regarding the non-stationarity problem inherent to incomplete asset market models.}

Second, NR’s specification of an I(1) consumption differential is inconsistent with a balanced growth path of the two-country model endowed with a single consumption good. The source of the non-stationary consumption differential is their presumption that the cross-country differential in the total factor productivity (TFP) is $I(1)$. NR stochastically detrend each country’s endogenous variables with the country’s own TFP. The de-trended market-clearing condition of a single consumption good, which is equivalent to the de-trended resource constraint for the global economy, then depends on the two-country TFP differential. In this case, the non-stationary TFP differential makes the de-trended resource constraint violate the balanced growth restriction.

Finally, the third argument is that NR omit the Euler equations for the optimal intertemporal consumption allocations of the two countries and treat the consumption differential as an exogenous I(1) random variable. This argument is particularly relevant once I recognize that each country’s consumption is determined by the permanent income hypothesis (PIH) and depends substantially on the I(1) endowment and the size of the market discount factor as well. The joint determination of nominal exchange rates and economic fundamentals within a single two-country model, hence, might lead to a statistical inference on the discount factor that is sharply different from those in the past studies. In fact, to my best knowledge, no past study in this literature of an equilibrium foundation of random-walk nominal exchange rates has taken into consideration the endogeneity of economic fundamentals when estimating the market discount factor.

To address the above three arguments, I investigate a neoclassical, two-country, single-good endowment economy model in which incomplete international financial markets are utilized as a device for intertemporal consumption-smoothing. The model used in this paper is quite stylized but similar to that of NR except with regard to two important aspects. The first is that the
model contains a debt-elastic risk premium. As characterized by Schmitt-Grohé and Uribe (2003) in a small open-economy model and Boileu and Normandin (2008) in a two-country international business cycle model, a debt-elastic risk premium has served as a popular instrument to induce the stationarity of the net foreign asset distribution.\(^5\)

The second aspect that differentiates this paper’s model from that of NR is that the stochastic trends in both countries appear to be independent in the short run but comove in the long run. In this model, the exogenous endowment processes of the two countries consist of both permanent and transitory components. I then allow the stochastic trends of the two countries, which are interpreted as their TFPs, to be cointegrated, as emphasized in recent papers by Mandelman et al. (2011), Rabanal et al. (2011), and Ireland (2013) in the context of international business cycles. In this case, because the TFP differential is stationary in population, a balanced growth path is guaranteed to exist in equilibrium.\(^6\)

Harnessing all the FONCs of the model to endogenously determine the nominal exchange rate along the unique equilibrium path, I theoretically show that the expected equilibrium currency return is characterized by a linear function of the de-trended net foreign asset position and other transitory components. When the market discount factor approaches one, this dependence of the expected currency return on the transitory components of the model vanishes asymptotically. Therefore, the near random-walk property of the equilibrium exchange rate indeed holds even after the two-country model is closed suitably. Importantly, the model generates a tractable approximated analytical solution of equilibrium random-walk exchange rates in cases with two symmetric countries. The resulting closed-form solution reveals that the exchange rate is primarily driven by a permanent shock to the money supply differential, among other stationary shocks. This stringent theoretical prediction echoes the findings of NR. However, in contrast to the claim of NR, a permanent but cointegrated TFP shock cannot be a significant driver of the random-walk nominal

\(^{5}\)A non-exhaustive list of studies that adopt a debt-elastic risk premium as a device to avoid the non-stationarity problem in open-economy DSGE models includes Nason and Rogers (2006), Adolffson et al. (2007), Kano (2009), Justiniano and Preston (2010), García-Cicco et al. (2010), and Bodenstein (2011).

\(^{6}\)It is worth noting that if the speed of technological diffusion the cointegration restriction reflects is set sufficiently slow, the TFP differential is empirically identified as an I(1) process with a finite sample. This conjecture is consistent with the empirical finding of NR that a unit root in the cross-country consumption differential cannot be rejected.
exchange rate because the TFP differential should be stationary to close the two-country model.

In addition, the investigation in this paper goes even further. I also characterize the equilibrium paths of two other endogenous relative variables, the consumption and interest rate differentials, through deriving approximated analytical solutions. The resulting closed-form representation of relative consumption reveals that at the limit of the unit market discount factor, the consumption differential is correlated perfectly with the PPP deviation, i.e., the real exchange rate (RER). This implication of perfect consumption risk sharing stems from two theoretically crucial facts. First, consumption in each country does not rely on any monetary shocks due to the classical dichotomy of this flexible price model. Second, at the limit of the unit discount factor, no country-specific endowment shock has a significant impact on the present discounted values of expected future endowment differentials because of the balanced growth restriction. The resulting homogeneity of the permanent income calculation across the two countries makes their consumption identical. Consequently, neither permanent nor transitory idiosyncratic endowment shock matters for the two-country consumption differential. Only the relative price, i.e., the RER, has an immediate effect on the consumption differential. The resulting perfect correlation between relative consumption and the RER has been recognized as a major empirical difficulty related to a broad class of international business cycle models since that of Backus and Smith (1993). This paper novelly proposes a theoretical possibility that cointegrated TFPs might result in perfect consumption risk-sharing even under incomplete financial markets at the limit of the unit discount factor.

The close-form solution of the interest rate differential uncovers that the relative interest rate is dominated by transitory monetary disturbances, i.e., transitory shocks to the money supply and

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7A caveat against this result is that in this model the RER is treated as an exogenous PPP deviation shock as in EW and BMW. Because the model in this paper includes none of price stickiness, non-tradable goods, home bias, and distributional margins, the RER is not determined endogenously in this model, as in the two-country models of Benigno (2004), Benigno and Thoenissen (2008) and Corsetti et al. (2008). The reasons for this simplification are that this paper tries to derive a tractable approximated analytical solution of the unique market equilibrium of the model in order to (i) identify the corresponding discount factor for the present value representation of nominal exchange rates and (ii) figure out as clearly as possible the equilibrium properties of nominal exchange rates and economic fundamentals when the identified discount factor approaches the limit of one.

8The online supplement of this paper theoretically shows that even when the model is extended to allow for endogenous fluctuations of real exchange rates by including non-tradable goods, the proposition of the equilibrium random-walk property of nominal exchange rates as well as the perfect consumption risk-sharing still hold at the limit of the unit discount factor.
demand differentials. Because in this model the unit discount factor means the zero nominal interest rate at the steady state, the money demand function becomes perfectly flat at the limit. To explain the actual data variations in the interest rate differential together with those in the exchange rate and the money supply differential, the implied perfectly flat money demand functions, however, require counterfactually large volatilities of transitory monetary disturbances.

A macroeconometrician who tries to fit the model to both nominal exchange rates and economic fundamentals then faces a theoretical trade-off. An obvious empirical question then is how seriously the statistical inference on the market discount factor is affected by this theoretical trade-off. To address this question, I estimate a UC model that is fully restricted by the proposed two-country model by a Bayesian posterior simulation method. Given relevant prior distributions of the model’s structural parameters, the same post-Bretton Woods data for Canada and the United States investigated in NR then finds that the market discount factor is \textit{a posteriori} distributed around 0.537. Notice that this size of the market discount factor is far below the size close to one that is statistically inferred by many recent empirical studies under different identification strategies. The empirical result of this paper, hence, uncovers difficulties and obstacles that the literature needs to overcome in explaining data variations in the nominal exchange rate and the corresponding macroeconomic fundamentals jointly and consistently through the lens of the proposed canonical two-country general equilibrium model.

The remainder of this paper is organized as follows. In the next section, I introduce the two-country incomplete market model employed in this paper. Section 3 then derives and discusses

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9That is to say, if she or he fits the model to the near random-walk exchange rate, the market discount factor should be close to one. The model, however, tends to impose three unrealistic theoretical restrictions on the data—a permanent money supply differential shock as the dominant driver of random-walk exchange rates, the infamous Backus and Smith problem of an implausibly strong connection between relative consumption and the RER, and counterfactually large volatilities of transitory nominal shocks required by the flat money demand functions. If she or he tries to avoid these counterfactual restrictions by sufficiently lowering the discount factor, the model loses its ability to mimic the near random-walk behavior of nominal exchange rates.

10As a subsequent research of this paper, Kano and Morita (2015) apply the model of this paper to post-Plaza Accord data of the Japanese yen/the U.S. dollar in order to understand the anecdotal “Soros chart”, i.e., the observed high correlation between the near random-walk Japanese yen/the U.S. dollar rate and the two-country differential in monetary base. Modeling the reserve markets and the money creation processes of the two countries, their Bayesian posterior simulation exercise finds a better match of the model to the data: the posterior mean of the subjective discount factor is 0.96.
the equilibrium random-walk property of nominal exchange rates and the Backus and Smith puzzle of a perfect correlation between relative consumption and the RER at the limit of the unit market discount factor. After reporting the main results of the Bayesian exercises in section 4, I conclude in section 5.

2. A two-country incomplete market model

2.1. The model

In this paper, I investigate a canonical incomplete market model with two countries, the home (h) and foreign (f) countries. Each country is endowed with a representative household whose objective is the lifetime money-in-utility

$$\sum_{j=0}^{\infty} \beta^j E_t \left\{ \ln C_{i,t+j} + \phi_{i,t+j} \ln \left( \frac{M_{i,t+j}}{P_{i,t+j}} \right) \right\}, \quad 0 < \beta < 1,$$

for $i = h, f$,

where $C_{i,t}$, $M_{i,t}$, and $P_{i,t}$ represent the $i$th country’s consumption, money stock, and price index, respectively. The money-in-utility function is subject to a persistent money demand shock $\phi_{i,t}$. The representative households in the home and foreign countries maximize their lifetime utility functions subject to the home budget constraint

$$B_{h,t} + S_t B_{h,t}^f + P_{h,t} C_{h,t} + M_{h,t} = (1 + r_{h,t}^h) B_{h,t-1}^h + S_t(1 + r_{h,t-1}^f) B_{h,t-1}^f + M_{h,t-1} + P_{h,t} Y_{h,t} + T_{h,t},$$

and its foreign counterpart

$$\frac{B_{f,t}^h}{S_t} + B_{f,t}^f + P_{f,t} C_{f,t} + M_{f,t} = (1 + r_{f,t}^h) B_{f,t-1}^h + (1 + r_{f,t-1}^f) B_{f,t-1}^f + M_{f,t-1} + P_{f,t} Y_{f,t} + T_{f,t},$$

respectively, where $B_{i,t}^l$, $r_{i,t}^l$, $Y_{i,t}$, $T_{i,t}$, and $S_t$ denote the $i$th country’s holdings of the $l$th country’s nominal bonds at the end of time $t$, the $i$th country’s returns on the $l$th country’s bonds, the $i$th country’s output level, the $i$th country’s government transfers, and the level of the bilateral nominal exchange rate, respectively. Each country’s output $Y_{i,t}$ is given as an exogenous endowment following a stochastic process $Y_{i,t} = y_{i,t} A_{i,t}$, where $y_{i,t}$ is the transitory component and $A_{i,t}$ is
the permanent component. Below, I interpret the permanent component \( A_{i,t} \) as the TFP in the underlying production technology.

The first-order necessary conditions (FONCs) of the home country’s household are given by the budget constraint, the Euler equation

\[
\frac{1}{P_{h,t}C_{h,t}} = \beta(1 + r_{h,t}^h)E_t \left( \frac{1}{P_{h,t+1}C_{h,t+1}} \right),
\]

the utility-based uncovered parity condition (UIP)

\[
(1 + r_{h,t}^h)E_t \left( \frac{1}{P_{h,t+1}C_{h,t+1}} \right) = \frac{(1 + r_{h,t}^h)}{S_t} E_t \left( \frac{S_{t+1}}{P_{h,t+1}C_{h,t+1}} \right),
\]

and the money demand function

\[
M_{h,t} = \phi_{h,t} \left( \frac{1 + r_{h,t}^h}{r_{h,t}^h} \right) C_{h,t}.
\]

The foreign country’s FONC counterparts are the budget constraint, the Euler equation

\[
\frac{1}{P_{f,t}C_{f,t}} = \beta(1 + r_{f,t}^f)E_t \left( \frac{1}{P_{f,t+1}C_{f,t+1}} \right),
\]

the utility-based uncovered parity condition (UIP)

\[
(1 + r_{f,t}^f)E_t \left( \frac{1}{P_{f,t+1}C_{f,t+1}} \right) = \frac{(1 + r_{f,t}^f)}{S_t} E_t \left( \frac{S_{t+1}}{P_{f,t+1}C_{f,t+1}} \right),
\]

and the money demand function

\[
M_{f,t} = \phi_{f,t} \left( \frac{1 + r_{f,t}^f}{r_{f,t}^f} \right) C_{f,t}.
\]

Each country’s government transfers the seigniorage to the household as a lump-sum. Hence, the government’s budget constraint is

\[
M_{i,t} - M_{i,t-1} = T_{i,t}, \quad \text{for } i = h, f.
\]

The money supply \( M_{i,t} \) is specified to consist of permanent and transitory components, \( M^\tau_{i,t} \) and \( m_{i,t}: M_{i,t} \equiv m_{i,t}M^\tau_{i,t} \) for \( i = h, f \).

To close the model within an incomplete international financial market, I allow for a debt-elastic risk premium in the interest rates faced only by the home country:

\[
r_{h,t}^l = r_{w,t}^l \{ 1 + \psi \{ \exp(-B_{h,t}^l/M^\tau_{l,t} + d) - 1 \} \}, \quad d \leq 0, \quad \psi > 0, \quad \text{for } l = h, f
\]

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where \( r^l_{w,t} \) is the equilibrium risk-free interest rate of the \( l \)th country’s bond. The risk premium is given as an externality: The household does not take into account the effect of the debt position on the risk premium when maximizing the lifetime utility function. On the other hand, I do not attach a risk premium to the foreign country’s interest rates: \( r^l_{f,t} = r^l_{w,t} \) for \( l = h, f \).

Following EW and BMW, I assume throughout this paper that purchasing power parity (PPP) holds only up to a persistent PPP deviation shock \( \ln q_t \):

\[
S_t P_{f,t} = P_{h,t} q_t.
\]

The market-clearing conditions of the two countries’ bond markets are

\[
B^h_{h,t} + B^h_{f,t} = 0 \quad \text{and} \quad B^f_{h,t} + B^f_{f,t} = 0,
\]

i.e., along an equilibrium path, the world net supply of nominal bonds is zero on a period-by-period basis.

As in NR, I assume that the logarithms of the total factor productivity (TFP) and the permanent component of the money supply, \( \ln A_{i,t} \) and \( \ln M^\tau_{i,t} \), are I(1) for \( i = h, f \), and the cross-country differential in the permanent component of money supply, \( \ln M^\tau_{h,t} - \ln M^\tau_{f,t} \), is also I(1):

**Assumption 1**: \( \ln A_{i,t} \) and \( \ln M^\tau_{i,t} \) are I(1) for \( i = h, f \).

**Assumption 2**: \( \ln M^\tau_{h,t} - \ln M^\tau_{f,t} \) is I(1).

Following Assumptions 1 and 2, I specify each country’s monetary growth rate \( \Delta \ln M^\tau_{i,t} \) to be an independent AR(1) process:

\[
\Delta \ln M^\tau_{i,t} = (1 - \rho_M) \ln \gamma_M + \rho_M \Delta \ln M^\tau_{i,t-1} + \epsilon^\tau_{M,t}, \quad \text{for } i = h, f.
\]

where \( \ln \gamma_M \) and \( \rho_M \) are the mean and AR root, respectively, of the money supply growth rate common to the two countries.

Importantly, I do not make NR’s assumption that the cross-country TFP differential, \( \ln a_t \equiv \ln A_{h,t} - \ln A_{f,t} \), is I(1). Rather, I assume that the TFP differential is integrated of order zero (I(0)). This deviation from NR’s key assumption stems from the fact that an I(1) TFP differential
is inconsistent with the stationarity of the stochastically de-trended model and the deterministic steady state of the resulting equilibrium-balanced growth path, as I will show below in more detail. Notice that Assumption 1 and the stationary TFP differential jointly imply that the TFP of the home country must be cointegrated with that of the foreign country:

**Assumption 3**: \( \ln A_{h,t} \) and \( \ln A_{f,t} \) are cointegrated with the cointegrated vector \([1, -1]\) and have the error correction models (ECMs)

\[
\Delta \ln A_{h,t} = \ln \gamma_A - \frac{\lambda}{2} (\ln A_{h,t-1} - \ln A_{f,t-1}) + \epsilon_{A,t}^h, \\
\Delta \ln A_{f,t} = \ln \gamma_A + \frac{\lambda}{2} (\ln A_{h,t-1} - \ln A_{f,t-1}) + \epsilon_{A,t}^f, \tag{1}
\]

where \( \gamma_A > 1 \) is the common drift term and \( \lambda \in [0, 1) \) is the adjustment speed of the error correction mechanism.

The cointegration restriction that Assumption 3 imposes on the two countries’ TFPs is adopted by recent open-economy DSGE studies by Mandelman et al. (2011), Rabanal et al. (2011), and Ireland (2013). ECMs (1) imply that the cross-country TFP differential is I(0) because

\[
\ln a_t = (1 - \lambda) \ln a_{t-1} + \epsilon_{A,t}^h - \epsilon_{A,t}^f.
\]

Importantly, if the adjustment speed \( \lambda \) is sufficiently close to zero, the cross-country TFP differential can be realized near I(1), as maintained by NR.

The stochastic process of the logarithm of the transitory output component for each country, \( \ln y_{i,t} \), is specified as the following AR(1) process:

\[
\ln y_{i,t} = (1 - \rho_y) \ln y_{i} + \rho_y \ln y_{i,t-1} + \epsilon_{y,t}^i, \quad \text{for } i = h, f.
\]

Similarly, the stochastic process of the logarithm of the transitory money supply component for each country, \( \ln m_{i,t} \), is specified as the following AR(1) process:

\[
\ln m_{i,t} = (1 - \rho_m) \ln m_{i} + \rho_m \ln m_{i,t-1} + \epsilon_{m,t}^i.
\]
for $i = h, f$. The three other structural shocks, the home and foreign money demand shocks $\phi_{h,t}$ and $\phi_{f,t}$, respectively, and the PPP shock $q_t$, follow persistent stationary processes. Specifically, they are characterized by AR(1) processes in terms of the following logarithm:

$$\ln \phi_{i,t} = (1 - \rho_{\phi}) \ln \phi + \rho_{\phi} \ln \phi_{i,t-1} + \epsilon_{i,t},$$

for $i = h, f$ and

$$\ln q_t = \rho_q \ln q_{t-1} + \epsilon_{q,t}.$$

Throughout this paper, I assume that all structural shocks are distributed independently.

2.2. The log-linear approximation of the stochastically de-trended system

Define stochastically de-trended variables as $c_{i,t} \equiv C_{i,t}/A_{i,t}$, $p_{i,t} \equiv P_{i,t}A_{i,t}/M_{i,t}^\tau$, $b_{i,t}^l \equiv B_{i,t}^l/M_{i,t}^\tau$, $\gamma_{A,t}^i \equiv A_{i,t}/A_{i,t-1}$, $\gamma_{M,t}^i \equiv M_{i,t}^\tau/M_{i,t-1}^\tau$, and $s_{i} \equiv S_tM_{f,t}^\tau/M_{h,t}^\tau$. Taking the stochastic de-trending of the FONCs, I construct the stochastically de-trended system of the FONCs, as reported in accompanying Appendix A. The resulting ten equations determine the ten endogenous variables $c_{h,t}$, $c_{f,t}$, $p_{h,t}$, $s_t$, $b_{h,t}^l$, $b_{f,t}^l$, $r_{h,t}^h$, $r_{h,t}^f$, $r_{w,t}$, and $r_{w,t}^f$, given nine exogenous variables $\gamma_{M,t}^h$, $\gamma_{M,t}^f$, $\gamma_{A,t}^h$, $\gamma_{A,t}^f$, $a_t$, $m_{h,t}$, $m_{f,t}$, $y_{h,t}$, and $y_{f,t}$.

Let $\hat{x}$ denote a percentage deviation of any variable $x_t$ from its deterministic steady state value $x^*$, $\hat{x} \equiv \ln x_t - \ln x^*$. Also, let $\tilde{x}$ denote a deviation of $x$ from its deterministic steady state, $\tilde{x} = x - x^*$. Appendix A reports the results of the ten equations of the log-linearized FONCs. Under the assumption of the symmetric two-countries with $\bar{d} = 0$, the resulting linear rational expectations system is further simplified as a three-equation representation with two-country relative variables.

To see this, let $c_t$, $y_t$, $m_t$, and $\phi_t$ denote the de-trended consumption ratio, the de-trended output ratio, the transitory money supply ratio, and the money demand shock ratio between the two countries, $c_t \equiv c_{h,t}/c_{f,t}$, $y_t \equiv y_{h,t}/y_{f,t}$, $m_t \equiv m_{h,t}/m_{f,t}$, and $\phi_t \equiv \phi_{h,t}/\phi_{f,t}$, respectively. Furthermore, let $M_t^\tau$ denote the ratio of the permanent money supplies of the home and foreign countries $M_{h,t}^\tau/M_{f,t}^\tau$; let $M_t$ foreign money supplies of the home to the foreign countries $M_{h,t}/M_{f,t} \equiv m_t M_t^\tau$; and $\gamma_{M,t}$ denote

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11 Appendix A provides the deterministic steady state.

12 In particular, for an interest rate $r_t$, $(1 + \hat{r}_t) = (r_t - r^*)/(1 + r^*)$. 

11
the ratio of the permanent money supply growth rate \( \gamma_{M,t} \equiv \frac{\gamma_{M,t}^h}{\gamma_{M,t}^f} \); let \( C_t \) denote the ratio of the consumptions of the home and foreign countries \( C_{h,t}/C_{f,t} \). Then the log-linearized FONCs are degenerated to the following three expectational difference equations with respect to the three endogenous variables \( \hat{s}_t, \hat{c}_t, \) and \( \tilde{b}_t \), given the six exogenous variables \( \hat{\gamma}_{M,t}, \hat{m}_t, \hat{\alpha}_t, \hat{y}_t, \hat{\phi}_t, \) and \( \hat{q}_t \):

\[
\begin{align*}
\hat{s}_t & = \kappa E_t \hat{s}_{t+1} - (1 - \kappa) \hat{c}_t + (1 - \kappa) (\hat{m}_t - \hat{\phi}_t + \hat{q}_t - \hat{\alpha}_t) + \kappa E_t \hat{\gamma}_{M,t+1} - \psi \kappa (1 - \kappa) \tilde{b}_t, \\
\hat{s}_t + \hat{c}_t - \hat{q}_t + \hat{\alpha}_t & = \kappa E_t (\hat{s}_{t+1} + \hat{c}_{t+1} - \hat{q}_{t+1} + \hat{\alpha}_{t+1}) + (1 - \kappa) (\hat{m}_t - \hat{\phi}_t) + \kappa E_t \hat{\gamma}_{M,t+1}, \\
\tilde{b}_t & = \beta^{-1} \tilde{b}_{t-1} + p^*_h y^*(\hat{y}_t - \hat{c}_t),
\end{align*}
\]

(2)

where \( y^* = y/4 \) and \( y = y_h = y_f \) by the symmetric two countries. In particular, the first equation of the linear rational expectations (LRE) system (2) represents the stochastically de-trended UIP; the second equation the cross-country difference in the Euler equation; and the third equation the law of motion of net foreign asset position after solving the interest rate differential through the money demand functions of the two countries. Importantly, as shown by the first equation above, the model identifies steady state nominal market discount factor \( \kappa \equiv 1/(1 + r^*) = \beta/\gamma_M \) as the discount factor that governs the stochastic process of the detrended nominal exchange rate \( \hat{s}_t \). As shown in the next section, \( \kappa \) is the most crucial parameter that characterizes the near random walk property of the nominal exchange rate.\(^{13}\)

3. A general equilibrium analysis of random-walk exchange rates

3.1. Equilibrium random-walk property of nominal exchange rates

I will now show that the equilibrium random-walk property of the exchange rate holds in this two-country model. After unwinding stochastic trends, the first equation of the LRE system (2) can be rewritten as

\(^{13}\text{Note that } \kappa \text{ is given as a function of structural parameters } \beta \text{ and } \gamma_M, \text{ the subjective discount factor and the deterministic mean of the gross money growth rate. In this paper, I assume that the limit of } \kappa \to 1 \text{ is well approximated by the limit of } \beta \to 1 \text{ because } \gamma_M \text{ takes a value that is very close to one, as found in the postwar data of money growth rates in Canada and the United States,}\)
\[ \ln S_t = \kappa E_t \ln S_{t+1} + (1 - \kappa) \ln M_t - (1 - \kappa) \ln C_t - (1 - \kappa)(\ln \phi_t - \ln q_t) - \psi \kappa (1 - \kappa) \tilde{b}_t. \]

Solving this expectational difference equation by forward iterations under a suitable transversality condition provides the DSGE-PVM of this model:

\[ \ln S_t = (1 - \kappa) \sum_{j=0}^{\infty} \kappa^j E_t \left( \ln M_{t+j} - \ln C_{t+j} - \psi \kappa \tilde{b}_{t+j} - \ln \phi_{t+j} + \ln q_{t+j} \right). \tag{3} \]

If the fundamental \( \ln M_t - \ln C_t \) is I(1), so is the exchange rate.\(^{14}\)

NR claim that the DSGE-PVM (3) implies an error-correction representation of the currency return \( \Delta \ln S_t \), in which \( \Delta \ln S_t \) depends on the lagged error correction term \( \ln S_{t-1} - \ln M_{t-1} + \ln C_{t-1} \). Their argument also holds even in this model. Appendix C shows that after rearranging the DSGE-PVM (3) in several steps, the currency return is

\[ \Delta \ln S_t = \frac{1 - \kappa}{\kappa} (\ln S_{t-1} - \ln M_{t-1} + \ln C_{t-1} + \ln \phi_{t-1} - \ln q_{t-1}) + \psi (1 - \kappa) \tilde{b}_{t-1} + u_{s,t}, \tag{4} \]

where \( u_{s,t} \) is the i.i.d., rational expectations error.

\(^{14}\)The exchange rate should be cointegrated with the fundamentals. To signify this property, the DSGE-PVM (3) can be rewritten as

\[ \ln S_t - \ln M_t + \ln C_t = \sum_{j=1}^{\infty} \kappa^j E_t (\Delta \ln M_{t+j} - \Delta \ln C_{t+j}) - (1 - \kappa) \sum_{j=0}^{\infty} \kappa^j E_t \left( \psi \kappa \tilde{b}_{t+j} + \ln \phi_{t+j} - \ln q_{t+j} \right). \]

Since the RHS of the above equation is I(0), the exchange rate \( \ln S_t \) and the I(1) fundamental \( \ln M_t - \ln C_t \) are cointegrated. NR hypothesize the cointegration relation among \( \ln S_t, \ln M_t, \) and \( \ln C_t \) based on their DSGE-PVM. The model in this paper theoretically restricts the stationarity of the consumption differential \( \ln C_t \) because of Assumption 3 due to the requirement of closing the two-country model. If the adjustment speed of the error correction mechanism of both countries’ TFPs, \( \lambda \), is sufficiently slow, the maintained stationarity of the consumption differential is unlikely to be detected with a finite sample.

EW and NR, however, reject the cointegration relation between the exchange rate and fundamentals in actual data for major currencies. In particular, EW suggest other unobservable I(1) components that the standard asset approach does not identify as primary reasons for the failure of the cointegration hypothesis. Notice that in the DSGE-PVM (3), the equilibrium exchange rate also depends on the present discounted values of expected future de-trended net foreign asset positions \( \tilde{b}_t \), the relative money demand shock \( \ln \phi_t \), and the PPP shock \( \ln q_t \). As shown in Appendix B in a case including symmetric countries, the stationarity of the de-trended international bond holding \( \tilde{b}_t \) relies on the sizes of the debt elasticity of the risk premium \( \psi \) as well as the market discount factor \( \kappa \); if either \( \psi \) is sufficiently close to zero or \( \kappa \) approaches one, \( \tilde{b}_t \) follows a near-I(1) process. Moreover, as stated by BMW, the relative money demand shock and the PPP shock could be unobservable near I(1) components.

13
\[ u_{s,t} = (1 - \kappa) \sum_{j=0}^{\infty} \kappa^j (E_t - E_{t-1}) (\ln M_{t+j} - \ln C_{t+j} - \psi \tilde{b}_{t+j} + \ln q_{t+j} - \ln \phi_{t+j}). \]

Recall that the DSGE-PVM (3) is constructed as an equilibrium condition from some of the model’s FONCs. The general equilibrium property of the model, however, imposes another restriction on the present value of the future fundamentals in the DSGE-PVM (3). Notice that after unwinding stochastic trends the second equation of the LRE system (2) yields the first-order expectational difference equation of \( \ln S_t - \ln M_t + \ln C_t - \ln q_t \):

\[
\ln S_t - \ln M_t + \ln C_t - \ln q_t = \kappa E_t (\ln S_{t+1} - \ln M_{t+1} + \ln C_{t+1} - \ln q_{t+1}) + \kappa \rho_M \hat{\gamma}_{M,t} + \kappa (\rho_m - 1) \ln m_t - (1 - \kappa) \ln \phi_t,
\]

where \( \hat{\gamma}_{M,t} \equiv \hat{\gamma}_{M,t}^h - \hat{\gamma}_{M,t}^f \) is the money supply growth rate differential. Because \( \kappa \) is less than one, the difference equation above has the unique forward solution

\[
\ln S_t = \ln M_t - \ln C_t + \ln q_t + \frac{\kappa \rho_M}{1 - \kappa \rho_M} \hat{\gamma}_{M,t} + \frac{\kappa (1 - \rho_m)}{1 - \kappa \rho_m} \ln m_t - \frac{1 - \kappa}{1 - \kappa \rho_\phi} \ln \phi_t \tag{5}
\]

under a suitable transversality condition.

Imposing the CER (5) on the error-correction process (4) provides the equilibrium currency return

\[
\Delta \ln S_t = \psi (1 - \kappa) \tilde{b}_{t-1} + \frac{(1 - \kappa) \rho_M}{1 - \kappa \rho_M} \hat{\gamma}_{M,t-1} + \frac{(1 - \kappa)(1 - \rho_\phi)}{1 - \kappa \rho_\phi} \ln \phi_{t-1} - \frac{(1 - \kappa)(1 - \rho_m)}{1 - \kappa \rho_m} \ln m_{t-1} + u_{s,t}. \tag{6}
\]

Equation (6) clearly shows that any dependence of the currency return on past information emerges through the persistence of the net foreign asset position, the money supply growth differential, the transitory money demand shock differential, and the transitory money supply differential.

The important implication of the equilibrium currency return equation (6) is that the logarithm of the exchange rate follows a Martingale difference sequence at the limit of \( \kappa \to 1 \) because

\[
\lim_{\kappa \to 1} E_t \Delta \ln S_{t+1} = 0.
\]
Therefore, in this paper, the exchange rate behaves like a random walk when the market discount factor approaches one along the equilibrium path of the two-country model. The equilibrium currency return equation (6) exhibits no dependence of the currency return on past information in this case. Hence, the equilibrium random walk property of the exchange rate, as found in EW and NR, is also preserved in this model.

In the limiting case with the unit market discount factor, the equilibrium currency return is dominated by the i.i.d. rational expectations error $u_{s,t}$. An advantage of working with a structural two-country model is that the rational expectations error $u_{s,t}$ is now fully interpretable as a linear combination of structural shocks. To see this, note that the rational expectations error $u_{s,t}$ in equilibrium is represented by $u_{s,t} = (E_t - E_{t-1}) \Delta \ln S_t = \epsilon_{M,t} + (E_t - E_{t-1}) \hat{s}_t$ where $\epsilon_{M,t} \equiv \epsilon^h_{M,t} - \epsilon^f_{M,t}$ denotes the relative permanent money supply shock. It is not straightforward, however, to calculate the equilibrium surprise of the de-trended exchange rate $(E_t - E_{t-1}) \hat{s}_t$. Appendix B shows that in the special case of two symmetric countries, assuming $\bar{\delta} = 0$ and $y_h = y_f$, the equilibrium de-trended exchange rate is determined by a linear function of $\tilde{b}_{t-1}$, $\ln a_t$, $\ln m_t$, $\ln \phi_t$, $\ln y_t$, $\ln q_t$, and $\hat{\gamma}_{M,t}$:

$$
\hat{s}_t = \frac{\beta \eta - 1}{\beta p^*_{y,t} \phi_t} \hat{b}_{t-1} - \frac{1 - \beta \eta}{1 - \beta \eta(1 - \lambda)} \ln a_t + \frac{1 - \kappa}{1 - \kappa \rho_m} \ln m_t - \frac{1 - \kappa}{1 - \kappa \rho_\phi} \ln \phi_t
$$

$$
- \frac{1 - \beta \eta}{1 - \beta \eta \rho_y} \ln y_t + \frac{1 - \beta \eta}{1 - \beta \eta \rho_q} \ln q_t + \frac{\kappa \rho M}{1 - \kappa \rho M} \hat{\gamma}_{M,t},
$$

where the constant $\eta$, which is less than one, approaches one at the limit of $\kappa \to 1$. The rational expectations error is then given as an explicit linear function of the structural shocks:

$$
u_{s,t} = \frac{1}{1 - \kappa \rho M} \epsilon^h_{M,t} - \frac{1 - \beta \eta}{1 - \beta \eta(1 - \lambda)} \epsilon^h_{A,t} + \frac{1 - \kappa}{1 - \kappa \rho_m} \epsilon^h_{m,t} - \frac{1 - \kappa}{1 - \kappa \rho_\phi} \epsilon^h_{\phi,t}
$$

$$
- \frac{1 - \beta \eta}{1 - \beta \eta \rho_y} \epsilon^h_{y,t} + \frac{1 - \beta \eta}{1 - \beta \eta \rho_q} \epsilon^h_{q,t},
$$

where $\epsilon_{A,t} \equiv \epsilon^h_{A,t} - \epsilon^f_{A,t}$, $\epsilon_{m,t} \equiv \epsilon^h_{m,t} - \epsilon^f_{m,t}$, $\epsilon_{\phi,t} \equiv (\epsilon^h_{\phi,t} - \epsilon^f_{\phi,t})$, and $\epsilon_{y,t} \equiv \epsilon^h_{y,t} - \epsilon^f_{y,t}$ denote shocks to the relative TFP, the relative transitory money supply, the relative transitory money demand, and the relative transitory income.

Notice that at the limit of $\kappa \to 1$, the model also implies the subjective discount factor

\[15\text{As defined in Appendix B, the constant } \eta \text{ is one of the two roots of the expectational difference equation of the de-trended net foreign asset position } \hat{b}_t. \text{ A simple calculation shows that the equilibrium currency return (6) can be derived directly from the CER once the approximated relation } \hat{s}_t \approx \ln S_t + \ln A_t - \ln M^*_t \text{ is recognized.} \]
\( \beta \to 1 \) under a positive deterministic money supply growth rate, \( \gamma_M > 1 \), which is close to one. In this limiting case, observe that the permanent monetary shock \( \epsilon_{M,t} \) surely dominates the rational expectations error \( u_{s,t} \) and, as a result, the random walk of the exchange rate.

\[
\lim_{\kappa \to 1} \Delta \ln S_t = \lim_{\kappa, \beta, \eta \to 1} u_{s,t} = \frac{1}{1 - \rho_M} \epsilon_{M,t}.
\]

Therefore, no transitory shock matters for the total variations in the random-walk exchange rate.

In contrast to the empirical result of NR, which depends on a more flexible reduced-form specification of the consumption differential, no permanent TFP shock can be a primary driver of the random-walk exchange rate. This result is due to the cointegration of the two-country TFPs: No discrepancy between the two countries’ TFPs can be permanent in order to guarantee the equilibrium-balanced growth path. The model’s theoretical implication of the dominant role of the permanent money supply shock on the random-walk exchange rate, hence, is too restrictive to trace out the actual data variations in the bilateral nominal exchange rate, at least between Canada and the United States.

### 3.2. Perfect consumption risk sharing at the limit

This model, moreover, has another unrealistic implication on the consumption differential equilibrium dynamics \( \ln C_t \) when the discount factor approaches one. At the limit of the unit discount factor, perfect consumption risk sharing emerges even under incomplete international financial markets. To observe this property, taking the first difference of the CER (5), substituting the equilibrium currency return (6) into the result, and exploiting the rational expectations error \( u_{s,t} \) yield the following consumption differential dynamics:

\[
\Delta \ln C_t = \Delta \ln q_t - \psi (1 - \kappa) \tilde{b}_{t-1} + \frac{1 - \beta \eta}{1 - \beta \eta (1 - \lambda)} \epsilon_{A,t} + \frac{1 - \beta \eta}{1 - \beta \eta \rho_y} \epsilon_{y,t} - \frac{1 - \beta \eta}{1 - \beta \eta \rho_q} \epsilon_{q,t}.
\]

Notice, therefore, that except through the net foreign asset position, no monetary shock directly matters for the change in the equilibrium consumption differential: As in the standard international business cycle model, only real shocks to the endowments and the PPP deviation affect the equilibrium consumption allocation between the two countries.
Taking the limit of equation (7) above with respect to $\kappa$ results in

$$\lim_{\kappa, \beta, \eta \to 1} \Delta \ln C_t = \Delta \ln q_t.$$ 

Thus, relative consumption becomes unrelated to any shocks to the endowments of the two countries but is rather perfectly correlated with the exogenous RER. The intuition behind this result is quite straightforward. In this incomplete market model with the PIH households, consumption in each country is determined by splitting the global aggregate endowment across both countries in each period. The portion of the global aggregate endowment allocated to one country is simply given as the present discounted values of the expected future relative endowments of this country to the other. Because the endowment differential is stationary due to the balanced growth restriction, the unit discount factor at the limit makes the portion converge to a constant; in particular, one-half in the case of two symmetric countries. Consumption in both countries, hence, responds to any endowment shocks in the same fashion. As the result, with the discount factor being close to one, relative consumption depends neither on permanent nor transitory endowment shocks. The only shock that can affect the relative consumption is in the corresponding relative price, i.e., the RER.\(^\text{16}\)

3.3. Inelastic money demand

\(^{16}\)More precisely, from Appendix B, the consumption logarithms of the home and foreign countries in terms of the home currency can be solved as

$$2\ln C_{h,t} = \ln Y_{h,t} + \ln q_t Y_{f,t} + \frac{1 - \beta\eta}{1 - \beta\eta(1 - \lambda)} \ln a_t + \frac{1 - \beta\eta}{1 - \beta\eta\rho_y} \ln y_t - \frac{1 - \beta\eta}{1 - \beta\eta\rho_y} \ln q_t + \frac{1 - \beta\eta}{\beta p^* h y^* b_{t-1}},$$

$$2\ln q_t C_{f,t} = \ln Y_{h,t} + \ln q_t Y_{f,t} - \frac{1 - \beta\eta}{1 - \beta\eta(1 - \lambda)} \ln a_t - \frac{1 - \beta\eta}{1 - \beta\eta\rho_y} \ln y_t - \frac{1 - \beta\eta}{1 - \beta\eta\rho_y} \ln q_t - \frac{1 - \beta\eta}{\beta p^* h y^* b_{t-1}}.$$

Each country’s consumption depends on the log-linearized global aggregate endowment $\ln Y_{h,t} + \ln q_t Y_{f,t}$, the log-linearized country-specific portion of the aggregate endowment $\frac{1 - \beta\eta}{1 - \beta\eta(1 - \lambda)} \ln a_t + \frac{1 - \beta\eta}{1 - \beta\eta\rho_y} \ln y_t - \frac{1 - \beta\eta}{1 - \beta\eta\rho_y} \ln q_t$, and the wealth effect of the net foreign asset position $\frac{1 - \beta\eta}{\beta p^* h y^* b_{t-1}}$. If the discount factor approaches one, both the log-linearized country-specific portion and the wealth effect of the net foreign asset position disappear and the log consumption levels become

$$\ln C_{h,t} = \frac{1}{2} (\ln Y_{h,t} + \ln Y_{f,t}) + \frac{1}{2} \ln q_t, \quad \ln C_{f,t} = \frac{1}{2} (\ln Y_{h,t} + \ln Y_{f,t}) - \frac{1}{2} \ln q_t.$$

Relative consumption then turns out to be correlated perfectly with the RER because

$$\ln C_{h,t} - \ln C_{f,t} = \ln q_t.$$
The two-country model of this paper also characterizes the analytical closed-form solutions of the nominal interest rates along the equilibrium path. Appendix B shows that the equilibrium home and foreign interest rates are

\[
(1 + \hat{r}_{h,t}^{h}) = (1 - \kappa) \left( \frac{\rho_{M}}{1 - \kappa \rho_{M}} \hat{\gamma}_{M,t}^{h} - \frac{1 - \rho_{m}}{1 - \kappa \rho_{m}} \hat{m}_{h,t} + \frac{1 - \rho_{\phi}}{1 - \kappa \rho_{\phi}} \hat{\phi}_{h,t} \right),
\]

\[
(1 + \hat{r}_{f,t}^{f}) = (1 - \kappa) \left( \frac{\rho_{M}}{1 - \kappa \rho_{M}} \hat{\gamma}_{M,t}^{f} - \frac{1 - \rho_{m}}{1 - \kappa \rho_{m}} \hat{m}_{f,t} + \frac{1 - \rho_{\phi}}{1 - \kappa \rho_{\phi}} \hat{\phi}_{f,t} \right).
\]

Hence, the home and foreign interest rates are determined by the money supply growth shock \( \hat{\gamma}_{M,t} \), the transitory money supply shock \( \hat{m}_{j,t} \), and the money demand shock \( \hat{\phi}_{j,t} \) for \( j = h, f \). Because the AR root of the money supply growth \( \rho_{M} \) is expected close to zero, the main determinants of the nominal interest rates are supposed to be transitory monetary shocks, \( \hat{m}_{j,t} \) and \( \hat{\phi}_{j,t} \).

The above equilibrium interest rates show that at the limit of the unit market discount factor, each of the home and foreign nominal interest rates \( r_{h,t}^{h} \) and \( r_{f,t}^{f} \) is insensitive to the domestic monetary shocks because money demand functions are perfectly flat.\(^{17}\) The difficulty due to the flat money demand functions should be that the monetary shocks have to have extremely large volatilities to explain the actual data variations in the nominal interest rates.

4. A Bayesian unobserved component approach

This section empirically explores the question of how significantly the tension emerging at the limit of the unit market discount factor among the three theoretical implications — the random-walk exchange rate, the dominance of permanent money supply differential shocks in the variations in the random-walk exchange rate, and the perfect correlation between the relative consumption and the RER — affects posterior inferences in relation to the market discount factor. For this specific purpose, I simplify the estimation exercise as much as possible by adopting the symmetric version of the two-country model, in which the same structural parameters are shared by both countries. This paper then takes a Bayesian UC approach to the proposed structural two-country model.

4.1. The restricted UC model and posterior simulation strategy

\(^{17}\)This is the situation of the Keynesian liquidity trap at the steady state.
Let \( X_t \) denote an unobserved state vector defined as

\[
X_t = [\hat{s}_t \hat{c}_t E_t \hat{s}_{t+1} E_t \hat{c}_{t+1} \tilde{b}_t \tilde{\gamma}_{M,t} \hat{m}_t \hat{q}_t \hat{\phi}_t]^\prime.
\]

Furthermore, let \( \epsilon_t \) and \( \omega_t \) denote random vectors consisting of structural shocks and rational expectations errors:

\[
\epsilon_t \equiv [\epsilon_{M,t} \epsilon_{A,t} \epsilon_{m,t} \epsilon_{q,t} \epsilon_{\phi,t}]^\prime \quad \text{and} \quad \omega_t \equiv [\hat{s}_t - E_{t-1} \hat{s}_t \hat{c}_t - E_{t-1} \hat{c}_t]^\prime,
\]

In particular, for empirical investigation purposes, I presume that the structural shock vector \( \epsilon_t \) is normally distributed, with a mean of zero and a diagonal variance-covariance matrix \( \Sigma \):

\[
\epsilon_t \sim i.i.d. N(0, \Sigma) \quad \text{with} \quad \text{diag}(\Sigma) = [\sigma^2_M \sigma^2_A \sigma^2_m \sigma^2_q \sigma^2_{\phi}]'.
\]

Accompanied by the stochastic processes of the exogenous forcing variables, the LRE model (2) then implies that

\[
\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Phi_0 \omega_t + \Phi_1 \epsilon_t,
\]

where \( \Gamma_0, \Gamma_1, \Phi_0, \) and \( \Phi_1 \) are the corresponding coefficient matrices. Applying Sims’s (2001) QZ algorithm to the linear rational expectations model above yields a unique solution as the following stationary transition equation of the unobservable state vector:

\[
X_t = FX_{t-1} + \Phi \epsilon_t, \quad (8)
\]

where \( F \) and \( \Phi \) are confirmable coefficient matrices.

To construct this paper’s UC model, I further expand the unobservable state vector \( X_t \) by the permanent money supply differential \( \ln M^*_t \) to obtain the augmented state vector \( Z_t \): 

\[
Z_t \equiv [X_t' \ln M^*_t]^\prime.
\]

The stochastic process of \( \ln M^*_t \) and the state transition (8) then imply the following non-stationary transition of the expanded state vector \( Z_t \):

\[
Z_t = GZ_{t-1} + \Psi \epsilon_t, \quad \epsilon_t \sim i.i.d. N(0, \Sigma), \quad (9)
\]

where \( G \) and \( \Psi \) are confirmable coefficient matrices.

In this paper, I explore time-series data on the log of the consumption differential \( \ln C_t \), the log of the output differential \( \ln Y_t \), the log of the money supply differential \( \ln M_t \), the interest rate differential \( r_t \equiv r^h_{i,t} - r^f_{i,t} \), and the log of the bilateral exchange rate \( \ln S_t \). Let \( Y_t \) denote the information set that consists of these five time series:

\[
Y_t \equiv [\ln C_t \ln Y_t \ln M_t \ln r_t \ln S_t]^\prime.
\]

It is then
straightforward to show that the information set $Y_t$ is linearly related to the unobservable state vector $Z_t$ as

$$Y_t = HZ_t,$$

where $H$ is a confirmable coefficient matrix. The transition equation, the unobserved state (9), and the observation equation (10) jointly consist of a non-stationary state-space representation of the two-country model, which is the restricted UC model estimated in this paper.\(^{18}\)

Given the data set $Y^T = \{Y_t\}_{t=0}^T$, applying the Kalman filter to the UC model provides model likelihood $L(Y^T|\theta)$, where $\theta$ is the structural parameter vector of the two-country model. Multiplying the likelihood by a prior probability of the structural parameters, $p(\theta)$, is proportional to the corresponding posterior distribution $p(\theta|Y^T) \propto p(\theta)L(Y^T|\theta)$ through the Bayes law. The posterior distribution $p(\theta|Y^T)$ is simulated by the random-walk Metropolis-Hastings algorithm, as implemented by Schorfheide (2000), Bouakez and Kano (2006), and Kano (2009).

4.2. Data and prior construction

The two countries that I empirically examine in this paper are Canada and the United States as the model’s home and foreign countries, respectively. I examine post-Bretton Woods quarterly data for these two countries because they satisfy the model assumptions. The data span the period from Q1:1973 to Q4:2007. All the data included in the information set $Y^T$, except nominal exchange rates, are seasonally adjusted annual rates.\(^{19}\)

Table 1 reports the prior distributions of the structural parameters of the two-country model, $p(\theta)$. Since the main goal of this paper’s empirical investigation is to draw a posterior inference on the market discount factor $\kappa \equiv \beta/\gamma_M$, I elicit a uniform prior distribution of $\kappa$ and let the data tell the posterior position of $\kappa$ given the identification of the restricted UC model. In so doing, on the one hand, the prior distribution of the mean gross monetary growth rate, $\gamma_M$, is intended to tightly

\(^{18}\)The state-space form of the model, (9) and (10), decomposes the I(1) difference-stationary information set $Y_t$ into permanent and transitory components exploiting the theoretical restrictions provided by the two-country model. Recursion of the Kalman filter for a non-stationary state-space model is explained in detail by Hamilton (1994).

\(^{19}\)Appendix D provides a detailed description of the source and construction of the data examined in this paper.
cover its sample counterparts in both countries through the Gamma distribution, with a mean of 1.015 and standard deviation of 0.005. On the other hand, the prior distribution of the subjective discount factor $\beta$ is uniformly distributed between zero and one. As a result, the prior distribution of the market discount factor $\kappa$ is well approximated as the uniform distribution spread over the support of the unit interval.

To guarantee the stationarity of the de-trended net foreign asset position $\tilde{b}_t$, the debt elasticity of the home risk premium $\psi$ should be positive. I therefore set the prior distribution of $\psi$ to the Gamma distribution, with a mean of 0.010 and standard deviation of 0.001. Closing the model also requires the technological diffusion speed $\lambda$ to be positive but less than one. This necessary condition for the equilibrium-balanced growth path elicits the prior distribution of $\lambda$ as the Beta distribution, with a mean of 0.010 and standard deviation of 0.001. The slow technological diffusion that the prior mean of $\lambda$ implies is intended to capture the slow-moving time-series properties observed in the actual consumption and output differentials between Canada and the United States. The prior distribution of the mean monetary demand shock $\phi$ follows the Gamma distribution, with a mean of 1.000 and small standard deviation of 0.010. By doing so, I assume a priori that the monetary demand shock has no effect on the deterministic steady state.

I admit a small persistence of the permanent money growth rate by setting the prior distribution of the AR(1) coefficient $\rho_M$ to the Beta distribution, with a mean of 0.100 and standard deviation of 0.010. The PPP deviation shock, i.e., the RER shock, is presumed to be very persistent, as observed by many past empirical studies on the RER. The AR(1) coefficient of the RER, $\rho_q$, is then accompanied by the Beta prior distribution, with a mean of 0.850 and standard deviation of 0.100. This prior distribution mimics fairly well the posterior distribution of the same structural parameter reported in Figure 3 of BMW, who used a long annual sample of data from the United Kingdom and the United States. On the other hand, there is no robust empirical consensus on the extent of the persistence of the money demand shock. Hence I allow the prior distribution of

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20 The sample mean of the M1 money supply’s gross growth rate is 1.016 for Canada and 1.014 for the United States.

21 In fact, the 95% interval of [0.607 0.983] includes the most inferences on RER persistence established in major past studies (see, e.g., Rogoff 1996 and Lothian and Taylor 2000).
the AR(1) coefficient of the money demand shock, $\rho_\phi$, to be distributed around 0.850 following the Beta distribution, with a mean of 0.850 and a large standard distribution of 0.100. The resulting 95% coverage, indeed, is $[0.607, 0.983]$, which also covers the corresponding posterior distribution displayed within Figure 3 of BMW. Furthermore, to better identify the permanent components of the money supplies and TFPs of both countries avoiding over-parametrizing the model, I assume that the corresponding transitory components are white noise by setting the prior mass points of the AR(1) coefficients $\rho_m$ and $\rho_y$ to zero. Following NR, I also allow for the deterministic time trend in the exchange rate, $\gamma_S$, with the normal prior distribution with the zero mean and the large standard deviation of 1.500. Finally, the prior standard deviations of all the structural shocks are assumed to share the identical inverse-Gamma distribution, with a mean of 0.010 and standard deviation of 0.010. This prior distribution of $\Sigma$ yields a higher marginal likelihood among small perturbations. Below, I refer to this prior specification as the Benchmark model.

### 4.3. Main Results

The second, third, and fourth columns of Table 2 describe the posterior distributions of the structural parameters under the Benchmark model. The most striking posterior inference conveyed by these columns is that the market discount factor $\kappa$ is identified as being far below one. As displayed in the first row, the data pin down the location of $\kappa$ very tightly around the posterior mean of 0.537, with a standard deviation of 0.041. This posterior distribution of the market discount factor is too low to guarantee the second necessary condition of the equilibrium random-walk exchange rate established by EW and NR, i.e., that the market discount factor is sufficiently close to one. The other significant result in Table 2 relates to the posterior inferences on the money demand differential shock, $\rho_\phi$ and $\sigma_\phi$: The data show a more persistent and volatile money demand differential shock compared to the prior specification of the Benchmark model. Notice that the posterior mean of $\rho_\phi$ is 0.997 and almost 10% larger than its prior mean value; the posterior mean of $\sigma_\phi$ is 0.027 and 17% larger than its prior mean value. The very persistent money demand differential shock provides evidence that such a structural shock could play a significant
role in actual exchange rate movements.

Does this lower market discount factor deteriorate the model’s fit to actual exchange rate movements? The answer is clearly no, although the equilibrium currency return depends slightly on past economic fundamentals. The estimated Benchmark model is indeed successful in explaining the historical trajectory of the exchange rate. Figure 1(a) plots the actual depreciation rate of the Canadian dollar against the United States dollar as the solid black line. The same figure also displays the 95 % Bayesian highest probability density (HPD) interval of the in-sample prediction of the depreciation rate by the Benchmark model (the dashed blue lines). The HPD interval is very narrow: the Benchmark model yields a sharp in-sample prediction of the depreciation rate. Indeed, the HPD interval includes the actual depreciation rate at almost all the sample periods. Hence, the model tracks the actual depreciation rate fairly well.

Which structural shocks are the main drivers of the successful in-sample fit of the Benchmark model to the depreciation rate? To answer this important question, I calculate the same in-sample prediction of the depreciation rate with the Benchmark model as in Figure 1(a), but shutting down one structural shock at a time. Along with the actual depreciation rate (the solid black line), each window in Figure 2 corresponding to a particular structural shock exhibits the HPD interval of the in-sample prediction of the depreciation rate with the Kalman smoother of the corresponding structural shock excluded (the dashed blue lines); the upper-left window corresponds to the prediction with the TFP differential shock $\epsilon_{A,t}$ excluded; the upper-middle window corresponds to the prediction with the transitory money supply differential shock $\epsilon_{m,t}$ excluded; the upper-right window corresponds to the prediction with the transitory output differential shock $\epsilon_{y,t}$ excluded; the lower-left window corresponds to the prediction with the PPP deviation shock $\epsilon_{q,t}$ excluded; the lower-center window corresponds to the prediction with the money demand differential shock $\epsilon_{\phi,t}$ excluded; and, finally, the lower-right window corresponds to the prediction with the permanent money supply differential shock $\epsilon_{M,t}$ excluded. Notice that if the corresponding structural shock

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22The in-sample prediction of the depreciation rate is calculated by feeding the Kalman smoothers of all the structural shocks into the restricted UC model (9) and (10) evaluated at each posterior draw of the structural parameters.
plays a major role in the successful in-sample prediction of the depreciation rate observed in Figure 1(a), shutting down such a shock will deteriorate the in-sample fit of the Benchmark model to the depreciation rate significantly.

The six windows in Figure 2(a) clearly reveal that the most important structural shock for the near-random-walk exchange rate between Canada and the United States is identified as the very persistent money demand differential shock in conjunction with the permanent money supply differential shock. This inference about the main driver of nominal exchange rates echoes the findings of the past studies by EW, BMW, and Sarno and Schmelling (2014): economic fundamentals of near random-walk exchange rates should be unobservable and nominal such as a money demand shock. Nevertheless, it is important to note that the estimated low discount factor allows the TFP differential shock to contribute to actual exchange rate fluctuations, although to a much smaller degree than the permanent money supply and the money demand differential shocks. In contrast to the observation of NR, the TFP shock plays only a minor role in data variations in the nominal exchange rate.

The same historical decomposition of the in-sample prediction of the Benchmark model into the structural shocks is also applicable to the two endogenous economic fundamentals, the consumption differential and the TB differential. Each window of Figure 3(a) (Figure 4a) corresponding to a particular structural shock displays the 95% HPD interval of the in-sample prediction of the consumption growth differential (the TB differential) with the Kalman smoother of the corresponding structural shock excluded (the dashed blue lines), respectively. Observe in the upper-left window of Figure 3(a) the dominant role that the TFP differential shock plays in the actual consumption growth differential. The Benchmark model identifies that the other structural shocks are unlikely to have any significant effect on the variations in the consumption growth differential at all. The upper-middle window of Figure 4(a) then shows evidence that the transitory money supply differential shock primarily drives the actual TB differential data. This result is consistent with the theoretical implication of the model for the equilibrium interest rate differential, given a highly persistent money demand differential shock.
4.4. Understanding the lower discount factor: the High Discount Factor model

Why does the Benchmark model result in such a lower discount factor? To understand this question, I conduct an alternative Bayesian posterior simulation exercise. In this exercise, I intend to fix the discount factor close to one and observe how the empirical performance of the model changes relative to that of the Benchmark model. In so doing, I replace the uniform prior distribution of \( \beta \) in the Benchmark model with a more informative Beta distribution, with a mean of 0.999 and standard deviation of 0.001, and stay with the same prior distributions of the remainder of the structural parameters as in the Benchmark model. I refer to this new specification as the High Discount Factor (HDF) model.\(^{23}\)

The fifth, sixth, and seventh columns of Table 2 correspond to the posterior distributions of the structural parameters under the HDF model. Observe that the resulting posterior distributions of both the market and subjective discount factors are much closer to one, with posterior means of 0.950 and 0.998, respectively. Crucial changes in the posterior distributions of the structural parameters from the Benchmark model, then, are recognized as significant increases in the posterior means of the standard deviations of the three monetary shocks, \( \sigma_M, \sigma_m, \) and \( \sigma_\phi \). The HDF model, which suffers from the perfectly flat money demand at the deterministic steady state, requires counterfactually greater volatilities in all the monetary shocks to explain the data.

An important difference between the Benchmark and HDF models is related to the overall fit to the data. First, the last row of Table 2 reports the estimated marginal likelihood for each model.\(^{24}\) The HDF model yields a smaller marginal likelihood of 1871.309 compared to that of the Benchmark model, which was 2148.572. The difference in the marginal likelihoods of the two models is so significant that I conclude that forcing the discount factor to be close to one makes the HDF model’s overall fit to the data much worse than that of the Benchmark model. Figure 5 more clearly reveals the source of this significant deterioration of the HDF model compared to the Benchmark model.

\(^{23}\text{Note that this prior construction of the subjective discount factor is almost the same exercise as calibrating the market discount factor to the unconditional mean of the nominal interest rates observed in actual data.}\)

\(^{24}\text{This paper estimates the marginal likelihoods by using Geweke’s (1999) modified harmonic mean estimator. A marginal likelihood is the probability of data } Y^T \text{ conditional on an underlying model. In general, the higher the marginal likelihood is, the better the underlying model’s overall fit to the data.}\)
model with respect to the marginal likelihood. This figure plots the 95 % HPD intervals of the one-period-ahead forecast errors of the Benchmark and HDF models toward the actual data as the dashed blue and dotted red lines, respectively.\textsuperscript{25} The figure clearly shows the greatest difficulty for the HDF model relative to the Benchmark model relates to its fit to the money supply differential.

Why does the HDF model fail to explain the money supply differential? Remember the model’s implication at the limit of the unit discount factor: in contrast to the Benchmark model with a lower discount factor, the exchange rate data should be explained exclusively by either the permanent money supply differential shock or the persistent money demand differential shock or both. Notice also that given the identification assumption of the white noise transitory money supply differential shock, the two shocks should have large volatilities to explain the actual data variations of the interest rate differential under the flat money demand functions. Because the HDF model’s forecast toward the money supply differential stems only from the permanent money supply differential shock, the large volatility of the corresponding shock leads to a large forecast error in this data dimension.\textsuperscript{26}

The dotted red line displayed in Figure 1(b) indicates the 95 % HPD interval of the in-sample prediction on the depreciation rate implied by the HDF model. Furthermore, Figures 2(b), 3(b), and 4(b) exhibit the historical decompositions of the in-sample predictions of the depreciation rate, the consumption growth differential, and the TB differential, as the counterparts of Figures 2(a), 3(b) and 4(b) for the Benchmark model, respectively. These in-sample predictions convey four properties of the HDF model; (i) the HDF model tracks the actual near random-walk exchange rate to the almost same degree as the Benchmark model; (ii) the permanent money supply differential shock and the persistent money demand differential shock jointly and dominantly explain actual exchange rate movements; (iii) the PPP deviation shock, not the TFP shock as in the Benchmark model, is the dominant driver of the consumption growth differential; and (iv) not only the transitory

\textsuperscript{25}The forecast errors of the two models are calculated through the Kalman filter forward recursion.

\textsuperscript{26}This result does not depend crucially on the identification assumption of the white noise transitory money supply differential shock. Even when I allow for a high serial correlation of this shock, the flat money demand functions require large volatilities of the monetary shocks and result in a worse fit of the HDF toward the money supply differential than the Benchmark model.
money supply differential shock but also the persistent money demand differential shock explains the TB differential.\textsuperscript{27} The first and second properties echo the main finding of the Benchmark model. The third property, however, represents the drawback of the HDF model. As seen in section 3, with a high discount factor, the consumption differential almost perfectly matches the exogenous PPP deviation shock, of which the exchange rate becomes independent. In the HDF model, the PPP deviation shock, hence, acts as a free latent variable to dominantly explain the consumption differential. The third property, i.e., perfect risk sharing, is counterfactual though.

5. Conclusion

In this paper, I try to reconcile the random-walk property of nominal exchange rates with a neoclassical two-country monetary model including incomplete international financial markets as a benchmark. The main challenge undertaken in this paper is to establish the joint equilibrium dynamics of nominal exchange rates and economic fundamentals, both of which should be endogenously determined by the two-country model. After closing the model correctly by allowing the TFPs of both countries to be cointegrated, the approximated analytical solution of the model discovers the equilibrium random-walk property of exchange rates when the cross-country money supply differential contains a permanent component and the market discount factor approaches one. The necessary assumption for the equilibrium random-walk exchange rate that the discount factor is close to one, however, implies unrealistic restrictions — permanent money supply differential shocks as the dominant driver of random-walk exchange rates, perfect consumption risk sharing, and the counterfactually large volatilities of monetary disturbances due to the flat money demands at the steady state.

Bayesian posterior simulation exercises based on post-Bretton Woods data from Canada and the United States reveal a major difficulty in reconciling the random-walk exchange rate and the economic fundamentals with the proposed two-country model. Indeed, under the benchmark iden-

\textsuperscript{27}The HDF model identifies a smaller AR coefficient of the money demand differential shock $\rho_{\phi}$. The implied smaller persistence of the money demand differential shock results in the larger role this shock plays in the TB differential than that identified by the Benchmark model.
tification of the model, the data updates the value of the market discount factor to far below one. Investigating the model with a specification in which the market discount factor is \textit{a priori} set sufficiently high, I empirically confirm the theoretical conjecture that the posterior inference of a low market discount factor stems from the fact that the model suffers from the Backus and Smith puzzle and that it fails to explain the actual money supply differential.

This paper’s finding of such a low discount factor is in sharp contrast to those of high market discount factors in past empirical studies such as NR, Sarno and Sojli (2009), and BMW. Because these past studies did not jointly consider the endogenous determination of economic fundamentals with nominal exchange rates, the general equilibrium exploration of this paper is relevant to better understanding of the near random-walk behavior of nominal exchange rates within structural open-economy models. Identifying an open economy DSGE model that can reconcile the joint equilibrium dynamics of random-walk exchange rates and economic fundamentals under an empirically plausible market discount factor value is a serious open question to be addressed.

Because the most crucial difference between the empirical exercise in this paper and that of NR’s is in the different stochastic treatments of the TFP differential and, as a result, the consumption differential, it would be a promising research direction to search for a model-consistent way of allowing the TFP differential to be $I(1)$ without violating the balanced growth restriction. A doubtlessly important future extension of the simple model in this paper should be implemented by including multiple goods and allowing for home bias towards home goods over preference. Corsetti et al. (2008) claim that the Backus and Smith observation could be replicated in the extended international real business cycle model with a high degree of the trade elasticity when each country’s productivity shock is nearly permanent. It, however, is not obvious that the same result will hold at the limit of the unit discount factor when cointegration is incorporated into the productivity processes of their model.

Furthermore, the model of this paper is absent from a more realistic specification of a monetary policy framework such as the inflation targeting policy introduced by the Bank of Canada in 1991. Because the inflation targeting policy affects the way of the market participants to form long-run
expectations of inflation, incorporating such a monetary policy framework into the model changes its CERs significantly. An open question, then, is how to admit an I(1) economic fundamental within the inflation targeting policy framework to preserve random walk exchange rates. A very persistent trend inflation, as investigated by Cogley and Sbordone (2008), might be a plausible candidate of an I(1) nominal economic fundamental. I leave these challenging questions as valuables for future studies on open-economy macroeconomics to undertake.

References


Appendix A. Stochastically de-trended system

The stochastically de-trended versions of the FONCs of the home country consist of the budget constraint

\[ p_{h,t}c_{h,t} + b_{h,t}^h + s_{t}b_{h,t}^f = \frac{(1 + r_{h,t-1}^h)b_{h,t-1}^h}{\gamma_{M,t}^h} + \frac{(1 + r_{h,t-1}^f)s_{t}b_{h,t-1}^f}{\gamma_{M,t}^f} + p_{h,t}y_{h,t}; \]

the Euler equation

\[ \frac{1}{p_{h,t}c_{h,t}} = \beta(1 + r_{h,t}^h)E_t \left( \frac{1}{p_{h,t+1}c_{h,t+1}^h}\right); \]

the UIP condition

\[ s_{t}(1 + r_{h,t}^f)E_t \left( \frac{1}{p_{h,t+1}c_{h,t+1}^h}\right) = (1 + r_{f,t}^f)E_t \left( \frac{s_{t+1}}{p_{h,t+1}c_{h,t+1}^f}\right); \]

the money demand function

\[ m_{h,t} = \phi_{h,t}c_{h,t} \left( \frac{1 + r_{h,t}^h}{r_{h,t}^h} \right); \]

the risk premiums

\[ r_{h,t}^h = r_{w,t}^h [1 + \psi\{\exp(-b_{h,t}^h + \bar{d}) - 1\}], \]

and

\[ r_{f,t}^f = r_{w,t}^f [1 + \psi\{\exp(-b_{f,t}^f + \bar{d}) - 1\}]. \]

Similarly, the stochastically de-trended versions of the FONCs of the foreign country consist of the budget constraint

\[ q_{f,t}c_{f,t} - s_{f,t}b_{f,t} - b_{f,t}^h = - \frac{(1 + r_{w,t-1}^f)s_{t}b_{h,t-1}^f}{\gamma_{M,t}^f} - \frac{(1 + r_{w,t-1}^h)b_{h,t-1}^h}{\gamma_{M,t}^h} + \frac{q_{f,t}y_{f,t}}{a_t}; \]

the Euler equation

\[ \frac{a_ts_{t}}{q_{f,t}c_{f,t}} = \beta(1 + r_{w,t}^f)E_t \frac{a_{t+1}s_{t+1}}{\gamma_{M,t+1}^f q_{t+1}p_{h,t+1}c_{f,t+1}^f}; \]

the UIP condition

\[ s_{t}(1 + r_{w,t}^f)E_t \left( \frac{a_{t+1}}{q_{t+1}p_{h,t+1}c_{f,t+1}^f}\right) = (1 + r_{w,t}^f)E_t \left( \frac{a_{t+1}s_{t+1}}{q_{t+1}p_{h,t+1}c_{f,t+1}^f}\right); \]
and the money demand function

$$\frac{a_t s_t M_{f,t}}{q_t p_{h,t}} = \phi_{f,t} c_{f,t} \left( \frac{1 + r_{w,t}^f}{r_{w,t}^f} \right).$$

Finally, the stochastically de-trended PPP condition is

$$s_t = p_{h,t} q_t / (a_t p_{f,t}).$$

If the TFP differential $a_t$ is I(1) as assumed in NR, the above system of stochastic difference equations becomes non-stationary through the home and foreign budget constraints and there is no deterministic steady state to converge. Notice that the cross-country permanent money supply differential $\ln M_{f,t}^s / M_{f,t}^f$ does not appear in the stochastically de-trended system of the FONCs. In contrast to the TFP differential $a_t$, the I(1) property of $\ln M_{f,t}^s / M_{f,t}^f$ in Assumption 2 does not matter for the closing of the model. This might be an obvious result of the model’s property that the super-neutrality of money holds in the money-in-utility model: Money growth does not matter for the deterministic steady state.

Notice that at the deterministic steady state, the TFP differential $a^*$ is one. Because of the stationarity of the above system of equations, the deterministic steady state is characterized by constants $c^*_h$, $c^*_f$, $p^*_h$, $s^*$, $b^*_h$, $b^*_f$, $r^*_h$, $r^*_f$, $r^*_w$, and $r^*_w$ that satisfy

$$b^*_h = b^*_h = \bar{d},$$
$$r^*_h = r^*_f = r^*_w = \gamma_M / \beta - 1,$$
$$s^* = y_f / (\phi \gamma_M)^{-1} + (1 - \beta^{-1}) \bar{d} / y_h / (\phi \gamma_M)^{-1} r^* + (1 - \beta^{-1}) \bar{d},$$
$$p^*_h y_h = (1 - \beta^{-1}) (1 + s^*) \bar{d} + (\phi \gamma_M)^{-1} r^*,$$
$$p^*_h c^*_h = (\phi \gamma_M)^{-1} r^*,$$
$$c^*_f = s^* c^*_h.$$

The log-linear approximation of the stochastically de-trended home budget constraint is

$$p^*_h (c^*_h - y_h) \hat{p}_{h,t} + p^*_h c^*_h \hat{c}_{h,t} - p^*_h y_h \hat{y}_{h,t} + \hat{b}^*_h + (1 - \beta^{-1}) s^* \hat{s}_t + s^* \hat{b}^*_h\]
$$= \beta^{-1} \bar{d} [(1 + r^*_h) - \hat{\gamma}_M \hat{\bar{y}}_{h,t} + (1 - \beta^{-1}) \hat{y}_{h,t} - \hat{\gamma}_M] + \beta^{-1} \hat{b}^*_h\]
$$= \beta^{-1} \bar{d} ((1 + r^*_h) - \hat{\gamma}_M \hat{\bar{y}}_{h,t} + (1 - \beta^{-1}) \hat{y}_{h,t} - \hat{\gamma}_M) + \beta^{-1} \hat{b}^*_h\]
$$\hat{y}_{h,t} + \hat{c}_{h,t} + (1 + r^*_h) = E_t (\hat{y}_{h,t+1} + \hat{c}_{h,t+1} + \hat{\gamma}_M);$$

(A.2)

that of the home Euler equation is

$$E_t \hat{s}_{t+1} - \hat{s}_t = (1 + r^*_h) - (1 + r^*_h) - E_t (\hat{\gamma}_M \hat{y}_{h,t+1} - \hat{r}_f);$$

(A.3)

and that of the home money demand function is

$$\hat{p}_{h,t} + \hat{c}_{h,t} - \hat{m}_{h,t} = \frac{1}{r^*} (1 + r^*_h) - \hat{\phi}_{h,t}.$$

(A.4)

The foreign country’s counterparts are the log-linear approximation of the stochastically de-trended foreign
Substituting the interest rate $d_i$ yields the following interest rate $d_i$.

I set the parameter $t$ to scrutinize a simpler version of the model that includes two symmetric countries. For this purpose, home and foreign bonds are perfectly substitutable along the equilibrium path. Hence, the equilibrium condition

\[ \text{that of the foreign Euler equation} \]

\[ \ddot{s}_t - \dot{p}_h,t - \dot{c}_{f,t} - \dot{q}_t + \dot{a}_t = (1 + \ddot{r}_{w,t}) - (1 + \dot{r}_{f,t} - \dot{c}_{f,t} - \dot{q}_t + \dot{a}_t) = -\frac{1}{\rho^*}(1 + \ddot{r}_{w,t}) + \dot{\phi}_{f,t}. \] (A.6)

The log-linear approximations of the home country’s interest rates are

\[ (1 + \dot{r}_{h,t}) = (1 + \dot{r}_{w,t}) - \psi(1 - \kappa)\bar{b}_{h,t}^h, \quad \text{and} \quad (1 + \dot{r}_{f,t}) = (1 + \dot{r}_{w,t}) - \psi(1 - \kappa)\bar{b}_{h,t}^f. \] (A.7)

Notice that the home interest rates (A.8) redefine the home UIP condition (A.3) as

\[ E_t\ddot{s}_{t+1} - \ddot{s}_t = (1 + \ddot{r}_{w,t}) - (1 + \dot{r}_{f,t}) - \psi(1 - \kappa)(\bar{b}_{h,t}^h - \bar{b}_{h,t}^f) - E_t(\ddot{s}_{M,t+1} - \ddot{y}_{M,t+1}). \] (A.9)

Comparing the above home UIP condition with the foreign UIP condition (A.6) implies that the home and foreign bonds are perfectly substitutable along the equilibrium path. Hence, the equilibrium condition $\bar{b}_t \equiv \bar{b}_{h,t}^h = \bar{b}_{h,t}^f$ holds.

**Appendix B. Solving the equilibrium with two symmetric countries**

To understand the equilibrium transitory dynamics of the exchange rate in this model, it is informative to scrutinize a simpler version of the model that includes two symmetric countries. For this purpose, I set the parameter $d$ to zero and assume that the transitory output components of the two countries, $y_h$ and $y_f$, are equal to $y$. Notice that the deterministic steady state in this case is characterized by $s^* = 1$, $c_h^* = c_f^* = y$, and $p_h^* = (\phi \gamma) \tau^{r*}$, where $r^* = \gamma / \beta - 1$.

The home and foreign money demand functions, (A.4) and (A.7), and the home interest rates (A.8) yield the following interest rate differential:

\[ (1 + \ddot{r}_{w,t}^h) - (1 + \dot{r}_{w,t}^f) = r^*(\ddot{s}_t + \dot{c}_t - \dot{m}_t + \dot{\phi}_t - \dot{q}_t + \dot{a}_t) + \psi(1 - \kappa)\bar{b}_t. \]

Substituting the interest rate differential into the foreign UIP condition (A.6) leads to the expectational difference equation of the de-trended exchange rate $\ddot{s}_t$:

\[ \ddot{s}_t = \kappa E_t\ddot{s}_{t+1} - (1 - \kappa)\dot{c}_t + (1 - \kappa)(\dot{m}_t - \dot{\phi}_t - \dot{q}_t + \dot{a}_t) + \kappa E_t(\ddot{s}_{M,t+1}^h - \ddot{y}_{M,t+1}^h) - \psi(1 - \kappa)\bar{b}_t. \]

I combine the log-linearized Euler equations of the home and foreign countries, (A.2) and (A.6), with those of the home country’s interest rates (A.8) to yield the first-order expectational difference equation...
of $\tilde{s}_t + \tilde{c}_t - \tilde{q}_t + \tilde{\alpha}_t$:

$$\tilde{s}_t + \tilde{c}_t - \tilde{q}_t + \tilde{\alpha}_t = \kappa E_t(\tilde{s}_{t+1} + \tilde{c}_{t+1} - \tilde{q}_{t+1} + \tilde{\alpha}_{t+1}) + \kappa E_t\tilde{\gamma}_{M,t+1} + (1 - \kappa)(\tilde{m}_t - \tilde{\phi}_t).$$

Since $\kappa$ takes a value between zero and one, the above expectational difference equation has a forward solution of $\tilde{s}_t + \tilde{c}_t - \tilde{q}_t + \tilde{\alpha}_t = \kappa \rho M (1 - \kappa \rho_M)^{-1} \tilde{\gamma}_{M,t} + (1 - \kappa)(1 - \kappa \rho_m)^{-1} \tilde{m}_t - (1 - \kappa)(1 - \kappa \rho_\phi)^{-1} \tilde{\phi}_t$ under a suitable transversality condition. By exploiting this forward solution and the stochastic processes of both countries’ TFPs (1), I rewrite the foreign UIP condition (A.6) as

$$E_t\tilde{s}_{t+1} - \tilde{s}_t = \psi(1 - \kappa)\tilde{b}_t - \frac{\kappa \rho M (1 - \rho_M)}{1 - \kappa \rho_M} \tilde{\gamma}_M \tilde{t} - \frac{(1 - \kappa)(1 - \rho_m)}{1 - \kappa \rho_m} \tilde{m}_t + \frac{(1 - \kappa)(1 - \rho_\phi)}{1 - \kappa \rho_\phi} \tilde{\phi}_t, \quad \text{(B.1)}$$

Furthermore, taking a difference between the log-linearized budget constraints of the home and foreign countries, (A.1) and (A.5), I find the law of motion of the international bond holdings

$$\tilde{b}_t = \beta^{-1}\tilde{b}_{t-1} + p_t^\ast y^\ast \tilde{s}_t - p_t^\ast y^\ast (\tilde{q}_t - \tilde{\alpha}_t) - p_t^\ast y^\ast \kappa \rho_M \tilde{\gamma}_{M,t} - \frac{p_t^\ast y^\ast (1 - \kappa)}{1 - \kappa \rho_m} \tilde{m}_t + \frac{p_t^\ast y^\ast (1 - \kappa)}{1 - \kappa \rho_\phi} \tilde{\phi}_t + p_t^\ast y^* \tilde{y}_t, \quad \text{(B.2)}$$

where $y^* = y/4$ and $\tilde{y}_t \equiv \tilde{y}_{h,t} - \tilde{y}_{f,t}$.

Combining equation (B.1) with equation (B.2) then yields the following second-order expectational difference equation with respect to international bond holdings:

$$E_t\tilde{b}_{t+1} - [1 + \beta^{-1} + p_t^\ast y^\ast \psi(1 - \kappa)]\tilde{b}_t + \beta^{-1}\tilde{b}_{t-1} = -\lambda p_t^\ast y^\ast \tilde{a}_t + p_t^\ast y^\ast (1 - \rho_y)\tilde{q}_t - p_t^\ast y^\ast (1 - \rho_y)\tilde{y}_t, \quad \text{(B.3)}$$

It is straightforward to show that equation (B.3) has two roots, one of which is greater than one and the other of which is less than one. To characterize the roots of the second-order expectational difference equation, see, for example, Sargent (1987).

Substituting equation (B.4) back into equation (B.2) provides the CER for the exchange rate (??):

$$\tilde{s}_t = \frac{\beta \eta - 1}{\beta p_t^\ast y^*} \tilde{b}_{t-1} - \frac{1 - \beta \eta}{1 - \beta \eta(1 - \lambda)} \tilde{a}_t + \frac{1 - \kappa}{1 - \kappa \rho_m} \tilde{m}_t - \frac{1 - \kappa}{1 - \kappa \rho_\phi} \tilde{\phi}_t - \frac{1 - \beta \eta}{1 - \beta \eta \rho_y} \tilde{y}_t + \frac{1 - \beta \eta}{1 - \beta \eta \rho_\phi} \tilde{\phi}_t + \frac{\kappa \rho M}{1 - \kappa \rho_M} \tilde{\gamma}_{M,t}. \quad \text{(B.4)}$$

Therefore, in this symmetric case, the competitive equilibrium along the balanced growth path is characterized by a lower dimensional dynamic system of $(\tilde{s}_t, \tilde{b}_t, \tilde{\alpha}_t, \tilde{\gamma}_{M,t}, \tilde{m}_t, \tilde{\phi}_t, \tilde{y}_t, \tilde{\phi}_t)$.

Adding the log-linearized home and foreign budget constraints together implies the resource constraint $\tilde{c}_h,t + \tilde{c}_{f,t} = \tilde{y}_{h,t} + \tilde{y}_{f,t}$. Since the equilibrium dynamics of the consumption differential follow $\tilde{c}_h,t - \tilde{c}_{f,t} = -\tilde{s}_t + \tilde{q}_t - \tilde{\alpha}_t + \kappa \rho M (1 - \kappa \rho_M)^{-1} \tilde{\gamma}_{M,t} + (1 - \kappa)(1 - \kappa \rho_m)^{-1} \tilde{m}_t - (1 - \kappa)(1 - \kappa \rho_\phi)^{-1} \tilde{\phi}_t$, the
home country’s consumption obeys $2\hat{c}_{h,t} = (\hat{y}_{h,t} + \hat{y}_{f,t}) - \hat{s}_t + \hat{q}_t - \hat{a}_t + \kappa\rho_M(1 - \kappa\rho_m)^{-1}\hat{\gamma}_{M,t} + (1 - \kappa)(1 - \kappa\rho_H)^{-1}\hat{\phi}_{f,t}$, while the foreign country’s is $2\hat{c}_{f,t} = (\hat{y}_{h,t} + \hat{y}_{f,t}) + \hat{s}_t - \hat{q}_t - \hat{a}_t - \kappa\rho_M(1 - \kappa\rho_m)^{-1}\hat{\gamma}_{M,t} - (1 - \kappa)(1 - \kappa\rho_H)^{-1}\hat{\phi}_{f,t}$. The home country’s price $\hat{p}_{h,t}$ then is determined as follows. The Euler equation and the money demand function of the foreign country, (A.6) and (A.7), imply the exponential difference equation of $\hat{s}_t - \hat{p}_{h,t} - \hat{c}_{f,t}$

$$\hat{s}_t - \hat{p}_{h,t} - \hat{c}_{f,t} - \hat{q}_t + \hat{a}_t = \kappa E_t(\hat{s}_{t+1} - \hat{p}_{h,t+1} - \hat{c}_{f,t+1} - \hat{q}_{t+1} + \hat{a}_{t+1} - \hat{\gamma}_{M,t+1} - (1 - \kappa)(\hat{m}_{f,t} - \hat{\phi}_{f,t}).$$

Solving the above equation by forward iterations and imposing a suitable transversality condition yields the CER $\hat{s}_t - \hat{p}_{h,t} - \hat{c}_{f,t} - \hat{q}_t + \hat{a}_t = -\kappa\rho_M(1 - \kappa\rho_m)^{-1}\hat{\gamma}_{M,t} - (1 - \kappa)(1 - \kappa\rho_H)^{-1}\hat{m}_{f,t} + (1 - \kappa)(1 - \kappa\rho_H)^{-1}\hat{\phi}_{f,t}$. This CER characterizes the equilibrium home price

$$2\hat{p}_{h,t} = \hat{s}_t - (\hat{y}_{h,t} + \hat{y}_{f,t}) - \hat{q}_t + \hat{a}_t + \frac{\kappa\rho_M}{1 - \kappa\rho_m}(\hat{\gamma}_{h,t} + \hat{\gamma}_{M,t}) + \frac{1 - \kappa}{1 - \kappa\rho_m}(\hat{m}_{h,t} + \hat{m}_{f,t}) - \frac{1 - \kappa}{1 - \kappa\rho_H}(\hat{\phi}_{h,t} + \hat{\phi}_{f,t}).$$

The money demand functions of both countries, eqs.(A.4) and (A.7), imply that the interest rates in the two countries are

$$(1 + \hat{r}_{h,t}^h) = (1 - \kappa)\left(\frac{\rho_M}{1 - \kappa\rho_m}\hat{\gamma}_{h,t} - \frac{1 - \rho_m}{1 - \kappa\rho_m} \hat{m}_{h,t} + \frac{1 - \rho_H}{1 - \kappa\rho_H} \hat{\phi}_{h,t}\right)$$

$$(1 + \hat{r}_{f,t}^f) = (1 - \kappa)\left(\frac{\rho_M}{1 - \kappa\rho_m}\hat{\gamma}_{M,t} - \frac{1 - \rho_m}{1 - \kappa\rho_m} \hat{m}_{f,t} + \frac{1 - \rho_H}{1 - \kappa\rho_H} \hat{\phi}_{f,t}\right).$$

Finally, as the last endogenous variable, the risk-free nominal interest rate of the home bonds then fluctuates in response to the risk premium, following $(1 + \hat{r}_{w,t}^h) = (1 + \hat{r}_{h,t}^h) + \psi(1 - \kappa)b_t$.

Suppose that $\psi = 0$: There is no debt elastic risk premium in the home country’s interest rate. It is easy to show that in this case, the second-order expectational difference equation (B.3) has a unit root, i.e., $\eta = 1$, and the resulting forward solution turns out to be

$$\hat{b}_t = \hat{b}_{t-1} + \frac{\beta\rho_H^*\gamma^*}{1 - \beta(1 - \lambda)} \hat{a}_t + \frac{\beta\rho_H^*y^*(1 - \rho_H)}{1 - \beta\rho_H} \hat{y}_t - \frac{\beta\rho_H^*y^*(1 - \rho_H)}{1 - \beta\rho_H} \hat{q}_t.$$

Hence, the stochastic process of the de-trended international bond holding $\hat{b}_t$ contains a permanent unit root component and never converges to the steady state. This lack of stationarity of the equilibrium balance growth path motivates this paper to allow for a positive elasticity of the risk premium with respect to the debt level.

Importantly, a permanent stochastic process of the de-trended international bond holding also emerges even when $\kappa = 1$. Because the log-linearized home country’s interest rates (A.8) imply that under $\kappa = 1$, the debt elastic risk premia in play no role in determining the interest rates faced by the home country. As a result, the de-trended international bond holding $\hat{b}_t$ contains a permanent unit root component, as in the case where $\psi = 0$. Hence, the closing of the two-country DSGE model in this paper requires the market discount factor to be strictly less than one.

Appendix C. Derivation of the error correction representation (4)

Let $\eta_t$ denote the fundamental of the DSGE-PVM (3): $\eta_t \equiv \ln M_t - \ln C_t - \psi\hat{b}_t - \ln \phi_t + \ln q_t$. Consider the currency return $\Delta \ln S_t$ adjusted by the fundamental $(1 - \kappa)\eta_{t-1}$: $\Delta \ln S_t + (1 - \kappa)\eta_{t-1}$. The
DSGE-PVM (3) then implies:

\[ \Delta \ln S_t + (1 - \kappa) n_{t-1} = (1 - \kappa) \sum_{j=0}^{\infty} \kappa^j (E_t - E_{t-1}) n_{t+j} + (1 - \kappa) \sum_{j=0}^{\infty} \kappa^j E_{t-1} n_{t+j} - (1 - \kappa) \sum_{j=0}^{\infty} \kappa^j E_{t-1} n_{t+j - 1} - (1 - \kappa) \sum_{j=0}^{\infty} \kappa^j E_{t-1} n_{t+j - 2} + \cdots 
\]

This result means that the currency return has the following error correction representation, given by equation (4):

\[ \Delta \ln S_t = \frac{1 - \kappa}{\kappa} (\ln S_{t-1} - \ln M_{t-1} + \ln C_{t-1} + \psi \tilde{b}_{t-1} + \ln \phi_{t-1} - \ln q_{t-1}) + (1 - \kappa) \sum_{j=0}^{\infty} \kappa^j (E_t - E_{t-1}) n_{t+j}. \]

Appendix D. Data description and construction

All data for the United States are distributed by Federal Reserve Economic Data (FRED), operated by the Federal Reserve Bank of St. Louis (http://research.stlouisfed.org/fred2/). The consumption data are constructed as the sum of the real personal consumption expenditures on non-durables and services. FRED, however, distributes only the nominal values of the two categories of personal consumption expenditures as Personal Consumption Expenditure on Non-Durables (PCND) and Personal Consumption Expenditure on Services (PCESV). To construct the real total personal consumption expenditure \( C_{us,t} \), I first calculate the share of the two nominal consumption categories in the nominal total personal consumption expenditure Personal Consumption Expenditure and then multiply the real total personal consumption expenditures, Real Personal Consumption Expenditures at Chained 2005 Dollars (PCECC96), by the calculated share. Following NR, I adopt the M1 money stock, \( M_{1SL} \), as the aggregate money supply \( M_{us,t} \).

The nominal interest rate \( r_{us,t} \) is provided by three-month Treasury Bill (TBM3MS). All the variables except the nominal interest rate are seasonally adjusted at annual rates and converted to the corresponding per capita terms by Total Population (POP).

All Canadian data are distributed by Statistics Canada (CANSIM) (http://www5.statcan.gc.ca/cansim/). The real consumption data \( C_{can,t} \) are constructed as the sum of Personal Consumption Expenditure on Non-Durables at Chained 2002 Dollars, Personal Expenditure on Semi-Durables at Chained 2002 Dollars, and Personal Expenditure on Services at Chained 2002 Dollars. I use the M1 money stock as the money supply \( M_{can,t} \). The nominal interest rate \( r_{can,t} \) is provided by three-month Treasury Bills. All the variables except the nominal interest rate are seasonally adjusted at annual rates and converted to the corresponding per capita terms by Estimate of Total Population.
The output measures for Canada and the United States, $Y_{can,t}$ and $Y_{us,t}$, are constructed as in a model-consistent way. In this two-country endowment economy model, a country’s output is given by the sum of consumption and the trade balance. To measure the bilateral trade balance between Canada and the United States, $TB_t$, I use the Canadian goods trade balance for the United States included in CANSIM’s balance of international payments data (CANSIM Table 376-0005). The Canadian output $Y_{can,t}$ is constructed by $C_{can,t} + TB_t$ and the United States output $Y_{us,t}$ is constructed by $C_{us,t} - TB_t/S_t$, where $S_t$ is the bilateral exchange rate between Canada and the United States.
<table>
<thead>
<tr>
<th>Parameters</th>
<th>Distribution</th>
<th>Mean</th>
<th>S.D.</th>
<th>95 % Coverage</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>Household Subjective Discount Factor</td>
<td>Uniform(0,1)</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$\gamma_M$</td>
<td>Deterministic (Gross) Money Growth</td>
<td>Gamma</td>
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<td>0.005</td>
</tr>
<tr>
<td>$\gamma_S$</td>
<td>Deterministic EX Trend</td>
<td>Normal</td>
<td>0.000</td>
<td>1.500</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Debt Elasticity of Risk Premium</td>
<td>Gamma</td>
<td>0.010</td>
<td>0.001</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Technology Diffusion Speed</td>
<td>Beta</td>
<td>0.010</td>
<td>0.001</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Mean Money Demand Shock</td>
<td>Gamma</td>
<td>1.000</td>
<td>0.010</td>
</tr>
<tr>
<td>$\rho_M$</td>
<td>Permanent Money Growth AR(1) Coef.</td>
<td>Beta</td>
<td>0.100</td>
<td>0.010</td>
</tr>
<tr>
<td>$\rho_q$</td>
<td>RER AR(1) Coef.</td>
<td>Beta</td>
<td>0.850</td>
<td>0.100</td>
</tr>
<tr>
<td>$\rho_\phi$</td>
<td>Money Demand AR(1) Coef.</td>
<td>Beta</td>
<td>0.850</td>
<td>0.100</td>
</tr>
</tbody>
</table>

Note 1. The AR(1) coefficients of the transitory money and output shocks, $\rho_m$ and $\rho_y$ respectively, have the mass points of zero for identification.

Note 2. The standard deviations of all the structural shocks, $\sigma_M$, $\sigma_A$, $\sigma_y$, $\sigma_q$, $\sigma_\phi$, have the identical inverse Gamma prior distribution, with a mean of 0.01 and standard deviation of 0.01 for the benchmark information set.

Note 3.: The prior distribution of $\beta$ is given by the Beta distribution, with a mean of 0.999 and standard deviation of 0.001 for the High Discount Factor model.
Table 2: Posterior Distributions of Structural Parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Benchmark Mean</th>
<th>S.D.</th>
<th>95 % Interval</th>
<th>HDF Mean</th>
<th>S.D.</th>
<th>95 % Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>0.537</td>
<td>0.041</td>
<td>[0.457 0.618]</td>
<td>0.950</td>
<td>0.001</td>
<td>[0.947 0.952]</td>
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<tr>
<td>$\beta$</td>
<td>0.547</td>
<td>0.042</td>
<td>[0.464 0.630]</td>
<td>0.998</td>
<td>0.000</td>
<td>[0.998 0.999]</td>
</tr>
<tr>
<td>$\gamma_M$</td>
<td>1.019</td>
<td>0.005</td>
<td>[1.009 1.027]</td>
<td>1.051</td>
<td>0.001</td>
<td>[1.050 1.054]</td>
</tr>
<tr>
<td>$\gamma_S$</td>
<td>-0.000</td>
<td>0.002</td>
<td>[-0.005 0.002]</td>
<td>0.002</td>
<td>0.000</td>
<td>[0.002 0.003]</td>
</tr>
<tr>
<td>$\psi$</td>
<td>0.010</td>
<td>0.001</td>
<td>[0.008 0.012]</td>
<td>0.011</td>
<td>0.000</td>
<td>[0.009 0.012]</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.009</td>
<td>0.001</td>
<td>[0.008 0.011]</td>
<td>0.007</td>
<td>0.000</td>
<td>[0.006 0.009]</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.999</td>
<td>0.010</td>
<td>[0.980 1.019]</td>
<td>0.999</td>
<td>0.001</td>
<td>[0.980 1.018]</td>
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<tr>
<td>$\rho_M$</td>
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<td>0.005</td>
<td>[0.083 0.097]</td>
<td>0.105</td>
<td>0.003</td>
<td>[0.099 0.108]</td>
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<tr>
<td>$\rho_q$</td>
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<td>0.044</td>
<td>[0.778 0.948]</td>
<td>0.982</td>
<td>0.004</td>
<td>[0.974 0.989]</td>
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<tr>
<td>$\rho_\phi$</td>
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<td>0.001</td>
<td>[0.995 0.999]</td>
<td>0.928</td>
<td>0.002</td>
<td>[0.924 0.931]</td>
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<tr>
<td>$\sigma_M$</td>
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<td>0.001</td>
<td>[0.016 0.019]</td>
<td>0.026</td>
<td>0.001</td>
<td>[0.024 0.028]</td>
</tr>
<tr>
<td>$\sigma_A$</td>
<td>0.006</td>
<td>0.000</td>
<td>[0.005 0.007]</td>
<td>0.006</td>
<td>0.000</td>
<td>[0.005 0.007]</td>
</tr>
<tr>
<td>$\sigma_m$</td>
<td>0.007</td>
<td>0.001</td>
<td>[0.005 0.009]</td>
<td>0.105</td>
<td>0.004</td>
<td>[0.098 0.111]</td>
</tr>
<tr>
<td>$\sigma_y$</td>
<td>0.003</td>
<td>0.000</td>
<td>[0.002 0.003]</td>
<td>0.003</td>
<td>0.000</td>
<td>[0.002 0.003]</td>
</tr>
<tr>
<td>$\sigma_q$</td>
<td>0.006</td>
<td>0.001</td>
<td>[0.004 0.009]</td>
<td>0.006</td>
<td>0.000</td>
<td>[0.005 0.006]</td>
</tr>
<tr>
<td>$\sigma_\phi$</td>
<td>0.027</td>
<td>0.002</td>
<td>[0.024 0.031]</td>
<td>0.066</td>
<td>0.003</td>
<td>[0.061 0.070]</td>
</tr>
</tbody>
</table>

Marginal Likelihood: 2148.572 1871.309

Note 1: The “Benchmark” represents the Benchmark specification of the two-country model and the “HDF” represents the High Discount Factor specification.

Note 2: The marginal likelihoods are estimated based on Geweke’s (1999) harmonic mean estimator.
Figure 1: Depreciation rates and in-sample predictions. Note: The solid black line represents the actual depreciation rate of the Canadian dollar against the US dollar. The dashed blue and dotted red lines in (a) and (b) respectively indicate the 95% HPD intervals of the in-sample predictions on the depreciation rate for the Benchmark and HDF models.
Figure 2: (a) Depreciation rates and historical decomposition of the in-sample prediction of the depreciation rate into structural shocks: the Benchmark model. Note: In each window, the solid black line represents the actual depreciation rate of the Canadian dollar against the US dollar. In each window with a particular structural shock, the dashed blue lines represents the 95% HPD interval of the in-sample prediction of the depreciation rate with the Kalman smoother of the corresponding structural shock excluded.
Figure 2: (b) Depreciation rates and historical decomposition of the in-sample prediction of the depreciation rate into structural shocks: the HDF model. Note: In each window, the solid black line represents the actual depreciation rate of the Canadian dollar against the US dollar. In each window with a particular structural shock, the dotted red lines represents the 95% HPD interval of the in-sample prediction of the depreciation rate with the Kalman smoother of the corresponding structural shock excluded.
Figure 3: (a) Consumption growth differential and historical decomposition of the in-sample prediction of the consumption growth differential into structural shocks: the Benchmark model. Note: In each window, the solid black line represents the actual consumption growth differential. In each window with a particular structural shock, the dashed blue lines represents the 95% HPD interval of the in-sample prediction of the consumption growth differential with the Kalman smoother of the corresponding structural shock excluded.
Figure 3. (b) Consumption growth differential and historical decomposition of the in-sample prediction of the consumption growth differential into structural shocks: the HDF model. Note: In each window, the solid black line represents the actual consumption growth differential. In each window with a particular structural shock, the dotted red lines represents the 95% HPD interval of the in-sample prediction of the consumption growth differential with the Kalman smoother of the corresponding structural shock excluded.
Figure 4: (a) TB differential and historical decomposition of the in-sample prediction of the TB differential into structural shocks: the Benchmark model. Note: In each window, the solid black line represents the actual TB differential. In each window with a particular structural shock, the dashed blue lines represent the 95% HPD interval of the in-sample prediction of the TB differential with the Kalman smoother of the corresponding structural shock excluded.
Figure 4: (b) TB differential and historical decomposition of the in-sample prediction of the TB differential into structural shocks: the HDF model. Note: In each window, the solid black line represents the actual TB differential. In each window with a particular structural shock, the dashed blue lines represents the 95% HPD interval of the in-sample prediction of the TB differential with the Kalman smoother of the corresponding structural shock excluded.
Figure 5: Forecast errors for observations. Note: In each window corresponding to a particular observation, the dashed blue and dotted red lines display the 95% HPD intervals of the one-period-ahead forecast error calculated through the Kalman forward recursion based on the state space representation of the Benchmark and HDF models, respectively.