Characterizing Social Value of Information

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Abstract

This paper characterizes the social value of information in Bayesian games with symmetric quadratic payoff functions and normally distributed public and private signals. The main result provides a necessary and sufficient condition for welfare to increase with public or private information. In so doing, we represent welfare as a linear combination of the variance of a common term in an equilibrium strategy and that of an idiosyncratic term, which are referred to as the common variance and the idiosyncratic variance of actions, respectively. The ratio of their coefficients is a key parameter in our condition. If the coefficient of the common variance is relatively large, welfare necessarily increases, but if it is relatively small, welfare can decrease. Using our condition, we find eight types of games with different welfare effects of information.

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1 Introduction

In multi-agent situations, more information is not necessarily valuable, thus raising doubts over the desirability of transparency. A notable example is a beauty contest game of Morris and Shin [32] (henceforth MS).[1] They show that increased precision of public information is detrimental to welfare if players have access to sufficiently precise private information. As explained by MS, the key factor underlying the anti-transparency result is a strategic complementarity, which induces players' overreaction to public information.

However, Angeletos and Pavan [3] (henceforth AP) make it clear that a strategic complementarity is neither necessary nor sufficient for the anti-transparency result. The key factor is what AP refer to as the equilibrium degree of coordination relative to the socially optimal degree of coordination. AP consider a general class of Bayesian games where a continuum of players have symmetric quadratic payoff functions and receive normally distributed public and private signals on the state of fundamentals,[2] which includes a beauty contest game. AP ask under what conditions welfare necessarily increases or decreases with public or private information and show the following among others.[3] In games with strategic complementarities, welfare necessarily increases with public information if the equilibrium degree of coordination is lower than the socially optimal degree of coordination. Symmetrically, in games with strategic substitutabilities, welfare necessarily increases with private information if the equilibrium degree of coordination is higher than the socially optimal degree of coordination.[4]

The comparison of the degrees of coordination is intuitive, insightful, and use-

[1]Earlier papers on this issue include Hirshleifer [23], Ho and Blau [24], Levine and Ponssard [29], Green and Stokey [17], Kamien et al. [27], Neyman [35], Bassan et al. [7], and Teoh [38].

[2]As a model of informationally decentralized organizations, Ui [39] independently proposes a more general class of quadratic Bayesian potential games with a finite number of players, allowing asymmetry of payoff functions and information structures.

[3]AP decompose an information structure into its accuracy and its commonality and study their social value, from which AP examine the social value of public and private information.

[4]More precisely, these results hold in games of which equilibria are efficient under complete information but inefficient under incomplete information.

ful in understanding the social value of information. A minor limitation is that it is not universally applicable. For example, it does not say much about the social value of public information in games with strategic substitutabilities and that of private information in games with strategic complementarities. Moreover, the following questions remain unanswered. Exactly in what games can welfare decrease with public or private information? In such games, exactly when does welfare decrease? Does complete information remain socially optimal? If not, what information structure maximizes welfare? Specifically, what is the optimal degree of transparency in public information?

The purpose of this paper is to figure out a universally applicable key factor determining the social value of information and to answer the above questions by using it. To this end, our main result provides a necessary and sufficient condition for welfare to increase with public or private information for given precision of information, by which we find out the key factor. Our condition is in contrast to AP's conditions ensuring that welfare necessarily increases or decreases with public or private information, regardless of precision of information.

In the main result, we consider a finite-player version of AP's model because of the following advantages. First, it is straightforward to extend the finite case to the continuum case. Next, we can conduct comparative statics with respect to the number of players. Finally, the assumption of a continuum of players is inappropriate in some cases for studying the social value of information. For example, in voluntary provision of public goods, each player would make no contribution facing an infinite number of opponents, where information has no influence on welfare.

Our measure of welfare is the ex ante expected payoff in the equilibrium, which is the same as AP's measure. The difference is that we represent it as a linear combination of the variance of a common term in the equilibrium strategy and that of an idiosyncratic term.[5] These variances are referred as the common variance and the idiosyncratic variance of actions, respectively.

[5]AP use a similar but different representation. See Section 5.1.

Our main result reveals that a key factor determining the social value of information is the ratio of the coefficients of the common variance and the idiosyncratic variance. For example, suppose that both coefficients are positive. Then, if the coefficient of the common variance is relatively large, welfare necessarily increases with both public and private information, but if it is relatively small, welfare can decrease with both public and private information. This is due to the following properties of the common variance and the idiosyncratic variance. The common variance equals the covariance of actions. Thus, it necessarily increases with both public and private information because more precise information causes more correlated actions. In contrast, the idiosyncratic variance equals the difference between the variance and covariance of actions. Thus, it can decrease with both public and private information because a higher correlation of actions brings the covariance and variance closer.

There are eight types of games with different welfare effects of information, which are determined by the coefficients of the idiosyncratic variance and the common variance in welfare. For example, consider a class of games in which the coefficient of the idiosyncratic variance is nonnegative. The following taxonomy is based upon the relative weight of the common variance in welfare. If the relative weight is sufficiently large, welfare necessarily increases with both public and private information as discussed above. We call this game type +I. If the relative weight is intermediate, welfare can decrease, but only with public information. We call this game type +II. If the relative weight is sufficiently small but still positive, welfare can decrease as well as increase with both public and private information. We call this game type +III. If the relative weight is negative, welfare can decrease with both, but can increase only with private information. We call this game type +IV. The remaining four types of games, which are referred to as types −I, −II, −III, and −IV, are the counterparts of types +I, +II, +III, and +IV with the opposite welfare effects of information, respectively.

In each type, we characterize the information structure that maximizes welfare.[6] Complete information is optimal in types +I, +II, and −IV. No information

[6]Bergemann and Morris [8] study a Cournot game with a continuum of players and identify

is optimal in types −I, −II, and −III. Incomplete information only with appropriate noisy private signals is optimal in types +III and +IV. We also identify the optimal precision of public information fixing the precision of private information and the optimal precision of private information fixing the precision of public information.

There are several applications. For example, a Cournot game with linear demand and cost functions is type +I with two players, type +II with three, and type +III with four or more. Thus, the expected profit can decrease with both public and private information if there are more than four players. In contrast, the expected total surplus necessarily increases, irrespectively of the number of players [43, 44]. We can verify it using our main result because the expected total surplus is represented as a linear combination of the common variance and the idiosyncratic variance. That is, we show that there exists a fictitious game such that the equilibrium coincides with that of a Cournot game and the expected payoff equals the expected total surplus, and that this game is type +I. Therefore, the expected total surplus necessarily increases with both public and private information.

We also consider public goods games with quadratic production and linear cost functions. If production is random, this game is type +I, but if cost is random, this game is type −I. Thus, the welfare effects of information can be opposite depending upon the source of uncertainty.

All the above results have their counterparts in games with a continuum of players, which are also classified into eight types with the same properties as those in the finite case. For example, MS's beauty contest game is type +I or +II, but its variant studied by Hellwig and Veldkamp [21] is type +I or −IV. Thus, welfare can decrease only with public information in MS's beauty contest game, but only with private information in Hellwig and Veldkamp's beauty contest game.

The aforementioned characterization of the socially optimal information structures also holds in the continuum case, and it is useful in identifying socially

its optimal information structure, which is a special case of our result in the continuum case. See Section 5.3. In an auction with many bidders, Bergemann and Pesendorfer [9] study its optimal information structure that maximizes revenue.

optimal Bayesian *correlated* equilibria. Consider a mediator who knows the true state and makes private action recommendations to players having no information about the state. If each player has an incentive to follow the mediator's recommendation, we say that the resulting joint action distribution is a Bayesian correlated equilibrium. Bergemann and Morris [8] show that, in games with a continuum of players, the set of all Bayesian correlated equilibria coincides with the set of all action distributions of Bayesian Nash equilibria generated by the bivariate (public and private) signal structures. This finding implies that the action distribution of the Bayesian Nash equilibrium under the aforementioned optimal information structure is the Bayesian correlated equilibrium that achieves the highest welfare. Thus, the recommended actions in the optimal Bayesian correlated equilibria are completely correlated in types +I, +II, and −IV, constant in types −I, −II, and −III, and conditionally independent given the state in types +III and +IV. This characterization of optimal Bayesian correlated equilibria in AP's model complements that in a large Cournot game due to Bergemann and Morris [8] because a large Cournot game is a special case of AP's model. Bergemann and Morris [8] show that the optimal recommended actions in a large Cournot game are either completely correlated or conditionally independent given the state, which is also implied by our result because a large Cournot game is type +I, +II, or +III.

Early studies on the social value of information in Bayesian games such as Levine and Ponssard [29] and Ho and Blau [24] consider 2×2 games with binary states and demonstrate that more information can be harmful. On the other hand, many recent studies including MS and AP adopt quadratic payoff functions and normally distributed signals, of which the study originates in the work of Radner [36] and is elaborated by Basar and Ho [6] and Vives [43, 44].[7] As demonstrated by Vives [43, 44], we can incorporate endogenous information structures into this framework such as information sharing and information acquisition.

Quadratic Bayesian games are also used in the following debates on MS's anti-transparency result. Svensson [37] points out that it is a consequence of unreason-

[7]For more details, see Ui [39] and references therein.

able parameter values. Angeletos and Pavan [2] and Hellwig [20] note that it stems from the particular payoff function, which leads to AP's general model. Several authors study more elaborated beauty contest models.[8] Cornand and Heinemann [13] argue that welfare increases when more precise public information reaches only a fraction of players. Colombo and Femminis [11] show that welfare necessarily increases with public information when players can choose the precision of private information and the marginal cost of private information exceeds that of public information. In contrast, James and Lawler [25] show that welfare necessarily decreases with public information when a policy maker has a direct influence on payoffs as well as announces public information. Morris and Shin [33] study semi-public information that is common knowledge among a fraction of players and demonstrate a trade-off between precision of information and fragmentation of that information. These results also stem from the particular payoff function, but Colombo et al. [12] adopt AP's model and study the difference between the social value of public information when the precision of private information is endogenously determined and that when it is exogenous.

In contrast to the above papers, this paper gives a complete characterization of the social value of information in AP's model under exogenous information structures, which offers another way to understand why more information can be harmful in terms of the common variance and the idiosyncratic variance. A limitation of our study is that the social value of public information under endogenous information structures can be different from that under exogenous information structures, as demonstrated by Colombo et al. [12].[9]

This paper is organized as follows. After introducing the model in Section 2, we present the main result in Section 3 and discuss applications in Section 4. Section 5 is devoted to the continuum case. We conclude the paper in Section 6.

[8] See also Arato and Nakamura [5].

[9] Ui [42] suggests one way to reconcile the difference using convexity of information acquisition costs.

2 The model

Consider a Bayesian game with n players. An individual player is indexed by $i \in N \equiv \{1, \dots, n\}$. Player i's action is a real number $a_i \in \mathbb{R}$. His payoff function is quadratic in an action profile $a \equiv (a_i)_{i\in N} \in \mathbb{R}^N$ and a payoff state $\theta \in \mathbb{R}$ and symmetric with respect to the permutation of players; that is,

$$\begin{aligned} u_i(a, \theta) = &- a_i^2 + 2\alpha a_i \sum_{j\neq i} a_j + 2\beta\theta a_i \\ &+ \kappa \sum_{j\neq i} a_j^2 + \lambda \sum_{j<k:j,k\neq i} a_j a_k + \mu \sum_{j\neq i} \theta a_j + \nu \sum_{j\neq i} a_j + f(\theta), \end{aligned} \tag{1}$$

where $\alpha, \beta, \kappa, \lambda, \mu, \nu \in \mathbb{R}$ are constants and $f : \mathbb{R} \to \mathbb{R}$ is a measurable function. Constants α and β are coefficients of terms including a_i, which determine player i's best response. This game exhibits strategic complementarity if $\alpha > 0$ and strategic substitutability if $\alpha < 0$. We assume $\beta > 0$ without loss of generality. Constants κ, λ, μ, and ν are coefficients of terms not including a_i, which have no influence on player i's best response. As we will see later, ν and $f(\theta)$ play no role in our welfare analysis.

Player i observes a private signal $x_i = \theta + \varepsilon_i$ and a public signal $y = \theta + \varepsilon_0$, where ε_i, ε_0, and θ are independently and normally distributed[10] with

$$E[\theta] = \bar{\theta},\ E[\varepsilon_i] = E[\varepsilon_0] = 0,\ \mathrm{var}[\theta] = \tau_\theta^{-1},\ \mathrm{var}[\varepsilon_i] = \tau_x^{-1},\ \mathrm{var}[\varepsilon_0] = \tau_y^{-1},$$

and ε_i and ε_j are independent for $i \neq j$. Player i's signal vector is denoted by $\mathbf{s}_i = (x_i, y)^\top$. We refer to τ_x, τ_y, and $\tau = (\tau_x, \tau_y)$ as the precision of private information, that of public information, and an information structure of the game, respectively.

Let $\sigma_i : \mathbb{R}^2 \to \mathbb{R}$ be player i's strategy for $i \in N$, which maps a signal vector $\mathbf{s}_i \in \mathbb{R}^2$ to an action $\sigma_i(\mathbf{s}_i) \in \mathbb{R}$. A strategy profile $(\sigma_i)_{i\in N}$ is a Bayesian Nash equilibrium if each player maximizes his interim expected payoff given the opponents' strategies; that is, $\sigma_i(\mathbf{s}_i) = \arg\max_{a_i} E[u_i((a_i, \sigma_{-i}), \theta)|\mathbf{s}_i]$ for all $\mathbf{s}_i \in \mathbb{R}^2$ and

[10] We can weaken the assumption of normal distributions using the result of Ericson [15]. See Vives [44].

$i \in N$, where $\sigma_{-i} = (\sigma_j(\mathbf{s}_j))_{j \neq i}$. The first order condition for equilibrium is

$$\sigma_i(\mathbf{s}_i) = \alpha \sum_{j \neq i} E[\sigma_j(\mathbf{s}_j)|\mathbf{s}_i] + \beta E[\theta|\mathbf{s}_i], \tag{2}$$

which is also player i's best response. Following AP, we confine our attention to a symmetric Bayesian Nash equilibrium.

As pointed out by Basar and Ho [6] and Vives [43, 44], players in this class of games behave as if they are in teams, where payoff functions are identical. In other words, this game is a Bayesian potential game [31, 22, 39]. Thus, we can calculate a unique symmetric Bayesian Nash equilibrium using the result of Radner [36, Theorem 5] on team decision problems.[11]

Lemma 1. *If* $\hat{\alpha} \equiv (n-1)\alpha < 1$, *then a Bayesian game with* (1) *has a unique symmetric Bayesian Nash equilibrium* $(\sigma_i)_{i \in N}$ *with*

$$\sigma_i(\mathbf{s}_i) = \mathbf{b}^\top (\mathbf{s}_i - E[\mathbf{s}_i]) + c \tag{3}$$

for all $\mathbf{s}_i \in \mathbb{R}^2$ *and* $i \in N$, *where*

$$\mathbf{b} = (b_x, b_y)^\top = \left(\frac{\beta \tau_x}{\tau_\theta + (1-\hat{\alpha})\tau_x + \tau_y}, \frac{\beta \tau_y}{(1-\hat{\alpha})(\tau_\theta + (1-\hat{\alpha})\tau_x + \tau_y)} \right)^\top, \ c = \frac{\beta \bar{\theta}}{1-\hat{\alpha}}.$$

Proof. See Appendix A. □

The ratio of the coefficient of a private signal to that of a public signal is

$$b_x / b_y = (1-\hat{\alpha})\tau_x / \tau_y. \tag{4}$$

Thus, if $\hat{\alpha}$ is close to one or τ_x/τ_y is small, the relative weight of a public signal is large, and if $\hat{\alpha}$ is small or τ_x/τ_y is large, the relative weight of a private signal is large.

To obtain the expected payoff, it is useful to rewrite the equilibrium strategy as

$$\begin{aligned} \sigma_i(\mathbf{s}_i) &= b_x(\theta + \varepsilon_i - \bar{\theta}) + b_y(\theta + \varepsilon_0 - \bar{\theta}) + c \\ &= b_x \varepsilon_i + \big(b_y \varepsilon_0 + (b_x + b_y)\theta\big) + \big(c - (b_x + b_y)\bar{\theta}\big), \end{aligned} \tag{5}$$

[11] See Vives [45, Chapter 8], Vives [46, Chapter 2], and Ui [39]. Radner [36] allows asymmetry of payoff functions and information structures. Ui [39] asks what games have the same best response correspondences as those of teams of Radner [36].

where $b_x\varepsilon_i$ is an idiosyncratic random term and $b_y\varepsilon_0 + (b_x + b_y)\theta$ is a common random term. We refer to the variances of these terms, $\mathrm{var}[b_x\varepsilon_i]$ and $\mathrm{var}[b_y\varepsilon_0 + (b_x + b_y)\theta]$, as the idiosyncratic variance and the common variance of actions, respectively. Because ε_i, ε_0, and θ are independent, it holds that

$$\mathrm{var}[b_y\varepsilon_0 + (b_x + b_y)\theta] = \mathrm{cov}[\sigma_i, \sigma_j],\ \mathrm{var}[b_x\varepsilon_i] = \mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i, \sigma_j].$$

That is, the common variance equals the covariance of actions and the idiosyncratic variance equals the difference between the variance and covariance of actions.

The next lemma represents the expected payoff as a linear function of the idiosyncratic variance and the common variance.

Lemma 2. *The ex ante expected payoff* $E[u_i(\sigma, \theta)]$ *equals*

$$\begin{aligned} W(\tau) \equiv& \zeta \mathrm{var}[b_x\varepsilon_i] + \eta \mathrm{var}[b_y\varepsilon_0 + (b_x + b_y)\theta] \\ =& \zeta(\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i, \sigma_j]) + \eta \mathrm{cov}[\sigma_i, \sigma_j] \end{aligned} \tag{6}$$

plus a constant independent of τ, *where*

$$\zeta = \hat{\mu}/\beta + \hat{\kappa} + 1,\ \eta = (1 - \hat{\alpha})\hat{\mu}/\beta + \hat{\kappa} + \hat{\lambda} + 1, \tag{7}$$

$$\hat{\kappa} = (n-1)\kappa,\ \hat{\lambda} = (n-1)(n-2)\lambda/2,\ \hat{\mu} = (n-1)\mu,\ \hat{\nu} = (n-1)\nu. \tag{8}$$

Proof. See Appendix B. □

We adopt $W(\tau)$ as a measure of welfare because the ex ante expected payoff equals $W(\tau)$ plus a constant independent of τ. Note that welfare under no information is normalized to zero. In fact, when players have no information, they choose a constant strategy (i.e., $b_x = b_y = 0$) and thus the variances are zero.

Remark 1. AP's measure of welfare is also the ex ante expected payoff, but AP use a different representation. See Section 5.1. MS's measure of welfare is the conditional expected payoff given the true state θ. However, the conditional expected payoff in MS's beauty contest game does not depend upon θ. Thus, the welfare analysis of MS's beauty contest game is essentially the same as that based upon the ex ante expected payoff. See Section 5.2.

3 Results

3.1 Social value of information

We study the social value of information in terms of the signs of $\partial W(\tau)/\partial\tau_x$ and $\partial W(\tau)/\partial\tau_y$. AP ask under what conditions $\partial W(\tau)/\partial\tau_x > 0$ for all τ or $\partial W(\tau)/\partial\tau_y > 0$ for all τ. In contrast, we ask under what conditions $\partial W(\tau)/\partial\tau_x > 0$ for given τ or $\partial W(\tau)/\partial\tau_y > 0$ for given τ. The main result of this paper is the following necessary and sufficient condition.

Proposition 1. *Assume that $\hat{\alpha} < 1$, $\beta > 0$, and $(\zeta, \eta) \neq (0, 0)$. Define*

$$X \equiv \begin{cases} (1-\hat{\alpha}) - 2\eta/\zeta & \text{if } \zeta \neq 0, \\ -\infty & \text{if } \zeta = 0, \end{cases}$$

$$Y \equiv (1-\hat{\alpha})\left(2(1-\hat{\alpha})\zeta/\eta - 3\right) \quad \text{if } \eta \neq 0.$$

Then, the following holds for $\tau_x, \tau_y, \tau_\theta > 0$.

(i) *In a game with $\zeta \geq 0$ and $\eta > 0$,*

$$\frac{\partial W(\tau)}{\partial \tau_x} \gtreqless 0 \;\Leftrightarrow\; X \lesseqgtr (\tau_y + \tau_\theta)/\tau_x,$$
$$\frac{\partial W(\tau)}{\partial \tau_y} \gtreqless 0 \;\Leftrightarrow\; Y \lesseqgtr (\tau_y + \tau_\theta)/\tau_x,$$

where $X \leq Y$ and the equality holds only if $X, Y < 0$.

(ii) *In a game with $\eta \leq 0 < \zeta$,*

$$\frac{\partial W(\tau)}{\partial \tau_x} \gtreqless 0 \;\Leftrightarrow\; X \lesseqgtr (\tau_y + \tau_\theta)/\tau_x,$$
$$\frac{\partial W(\tau)}{\partial \tau_y} < 0 \text{ for all } \tau,$$

where $X > 0$.

(iii) *In a game with $\zeta \leq 0$ and $\eta < 0$,*

$$\frac{\partial W(\tau)}{\partial \tau_x} \gtreqless 0 \;\Leftrightarrow\; X \gtreqless (\tau_y + \tau_\theta)/\tau_x,$$
$$\frac{\partial W(\tau)}{\partial \tau_y} \gtreqless 0 \;\Leftrightarrow\; Y \gtreqless (\tau_y + \tau_\theta)/\tau_x,$$

where $X \leq Y$ and the equality holds only if $X, Y < 0$.

(iv) *In a game with* $\zeta < 0 \leq \eta$,

$$\frac{\partial W(\tau)}{\partial \tau_x} \gtreqless 0 \;\Leftrightarrow\; X \gtreqless (\tau_y + \tau_\theta)/\tau_x,$$
$$\frac{\partial W(\tau)}{\partial \tau_y} > 0 \textit{ for all } \tau,$$

where $X > 0$.

Proof. See Appendix C. □

We can check whether welfare increases with public or private information by comparing $(\tau_y + \tau_\theta)/\tau_x$ with X or Y. Note that X and Y are functions of ζ/η and $\hat{\alpha}$. Thus, the social value of information is determined by the ratio of the coefficients of the idiosyncratic variance and the common variance (i.e. ζ/η) and the slope of best responses with respect to average actions (i.e. $\hat{\alpha}$).

To see the roles of ζ/η and $\hat{\alpha}$, consider the case of (i) with $\zeta \geq 0$ and $\eta > 0$. Welfare increases with private information if and only if $X < (\tau_y + \tau_\theta)/\tau_x$, and with public information if and only if $Y < (\tau_y + \tau_\theta)/\tau_x$. If ζ/η is sufficiently small or $\hat{\alpha}$ is sufficiently close to one, then $X, Y < 0$. In this case, welfare necessarily increases with both public and private information because $\max\{X, Y\} < (\tau_y + \tau_\theta)/\tau_x$ always holds. In contrast, if ζ/η is sufficiently large or $\hat{\alpha}$ is sufficiently small, then $X, Y > 0$. In this case, welfare decreases with both public and private information if $\min\{X, Y\} > (\tau_y + \tau_\theta)/\tau_x$.

The following properties of the common variance and the idiosyncratic variance help us to understand the intuition. The common variance, which equals the covariance of actions, necessarily increases with both public and private information because more precise information causes more correlated actions. In contrast, the idiosyncratic variance, which equals the difference between the variance and covariance of actions, can decrease because a higher correlation of actions brings the covariance and variance closer.

Therefore, welfare necessarily increases with both public and private information if the common variance is relatively large in welfare, which is true when ζ/η is sufficiently small or $\hat{\alpha}$ is sufficiently close to one. When $\hat{\alpha}$ is sufficiently close to one, the relative weight of a public signal in the equilibrium strategy is very large by (4). In this case, the common variance is relatively large in welfare because it is the variance of the common random term in the equilibrium strategy.

In contrast, welfare can decrease with both public and private information if the idiosyncratic variance is relatively large in welfare, which is true when ζ/η is sufficiently large or $\hat{\alpha}$ is sufficiently small. When $\hat{\alpha}$ is sufficiently small, the relative weight of a private signal in the equilibrium strategy is very large by (4). In this case, the idiosyncratic variance is relatively large in welfare because it is the variance of the idiosyncratic random term in the equilibrium strategy.

We can verify the above properties of the common variance and the idiosyncratic variance by applying Proposition 1 to the following two games: a game with $(\zeta, \eta) = (0, 1)$ and a game with $(\zeta, \eta) = (1, 0)$.

Corollary 2. *The common variance necessarily increases with τ_x and τ_y for all τ. The idiosyncratic variance decreases with τ_y for all τ, and with τ_x if and only if $\tau_x > (\tau_y + \tau_\theta)/(1 - \hat{\alpha})$.*

Remark 2. The idiosyncratic variance not only decreases but also increases with private information if its precision τ_x is sufficiently small. When $\tau_x = 0$, the coefficient of a private signal in the equilibrium strategy equals zero, i.e., $b_x = 0$, because a player ignores his private signal. This implies that the idiosyncratic variance $\text{var}[b_x \varepsilon_i]$ equals zero. In this case, as τ_x increases, b_x increases, and thus the idiosyncratic variance also increases.

Remark 3. Recall that the common variance is the covariance of actions and the idiosyncratic variance is the difference between the variance and covariance of actions. Thus, even if signals are not normally distributed or payoff functions are not quadratic, the common variance and the idiosyncratic variance are well defined and may have properties similar to the above. This suggests that the most crucial assumption in this paper is a quadratic welfare function. In Section 3.4, we consider a general quadratic welfare function.

3.2 Types of games

To check whether $W(\tau)$ is monotone in τ, it is enough to find the sign combinations of ζ, η, X, and Y. By Proposition 1, there are eight combinations.

(i) In a game with $\zeta \geq 0$ and $\eta > 0$, all possible sign combinations of X and Y are $X \leq Y \leq 0$, $X \leq 0 < Y$, and $0 < X < Y$. We call a game with each combination type +I, type +II, and type +III, respectively.

(ii) In a game with $\eta \leq 0 < \zeta$, it holds that $X > 0$. We call this game type +IV.

(iii) In a game with $\zeta \leq 0$ and $\eta < 0$, all possible sign combinations of X and Y are $X \leq Y \leq 0$, $X \leq 0 < Y$, and $0 < X < Y$. We call a game with each combination type −I, type −II, and type −III, respectively.

(iv) In a game with $\zeta < 0 \leq \eta$, it holds that $X > 0$. We call this game type −IV.

To identify types +IV and −IV, it is sufficient to find the signs of ζ and η, but to identify the other types, we must also calculate X and Y. We provide a simpler characterization of these types by the value of $(1-\hat{\alpha})\zeta/\eta$, which follows immediately from the definitions of X and Y.

Corollary 3. *A game with $\zeta \geq 0$ and $\eta > 0$ is type +I if $0 \leq (1-\hat{\alpha})\zeta/\eta \leq 3/2$, type +II if $3/2 < (1-\hat{\alpha})\zeta/\eta \leq 2$, and type +III if $2 < (1-\hat{\alpha})\zeta/\eta$. A game with $\zeta \leq 0$ and $\eta < 0$ is type −I if $0 \leq (1-\hat{\alpha})\zeta/\eta \leq 3/2$, type −II if $3/2 < (1-\hat{\alpha})\zeta/\eta \leq 2$, and type −III if $2 < (1-\hat{\alpha})\zeta/\eta$.*

For example, consider a game with $\zeta \geq 0$ and $\eta > 0$. If $(1-\hat{\alpha})\zeta/\eta$ is sufficiently small, then this game is type +I, where welfare necessarily increases with both public and private information because $X, Y < 0$. If $(1-\hat{\alpha})\zeta/\eta$ is sufficiently large, then this game is type +III, where welfare can decrease with both public and private information because $X, Y > 0$. Clearly, this result is consistent with the previous discussion on the roles of ζ/η and $\hat{\alpha}$.

For each type, Table 1 summarizes the signs of $\partial W/\partial \tau_x$ and $\partial W/\partial \tau_y$, which are illustrated in Figure 1. In each graph of Figure 1, the horizontal axis is the

type	τ	$\partial W/\partial \tau_x$	$\partial W/\partial \tau_y$
+I	all τ	+	+
+II	$\tau_x < (\tau_y + \tau_\theta)/Y$	+	+
	$\tau_x > (\tau_y + \tau_\theta)/Y$	+	−
+III	$\tau_x < (\tau_y + \tau_\theta)/Y$	+	+
	$(\tau_y + \tau_\theta)/Y < \tau_x < (\tau_y + \tau_\theta)/X$	+	−
	$\tau_x > (\tau_y + \tau_\theta)/X$	−	−
+IV	$\tau_x < (\tau_y + \tau_\theta)/X$	+	−
	$\tau_x > (\tau_y + \tau_\theta)/X$	−	−
−I	all τ	−	−
−II	$\tau_x < (\tau_y + \tau_\theta)/Y$	−	−
	$\tau_x > (\tau_y + \tau_\theta)/Y$	−	+
−III	$\tau_x < (\tau_y + \tau_\theta)/Y$	−	−
	$(\tau_y + \tau_\theta)/Y < \tau_x < (\tau_y + \tau_\theta)/X$	−	+
	$\tau_x > (\tau_y + \tau_\theta)/X$	+	+
−IV	$\tau_x < (\tau_y + \tau_\theta)/X$	−	+
	$\tau_x > (\tau_y + \tau_\theta)/X$	+	+

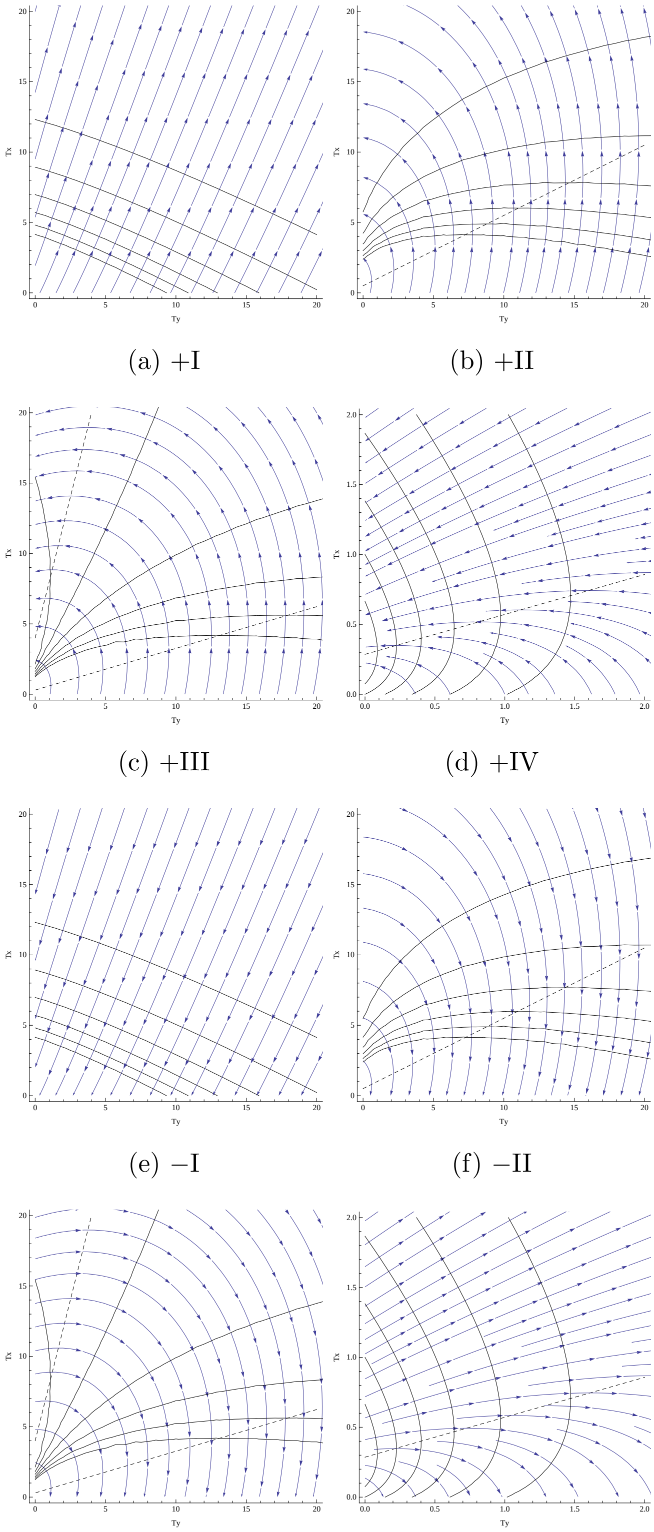
Table 1: Eight types of games.

τ_y-axis (the precision of public information) and the vertical axis is the τ_x-axis (the precision of private information).[12] Arrows indicate the direction in which $W(\tau)$ increases. Black lines are contour lines of $W(\tau)$. Each type has the following properties.

+I (−I) Welfare increases (decreases) with the precision of both public and private information at any information structure. See Figure 1a (Figure 1e).

+II (−II) If $\tau_x < (\tau_y + \tau_\theta)/Y$, welfare increases (decreases) with the precision of both public and private information. If $\tau_x > (\tau_y + \tau_\theta)/Y$, welfare increases (decreases) with the precision of private information and decreases

[12]This follows the choice of the axes in Figure 1 of MS.



(a) +I (b) +II

(c) +III (d) +IV

(e) −I (f) −II

(g) −III (h) −IV

Figure 1: Welfare and information structures in the (τ_y, τ_x)-plane. The horizontal axis is the τ_y-axis (the precision of public information) and the vertical axis is the τ_x-axis (the precision of private information). Arrows indicate the direction in which $W(\tau)$ increases.

(increases) with that of public information. See Figure 1b (Figure 1f), where the dashed line is a graph of $\tau_x = (\tau_y + \tau_\theta)/Y$.

+III (−III) If $\tau_x < (\tau_y + \tau_\theta)/Y$, welfare increases (decreases) with the precision of both public and private information. If $(\tau_y + \tau_\theta)/Y < \tau_x < (\tau_y + \tau_\theta)/X$, welfare increases (decreases) with the precision of private information and decreases (increases) with that of public information. If $\tau_x > (\tau_y + \tau_\theta)/X$, welfare decreases (increases) with the precision of both public and private information. See Figure 1c (Figure 1g), where the lower and upper dashed lines are graphs of $\tau_x = (\tau_y + \tau_\theta)/Y$ and $\tau_x = (\tau_y + \tau_\theta)/X$, respectively.

+IV (−IV) If $\tau_x < (\tau_y + \tau_\theta)/X$, welfare increases (decreases) with the precision of private information and decreases (increases) with that of public information. If $\tau_x > (\tau_y + \tau_\theta)/X$, welfare decreases (increases) with the precision of both public and private information. See Figure 1d (Figure 1h), where the dashed line is a graph of $\tau_x = (\tau_y + \tau_\theta)/X$.

3.3 Optimal information structures

As a corollary of Proposition 1, we obtain the information structure that maximizes welfare in each type. Clearly, the most precise information is optimal in type +I, but it is not necessarily so in the other types.

Corollary 4. *In types +I, +II, and −IV,* $\sup_\tau W(\tau) = W(\tau_x, \infty) = W(\infty, \tau_y)$. *In types +III and +IV,* $\sup_\tau W(\tau) = W(\tau_\theta/X, 0)$. *In types −I, −II, and −III,* $\sup_\tau W(\tau) = W(0, 0)$.

Proof. See Appendix D. □

The highest precision is optimal not only in type +I but also in types +II and −IV, whereas the lowest precision is optimal in types −I, −II, and −III. In contrast, it is optimal to receive only noisy private signals in types +III and +IV. In Figures 1c and 1d, the optimal information structure is depicted as the intercept of the dashed line $\tau_x = (\tau_y + \tau_\theta)/X$. In these types, the idiosyncratic variance term is dominant in $W(\tau)$. The idiosyncratic variance is maximized when $\tau_y = 0$ and $\tau_x < \infty$ because it decreases with τ_y for all τ and with τ_x if τ_x is sufficiently large, as shown by Corollary 2. This is why it is optimal to receive only noisy private signals in types +III and +IV.

Note that complete information is optimal if and only if welfare never decreases with both public and private information (i.e., types +I, +II, and −IV), which gives a simple test of optimality of the highest degree of transparency. For example, MS show that welfare can decrease with public information but not with private information in their beauty contest game, which implies that the highest degree of transparency is optimal.[13] Svensson [37] elaborates this point and argues that MS's result is not an anti-transparency result because welfare increases with public information as long as the precision of public information is not implausibly low.

In contrast, no information is optimal if and only if welfare decreases with both public and private information when the precision of private information is sufficiently low (i.e., types −I, −II, and −III).[14] In a public goods game studied by Teoh [38], which does not conform to our formulation, welfare is highest when there is no information. We will study a public goods game in Section 4.3.

Next, we obtain the optimal precision of public information, fixing the precision of private information.

Corollary 5. *In types +I, +II, and −IV,* $\sup_{\tau_y} W(\tau) = W(\tau_x, \infty)$. *In type +III,*

$$\sup_{\tau_y} W(\tau) = \begin{cases} W(\tau_x, \infty) & \text{if } \tau_x < \eta\tau_\theta/((1-\hat{\alpha})X\zeta), \\ W(\tau_x, 0) & \text{if } \tau_x \geq \eta\tau_\theta/((1-\hat{\alpha})X\zeta). \end{cases}$$

In types +IV and −I, $\sup_{\tau_y} W(\tau) = W(\tau_x, 0)$. *In types −II and −III,*

$$\sup_{\tau_y} W(\tau) = \begin{cases} W(\tau_x, 0) & \text{if } \tau_x < \tau_\theta/Y, \\ W(\tau_x, Y\tau_x - \tau_\theta) & \text{if } \tau_x \geq \tau_\theta/Y. \end{cases}$$

Proof. See Appendix E. □

[13] In fact, this game is type +II. See in Section 5.2.

[14] In the two-player case, Ui [39] gives a sufficient condition for complete information to achieve the lowest welfare.

This result provides the optimal degree of transparency in public information. In types +I, +II, and −IV, the highest precision is optimal, while in types +IV and −I, the lowest precision is optimal.

In the other types, the optimal precision depends upon the precision of private information. In type +III, the highest precision is optimal if private information has low precision, and the lowest precision is optimal if private information has high precision. That is, if players have sufficiently precise private information, no disclosure of public information is optimal.

In types −II and −III, the lowest precision is optimal if private information has low precision, and intermediate precision $Y\tau_x - \tau_\theta$ is optimal if private information has high precision, which corresponds to a point $(Y\tau_x - \tau_\theta, \tau_x)$ on the dashed line $\tau_x = (\tau_y + \tau_\theta)/Y$ in Figures 1f and 1g. That is, if players have sufficiently precise private information, a disclosure of a noisy public signal is optimal, and the optimal precision of public information is increasing in the precision of private information.

Finally, we obtain the optimal precision of private information, fixing the precision of public information.

Corollary 6. *In types +I, +II, and −IV,* $\sup_{\tau_x} W(\tau) = W(\infty, \tau_y)$*. In types +III and +IV,* $\sup_{\tau_x} W(\tau) = W((\tau_y + \tau_\theta)/X, \tau_y)$*. In types −I, −II, and −III,* $\sup_{\tau_x} W(\tau) = W(0, \tau_y)$*.*

Proof. See Appendix F. □

In types +I, +II, and −IV, the highest precision is optimal, while in types −I, −II, and −III, the lowest precision is optimal. In types +III and +IV, the optimal precision is $(\tau_y + \tau_\theta)/X$, which corresponds to a point $(\tau_y, (\tau_y + \tau_\theta)/X)$ on the dashed line $\tau_x = (\tau_y + \tau_\theta)/X$ in Figures 1c and 1d.

To see what this implies, imagine that each player is allowed to choose the precision of his private information as well as his strategy given the opponents' precision and strategies. The cost of information acquisition is linear in the precision. This is a one-stage model of information acquisition introduced by Hauk and Hurkens [19] in the context of a Cournot game.[15] If the marginal cost is suffi-

[15] Li et al. [30] and Vives [44] consider a two-stage model of information acquisition in the

ciently small, players choose strictly positive precision. However, if the underlying game is type −I, −II, or −III, the lowest precision is socially optimal. In such a case, an increase in the marginal cost can improve welfare.[16]

3.4 Quadratic welfare functions

Using Proposition 1, we study a general quadratic welfare function

$$E\Big[c_1\sum_{i=1}^{n}\sigma_i^2 + c_2\sum_{i<j}^{n}\sigma_i\sigma_j + c_3\sum_{i=1}^{n}\theta\sigma_i + c_4\sum_{i=1}^{n}\sigma_i\Big] + c_5, \tag{9}$$

where $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$ are constant.[17] For example, the expected total surplus in a Cournot game with a linear demand function has the above representation (see Section 4.1). A quadratic welfare function also appears in the literature of monetary policy [1].

This measure of welfare is represented as a linear combination of the idiosyncratic variance and the common variance.

Lemma 3. *The measure of welfare* (9) *equals*

$$W^*(\tau) = \zeta^*(\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i, \sigma_j]) + \eta^*\mathrm{cov}[\sigma_i, \sigma_j]$$

plus a constant independent of τ*, where* $\zeta^* = n(c_1 + c_3/\beta)$ *and* $\eta^* = n(c_1 + (n-1)c_2/2 + (1-\hat{\alpha})c_3/\beta)$.

Proof. See Appendix G. □

Moreover, we can find the signs of $\partial W^*(\tau)/\partial\tau_x$ and $\partial W^*(\tau)/\partial\tau_y$ by replacing (ζ, η) in Proposition 1 with (ζ^*, η^*).

Corollary 7. *For a given game with a payoff function* (1) *and a given measure of welfare* $W^*(\tau)$ *in Lemma 3, the signs of* $\partial W^*(\tau)/\partial\tau_x$ *and* $\partial W^*(\tau)/\partial\tau_y$ *are given*

context of Cournot games. Colombo and Femminis [11], Hellwig and Veldkamp [21], and Myatt and Wallace [34] consider a two-stage model of information acquisition in the context of beauty contest games with a continuum of players. Equilibria in one-stage and two-stage models are the same if the number of players is infinite. See Hauk and Hurkens [19].

[16] See Ui [41] for more details.

[17] We thank a referee for suggesting a formal discussion of this issue.

by those of $\partial W(\tau)/\partial\tau_x$ and $\partial W(\tau)/\partial\tau_y$ in Proposition 1, respectively, where (ζ, η) is replaced with (ζ^*, η^*).

Proof. See Appendix H. □

Note that parameters used in Proposition 1 are α, β, ζ, and η, where (α, β) determines an equilibrium and (ζ, η) determines a measure of welfare. Corollary 7 means that we can apply Proposition 1 for any combination of an equilibrium determined by (α, β) and a measure of welfare determined by (ζ, η).

4 Applications

4.1 Cournot competition

Consider a Cournot game with a homogeneous product. Player i produces a_i units of the product. We assume a constant marginal cost $c > 0$ and a linear inverse demand function $\theta' - \rho \sum_i a_i$, where θ' is normally distributed and $\rho > 0$ is constant. Thus, there is uncertainty about the demand intercept. Then, player i's profit is

$$(\theta' - \rho \sum_{j\in N} a_j) a_i - c a_i = \rho(-a_i^2 - a_i \sum_{j\neq i} a_j + (\theta' - c)\rho^{-1} a_i). \qquad (10)$$

The type of this game is summarized as follows by Corollary 3.

Corollary 8. *This game is type +I if $n = 2$, type +II if $n = 3$, and type +III if $n \geq 4$.*

Proof. See Appendix J. □

This result says that the expected profit can decrease with the precision of both public and private information if $n \geq 4$. In this case, the slope of best responses with respect to average actions, $\hat{\alpha} = -(n-1)/2$, is very small. Thus, the relative weight of a private signal in the equilibrium strategy is very large, and so is that of the idiosyncratic variance in the expected profit, by which more precise information can reduce welfare.

While the expected producer surplus can decrease with the precision of both public and private information if $n \geq 4$, the expected total surplus necessarily increases with the precision of both public and private information as shown by Vives [43, 44], which we can confirm using Corollaries 3 and 7. A direct calculation shows that the expected total surplus is

$$W^*(\tau) = \frac{3n\rho}{2}(\text{var}[\sigma_i] - \text{cov}[\sigma_i, \sigma_j]) + \frac{n(n+2)\rho}{2}\text{cov}[\sigma_i, \sigma_j]$$

plus a constant.[18] By Corollaries 3 and 7, it follows that $\partial W^*(\tau)/\partial \tau_x > 0$ and $\partial W^*(\tau)/\partial \tau_y > 0$ for all τ because $\zeta^* = 3n\rho/2$, $\eta^* = n(n+2)\rho/2$, and $(1-\hat{\alpha})\zeta^*/\eta^* = 3/2 \times (n+1)/(n+2) < 3/2$.

4.2 Bertrand competition

Consider a Bertrand game with differentiated products. Player i produces good i and chooses its price a_i. We assume a linear demand function $1 - a_i + \rho \sum_{j \neq i} a_j$, where ρ is constant. The goods are gross substitutes if $\rho > 0$ and gross complements if $\rho < 0$. The marginal cost is a normally distributed random variable θ'. Then, player i's profit is

$$(1 - a_i + \rho \sum_{j \neq i} a_j)(a_i - \theta') = -a_i^2 + \rho a_i \sum_{j \neq i} a_j + (\theta' + 1)a_i - \rho \sum_{j \neq i} \theta' a_j - \theta'. \quad (11)$$

We assume $\hat{\rho} \equiv (n-1)\rho < 2$ to guarantee the uniqueness of symmetric equilibrium.

The type of this game is summarized as follows by Corollary 3.

Corollary 9. *This game is type +I if $\hat{\rho} \leq 1/2$ and type −IV if $\hat{\rho} > 1/2$.*

Proof. See Appendix J. □

This result says that the expected profit can decrease with the precision of private information if ρ is sufficiently large, that is, if the goods are gross substitutes and the cross price effect is large enough. When ρ is sufficiently large, the slope of best responses, $\hat{\alpha} = \hat{\rho}/2$, is close to one, which means that the common variance is

[18] The expected producer surplus is $n\rho\text{var}[\sigma_i]$ plus a constant by (6) and the expected consumer surplus is $E\Big[\frac{\rho}{2}\big(\sum_{i \in N} \sigma_i\big)^2\Big] = \frac{\rho}{2}\left(nE[\sigma_i^2] + n(n-1)E[\sigma_i \sigma_j]\right)$.

relatively large. However, the coefficient of the idiosyncratic variance, $\zeta = 1 - 2\hat{\rho}$, is negative, which plays a dominant role. Thus, the expected profit decreases with the precision of private information if the precision is sufficiently low.

4.3 Voluntary provision of public goods

We consider two different formulations of public goods games, where player i chooses his contribution level a_i.

In the first game, each player receives a common benefit $-(\sum_j a_j)^2 + \theta' \sum_j a_j$, where θ' is normally distributed. Note that the common benefit is a quadratic function of the total contribution $\sum_j a_j$. The marginal cost of a player's contribution is a constant $c > 0$. Then, player i's payoff is

$$\left(-\Big(\sum_j a_j\Big)^2 + \theta' \sum_j a_j\right) - c a_i. \tag{12}$$

In the second game, each player receives a common benefit $-(\sum_j a_j)^2 + c \sum_j a_j$, where $c > 0$ is constant. The marginal cost of a player's contribution is a normally distributed random variable θ'. Then, player i's payoff is

$$\left(-\Big(\sum_j a_j\Big)^2 + c \sum_j a_j\right) - \theta' a_i. \tag{13}$$

The difference is the source of uncertainty: the production is random in the first game and the cost is random in the second game, which results in the opposite welfare effects of information.

Corollary 10. *Suppose that $n \geq 3$. The first game with a payoff function* (12) *is type $+I$. The second game with a payoff function* (13) *is type $-I$.*

Proof. See Appendix J. □

In the second game, the common benefit is independent of the state. Because it is concave in the total contribution $\sum_j a_j$, the expected common benefit is greater when players choose the constant expected action $E[\sigma_i]$ than when they follow the equilibrium strategy σ_i by Jensen's inequality. However, $E[\sigma_i]$ is not a best response when a player has some information about the marginal cost θ'. This is why information is harmful in the second game. In contrast, the common benefit in the first game depends upon the state, and information is useful in adjusting actions to increase the common benefit. This is why information is beneficial in the first game.

Teoh [38] studies the social value of public information in a public goods game with binary states and random production, which is not a quadratic Bayesian game. Teoh [38] shows that welfare is higher without public information about the state than with it, where concavity also plays an essential role. Our result on the second game paraphrases the effects of concavity on the social value of information in terms of quadratic Bayesian games.

5 A continuum of players

This section considers AP's model with a continuum of players and discusses how to apply Proposition 1 and its consequences to the continuum model.

5.1 The model

Let $[0, 1]$ be a set of players with an individual player indexed by $i \in [0, 1]$. Player i chooses an action $a_i \in \mathbb{R}$ and an action profile is denoted by $a = (a_i)_{i\in[0,1]}$. Player i's payoff function is

$$\begin{aligned}\bar{u}_i(a, \theta) = &- a_i^2 + 2\alpha a_i \int_0^1 a_j dj + 2\beta\theta a_i \\ &+ \kappa \int_0^1 a_j^2 dj + \lambda \left(\int_0^1 a_j dj\right)^2 + \mu\theta \int_0^1 a_j dj + \nu \int_0^1 a_j dj + f(\theta). \quad (14)\end{aligned}$$

Player i observes the same bivariate signal vector as that in Section 2.

Angeletos and Pavan [3, 4] show that a unique symmetric equilibrium exists if $\alpha < 1$ and obtain it. AP also obtain the socially optimal strategy profile, which coincides with a unique symmetric equilibrium of a certain fictitious game. The slope of best responses in the original game and that of the fictitious game are referred to as the equilibrium degree of coordination and the optimal degree of coordination, respectively. AP show that the equilibrium degree of coordination relative to the optimal degree of coordination is useful in studying the social value of information.

Let us connect the continuum model with the finite model. The first order condition for a symmetric equilibrium in the continuum model coincides with that in the finite model by the replacement of $\hat{\alpha}$ with α.[19] Thus, the equilibrium strategy in the continuum model is given by Lemma 1 with $\hat{\alpha} = \alpha$. Moreover, for any continuum model, we can construct a finite model such that not only the equilibrium but also the expected payoff coincides with that in the continuum model by replacing $\hat{\kappa}$, $\hat{\lambda}$, and $\hat{\mu}$ in the finite model with κ, λ, and μ.[20] That is, we can represent the expected payoff in the continuum model as follows.

Lemma 4. *The expected payoff* $E[\bar{u}_i(\sigma, \theta)]$ *equals*

$$\begin{aligned} \overline{W}(\tau) &\equiv \zeta \mathrm{var}[b_x \varepsilon_i] + \eta \mathrm{var}[b_y \varepsilon_0 + (b_x + b_y)\theta] \\ &= \zeta(\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i, \sigma_j]) + \eta \mathrm{cov}[\sigma_i, \sigma_j] \end{aligned} \tag{15}$$

plus a constant independent of τ, *where* $\zeta = \mu/\beta + \kappa + 1$ *and* $\eta = (1-\alpha)\mu/\beta + \kappa + \lambda + 1$.

Proof. See Appendix I. □

Therefore, we can apply Proposition 1 and its consequences to the continuum model by replacing $(\hat{\alpha}, \zeta, \eta) = (\hat{\alpha}, \hat{\mu}/\beta + \hat{\kappa} + 1, (1-\hat{\alpha})\hat{\mu}/\beta + \hat{\kappa} + \hat{\lambda} + 1)$ in the finite model with $(\alpha, \zeta, \eta) = (\alpha, \mu/\beta + \kappa + 1, (1-\alpha)\mu/\beta + \kappa + \lambda + 1)$.

In the continuum model, there is another interpretation of the common variance and the idiosyncratic variance. Bergemann and Morris [8] consider the variance of the average action $\int a_j dj$ and that of the idiosyncratic difference $a_i - \int a_j dj$

[19]This implies that the theorem of Radner [36] is useful not only in the finite model but also in the continuum model, though Angeletos and Pavan [3, 4] do not use it. See Ui and Yoshizawa [40].

[20]Vives [44] compares the equilibrium and welfare with a finite number of players and those with a continuum of players in Cournot games. He shows that the former converges to the latter as the number of players goes to infinity. In contrast, we consider finite and continuum models possessing the same equilibrium and welfare. See the online appendix for a more detailed comparison.

in the continuum model and refer to them as volatility and dispersion, respectively. They show that the volatility equals $\mathrm{cov}[\sigma_i, \sigma_j]$ and the dispersion equals $\mathrm{var}[\sigma_i] - \mathrm{cov}[\sigma_i, \sigma_j]$. Thus, the common variance and the idiosyncratic variance equal the volatility and the dispersion, respectively.

This implies that $\overline{W}(\tau)$ is a linear combination of the volatility and the dispersion. Note that, in the finite case, $W(\tau)$ is not a linear combination of volatility and dispersion in the corresponding sense.[21]

AP are the first to use the terms "volatility" and "dispersion" in this context. Dispersion is the same, but volatility is different. AP refer to the variance of $\int a_j dj - x(\theta)$ as volatility, where $x(\theta)$ is an action in the equilibrium under complete information, and write welfare as a linear combination of the volatility, the dispersion, and the other term, the last of which plays an important role in their analysis.

In contrast, we write welfare as a linear combination of the volatility and the dispersion in the sense of Bergemann and Morris [8], though Bergemann and Morris [8] do not use the volatility and the dispersion for this purpose. This is a major methodological difference in the continuum case between AP and this paper.

We emphasize the following advantages of our representation. First, and most importantly, the ratio of the coefficients of the volatility and the dispersion determines the social value of information. Our representation of welfare leads us to this result. Next, we can study any quadratic welfare function. Finally, we can study Bayesian correlated equilibria, as we will discuss later in this section. For other differences between this paper and AP, see the online appendix.

5.2 Beauty contest

Let $\alpha = r \in (0, 1)$ and $\beta = 1 - r$ in (14). By the first order condition for equilibrium, a player's best response is the weighted mean of the conditional expectation

[21] The variance of the average action $\sum_i a_i/n$ is $\mathrm{var}[\sigma_i]/n + (n-1)\mathrm{cov}[\sigma_i, \sigma_j]/n$, which converges to $\mathrm{cov}[\sigma_i, \sigma_j]$ as $n \to \infty$.

of the state and that of the opponents' actions, i.e., $(1-r)E[\theta|\mathbf{s}_i] + rE[\sigma_j|\mathbf{s}_i]$. Because it induces strategic behavior in the spirit of a Keynesian beauty contest, this game is referred to as a beauty contest game.

MS consider a beauty contest game with a payoff function

$$-(1-r)(a_i-\theta)^2 - r\Big(\int (a_j-a_i)^2 dj - \int (a_j-a_k)^2 dj dk\Big). \tag{16}$$

In this formulation, welfare is measured by the mean squared error of an action from the state because the expected payoff equals $-(1-r)E[(\sigma_i-\theta)^2]$. As argued by James and Lawler [26], this formulation is appropriate to models of asset markets.

Hellwig and Veldkamp [21] consider a beauty contest game with a payoff function

$$-\Big(a_i - r\int a_j dj - (1-r)\theta\Big)^2, \tag{17}$$

which equals the squared error of an action from the weighted mean of the state and the average action of the opponents. This formulation is appropriate to models where both the aggregate action and the state impact on some macroeconomic variable that is important for individual agents' optimal choices.

Myatt and Wallace [34] consider a beauty contest game with a payoff function

$$-r\Big(a_i - \int a_j dj\Big)^2 - (1-r)\Big(a_i-\theta\Big)^2, \tag{18}$$

which equals the weighted mean of the squared error of an action from the state and that from the average action of the opponents. This formulation is appropriate to models where the aggregate action and the state separately impact on two different macroeconomic variables that are important for individual agents' optimal choices.

The types of these games are summarized as follows by Corollary 3.

Corollary 11. *MS's beauty contest game is type +I if $r \le 1/2$ and type +II if $r > 1/2$. Hellwig and Veldkamp's beauty contest game is type +I if $r \le 1/2$ and type $-IV$ if $r > 1/2$. Myatt and Wallace's beauty contest game is type +I for each $r \in (0,1)$.*

Proof. See Appendix J. □

If r is sufficiently large, welfare can decrease with public information in MS's beauty contest game, as shown by MS, whereas it can decrease with private information in Hellwig and Veldkamp's beauty contest game. In contrast, welfare necessarily increases with both public and private information in Myatt and Wallace's beauty contest game.[22] When r is sufficiently large, the slope of best responses $\alpha = r$ is close to one, which means that the relative weight of the common variance is large. However, in MS's beauty contest game, the coefficient of the idiosyncratic variance equals $\zeta = 1+r$ and it is very large, generating the harmful effect of public information. On the other hand, in Hellwig and Veldkamp's beauty contest game, the coefficient of the idiosyncratic variance equals $\zeta = 1 - 2r$ and it is negative, generating the harmful effect of private information.

Finally, we consider a variant of Hellwig and Veldkamp's beauty contest game with a payoff function

$$-\left(\int a_j dj\right)^2 + \rho \int a_j dj - c_1 a_i - c_2\left(a_i - r\int a_j dj - (1-r)\theta\right)^2, \qquad (19)$$

where $\rho, c_1, c_2 > 0$ are constant. We can interpret this as a model of conditional cooperation in a public goods game. When player i's contribution level is a_i for each i, every player receives a common benefit $-\left(\int a_j dj\right)^2 + \rho \int a_j dj$, which is a quadratic function of the total contribution $\int a_j dj$. The marginal cost of each player's contribution is a constant c_1.

To introduce the last term in (19), we assume that players are conditional cooperators who are willing to contribute more to a public good when the opponents contribute more. Evidence for conditional cooperation comes from several experiments [10, 16, 28]: even though no contribution is rational, a majority of the subjects increase their contribution as the other subjects increase their contribution on average. We model incentive for conditional cooperation by incorporating a cost of deviation $c_2(a_i - r\int a_j dj - (1-r)\theta)^2$ from a target contribution level $r\int a_j dj + (1-r)\theta$.

[22] We can also obtain this result using Proposition 6 in AP, which provides a sufficient condition for welfare to increase with the precision of both pubic and private information for all information structures.

The type of this game is summarized as follows by Corollary 3.

Corollary 12. *Assume that $c_2 < 1/(1-r)^2$; that is, the incentive for conditional cooperation is not too large. This game is type +IV if $r < 1/2$, type $-I$ if $r = 1/2$ or if $r > 1/2$ and $c_2 \leq 3/(1-r^2)$, type $-II$ if $r > 1/2$ and $3/(1-r^2) < c_2 \leq 2/(1-r)$, and type $-III$ if $r > 1/2$ and $2/(1-r) < c_2 < 1/(1-r)^2$.*

Proof. See Appendix J. □

Welfare necessarily decreases with both public and private information if r is sufficiently large and c_2 is sufficiently small, that is, if the slope of best responses is close to one and the incentive for conditional cooperation is not too large. The intuition is similar to that in Section 4.3. When the incentive for conditional cooperation is not too large, the common benefit is dominant in the payoff function. The common benefit is a concave function of the total contribution $\int a_j dj$ and thus its expected value decreases as the variance of $\int a_j dj$ increases. Recall that the variance of $\int a_j dj$ equals the common variance of actions. Because the common variance increases with both public and private information and it is large when the slope of best responses is close to one, welfare necessarily decreases with both public and private information.

5.3 Optimal Bayesian correlated equilibrium

Consider a mediator who knows the true state and makes private, perhaps correlated, action recommendations to players who have no information about the state. If each player has an incentive to follow the mediator's recommendation, we say that the resulting action distribution is a Bayesian correlated equilibrium. We are interested in the Bayesian correlated equilibrium that achieves the highest welfare.

Bergemann and Morris [8] study a game with (14) and characterize the set of all Bayesian correlated equilibria with normally distributed action recommendations. It is known that the set of all Bayesian correlated equilibria coincides with the set of all action distributions of Bayesian Nash equilibria generated by all possible signal structures. Clearly, action distributions of Bayesian Nash equilibria generated by the bivariate (public and private) signal structures are Bayesian correlated equilibria. Bergemann and Morris [8] show that the converse is also true in games with a continuum of players; that is, the set of all Bayesian correlated equilibria coincides with the set of all action distributions of Bayesian Nash equilibria generated by the bivariate signal structures.

The finding of Bergemann and Morris [8] implies that the action distribution of the Bayesian Nash equilibrium under the optimal information structure in Corollary 4 is the optimal Bayesian correlated equilibrium. The following corollary calculates it by plugging the optimal information structure in Corollary 4 into the equilibrium strategy (3). In the optimal Bayesian correlated equilibrium of types +I, +II, and −IV, actions are completely correlated with θ; in that of types −I, −II, and −III, actions are constant; in that of types +III and +IV, actions are conditionally independent given θ.

Corollary 13. *Consider a Bayesian correlated equilibrium that achieves the highest welfare. If $\zeta \le 2\eta/(1-\alpha)$ and $\eta \ge 0$, then the recommended action for all players is $\beta\theta/(1-\alpha)$. If $\zeta \le 0$ and $\eta < 0$, then the recommended action for all players is $\beta\bar{\theta}/(1-\alpha)$. If $\zeta > \max\{0, 2\eta/(1-\alpha)\}$, then the recommended action for player i is $\beta(\theta+\varepsilon_i-\bar{\theta})/(1-\alpha+X)+\beta\bar{\theta}/(1-\alpha)$, where ε_i is an i.i.d. normally distributed random variable with mean zero and variance X/τ_θ.*

This corollary complements the following result of Bergemann and Morris [8] on a large Cournot game.[23] Player i produces a_i units of a homogeneous product. The inverse demand function is $\theta + \alpha \int a_j dj$, where $\alpha < 0$ is constant and θ is normally distributed, and the cost function is $a_i^2/2$. Then, player i's profit is

$$\Big(\theta + \alpha \int a_j dj\Big) a_i - a_i^2/2. \tag{20}$$

Bergemann and Morris [8] show that, in the optimal Bayesian equilibrium, actions are completely correlated with θ if $\alpha \ge -1$ and conditionally independent given

[23] This game can be derived as the limit of Cournot games with a finite number of players as shown by Vives [44]. See the online appendix. Vives [45, Section 8.4.4] studies a related but different class of action recommendations in large markets.

θ if $\alpha < -1$. Using this result and the equivalence of Bayesian correlated and Nash equilibria, they show that the optimal information structure is complete information if $\alpha \geq -1$ and incomplete information only with appropriate noisy private signals if $\alpha < -1$. These results are special cases of Corollaries 4 and 13 because the type of this game is summarized as follows by Corollary 3.[24]

Corollary 14. *This game is type +I if $\alpha \geq -1/2$, type +II if $-1 \leq \alpha < -1/2$, and type +III if $\alpha < -1$.*

Proof. See Appendix J. □

The difference between Bergemann and Morris [8] and this paper is that the focus of the former is equilibria determined by (α, β),[25] whereas that of the latter is welfare determined by (ζ, η) given (correlated or Nash) equilibria. Combining the results of both papers, we can identify all the optimal Bayesian correlated equilibria in terms of $(\alpha, \beta, \zeta, \eta)$ as in Corollary 13.

6 Concluding remarks

This paper characterizes the social value of information in AP's model in terms of the relative weights of the common variance and the idiosyncratic variance in welfare and finds that AP's model is classified into eight types of games with different welfare effects of information. If the relative weight of the common variance is sufficiently large, welfare necessarily increases with both public and private information, but if that of the idiosyncratic variance is sufficiently large, welfare can decrease. For example, in a Cournot game with more than four players, welfare can decrease with both public and private information because this game exhibits strong strategic substitutability, which induces a large weight on a private signal in the equilibrium strategy and thus a large weight on the idiosyncratic variance in welfare.

[24] This result is inconsistent with AP's Corollary 10. See the online appendix.

[25] Their welfare analysis is also determined by (α, β) because they restrict attention to a large Cournot game with $\kappa = \lambda = \mu = 0$.

Our approach based upon the common variance and the idiosyncratic variance has the following advantages. Firstly and most importantly, the social value of information is completely characterized by the ratio of their coefficients in welfare together with the slope of best responses. Secondly, we can apply our result to the study of all quadratic welfare functions. Put it differently, we can regard our contribution as a characterization of the effects of public and private information on quadratic welfare functions, of which special case is the expected payoff. Thirdly, our representation of welfare and classification of games are also useful in studying Bayesian correlated equilibria. Finally, insofar as welfare functions are quadratic, a similar analysis could be possible even if signals are not normally distributed.

A limitation of our study is that information structures are exogenously given. In recent years, a growing number of researchers have studied endogenous information structures using MS's or AP's model. Examples include Colombo and Femminis [11], Dewan and Myatt [14], Hellwig and Veldkamp [21], Hagenbach and Koessler [18], Myatt and Wallace [34], Colombo et al. [12], and Ui [42]. Clearly, private collection of information and aggregation of information may have different effects on the expected payoff depending upon the types of games. Thus, comparing endogenous information structures and their welfare properties in different types would be an interesting topic for future research. In so doing, our representation of welfare could also be useful in more elaborated models.[26]

Appendix

A Proof of Lemma 1

The use of Radner's theorem is standard in the literature of information sharing, but it is not necessarily so in the literature of social value of information. Thus, we give a proof for completeness.

Consider a game with $\kappa = -1$, $\lambda = 2\alpha$, $\mu = 2\beta$, $\nu = 0$, and $f(\theta) = 0$, where every player has an identical payoff function $v(a, \theta) \equiv -\sum_j a_j^2 + 2\alpha \sum_{j<k} a_j a_k +$

[26]Such an example is found in Ui [41].

$2\beta \sum_j \theta a_j$. Theorem 5 of Radner [36] states that if $v(a,\theta)$ is strictly concave in a then there exists a unique equilibrium and that each strategy in the equilibrium is a linear function. Because each player's best response is independent of κ, λ, μ, ν, and $f(\theta)$, a game with an identical payoff function $v(a,\theta)$ has a unique equilibrium if and only if a game with a payoff function (1) has the same unique equilibrium.

The leading minors of the Hessian matrix of $v(a,\theta)$ are $-(1+\alpha)^{k-1}(1-(k-1)\alpha)$ for $k = 1,\ldots,n$. This implies that $v(a,\theta)$ is strictly concave in a if and only if $-(n-1) < \hat{\alpha} < 1$. Thus, the first order condition (2) has a unique solution if $-(n-1) < \hat{\alpha} < 1$. In the following, we show that a weaker condition $\hat{\alpha} < 1$ suffices because we confine our attention to a symmetric equilibrium following AP.

First, we show that if $-(n-1) < \hat{\alpha} < 1$ then a unique equilibrium is symmetric. Let $(\sigma_i)_{i\in N}$ be a unique solution of (2). Because the joint probability distribution of $(\mathbf{s}_1,\ldots,\mathbf{s}_n)$ is symmetric, for any permutation $\pi : N \to N$, (2) is equivalent to

$$\sigma_i(\mathbf{s}_{\pi(i)}) = \alpha \sum_{j\neq i} E[\sigma_j(\mathbf{s}_{\pi(j)})|\mathbf{s}_{\pi(i)}] + \beta E[\theta|\mathbf{s}_{\pi(i)}],$$

which implies that a strategy profile $(\sigma'_i)_{i\in N}$ with $\sigma'_{\pi(i)} = \sigma_i$ is also a unique solution of (2).

Next, we show that $\hat{\alpha} < 1$ guarantees the existence and uniqueness of a symmetric equilibrium. Let $(\sigma_i)_{i\in N}$ be a symmetric equilibrium with $\sigma_i = \sigma_j$ for all i, j. Then, (2) is reduced to

$$\sigma_i(\mathbf{s}_i) = \alpha \sum_{j\neq i} E[\sigma_i(\mathbf{s}_j)|\mathbf{s}_i] + \beta E[\theta|\mathbf{s}_i].$$

Because $E[\sigma_i(\mathbf{s}_j)|\mathbf{s}_i] = E[\sigma_i(\mathbf{s}_k)|\mathbf{s}_i]$ for all $j, k \neq i$, this is rewritten as

$$\sigma_i(\mathbf{s}_i) = \hat{\alpha} E[\sigma_i(\mathbf{s}_j)|\mathbf{s}_i] + \beta E[\theta|\mathbf{s}_i]. \tag{A1}$$

If $-(n-1) < \hat{\alpha} < 1$, (A1) has a unique solution by Theorem 5 of Radner [36]. Because n is an arbitrary positive integer, (A1) has a unique solution if $\hat{\alpha} < 1$.

Finally, we obtain $\mathbf{b}$ and c. Plugging (3) into (A1), we have

$$\mathbf{b}^\top(\mathbf{s}_i - E[\mathbf{s}_i]) + c = \hat{\alpha}(\mathbf{b}^\top(E[\mathbf{s}_j|\mathbf{s}_i] - E[\mathbf{s}_j]) + c) + \beta E[\theta|\mathbf{s}_i]. \tag{A2}$$

Let us write $\bar{\mathbf{s}} = E[\mathbf{s}_i]$, $C = \text{var}[\mathbf{s}_i]$, $D = \text{cov}[\mathbf{s}_i, \mathbf{s}_j]$, and $\mathbf{g} = \text{cov}[\theta, \mathbf{s}_i]$. Then, the property of multivariate normal distributions[27] means $E[\mathbf{s}_j|\mathbf{s}_i] = \bar{\mathbf{s}} + DC^{-1}(\mathbf{s}_i - \bar{\mathbf{s}})$ and $E[\theta|\mathbf{s}_i] = \bar{\theta} + \mathbf{g}^\top C^{-1}(\mathbf{s}_i - \bar{\mathbf{s}})$. Plugging these into (A2), we have

$$-\Big(\mathbf{b}^\top(I - \hat{\alpha}DC^{-1}) - \beta\mathbf{g}^\top C^{-1}\Big)(\mathbf{s}_i - \bar{\mathbf{s}}) - (1-\hat{\alpha})c + \beta\bar{\theta} = 0$$

for all $\mathbf{s}_i \in \mathbb{R}^2$. This implies that $\mathbf{b}^\top = \beta\mathbf{g}^\top(C - \hat{\alpha}D)^{-1}$ and $c = \beta\bar{\theta}/(1-\hat{\alpha})$.

B Proof of Lemma 2

Using the symmetry $\sigma_i(\cdot) = \sigma_j(\cdot)$ for $i \neq j$, we have

$$\begin{aligned}
E[u_i(\sigma,\theta)] =& - E[{\sigma_i}^2] + 2\alpha E[\sigma_i \sum_{j\neq i}\sigma_j] + 2\beta E[\theta\sigma_i] \\
&+ \kappa E[\sum_{j\neq i}{\sigma_j}^2] + \lambda E[\sum_{j<k:j,k\neq i}\sigma_j\sigma_k] + \mu E[\sum_{j\neq i}\theta\sigma_j] + \nu E[\sum_{j\neq i}\sigma_j] + E[f(\theta)] \\
=&(\hat{\kappa}-1)\text{var}[\sigma_i] + (2\hat{\alpha}+\hat{\lambda})\text{cov}[\sigma_i,\sigma_j] + (2\beta+\hat{\mu})\text{cov}[\theta,\sigma_i] \\
&+ (2\hat{\alpha}+\hat{\kappa}+\hat{\lambda}-1)c^2 + ((2\beta+\hat{\mu})\bar{\theta}+\hat{\nu})c + E[f(\theta)].
\end{aligned}$$

Thus, $E[u_i(\sigma,\theta)]$ is the sum of

$$(\hat{\kappa}-1)\text{var}[\sigma_i] + (2\hat{\alpha}+\hat{\lambda})\text{cov}[\sigma_i,\sigma_j] + (2\beta+\hat{\mu})\text{cov}[\theta,\sigma_i] \tag{B1}$$

and a constant independent of τ. We show that (6) equals (B1).

Multiplying the first order condition by $\sigma_i(\mathbf{s}_i)$ and taking the expectation, we have

$$\begin{aligned}
-E[\sigma_i^2]+&\hat{\alpha}E[\sigma_i\sigma_j] + \beta E[\theta\sigma_i] \\
&= -\text{var}[\sigma_i] + \hat{\alpha}\text{cov}[\sigma_i,\sigma_j] + \beta\text{cov}[\theta,\sigma_i] - (1-\hat{\alpha})c^2 + \beta\bar{\theta}c \\
&= -\text{var}[\sigma_i] + \hat{\alpha}\text{cov}[\sigma_i,\sigma_j] + \beta\text{cov}[\theta,\sigma_i] = 0
\end{aligned}$$

for $i \neq j$ because $c = \beta\bar{\theta}/(1-\hat{\alpha})$ by Lemma 1, and thus

$$\text{cov}[\theta,\sigma_i] = \beta^{-1}\text{var}[\sigma_i] - \hat{\alpha}\beta^{-1}\text{cov}[\sigma_i,\sigma_j]. \tag{B2}$$

Plugging this into (B1), we obtain $\zeta(\text{var}[\sigma_i] - \text{cov}[\sigma_i,\sigma_j]) + \eta\text{cov}[\sigma_i,\sigma_j]$.

[27] Let $X = (X_1, X_2)$ be a random vector whose distribution is multivariate normal with $\mu_i = EX_i$ and $C_{ij} = \text{cov}(X_i, X_j)$ for $i,j = 1,2$. Then, $E[X_2|X_1] = \mu_2 + C_{21}C_{11}^{-1}(X_1 - \mu_1)$.

C Proof of Proposition 1

By (6), we have

$$W(\tau) = \beta^2 \Big(\frac{\tau_x}{(\tau_\theta + (1-\hat{\alpha})\tau_x + \tau_y)^2} \zeta + \frac{((1-\hat{\alpha})^2 \tau_x^2 + 2(1-\hat{\alpha})\tau_x \tau_y + \tau_y (\tau_\theta + \tau_y))}{(1-\hat{\alpha})^2 \tau_\theta (\tau_\theta + (1-\hat{\alpha})\tau_x + \tau_y)^2} \eta \Big). \tag{C1}$$

By differentiating $W(\tau)$ with respect to τ_x, we get

$$\frac{\partial W}{\partial \tau_x} = \frac{\beta^2 (-\tau_x((1-\hat{\alpha})\zeta - 2\eta) + \zeta(\tau_\theta + \tau_y))}{(\tau_\theta + (1-\hat{\alpha})\tau_x + \tau_y)^3}. \tag{C2}$$

Because the denominator is positive, $\partial W/\partial \tau_x > 0$ if and only if

$$-\tau_x((1-\hat{\alpha})\zeta - 2\eta) + \zeta(\tau_\theta + \tau_y) > 0. \tag{C3}$$

If $\zeta = 0$, (C3) is rewritten as $2\tau_x \eta > 0$, and thus $\partial W/\partial \tau_x > 0$ if and only if $\eta > 0$, which establishes the signs of $\partial W/\partial \tau_x$ in (i) and (iii) with $\zeta = 0$ because we set $X = -\infty$. If $\zeta \gtrless 0$, (C3) is rewritten as

$$X = ((1-\hat{\alpha})\zeta - 2\eta)/\zeta \lessgtr (\tau_y + \tau_\theta)/\tau_x.$$

Thus, $\partial W/\partial \tau_x > 0$ if and only if either $X < (\tau_y + \tau_\theta)/\tau_x$ and $\zeta > 0$ or $X > (\tau_y + \tau_\theta)/\tau_x$ and $\zeta < 0$, which establishes the signs of $\partial W/\partial \tau_x$ in (i), (ii), (iii), and (iv) with $\zeta \neq 0$.

By differentiating $W(\tau)$ with respect to τ_y, we get

$$\frac{\partial W}{\partial \tau_y} = \frac{\beta^2 (-(1-\hat{\alpha})\tau_x(2(1-\hat{\alpha})\zeta - 3\eta) + \eta(\tau_\theta + \tau_y))}{(1-\hat{\alpha})^2 (\tau_\theta + (1-\hat{\alpha})\tau_x + \tau_y)^3}. \tag{C4}$$

Because the denominator is positive, $\partial W/\partial \tau_y > 0$ if and only if

$$-(1-\hat{\alpha})\tau_x(2(1-\hat{\alpha})\zeta - 3\eta) + \eta(\tau_\theta + \tau_y) > 0. \tag{C5}$$

If $\eta = 0$, (C5) is rewritten as $-2\tau_x (1-\hat{\alpha})^2 \zeta > 0$, and thus $\partial W/\partial \tau_y > 0$ if and only if $\zeta < 0$, which establishes the signs of $\partial W/\partial \tau_y$ in (ii) and (iv) with $\eta = 0$. If $\eta \gtrless 0$, (C5) is rewritten as

$$Y = (1-\hat{\alpha})(2(1-\hat{\alpha})\zeta - 3\eta)/\eta \lessgtr (\tau_y + \tau_\theta)/\tau_x.$$

Thus, $\partial W/\partial \tau_y > 0$ if and only if either $Y < (\tau_y + \tau_\theta)/\tau_x$ and $\eta > 0$ or $Y > (\tau_y + \tau_\theta)/\tau_x$ and $\eta < 0$, which establishes the signs of $\partial W/\partial \tau_y$ in (i) and (iii) with $\eta \neq 0$. This also establishes the signs of $\partial W/\partial \tau_y$ in (ii) and (iv) with $\eta \neq 0$ because $\zeta\eta < 0$ implies $Y = (1 - \hat{\alpha})(2(1 - \hat{\alpha})\zeta/\eta - 3) < 0$.

Finally, if $\zeta\eta > 0$, then

$$Y - X = 2((1 - \hat{\alpha})\zeta - \eta)^2/(\zeta\eta) \geq 0,$$

which establishes $X \leq Y$ in (i) and (iii). If $X = Y$, then $(1 - \hat{\alpha})\zeta = \eta$, and thus $X = Y = \hat{\alpha} - 1 < 0$.

D Proof of Corollary 4

In types +I and +II, $\partial W(\tau)/\partial \tau_x > 0$ for all τ, and thus $W(\tau_x, \tau_y) < W(\infty, \tau_y) = W(\tau_x, \infty)$. In type −IV, $\partial W(\tau)/\partial \tau_y > 0$ for all τ, and thus $W(\tau_x, \tau_y) < W(\tau_x, \infty) = W(\infty, \tau_y)$.

In types +III and +IV, if $\tau_x \geq (\tau_y + \tau_\theta)/X \geq \tau_\theta/X$, then $\partial W(\tau)/\partial \tau_x \leq 0$ and $\partial W(\tau)/\partial \tau_y < 0$, and thus $W(\tau_x, \tau_y) \leq W(\tau_x, 0) \leq W(\tau_\theta/X, 0)$. If $\tau_x < (\tau_y + \tau_\theta)/X$, then $\partial W(\tau)/\partial \tau_x > 0$, and thus $W(\tau_x, \tau_y) < W((\tau_y + \tau_\theta)/X, \tau_y) \leq W(\tau_\theta/X, 0)$, where the last inequality holds by the case with $\tau_x \geq (\tau_y + \tau_\theta)/X$.

In type −I, $\partial W(\tau)/\partial \tau_x < 0$ and $\partial W(\tau)/\partial \tau_y < 0$ for all τ, and thus $W(\tau_x, \tau_y) \leq W(0, 0)$.

In types −II and −III, if $\tau_x \leq (\tau_y + \tau_\theta)/Y$, then $\partial W(\tau)/\partial \tau_x < 0$ and $\partial W(\tau)/\partial \tau_y \leq 0$, and thus $W(\tau_x, \tau_y) \leq W(0, \tau_y) \leq W(0, 0)$. If $\tau_x > (\tau_y + \tau_\theta)/Y$, then $\partial W(\tau)/\partial \tau_y > 0$, and thus $W(\tau_x, \tau_y) < W(\tau_x, Y\tau_x - \tau_\theta) \leq W(0, 0)$, where the last inequality holds by the case with $\tau_x \leq (\tau_y + \tau_\theta)/Y$.

E Proof of Corollary 5

In types +I, +II, and −IV, $W(\tau_x, \tau_y) < W(\tau_x, \infty)$ by Corollary 4.

In type +III, if $\tau_y < Y\tau_x - \tau_\theta$, then $\partial W(\tau)/\partial \tau_y < 0$, and if $\tau_y > Y\tau_x - \tau_\theta$, then

$\partial W(\tau)/\partial\tau_y > 0$. Thus,

$$\sup_{\tau_y} W(\tau) = \max\{W(\tau_x, 0), W(\tau_x, \infty)\}.$$

Because $W(\tau_x, \infty) = W(\infty, 0)$, we compare $W(\tau_x, 0)$ and $W(\infty, 0)$. Note that if $\tau_x > \tau_\theta/X$, then $\partial W(\tau_x, 0)/\partial\tau_x < 0$, and thus $W(0, 0) < W(\infty, 0) < W(\tau_\theta/X, 0)$. Hence, there exists a unique $\tau_x^* < \tau_\theta/X$ such that $W(\tau_x^*, 0) = W(\infty, 0)$ since $\partial W(\tau_x, 0)/\partial\tau_x > 0$ for $\tau_x < \tau_\theta/X$. Note that $W(\tau_x, 0) < W(\infty, 0)$ if $\tau_x < \tau_x^*$ and $W(\tau_x, 0) \geq W(\infty, 0)$ if $\tau_x \geq \tau_x^*$. Therefore,

$$\sup_{\tau_y} W(\tau) = \begin{cases} W(\tau_x, \infty) & \text{if } \tau_x < \tau_x^*, \\ W(\tau_x, 0) & \text{if } \tau_x \geq \tau_x^*. \end{cases}$$

To find τ_x^*, we solve

$$W(\tau_x^*, 0) - W(\infty, 0) = \frac{\beta^2\left((1-\hat{\alpha})\tau_x^*((1-\hat{\alpha})\zeta - 2\eta) - \eta\tau_\theta\right)}{(1-\hat{\alpha})^2\left(\tau_\theta + (1-\hat{\alpha})\tau_x^*\right)^2} = 0,$$

and obtain $\tau_x^* = \eta\tau_\theta/((1-\hat{\alpha})X\zeta)$.

In types +IV and −I, $\partial W(\tau)/\partial\tau_y < 0$ for all τ, and thus $W(\tau_x, \tau_y) \leq W(\tau_x, 0)$.

In types −II and −III, $\partial W(\tau)/\partial\tau_y > 0$ if $\tau_y < Y\tau_x - \tau_\theta$ and $\partial W(\tau)/\partial\tau_y < 0$ if $\tau_y > Y\tau_x - \tau_\theta$. Thus, $W(\tau_x, \tau_y) \leq W(\tau_x, \max\{Y\tau_x - \tau_\theta, 0\})$.

F Proof of Corollary 6

In types +I, +II, and −IV, $W(\tau_x, \tau_y) < W(\infty, \tau_y)$ by Corollary 4.

In types +III and +IV, $\partial W(\tau)/\partial\tau_x > 0$ if $\tau_x < (\tau_y + \tau_\theta)/X$ and $\partial W(\tau)/\partial\tau_x < 0$ if $\tau_x > (\tau_y + \tau_\theta)/X$. Thus, $W(\tau_x, \tau_y) \leq W((\tau_y + \tau_\theta)/X, \tau_y)$.

In types −I and −II, $\partial W(\tau)/\partial\tau_x < 0$ for all τ, and thus $W(\tau_x, \tau_y) \leq W(0, \tau_y)$.

In type −III, $\partial W(\tau)/\partial\tau_x < 0$ if $\tau_x < (\tau_y + \tau_\theta)/X$ and $\partial W(\tau)/\partial\tau_x > 0$ if $\tau_x > (\tau_y + \tau_\theta)/X$. Thus,

$$\sup_{\tau_x} W(\tau) = \max\{W(0, \tau_y), W(\infty, \tau_y)\} = W(0, \tau_y)$$

because $W(\infty, \tau_y) = W(0, \infty) < W(0, \tau_y)$ by $\partial W(0, \tau_y)/\partial\tau_y < 0$.

G Proof of Lemma 3

Note that (9) equals $nc_1\mathrm{var}[\sigma_i] + n(n-1)c_2\mathrm{cov}[\sigma_i, \sigma_j]/2 + nc_3\mathrm{cov}[\theta, \sigma_i]$ plus a constant independent of τ. Plugging (B2) into the above, we obtain Lemma 3.

H Proof of Corollary 7

It is enough to show that there exists a fictitious game such that it has the same equilibrium as that of the given game and the ex ante expected payoff equals $W^*(\tau)$ plus a constant independent of τ.

Let α', β', κ', λ', and μ' be the coefficients of the payoff function in the fictitious game. Put $\alpha' = \alpha$, $\beta' = \beta$, $\kappa' = (\zeta^* - 1)/(n-1)$, $\lambda' = 2(\eta^* - \zeta^*)/((n-1)(n-2))$, and $\mu' = 0$. Then, the fictitious game has the same equilibrium as that of the given game because α and β determine players' best responses. Moreover, the ex ante expected payoff equals $W^*(\tau)$ plus a constant independent of τ by Lemma 2.

I Proof of Lemma 4

Using the symmetry $\sigma_i(\cdot) = \sigma_j(\cdot)$ for $i \neq j$, we have

$$\begin{aligned}
E[\bar{u}_i(\sigma, \theta)] =& - E[{\sigma_i}^2] + 2\alpha E[\sigma_i\sigma_j] + 2\beta E[\theta\sigma_i] \\
&+ \kappa E\Big[\int \sigma_j^2 dj\Big] + \lambda E\Big[\Big(\int \sigma_j dj\Big)^2\Big] + \mu E\Big[\theta \int \sigma_j dj\Big] \\
&+ \nu E\Big[\int \sigma_j dj\Big] + E[f(\theta)] \\
=&(\kappa - 1)\mathrm{var}[\sigma_i] + (2\alpha + \lambda)\mathrm{cov}[\sigma_i, \sigma_j] + (2\beta + \mu)\mathrm{cov}[\theta, \sigma_i] \\
&+ (2\alpha + \kappa + \lambda - 1)c^2 + ((2\beta + \mu)\bar{\theta} + \nu)c + E[f(\theta)].
\end{aligned}$$

This is because we have

$$
\begin{aligned}
E\Big[\int \sigma_j^2 dj\Big] &= E\Big[E\Big[\int \sigma_j^2 dj\Big|\theta, y\Big]\Big] = E[\sigma_j^2], \\
E\Big[\Big(\int \sigma_j dj\Big)^2\Big] &= E\Big[E\Big[\Big(\int \sigma_j dj\Big)^2\Big|\theta, y\Big]\Big] = E[\sigma_i\sigma_j], \\
E\Big[\theta\int \sigma_j dj\Big] &= E\Big[E\Big[\theta\int \sigma_j dj\Big|\theta, y\Big]\Big] = E[\theta\sigma_j], \\
E\Big[\int \sigma_j dj\Big] &= E\Big[E\Big[\int \sigma_j dj\Big|\theta, y\Big]\Big] = E[\sigma_j]
\end{aligned}
$$

by the following consequences of the law of large numbers:

$$
\begin{aligned}
E\Big[\int \sigma_j dj\Big|\theta, y\Big] &= E[E[\sigma_j|\theta, y]|\theta, y] = E[\sigma_j|\theta, y], \\
E\Big[\int \sigma_j^2 dj\Big|\theta, y\Big] &= E[E[\sigma_j^2|\theta, y]|\theta, y] = E[\sigma_j^2|\theta, y], \\
E\Big[\Big(\int \sigma_j dj\Big)^2\Big|\theta, y\Big] &= E[(E[\sigma_j|\theta, y])^2|\theta, y] = E[\sigma_i\sigma_j|\theta, y].
\end{aligned}
$$

The last equality follows from (5) because

$$
(E[\sigma_j|\theta, y])^2 = (b_y\varepsilon_0 + (b_x + b_y)(\theta - \bar{\theta}) + c)^2 = E[\sigma_i\sigma_j|\theta, y].
$$

Thus, $E[\bar{u}_i(\sigma, \theta)]$ is the sum of $(\kappa-1)\text{var}[\sigma_i]+(2\alpha+\lambda)\text{cov}[\sigma_i, \sigma_j]+(2\beta+\mu)\text{cov}[\theta, \sigma_i]$ and a constant independent of τ. By the same argument as that in Lemma 2, we obtain (15).

J Proofs of Corollaries 8, 9, 10, 11, 12, and 14

This appendix collects proofs of results which are implied by Corollary 3.

Proof of Corollary 8. Dividing (10) by ρ and setting $\theta = (\theta' - c)\rho^{-1}$, we can obtain (1) with $\alpha = -1/2$, $\beta = 1/2$, and $\kappa = \lambda = \mu = 0$, which implies that $\zeta = \eta = 1$ and $(1 - \hat{\alpha})\zeta/\eta = (n + 1)/2$. □

Proof of Corollary 9. The payoff function (11) is (1) with $\theta = \theta' + 1$, $\alpha = \rho/2$, $\beta = 1/2$, $\kappa = \lambda = 0$, and $\mu = -\rho$, which implies that $\zeta = 1 - 2\hat{\rho}$ and $\eta = (\hat{\rho} - 1)^2 \geq 0$. If $\hat{\rho} > 1/2$ then $\zeta < 0$. If $\hat{\rho} \leq 1/2$ then we can verify that $\zeta \geq 0$, $\eta > 0$, and $(1 - \hat{\alpha})\zeta/\eta < 3/2$. □

Proof of Corollary 10. The payoff function (12) is (1) with $\theta = \theta' - c$, $\alpha = -1$, $\beta = 1/2$, $\kappa = -1$, $\lambda = -2$, and $\mu = 1$, which implies that $\zeta = n > 0$, $\eta = n^2 > 0$, and $(1 - \hat{\alpha})\zeta/\eta = 1$. The payoff function (13) is (1) with $\theta = c - \theta'$, $\alpha = -1$, $\beta = 1/2$, $\kappa = -1$, $\lambda = -2$, and $\mu = 0$, which implies that $\zeta = -(n-2) < 0$, $\eta = -n(n-2) < 0$, and $(1 - \hat{\alpha})\zeta/\eta = 1$. □

Proof of Corollary 11. In MS's beauty contest game, we have $\kappa = r$, $\lambda = -2r$, and $\mu = 0$, which implies that $\zeta = 1 + r > 0$, $\eta = 1 - r > 0$, and $(1 - \alpha)\zeta/\eta = 1 + r$. In Hellwig and Veldkamp's beauty contest game, we have $\kappa = 0$, $\lambda = -r^2$, and $\mu = -2r(1-r)$, which implies that $\zeta = 1 - 2r$, $\eta = (1-r)^2 > 0$, and $(1-\alpha)\zeta/\eta = (1-2r)/(1-r) < 1$. In Myatt and Wallace's beauty contest game, we have $\kappa = 0$, $\lambda = -r$, and $\mu = 0$, which implies that $\zeta = 1 > 0$, $\eta = 1 - r > 0$, and $(1-\alpha)\zeta/\eta = 1$. □

Proof of Corollary12. By normalizing the payoff function, we have a beauty contest game with $\kappa = 0$, $\lambda = -r^2 - 1/c_2$, and $\mu = -2r(1-r)$, which implies that $\zeta = 1-2r$, $\eta = (1-r)^2 - 1/c_2 < 0$, and $(1-\alpha)\zeta/\eta = (1-r)(1-2r)/((1-r)^2 - 1/c_2)$. □

Proof of Corollary14. Multiplying (20) by 2, we have (14) with $\kappa = \lambda = \mu = 0$. Thus, $\zeta = \eta = 1$ and $(1-\alpha)\zeta/\eta = 1 - \alpha$. □

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