Discussion Paper Series A No.662

Partially-Honest Nash Implementation: A Full Characterization

Michele Lombardi
(Adam Smith Business School, University of Glasgow)
and
Naoki Yoshihara
(Department of Economics, University of Massachusetts Amherst
Institute of Economic Research, Hitotsubashi University)

August 2017

Institute of Economic Research
Hitotsubashi University
Kunitachi, Tokyo, 186-8603 Japan

Partially-honest Nash implementation: a full characterization

Michele Lombardi*
E-mail: michele.lombardi@glasgow.ac.uk

Naoki Yoshihara†
E-mail: nyoshihara@econs.umass.edu

August 3, 2017

Abstract

A partially-honest individual is a person who follows the maxim, "Do not lie if you do not have to" to serve your material interest. By assuming that the mechanism designer knows that there is at least one partially-honest individual in a society of $n \geq 3$ individuals, a social choice rule (SCR) that can be Nash implemented is termed partially-honestly Nash implementable. The paper offers a complete characterization of the n-person SCRs that are partially-honestly Nash implementable. It establishes a condition which is *both* necessary *and* sufficient for the partially-honest Nash implementation. If all individuals are partially-honest, then all SCRs that satisfy the property of unanimity are partially-honestly Nash implementable. The partially-honest Nash implementation of SCRs is examined in a variety of environments.

JEL classification: C72; D71.
Keywords: Nash implementation, pure strategy Nash equilibrium, partial-honesty, Condition μ^*.

*Corresponding author: Michele Lombardi, Adam Smith Business School, University of Glasgow, Glasgow, G12 8QQ, United Kingdom.

†Department of Economics, University of Massachusetts Amherst, 412 North Pleasant Street, Amherst, MA 01002, USA; The Institute of Economic Research, Hitotsubashi University, Kunitachi, Tokyo 186-0004 Japan; and School of Management, Kochi University of Technology, Kochi 782-8502, Japan.

1. Introduction

The implementation problem is the problem of designing a mechanism or game form with the property that, for each state of the world, the equilibrium outcomes of the mechanism played in that state coincide with the recommendations that a given social choice rule (SCR) would prescribe for that state. If that mechanism design exercise can be accomplished, the SCR is said to be implementable. The fundamental paper on implementation in Nash equilibrium is thanks to Maskin (1999; circulated since 1977), who proves that any SCR that can be Nash implemented satisfies a remarkably strong invariance condition, now widely referred to as Maskin monotonicity. Moreover, he shows that when the mechanism designer faces $n \geq 3$ individuals, a SCR is Nash implementable if it is Maskin monotonic and satisfies the condition of no veto-power, subsequently, *Maskin's theorem*.[1]

Since the introduction of Maskin's theorem, economists have been interested in understanding how to circumvent the limitations imposed by Maskin monotonicity by exploring the possibilities offered by approximate (as opposed to exact) implementation (Matsushima, 1988; Abreu and Sen, 1991), as well as by implementation in refinements of Nash equilibrium (Moore and Repullo, 1988; Abreu and Sen, 1990; Palfrey and Srivastava, 1991; Jackson, 1992; Vartiainen, 2007a) and by repeated implementation (Kalai and Ledyard, 1998; Lee and Sabourian, 2011; Mezzetti and Renou, 2012). One additional way around those limitations is offered by implementation with partially-honest individuals.

A *partially-honest individual* is an individual who deceives the mechanism designer when the truth poses some obstacle to her material well-being. Thus, she does not deceive when the truth is equally efficacious. Simply put, a partially-honest individual follows the maxim, "Do not lie if you do not have to" to serve your material interest.

In a general environment, a seminal paper on Nash implementation problems involving partially-honest individuals is Dutta and Sen (2012), which shows that for implementation problems involving $n \geq 3$ individuals and in which there is at least one partially-honest individual, the Nash implementability is assured by no veto-power. Similar positive results are uncovered in other environments by Matsushima (2008a,b), Kartik and Tercieux (2012), Kartik et al. (2014), Saporiti (2014), Ortner (2015) and Mukherjee et al. (2017). Thus, there are far fewer limitations for Nash implementation when there are partially-honest individuals.[2]

A natural question, then, is: Where do the exact boundaries of those limitations lie? This paper answers that question by providing a complete characterization of the n-person SCRs that are Nash implementable when there is at least one partially-honest individual in a society of $n \geq 3$ individuals. Thus, it provides the counterpart to Moore and Repullo

[1] Moore and Repullo (1990), Dutta and Sen (1991), Sjöström (1991) and Lombardi and Yoshihara (2013) refined Maskin's theorem by providing necessary and sufficient conditions for a SCR to be implementable in (pure strategies) Nash equilibrium. For two recent excellent surveys on the subject of implementation see Jackson (2001) and Maskin and Sjöström (2002). For a concise and elegant collection of seminal results on the subject of mechanism design, the reader should consult Dasgupta et al. (1979).

[2] A pioneering work on the impact of decency constraints on Nash implementation problems is Corchón and Herrero (2004). These authors propose restrictions on sets of strategies available to agents that depend on the state of the world. They refer to these strategies as *decent* strategies and study Nash implementation problems in them. For a particular formulation of decent strategies, they are also able to circumvent the limitations imposed by Maskin monotonicity.

(1990)'s conditions for a many-person setting with partially-honest individuals.

The necessary and sufficient conditions are derived by using the approach developed by Moore and Repullo (1990). Moreover, given the positive result provided by Dutta and Sen (2012), no Maskin monotonicity whatsoever is used to derive our characterization result for SCR satisfying the standard condition of unanimity. Consequently, it consists of a weakened version of the Condition μ(ii) of Moore and Repullo (1990), which we name Condition μ^*(ii).[3] Furthermore, this condition, when combined with other two necessary conditions, called Condition μ^*(i) and Condition μ^*(iii), provide also a full characterization of the class of SCRs that are Nash implementable with partially-honest individuals. Condition μ^*(i) is a weak variant of Maskin monotonicity, which is trivially satisfied by any SCR satisfying the unanimity condition, and so it is dispensable for the characterization of the class of unanimous SCRs that partially honest Nash implementable.

It is shown that if a SCR can be implemented in Nash equilibrium when there are partially-honest individuals, then it satisfies our properties. To prove the other direction, we construct a canonical mechanism which involves partially-honest individuals and in which each participant chooses the information about a state of the world as part of her strategy choice. By assuming that a participant's play is honest if she plays a strategy choice which is veracious in its state announcement component and that the mechanism designer knows that there is at least one partially-honest participant, it is shown that if a SCR satisfies our properties, then it can be implemented in Nash equilibrium in any many-person setting.[4] By employing this mechanism, we also show that if all individuals are partially-honest, then any SCR satisfying the unanimity condition is Nash implementable. This is so because Condition μ^*(ii) applies only to cases where not all individuals have a taste for honesty.

To help understand the content of our conditions, we present the characterization in two parts: first, by showing in section 3 that Condition μ^*(ii) is a necessary and sufficient condition for the Nash implementation of SCRs that satisfy the standard unanimity condition, then demonstrating in section 4 that Condition μ^*(ii), when combined with Condition μ^*(i) and Condition μ^*(iii), also completely characterizes the class of SCRs that are Nash implementable with partially-honest individuals.

1.2 *Condition μ^*(ii) at work*

The importance and usefulness of Condition μ^*(ii) is underlined in four applications: coalitional games, marriage games, rationing problems with single-peaked preferences and bargaining games. In these contexts, we study SCRs that satisfy the unanimity condition but fail both Maskin monotonicity and the condition of no veto-power.

For the coalitional game environment, we present the *core* solution, which is the main set solution used for coalitional games. We show that this solution is not Nash implementable with partially-honest individuals when the mechanism designer knows the coalitional function

[3] Recall that Condition μ(ii) is a weakened version of the no veto-power condition, whereas Condition μ(iii) is a weakened version of the unanimity condition. Finally, Condition μ(i) of Moore and Repullo (1990) is equivalent to Maskin monotonicity if the SCR satisfies the condition of no veto-power.

[4] The canonical mechanism is subject to standard criticisms (see, Jackson (1992; 2001), for discussion). The usual counterargument to these criticisms is that general results need to rely on canonical mechanisms (see, again Jackson (1992), for discussion).

of the games, who, however, does not know the prevailing state. However, as already noted earlier, this solution becomes Nash implementable when all individuals are partially-honest. This means that the informational assumption of what the mechanism designer knows of the identity (or identities) of the partially-honest individual(s) can have profound effects on the limitations imposed by Condition μ^*(ii).

As a second application, we consider the classical model of matching men to women (Gale and Shapley; 1962). We study the so-called *man-optimal stable* solution, which selects the stable matching produced by the deferred acceptance algorithm when men propose to women: it is the best stable matching from the perspective of every man.[5] When the mechanism designer does not know the prevailing state, it is shown that this solution can be successfully Nash implemented with partially-honest individuals. This result is in contrast to the literature on Nash implementation of matching solutions where no proper sub-solution of the stable solution is Nash implementable in the class of marriage games with singles - as per Kara and Sönmez (1996) - and where no single-valued sub-solution of the stable solution is Nash implementable in the class of pure marriage games, where being single is not a feasible choice or it is always the last choice of every individual – as per Tadenuma and Toda (1998).

As a third application, we consider problems of fully allocating a perfectly divisible commodity among a group of individuals who have single-peaked preferences over all possible partitions of the commodity (Sprumont, 1991; Thomson, 1994a). Individual's preferences are single-peaked if there is a fraction of the commodity, named the peak amount, which is judged to be better than any other fraction and if her preferences, on each side of the peak, are strictly monotonic, increasing on its left and decreasing on its right.

In this rationing environment, we consider the *equal-distance* solution, which selects the partition whose fractions are equally far from the peak amounts of individuals (subject to non-negativity). We show that this solution can be Nash implemented with partially-honest individuals. Though we do not provide formal arguments, those offered for the equal-distance solution apply entirely to the so-called *equal-sacrifice* solution, which divides the commodity so that all upper contour sets are of the same size (subject to non-negativity).

Two other remarks on the Nash implementability in this setting are worth mentioning. In the first place, the *proportional* solution, which partitions the commodity proportionally to the peak amounts is not Nash implementable with partially-honest individuals. The reason for this is that this solution is not continuous: a discontinuity occurs when all peak amounts are equal to zero. Secondly, none of the solutions we study in this environment are Nash implementable with partially-honest individuals in a two-person setting. This is so because we pay attention to non-wasteful divisions of the commodity.

Last but not least, we look at the Nash implementability of the Nash (bargaining) solution. In the classical cooperative bargaining theory, initiated in Nash (1950), a number

[5] A matching is *stable* if no individual prefers being single to her or his mate under this matching and, moreover, in each case where a woman/man prefers another man/woman to the man/woman to whom she/he is matched under this matching, that man/woman prefers the woman/man to whom he/she is matched to her/him. Note that the roles of men and women can be interchanged in the deferred acceptance algorithm, and this produces the so-called *woman-optimal stable* solution to marriage problems, which selects the best stable matching from the perspective of every woman. In the paper, we focus our discussion on the man-optimal stable solution since the arguments for the woman-optimal stable rule are entirely symmetric. The stable solution of a marriage game consists only of stable matchings of the game.

of individuals face the task of finding a unanimous agreement over the (expected) utility allocations resulting from the lotteries over a set of physical objects. The *Nash* solution, due to Nash (1950), selects the utility allocation that maximizes the product of the utilities over the feasible utility allocations. This allocation is now widely referred to as the Nash point.[6]

The normative evaluation of the Nash solution is thus done entirely in utility space, based on the expected utility functions of the individuals. On the other hand, the objective of the abstract theory of Nash implementation is to help a uninformed mechanism designer to Nash implement outcomes satisfying certain desirable welfare criteria. This means that the shape of the utility space is unknown to the mechanism designer. One way to get these two classic areas of study closer has recently been suggested by Vartiainen (2007b) in the canonical cake sharing setting, which we follow in this last application.

We consider a situation where individuals bargain over the partition of one unit of a perfectly divisible commodity. Additionally, we assume that at each state every individual's preference over the set of possible agreements is represented by a continuous and increasing expected utility function.[7] With these specifications, and when lotteries are feasible, every state generates a classic (non-empty, convex, compact and comprehensive) utility space. We thus require that the Nash solution associates, with each state, the set of all lotteries that generate the Nash point of the utility space generated by the state.

When both individuals and the mechanism designer know the size of the commodity and the space of lotteries but only individuals know the prevailing state, it is shown that the Nash solution can be Nash implemented in a setting with partially-honest individuals, though it violates the condition of no veto-power. This is a rather significant permissive result because several attempts have been made to give a non-cooperative foundation to the Nash solution since Nash (1953). With the exception of Naeve (1999),[8] reconstructions of the Nash point as an equilibrium point of a mechanism are based on refinements of Nash equilibrium as solution concepts. See, e.g., Howard (1992) and Miyagawa (2002).[9]

The remainder of this paper is divided into 4 sections. Section 2 sets out the theoretical framework and outlines the basic model. Section 3 completely characterizes the class of Nash implementable SCRs satisfying the unanimity condition and assesses its implications in a variety of environments. Section 4 offers a complete characterization. Section 5 concludes. Appendices include proofs not in the main body.

[6] For an excellent survey on the subject of cooperative bargaining theory see Thomson (1994b).

[7] If expected utility functions representing individuals' preferences are strictly increasing, it follows from the result of Dutta and Sen (2012) that the Nash solution is Nash implementable with partially-honest individuals.

[8] In a variant of the model of Serrano (1997), Naeve (1999) shows that the Nash bargaining solution can be Nash implemented. However, this could be purchased at the cost of a strong domain restriction of individuals' preferences. For instance, the set of states cannot take the structure of the Cartesian product of allowable independent characteristics for individuals (see Naeve, 1999; p. 24).

[9] Moulin (1984) constructs a mechanism that implements the so-called Kalai–Smorodinsky bargaining solution in subgame perfect Nash equilibrium.

2. Preliminaries

2.1 Basic framework

We consider a finite set of individuals indexed by $i \in N = \{1, \cdots, n\}$, which we will refer to as a society. The set of outcomes available to individuals is X. The information held by the individuals is summarized in the concept of a state, which is a complete description of the variable characterizing the world. Write Θ for the domain of possible states, with θ as a typical state. In the usual fashion, individual i's preferences in state θ are given by a complete and transitive binary relation, subsequently an ordering, $R_i(\theta)$ over the set X. The corresponding strict and indifference relations are denoted by $P_i(\theta)$ and $I_i(\theta)$, respectively. The statement $xR_i(\theta)y$ means that individual i judges x to be at least as good as y. The statement $xP_i(\theta)y$ means that individual i judges x better than y. Finally, the statement $xI_i(\theta)y$ means that individual i judges x and y as equally good, that is, she is indifferent between them.

We assume that the mechanism designer does not know the true state, that there is complete information among the individuals in N and that the mechanism designer knows the preference domain consistent with the domain Θ. We shall sometimes identify states with preference profiles.

The goal of the mechanism designer is to implement a SCR F, which is a correspondence $F : \Theta \twoheadrightarrow X$ such that $F(\theta)$ is non-empty for every $\theta \in \Theta$. We shall refer to $x \in F(\theta)$ as an F-optimal outcome at θ. The image or range of the SCR F is the set $F(\Theta) \equiv \{x \in X | x \in F(\theta) \text{ for some } \theta \in \Theta\}$.

Given that individuals will have to be given the necessary incentives to reveal the state truthfully, the mechanism designer delegates the choice to individuals according to a mechanism $\Gamma \equiv \left(\prod_{i \in N} M_i, g\right)$, where M_i is the strategy space of individual i and $g : M \to X$, the outcome function, assigns to every strategy profile $m \in M \equiv \prod_{i \in N} M_i$ a unique outcome in X. The strategy profile m_{-i} is obtained from m by omitting the ith component, that is, $m_{-i} = (m_1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_n)$, and we identify (m_i, m_{-i}) with m.

2.2 Intrinsic preferences for honesty

An individual who has an intrinsic preference for truth-telling can be thought of as an individual who is torn by a fundamental conflict between her deeply and ingrained propensity to respond to material incentives and the desire to think of herself as an honest person. In this paper, the theoretical construct of the balancing act between those contradictory desires is based on two ideas.

First, the pair (Γ, θ) acts as a "context" for individuals' conflicts. The reason for this is that an individual who has an intrinsic preference for honesty can categorize her strategy choices as truthful or untruthful relative to the state θ and the mechanism Γ designed by the mechanism designer to govern the communication with individuals. That categorization can be captured by the following notion of truth-telling correspondence:

Definition 1 For each Γ and each individual $i \in N$, individual i's *truth-telling correspondence* is a (non-empty) correspondence $T_i^{\Gamma} : \Theta \twoheadrightarrow M_i$ such that, for each $\theta \in \Theta$ and $m_i \in T_i^{\Gamma}(\theta)$, the strategy choice m_i encodes information that is consistent with the state θ. Strategy choices in $T_i^{\Gamma}(\theta)$ will be referred to as truthful strategy choices for θ.

Second, in modeling intrinsic preferences for honesty, we endorse the notion of partially-honest individuals introduced by Dutta and Sen (2012). First, a partially-honest individual is an individual who responds primarily to material incentives. Second, she strictly prefers to tell the truth whenever lying has no effect on her material well-being. That behavioral choice of a partially-honest individual can be modeled by extending an individual's ordering over X to an ordering over the strategy space M because that individual's preference between being truthful and being untruthful is contingent upon announcements made by other individuals as well as the outcome(s) obtained from them. By following standard conventions of orderings, write $\succcurlyeq_i^{\Gamma,\theta}$ for individual i's ordering over M in state θ whenever she is confronted with the mechanism Γ. Formally, our notion of a partially-honest individual is as follows:

Definition 2 For each Γ, individual $i \in N$ is *partially-honest* if for all $\theta \in \Theta$ individual i's intrinsic preference for honesty $\succcurlyeq_i^{\Gamma,\theta}$ on M satisfies the following properties: for all m_{-i} and all $m_i, m_i' \in M_i$ it holds that:

(i) If $m_i \in T_i^{\Gamma}(\theta)$, $m_i' \notin T_i^{\Gamma}(\theta)$ and $g(m)\, R_i(\theta)\, g(m_i', m_{-i})$, then $m \succ_i^{\Gamma,\theta} (m_i', m_{-i})$.

(ii) In all other cases, $m \succcurlyeq_i^{\Gamma,\theta} (m_i', m_{-i})$ if and only if $g(m)\, R_i(\theta)\, g(m_i', m_{-i})$.

An intrinsic preference for honesty of individual i is captured by the first part of the above definition, in that, for a given mechanism Γ and state θ, individual i strictly prefers the strategy profile (m_i, m_{-i}) to (m_i', m_{-i}) provided that the outcome $g(m_i, m_{-i})$ is at least as good as $g(m_i, m_{-i})$ according to her ordering $R_i(\theta)$ and that m_i is truthful for θ and m_i' is not truthful for θ.

If individual i is *not* partially-honest, this individual cares for her material well-being associated with outcomes of the mechanism and nothing else. Then, individual i's ordering over M is just the transposition into space M of individual i's relative ranking of outcomes. More formally:

Definition 3 For each Γ, individual $i \in N$ is *not partially-honest* if for all $\theta \in \Theta$, individual i's intrinsic preference for honesty $\succcurlyeq_i^{\Gamma,\theta}$ on M satisfies the following property:

$$m \succcurlyeq_i^{\Gamma,\theta} m' \iff g(m)\, R_i(\theta)\, g(m'), \quad \text{for all } m, m' \in M.$$

2.3 Implementation problems

In formalizing the mechanism designer's problem with partially-honest individuals, we first introduce an informational assumption and discuss its implications for our analysis. It is:

Assumption 1 There exists at least one partially-honest individual in the society N.

Thus, in our setting, the mechanism designer does not know the true state and, moreover, he does not know neither the identity (or identities) nor the number of the partially-honest individual(s). Indeed, the mechanism designer cannot exclude any member(s) of society from being partially-honest purely on the basis of Assumption 1. Therefore, the following considerations are in order from the viewpoint of the mechanism designer.

An environment is described by two parameters, (θ, H): a state θ and a conceivable set of partially-honest individuals H. We denote by H a typical conceivable set of partially-honest individuals in N, with h as a typical element, and by $\mathcal{H}$ the class of conceivable sets of partially-honest individuals.

A mechanism Γ and an environment (θ, H) induce a strategic game $\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$, where:

$$\succsim^{\Gamma,\theta,H} \equiv \left(\succsim_i^{\Gamma,\theta}\right)_{i \in N}$$

is a profile of orderings over the strategy space M as formulated in Definition 2 and in Definition 3. Specifically, $\succsim_i^{\Gamma,\theta}$ is individual i's ordering over M as formulated in Definition 2 if individual i is in H, whereas it is the individual i's ordering over M as formulated in Definition 3 if individual i is not in H.

A (pure strategy) Nash equilibrium of the strategic game $\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$ is a strategy profile m such that for all $i \in N$, it holds that

$$m \succsim_i^{\Gamma,\theta} \left(m_i', m_{-i}\right), \quad \text{for all } m_i' \in M_i.$$

Write $NE\left(\Gamma, \succsim^{\Gamma,\theta H}\right)$ for the set of Nash equilibrium strategies of the strategic game $\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$ and $NA\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$ for its corresponding set of Nash equilibrium outcomes.

The following definition is to formulate the designer's Nash implementation problem involving partially-honest individuals.

Definition 4 Let Assumption 1 hold. A mechanism Γ *partially-honestly Nash implements* the SCR $F : \Theta \twoheadrightarrow X$ provided that for all $\theta \in \Theta$ there exists a truth-telling correspondence $T_i^{\Gamma}(\theta)$ as formulated in Definition 1 for every $i \in N$ and, moreover, it holds that

$$F(\theta) = NA\left(\Gamma, \succsim^{\Gamma,\theta,H}\right), \quad \text{for every pair } (\theta, H) \in \Theta \times \mathcal{H}.$$

If such a mechanism exists, F is said to be *partially-honestly Nash implementable.*

The objective of the mechanism designer is thus to design a mechanism whose Nash equilibrium outcomes coincide with $F(\theta)$ for each state θ as well as each set H. Note that there is no distinction between the above formulation and the standard Nash implementation problem as long as Assumption 1 is discarded.

3. The characterization theorem for unanimous SCRs

In this section, we provide a full characterization of the class of n-person SCRs which are partially-honestly Nash implementable SCRs as well as satisfy the property of unanimity:

Definition 5 The SCR $F : \Theta \twoheadrightarrow X$ satisfies *unanimity* provided that for all $\theta \in \Theta$ and all $x \in X$ if $xR_i(\theta)\, y$ for all $i \in N$ and all $y \in X$, then $x \in F(\theta)$. A SCR that satisfies this property is said to be a unanimous SCR.

In other words, it states that if an outcome is at the top of the preferences of all individuals, then that outcome should be selected by the SCR. Unanimity is a property satisfied, for example, by the Pareto rule and, in the market contexts, by the rule which selects all core allocations. However, some interesting SCRs are not unanimous. For instance, the egalitarian-equivalence rule from equal division is not a unanimous SCR. Therefore, given the characterization results presented below, there will be limits to the success of partially-honest implementability: A characterization of the class of n-person SCRs with $n \geq 3$ which are partially-honestly Nash implementable is presented in the next section.

We introduce below Condition μ^*(ii), which is necessary and sufficient for partially-honest implementation of unanimous SCRs in many-individual settings. Let us formalize the condition as follows. Given a state θ, an individual i, a set of outcomes $A \subseteq X$ and an outcome $x \in X$, the *contour set of* $R_i(\theta)$ *through* $x \in X$ *restricted to* A is $I_i(\theta, x, A) = \{x' \in A | xI_i(\theta)\, x'\}$; the *weak lower contour set of* $R_i(\theta)$ *at* x is $L_i(\theta, x) = \{x' \in X | xR_i(\theta)\, x'\}$; and the *strict lower contour set of* $R_i(\theta)$ *at* x is $SL_i(\theta, x) = \{x' \in X | xP_i(\theta)\, x'\}$. Therefore:

Definition 6 The SCR $F : \Theta \to X$ satisfies *Condition* μ^*(ii) with respect to $Y \subseteq X$ if $F(\Theta) \subseteq Y$, and if for every $(i, \theta, x) \in N \times \Theta \times Y$ with $x \in F(\theta)$, there exists a set $C_i(\theta, x) \subseteq Y$ with $x \in C_i(\theta, x) \subseteq L_i(\theta, x)$, such that for every pair $(\theta', H) \in \Theta \times \mathcal{H}$ we have:
(1) (a) For all $i \in N$, there exists a non-empty set $S_i(\theta'; x, \theta)$ such that $S_i(\theta'; x, \theta) \subseteq C_i(\theta, x)$.
(b) For all $h \in H$, if $\theta = \theta'$ and $x \notin S_h(\theta'; x, \theta)$, then $S_h(\theta'; x, \theta) \subseteq SL_h(\theta, x)$.
(2) For all $i \in N$, if $y \in C_i(\theta, x) \subseteq L_i(\theta', y)$ and $Y \subseteq L_j(\theta', y)$ for all $j \in N \setminus \{i\}$, and if $y \notin F(\theta')$, then:
(a) the intersection $S_i(\theta'; x, \theta) \cap I_i(\theta', y, Y)$ is not empty and $y \notin S_i(\theta'; x, \theta)$ if $H = \{i\}$.
(b) $x \notin S_j(\theta'; x, \theta)$ for some $j \in H$ if $i \notin H$ and $\theta = \theta'$.

Moore and Repullo (1990) showed that a necessary and sufficient condition for implementation needs to require the existence of the set Y as well as the existence of the set $C_i(\theta, x)$ for each triplet (i, x, θ) with $x \in F(\theta)$. Condition μ^*(ii) requires the existence of those sets as well. In addition to this, part (1)(a) of Condition μ^*(ii) requires the existence of a set $S_i(\theta'; x, \theta) \subseteq C_i(\theta, x)$ for every quadruplet (i, x, θ, θ') with $x \in F(\theta)$. Let us give an intuitive explanation of this set.

Suppose that F is partially-honestly implementable by a mechanism Γ. Thus, if $x = g(m)$ is F-optimal at θ, that is, $x \in F(\theta)$, whilst the set $C_i(\theta, x) = g(M_i, m_{-i})$ represents the set of outcomes that individual i can generate by varying her own strategy, keeping the other individuals' equilibrium strategy choices fixed at m_{-i}, the set $S_i(\theta'; x, \theta) = g\left(T_i^{\Gamma}(\theta'), m_{-i}\right)$ represents the set of outcomes that this individual can attain by playing truthful strategy choices for θ' when the state moves from θ to θ', keeping the other individuals' equilibrium strategy choices fixed at m_{-i}.

Given this idea of the set of $S_i(\theta'; x, \theta)$, we refer to elements of $S_i(\theta'; x, \theta)$ as *truthful outcomes* for individual i at the state θ' when the state moves from θ to θ' and x is an F-optimal outcome at θ.

Part (1)(b) of Condition μ^*(ii) follows the reasoning that if x is F-optimal at θ but x is not a truthful outcome for the partially-honest individual $h \in H$ at this θ, then, in order not to break the Nash equilibrium via a unilateral deviation of a partially-honest individual h, it must be the case that this x is strictly preferred to any truthful outcome in $S_h(\theta; x, \theta)$ according to her ordering $R_h(\theta)$.

To give an intuitive overview of part (2) of Condition μ^*(ii), suppose that x is F-optimal at θ and when the state moves from θ to θ' it happens that an outcome $y \in C_i(\theta, x)$ is $R_i(\theta')$-maximal for some individual i in the set $C_i(R, x)$ and that this y is also $R_j(\theta')$-maximal for any other individual j in the set Y but y is not F-optimal at θ'. Thus, only a partially-honest individual h can find it profitable unilaterally to deviate from a strategy profile supporting the outcome y as the outcome of the mechanism Γ.

Part (2)(a) specifies that if individual i can be identified as the only individual who can find a unilateral profitable deviation from the strategy profile supporting the outcome y, then the y is not a truthful outcome for this individual i at the state θ', that is, $y \notin S_i(\theta'; x, \theta)$. In addition, individual i needs to find a truthful outcome $z \in S_i(\theta'; x, \theta)$ that is equally good to y according to her ordering $R_i(\theta')$ in order to have a unilateral non-material profitable deviation, that is, the outcome z is an element of $S_i(\theta'; x, \theta) \cap I_i(\theta', y, Y)$.

Part (ii)(b) specifies that if the state θ coincides with the state θ' and individual i is not a partially-honest individual, that is, $i \notin H$, then it cannot be that the deviant partially-honest individual $h \in H$ played a truthful strategy choice at equilibrium strategy profile supporting x as a Nash equilibrium outcome of $\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$.

We are now ready to present our characterization result for the partially-honest implementability of unanimous SCRs. However, before stating it, we restrict the structure of the family $\mathcal{H}$ to the following specification:

Assumption 2 The family $\mathcal{H}$ has as elements all non-empty subsets of the set N.

This requirement is consistent with Assumption 1 since the mechanism designer cannot exclude any member(s) of the society from being partially-honest purely on the basis of that assumption. This is sufficient for our characterization theorem:

Theorem 1 Let $n \geq 3$. Suppose that assumptions 1-2 hold. The unanimous SCR $F : \Theta \twoheadrightarrow X$ satisfies Condition μ^*(ii) with respect to $Y \subseteq X$ if and only if it is partially-honestly Nash implementable.

Proof. See Appendix A. ■

We make several remarks below regarding Theorem 1.

Remark 1 The "if" part of the theorem continues to hold if Assumption 2 is replaced with the requirement that the family $\mathcal{H}$ is *closed under union*, that is, with the requirement that if H is an element of $\mathcal{H}$ and if H' is another of its elements, then the union of these sets is also an element of $\mathcal{H}$. This specification of the family $\mathcal{H}$ is by far weaker than Assumption 2 and has an obvious expansion-consistency interpretation: If the mechanism designer views H as a conceivable set of partially-honest individuals and he also views H' as another conceivable set, then there is no reason for him to exclude their union from $\mathcal{H}$ purely on the basis of Assumption 1. This specification is the minimal restriction on the family $\mathcal{H}$ that allows part (1)(b) of Condition μ^*(ii) to still be a necessary condition for partially-honest implementation. The reason is that to assure it we need to be able to select a strategy profile m that generates the F-optimal outcome x at θ as a Nash equilibrium outcome for this θ and for a set of partially-honest individuals $\hat{H}$ which contains all elements of the family $\mathcal{H}$, that is, $H \subseteq \hat{H}$ for every $H \in \mathcal{H}$. This is because if $\hat{H}$ is an element of the family $\mathcal{H}$, then the strategy profile m supporting the F-optimal x at the state θ as a Nash equilibrium of $\left(\Gamma, \succcurlyeq^{\Gamma,\theta,\hat{H}}\right)$ is also a Nash equilibrium of $\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$ for every other allowable set H. This allows us to show that part (1)(b) of Condition μ^*(ii) applies to whatever conceivable set of partially-honest individuals.

Remark 2 Condition μ^*(ii) is a necessary condition for the class of n-person SCRs with $n \geq 2$ which are partially-honestly Nash implementable when the family $\mathcal{H}$ is closed under union.

Remark 3 The "only if" part of the theorem continues to hold if Assumption 2 is replaced with the requirement that the family $\mathcal{H}$ includes *all singletons of the set* N. This is because if m is a Nash equilibrium of some strategic game $\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$ and if individual i's strategy choice m_i is a truthful one for the state θ, then this m is also a Nash equilibrium of the strategic game $\left(\Gamma, \succcurlyeq^{\Gamma,\theta,\{i\}}\right)$ provided that the singleton $\{i\}$ is an element of $\mathcal{H}$.

Common to the literature of implementation with partially-honest individuals is also the requirement that every member of society has a taste for honesty, as per Matsushima (2007), Dutta and Sen (2012), Saporiti (2015) and Mukherjee et al. (2017). Thus, if we follow these authors and confine our analysis to this case, we have the following characterization theorem as well:

Theorem 2 Let $n \geq 3$ and let all individuals in N be partially-honest. The unanimous SCR $F : \Theta \twoheadrightarrow X$ satisfies Condition μ^*(ii) with respect to $Y \subseteq X$ if and only if it is partially-honestly Nash implementable.

Proof. It follows from the proof of Theorem 1, with the observation that in this case $\mathcal{H} = \{N\}$ and no Nash equilibrium strategy profile can fall into *Rule 2.2* as well as into *Rule 2.3* of the constructed mechanism. ■

Based on this result, when there are more than two individuals, the class of n-person unanimous SCRs with $n \geq 3$, which are partially-honestly Nash implementable, coincides with the collection of all unanimous SCRs. This is because every unanimous SCR satisfies Condition μ^*(ii) under the specification that the set $Y = X$ and that $S_i(\theta'; x, \theta) = C_i(\theta, x) = L_i(\theta, x)$ for every quadruplet (i, θ, θ', x) such that x is an F-optimal outcome at θ.[10]

In the following subsections, we propose several settings where Theorem 1 is applied.

[10]Note that part (2) of Condition μ^*(ii) is satisfied vacuously.

3.1 Applications to coalitional games

This subsection presents the *core* solution, which is the main set solution used for coalitional games, and it shows that this solution is not partially-honestly Nash implementable.

A *coalitional game* is a quadruplet $(N, X, \theta; v)$ such that:

- N is a finite set of individuals. A subset of N is called a coalition. The class of all non-empty coalitions is denoted by $\mathcal{P}(N)$.
- X is a set of outcomes.
- θ is a state in Θ.
- $v : \mathcal{P}(N) \to 2^X$ is a function associating every element of class $\mathcal{P}(N)$ with a subset of the set X, where 2^X is a family that has as elements all subsets of X. This function is called the coalitional function of the game.

Let $(N, X, \theta; v)$ be a coalitional game. An outcome $x \in X$ is *weakly blocked by a coalition* $S \in \mathcal{P}(N)$ if there is an outcome $y \in v(S)$ such that $yR_j(\theta)x$ for every member j of S, with $yP_j(\theta)x$ for at least one of its members.

Definition 7 The *core solution* of a coalitional game $(N, X, \theta; v)$, denoted by $\mathcal{C}$, is the collection of all outcomes that are not weakly blocked by any coalition S,

$$\mathcal{C}(\theta) \equiv \{x \in X | \text{for every } S \in \mathcal{P}(N) \text{ and every } y \in v(S) : xR_j(\theta)y \text{ for every } j \in S\}.$$

The following claim establishes the failure of partially-honestly Nash implementing the core solution when the mechanism designer knows what is feasible for every element of $\mathcal{P}(N)$, that is, he knows the coalitional function, and he does not know the true state.

Claim 1 Let $n \geq 3$. Let Assumption 2 be given. Then, the core solution does not satisfy Condition μ^*(ii) with respect to $Y \subseteq X$.

Proof. Let the premises hold and assume, to the contrary, that the core solution satisfies Condition μ^*(ii) with respect to $Y \subseteq X$.

Since the core solution is unanimous, the set Y coincides with the set X as per Sjöström (1991), and so Y contains the range of $\mathcal{C}$.

Suppose that there are three individuals and two states θ and θ'. Individuals' preferences are represented in the table below:

θ			θ'		
1	2	3	1	2	3
y, z	x	w	y	w, x, y, z	w, x, y, z
x	$w.y, z$	x, z	x		
w		y	w, z		

where, as usual, $\substack{a \\ b}$ for individual i means that she strictly prefers a to b, while a, b means that this i is indifferent between a and b. Suppose that the coalitional function is defined as follows:

$$\begin{aligned} \upsilon(\{1,2\}) &= \{x,z\}, \upsilon(\{1,3\}) = \{w,y\}, \upsilon(\{2,3\}) = \{w,z\}, \\ \upsilon(N) &= X \text{ and } \upsilon(S) = \varnothing \text{ for every other } S \in \mathcal{P}(N). \end{aligned}$$

In the coalitional game $(N, X, \theta; \upsilon)$, the core solution contains only the outcome x. To see this, note that w is weakly blocked by the coalition $\{1,2\}$ and that y and z are both weakly blocked by the coalition $\{2,3\}$. However, in the coalitional game $(N, X, \theta'; \upsilon)$, the core solution contains only the outcome y since every other outcome is weakly blocked by the coalition $\{1,3\}$.

Thus, by construction, we have that $C_1(\theta, x) \subseteq L_1(\theta', x)$, that $Y \subseteq L_j(\theta', x)$ for every individual $j \neq 1$ and that the intersection $S_1(\theta'; x, \theta) \cap I_1(\theta', x, Y)$ is empty if $x \notin S_1(\theta'; x, \theta)$. However, part (2)(a) of Condition μ^*(ii) implies for $H = \{1\}$ that $x \notin S_1(\theta'; x, \theta)$ and that the intersection $S_1(\theta'; x, \theta) \cap I_1(\theta', x, Y)$ is not empty, which is a contradiction.

We have proved the claim by assuming that $n = 3$. The proof will be identical for $n > 3$: just endow individual $k > 3$ with the same preferences of individual 3 considered above and just change the coalitional function as follows: $\upsilon(\{1,3\}) = \upsilon(\{1,k\})$ and $\upsilon(\{2,3\}) = \upsilon(\{2,k\})$. ■

As noted above, Condition μ^*(ii) is also a necessary condition for partially-honest Nash implementation when $n = 2$ and when the family $\mathcal{H}$ has as elements all non-empty subsets of the set N. Therefore, by a reasoning like that used in the above claim one can also show that the core solution violates Condition μ^*(ii) when $n = 2$.[11]

3.2 Applications to marriage problems

This section presents the basic model of matching men to women and shows that the *man-optimal stable* solution can be successfully partially-honestly Nash implemented.

A *marriage problem* is a quadruplet $(M, W, \theta, \mathcal{M})$ such that:

- M is a finite non-empty set of men, with m as a typical element.

[11] To see this, suppose that there are two individuals and two states θ and θ'. Individuals' preferences are represented in the table below:

θ		θ'	
1	2	1	2
y	z	y	x, y, z
x	x	x	
z	y	z	

Suppose that the coalitional function is defined as follows:

$$\upsilon(\{1,2\}) = X \text{ and } \upsilon(S) = \varnothing \text{ for every other } S \in \mathcal{P}(N).$$

In the coalitional game $(N, X, \theta; \upsilon)$, the core solution contains only the outcome x. In addition, in the coalitional game $(N, X, \theta'; \upsilon)$, the core solution contains only the outcome y. Thus, as in the above claim, $x \in \mathcal{C}(\theta)$ but $x \notin \mathcal{C}(\theta')$ and, moreover, $L_1(\theta, x) = L_1(\theta', x)$ and $X \subseteq L_2(\theta', x)$. Since the singleton $\{1\}$ is an element of the family $\mathcal{H}$, one can now easily check that the core solution violates part (2)(a) of Condition μ^*(ii) under the specification that $Y = X$. The reason is that there cannot exist any outcome $z \neq x$ in the set $L_1(\theta, x)$ such that individual 1 is indifferent between this z and x according to her ordering $R_1(\theta')$.

- W is a finite non-empty set of women, , with w as a typical element.

- θ is a state such that (i) every man $m \in M$'s preferences are represented by a linear ordering $P_m(\theta)$ over the set $W \cup \{m\}$ and (ii) every woman $w \in W$'s preferences are represented by a linear ordering $P_w(\theta)$ over the set $M \cup \{w\}$.

- $\mathcal{M}$ is a collection of all matchings, with μ as a typical element. $\mu : M \cup W \to M \cup W$ is a bijective function matching every individual $i \in M \cup W$ either with a partner of the opposite sex or with herself. If an individual i is matched with herself, we say that this i is *single* under μ.

Let $(M, W, \theta, \mathcal{M})$ be a marriage problem. Every man $m \in M$'s preferences over the set $W \cup \{m\}$ in the state θ can be extended to an ordering over the collection $\mathcal{M}$ in the following way:

$$\mu R_m(\theta)\, \mu' \Leftrightarrow \text{either } \mu(m)\, P_m(\theta)\, \mu'(m) \text{ or } \mu(m) = \mu'(m), \quad \text{for every } \mu, \mu' \in \mathcal{M}.$$

Likewise, this can be done for every woman $w \in W$.

Let $(M, W, \theta, \mathcal{M})$ be a marriage problem. A matching μ is *individually rational* in state θ if no individual $i \in M \cup W$ prefers strictly being single to being matched with the partner assigned by the matching μ; that is, for every individual i, either $\mu(i)\, P_i(\theta)\, i$ or $\mu(i) = i$. Furthermore, a matching μ is *blocked* in state θ if there are two individuals m and w of the opposite sex who would each prefer strictly to be matched with the other rather than with the partner assigned by the matching μ; that is, there is a pair (m, w) such that

$$w P_m(\theta)\, \mu(m) \text{ and } m P_w(\theta)\, \mu(w).$$

A matching μ is *stable* in state θ if it is individually rational and unblocked in state θ. A matching μ is *man-optimal stable* in state θ if it is the best stable matching from the perspective of all the men; that is, m is stable in state θ and for every man $m \in M$, $\mu R_m(\theta)\, \mu'$ for every other stable matching μ' in state θ. The man-optimal stable matching in state θ is denoted by μ^θ.

Definition 8 The *man-optimal stable* solution of a marriage problem $(M, W, \theta, \mathcal{M})$, denoted by $\mathcal{O}_M$, is a function associating the state θ with its man-optimal stable matching μ^θ,

$$\mathcal{O}_M(\theta) \equiv \left\{\mu^\theta\right\}, \quad \text{for every } \theta \in \Theta.$$

The following result shows that this solution is partially-honestly Nash implementable when the mechanism designer does not know the true state. We refer to $(M, W, \Theta, \mathcal{M})$ as a *class of marriage problems*, with $(M, W, \theta, \mathcal{M})$ as typical marriage problem.

Proposition 1 Let $(M, W, \Theta, \mathcal{M})$ be a class of marriage problems with $|M \cup W| \geq 3$. Let Assumption 1 and Assumption 2 be given. Then, the man-optimal stable solution is partially-honestly Nash implementable.

Proof. Let the premises hold. In the context of matching problems, the set X coincides with the collection $\mathcal{M}$, and N is the set $M \cup W$. We show that the man-optimal stable solution satisfies Condition μ^*(ii) with respect to $Y = X$.

Since the man-optimal stable solution is unanimous, we can set $Y = X$ as per Sjöström (1991), and so Y contains the range of $\mathcal{O}_M$. In addition, for every triplet (i, θ, θ'), let

$$C_i\left(\theta, \mu^\theta\right) \equiv L_i\left(\theta, \mu^\theta\right) \text{ and } S_i\left(\theta'; \mu^\theta, \theta\right) \equiv C_i\left(\theta, \mu^\theta\right).$$

One can check that for every state θ, it holds that $\mu^\theta \in C_i\left(\theta, \mu^\theta\right) \subseteq L_i\left(\theta, \mu^\theta\right) \subseteq Y$ for every individual i. Moreover, for every triplet (i, θ, θ'), one can also check that the set $S_i\left(\theta'; \mu^\theta, \theta\right)$ is non-empty and that $\mathcal{O}_M\left(\theta'\right) \in S_i\left(\theta'; \mu^\theta, \theta\right)$ if $\theta' = \theta$, establishing part (1) of Condition μ^*(ii). Finally, let us show that the man-optimal stable solution satisfies part (2) of Condition μ^*(ii).

For every quadruplet $(i, \theta, \theta', \mu)$ with $\mu \in C_i\left(\theta, \mu^\theta\right)$, suppose that $C_i\left(\theta, \mu^\theta\right) \subseteq L_i\left(\theta', \mu\right)$ and that $Y \subseteq L_j\left(\theta', \mu\right)$ for every individual $j \neq i$. By construction, the man-optimal stable solution satisfies part (2) of Condition μ^*(ii) if we show that μ is the man-optimal matching in state θ'; that is, $\mu = \mu^{\theta'}$.

Assume, to the contrary, that $\mu \neq \mu^{\theta'}$. Note that the matching μ is stable in state θ'. So, by Theorem 2.13 in Roth and Sotomayor (1990; p. 33), which is due to Knuth (1976), it follows that $\mu^{\theta'} R_m\left(\theta'\right) \mu$ for every man $m \in M$ and that $\mu R_w\left(\theta'\right) \mu^{\theta'}$ for every woman $w \in W$. From this and the fact that the matching μ is also $R_j\left(\theta'\right)$-maximal for every individual $j \neq i$ in the set Y, it follows that $\mu\left(j\right) = \mu^{\theta'}\left(j\right)$ if individual j is a man. Therefore, it must be the case that individual i is a man and the mate of the man i under $\mu^{\theta'}$ differs from that under μ, that is, $\mu\left(i\right) \neq \mu^{\theta'}\left(i\right)$; otherwise, $\mu = \mu^{\theta'}$, which is a contradiction.

Since $\mu\left(i\right) \neq \mu^{\theta'}\left(i\right)$ and since, moreover, $\mu^{\theta'} R_i\left(\theta'\right) \mu$, it follows from the definition of $R_i\left(\theta'\right)$ that $\mu^{\theta'} P_i\left(\theta'\right) \mu$. From this and the fact that the matching μ is stable in state θ', we have that the man i must be matched with a partner of the opposite sex under $\mu^{\theta'}$; that is, $\mu^{\theta'}\left(i\right) = w$. Moreover, it must be the case that the mate of the woman w under $\mu^{\theta'}$ differs from that under μ, that is, $\mu\left(w\right) \neq \mu^{\theta'}\left(w\right) = i$; otherwise, the man i is matched with the same mate under μ and under $\mu^{\theta'}$, which contradicts that $\mu\left(i\right) \neq \mu^{\theta'}\left(i\right)$.

Since $\mu\left(w\right) \neq \mu^{\theta'}\left(w\right) = i$ and the matching μ is $R_w\left(\theta'\right)$-maximal in the set Y for the woman w and since, moreover, $\mu^{\theta'}$ is stable in state θ', it follows that $\mu P_w\left(\theta'\right) \mu^{\theta'}$ and that the mate of the woman w under μ is a man $m \neq i$. However, since the matching μ is $R_m\left(\theta'\right)$-maximal in the set Y for the man $m \neq i$ and since, moreover, $\mu^{\theta'} R_m \mu$, it must be the case that the man m is matched with the same woman w under μ and under $\mu^{\theta'}$, that is, $\mu\left(m\right) = \mu^{\theta'}\left(m\right) = w$. This implies that the woman w is matched with the same mate under μ and under $\mu^{\theta'}$, that is, $\mu\left(w\right) = \mu^{\theta'}\left(w\right)$, which is a contradiction. Thus, we conclude that $\mu = \mu^{\theta'}$.

Since the man-optimal stable solution satisfies Condition μ^*(ii) with respect to $Y \subseteq X$, Theorem 1 implies that this solution is partially-honestly Nash implementable. ■

3.3 Applications to rationing problems with single-peaked preferences

This subsection applies Theorem 1 to the problem of fairly allocating an infinitely divisible and non-disposable commodity among a group of individuals with single-peaked preferences and shows that the *equal-distance* solution is partially-honestly Nash implementable. It also establishes the failure of partially-honestly Nash implementing the *proportional* solution.

A *rationing problem with single-peaked preferences* (Sprumont, 1991; Thomson, 1994a) is a triplet $(N, X(M), \theta)$, where:

- N is a finite set of individuals, with $n \geq 2$.
- $X(M)$ consists of all different non-wasteful ways - *allocations* - of dividing an infinitely divisible commodity of a finite size $M > 0$ among the n individuals, where the allocation x assigns the non-negative fraction x_j of M to individual j.[12]
- θ is a state at which every individual j has a (self-regarding) continuous single-peaked preference relation $R_j(\theta)$ over the consumption space $[0, M]$. Individual j's preference relation $R_j(\theta)$ is *single-peaked* if there is a fraction $p_j(\theta)$ of M, called the peak amount of this individual in state θ, such that she judges the fraction x_j better than the fraction y_j if $p_j(\theta) \geq x_j > y_j$ or $y_j > x_j \geq p_j(\theta)$. Furthermore, for every individual j and fraction x_j of M, let $r_j(x_j)$ be the fraction of M on the other side of j's peak amount $p_j(\theta)$ that she finds equally good to x_j according to $R_j(\theta)$, if such fraction exists, and the end point of $[0, M]$ on the other side of her peak amount, otherwise.[13]

Let $(N, X(M), \theta)$ be a rationing problem with single-peaked preferences. In state θ, every individual j is equipped with preferences over her consumption space, not over the collection of allocations $X(M)$. However, her preferences can be extended to $X(M)$ in the following standard and natural way: Individual j judges the allocation x to be at least as good as the allocation y if she judges the fraction x_j to be at least as good as the fraction y_j according to $R_j(\theta)$. We use $R_j(\theta)$ to represent both.

Definition 9 The *proportional* solution of a rationing problem with single-peaked preferences $(N, X(M), \theta)$, denoted by $\mathcal{P}$, is a function associating the state θ with the allocation $\mathcal{P}(\theta) = x$ provided that this $x \in X(M)$ satisfies the following properties for some positive real number $\lambda > 0$: (i) $x_j = \lambda p_j(\theta)$ for every individual $j \in N$ if $\sum_{j \in N} p_j(\theta) > 0$; and (ii) $x_j = \left(\frac{M}{n}\right)$ for every individual $j \in N$ if $\sum_{j \in N} p_j(\theta) = 0$.

The next claim shows the failure of partially-honestly Nash implementing the proportional solution when the mechanism designer does not know the true state. This is so because this solution is not continuous at any state in which all peak amounts are zero.

[12] In symbols, $X(M) \equiv \left\{x \in \mathbb{R}^n_+ \mid \sum_{i \in N} x_i = M\right\}$ for every number $M > 0$.

[13] In other words, if $x_i \leq p_i(\theta)$, then $p_i(\theta) \leq r_i(x_i)$ and $x_i I_i(\theta) r_i(x_i)$ if such an amount exists, or else $r_i(x_i) \equiv M$; and if $x_i \geq p_i(\theta)$, then $p_i(\theta) \geq r_i(x_i)$ and $x_i I_i(\theta) r_i(x_i)$ if such an amount exists, or else $r_i(x_i) \equiv 0$.

Claim 2 Let $n \geq 3$. Let Assumption 2 be given. Then, the proportional solution does not satisfy Condition μ^*(ii) with respect to $Y \subseteq X$.

Proof. Let the premises hold. Assume, to the contrary, that the proportional solution satisfies Condition μ^*(ii) with respect to $Y \subseteq X$.

In the context of rationing problems with single-peaked preferences, the set X coincides with the collection $X(M)$. Since, moreover, this solution is unanimous, the set Y coincides with the set X as per Sjöström (1991), and so Y contains the range of $\mathcal{P}$.

Suppose that there are two states θ and θ' such that for some individual $i \in N$, it holds that M is the unique worst possible fraction for individual i in state θ, that individual i's peak amount is a positive fraction of M in state θ but it reduces to zero in state θ', and that every other individual j has the same preferences in both states with a peak amount equal to zero.

For the rationing problem with single-peaked preferences $(N, X(M), \theta)$, one can check that the proportional solution assigns the fraction $x_i = M$ to individual i and the fraction $x_j = 0$ to every individual $j \neq i$. However, for the rationing problem with single-peaked preferences $(N, X(M), \theta')$, the proportional solution assigns the fraction $y_p = \frac{M}{n}$ to every individual $p \in N$.

Thus, by construction, we have that $C_i(\theta, x) = L_i(\theta, x) = \{x\}$ and $C_i(\theta, x) \subseteq L_i(\theta', x)$, that $Y \subseteq L_j(\theta', x)$ for every individual $j \neq i$ and that the intersection $S_i(\theta'; x, \theta) \cap I_i(\theta', x, Y)$ is empty if $x \notin S_i(\theta'; x, \theta)$. However, part (2)(a) of Condition μ^*(ii) implies for $H = \{i\}$ that $x \notin S_i(\theta'; x, \theta)$ and that the intersection $S_i(\theta'; x, \theta) \cap I_i(\theta', x, Y)$ is not empty, which is a contradiction.[14] ■

Therefore, the prospects for Nash implementing the solution which divides the commodity M proportionally to the peak amounts are quite bleak. However, this is not the case when the objective is to allocate fractions that are equally far from their peak amounts (subject to non-negativity), and so the equal-distance solution becomes the focus of the mechanism designer.

Definition 10 The *equal-distance* solution of a rationing problem with single-peaked preferences $(N, X(M), \theta)$, denoted by $\mathcal{ED}$, is a function associating the state θ with the allocation $\mathcal{ED}(\theta) = x$ provided that this $x \in X(M)$ satisfies the following properties for some real number $d \geq 0$: *(i)* $x_j = \max\{0, p_j(\theta) - d\}$ for every individual $j \in N$ if $\sum_{j \in N} p_j(\theta) \geq M$; and *(ii)* $x_j = p_j(\theta) + d$ for every individual $j \in N$ if $\sum_{j \in N} p_j(\theta) \leq M$.

The following proposition substantiates this claim: We refer to $(N, X(M), \Theta)$ as a *class of rationing problems with single-peaked preferences*, where $(N, X(M), \theta)$ is a rationing problem with single-peaked preferences for every θ in Θ.

Proposition 2 Let $(N, X(M), \Theta)$ be a class of rationing problems with single-peaked preferences with $n \geq 3$. Let Assumption 1 and Assumption 2 be given. Then, the equal-distance solution is partially-honestly Nash implementable.

[14] This proof also holds in the case where $n = 2$, establishing that the proportional solution is not partially-honestly Nash implementable as along as $n \geq 2$ and the family $\mathcal{H}$ has as elements all nonempty subsets of the set N.

Proof. Let the premises hold. In the context of rationing problems with single-peaked preferences, the set X coincides with the set $X(M)$. We show that the equal-distance solution satisfies Condition μ^*(ii) with respect to $Y = X$. The equal-distance allocation in state θ is denoted by x^θ.

Since this solution is unanimous, we can set $Y = X$ as per Sjöström (1991), and so Y contains the range of $\mathcal{ED}$. In addition, for every pair (i, θ), let

$$S_i\left(\theta; x^\theta, \theta\right) \equiv \left\{x^\theta\right\},$$

and

$$C_i\left(\theta, x^\theta\right) \equiv \begin{cases} L_i\left(\theta, x^\theta\right) \setminus \left\{\left(r_i\left(x_i^\theta\right), 0_{-i}\right)\right\} & \text{if } x_i^\theta < p_i(\theta) < r_i\left(x_i^\theta\right) = M \text{ and} \\ & x^\theta I_i(\theta)\left(r_i\left(x_i^\theta\right), 0_{-i}\right); \\ L_i\left(\theta, x^\theta\right) & \text{otherwise,} \end{cases}$$

where 0_{-i} is obtained from the n-dimensional zero vector by omitting the ith component.

One can check that for every state θ, it holds that $x^\theta \in C_i\left(\theta, x^\theta\right) \subseteq L_i\left(\theta, x^\theta\right) \subseteq Y$ for every individual i. Moreover, one can also check that $x^\theta \in S_i\left(\theta; x^\theta, \theta\right)$, establishing part (1) of Condition μ^*(ii) when $\theta' = \theta$. Next, let us show that the equal-distance solution satisfies part (2) of Condition μ^*(ii) when $\theta' = \theta$. We do it by proving that $x^\theta = y$ provided that the allocation $y \in X(M)$ is $R_i(\theta)$-maximal for some individual i in the set $L_i\left(\theta, x^\theta\right)$, that this individual i judges x^θ to be at least as good as this y according to $R_i(\theta)$ and that this y is also $R_j(\theta)$-maximal for every other individual j in the set Y.

To this end, let the premises hold and assume, to the contrary, that $x^\theta \neq y$. Note that $xI_i(\theta)\, y$ and that the fraction y_j coincides with the peak $p_j(\theta)$ for every individual $j \neq i$. It follows from the efficiency of the equal-distance allocation x^θ that every individual judges this x^θ to be at least as good as y in state θ. Thus, individual $j \neq i$'s fraction x_j^θ at x^θ coincides with her fraction y_j at y, and this follows from the single-peakedness of $R_j(\theta)$. Since y is an element of $X(M)$, it follows that individual i's fraction at x^θ coincides with her fraction y_i at y, resulting in $x^\theta = y$, which is a contradiction.[15] In summary, by construction, Condition μ^*(ii) is satisfied when $\theta' = \theta$.

We next turn to deal with the case where $\theta \neq \theta'$. Let us then first provide a construction of the set $S_i\left(\theta'; x^\theta, \theta\right)$ for every individual i when $\theta \neq \theta'$. To this end, for every triplet (i, θ, θ') with $\theta \neq \theta'$, define the set $S_i\left(\theta'; x^\theta, \theta\right)$ as follows:

- For all $y \in Y$, if $y \in C_i\left(\theta, x^\theta\right) \subseteq L_i(\theta', y)$ and $Y \subseteq L_j(\theta', y)$ for every other individual j and if $y \neq x^{\theta'}$, then:

$$S_i\left(\theta'; x^\theta, \theta\right) \equiv \left\{ \begin{array}{r} z \in C_i\left(\theta, x^\theta\right) | z_i = y_i,\ z_p \neq p_p(\theta') \text{ and } z_q \neq p_q(\theta') \\ \text{for some } p, q \in N \setminus \{i\} \text{ with } p \neq q \end{array} \right\}.$$

- In all other cases, $S_i\left(\theta'; x^\theta, \theta\right) \equiv C_i\left(\theta, x^\theta\right)$.

One can check that Condition μ^*(ii) is satisfied provided that the constructed set

[15] An allocation w is (Pareto) *efficient* in state θ if it is feasible and there is no other allocation that every individual judges to be at least as good as w and at least one individual judges it better than w.

$S_i\left(\theta';x^{\theta},\theta\right)$ is not empty. To see this, take any triplet (i,θ,θ') with $\theta \neq \theta'$. Two cases need to be checked.

Firstly, suppose that the premises of part (2)(a) of Condition μ^*(ii) never apply to outcomes in $C_i\left(\theta,x^{\theta}\right)$. Then, $S_i\left(\theta';x^{\theta},\theta\right)$ coincides with the non-empty set $C_i\left(\theta,x^{\theta}\right)$, which shows that part (1)(a) as well as part (2)(a) of Condition μ^*(ii) are satisfied for this i.

Secondly, suppose that the premises of part (2)(a) of the condition apply to at least one outcome $y \in C_i\left(\theta,x^{\theta}\right)$. Consequently, by the single-peakedness of preferences and by the fact that the allocation y is $R_j(\theta)$-maximal for every individual $j \neq i$ in the set Y, we have that this y is such that the fraction y_j coincides with the peak $p_j(\theta')$ of every $j \neq i$. Now, to satisfy part (2)(a) of Condition μ^*(ii) we need to have that this y is not an element of $S_i\left(\theta';x^{\theta},\theta\right)$ and, moreover, that the intersection $S_i\left(\theta';x^{\theta},\theta\right) \cap I_i(\theta',y,Y)$ is not empty. This is the case by construction of the set $S_i\left(\theta';x^{\theta},\theta\right)$ provided that this set is not empty. Indeed, if this set is not empty, then Condition μ^*(ii) is satisfied for two reasons: (1) there would exist an allocation z in $S_i\left(\theta';x^{\theta},\theta\right)$ which assigns the fraction $y_i = z_i$ to individual i, establishing that the intersection $S_i\left(\theta';x^{\theta},\theta\right) \cap I_i(\theta',y,Y)$ is not empty; and (2) every element w of $S_i\left(\theta';x^{\theta},\theta\right)$ would assign a fraction w_p of M to some individual $p \neq i$, w_q to some other individual $q \neq i$ and, moreover, each of the fractions w_p and w_q would differ, respectively, from the peak amounts $p_p(\theta')$ and $p_q(\theta')$ in state θ', establishing that the allocation y cannot be an element of $S_i\left(\theta';x^{\theta},\theta\right)$.

Thus, to show that the set $S_i\left(\theta';x^{\theta},\theta\right)$ is not empty, it suffices to show that this set is not empty for every triplet (i,θ,θ') for which the premises of part (2)(a) of Condition μ^*(ii) apply to some $y \in C_i\left(\theta,x^{\theta}\right)$. To this end, take any of these triplets and denote it by (i,θ,θ').

Then, every individual $j \neq i$ receives her peak amount in state θ' at y; that is, $y_j = p_j(\theta')$. Also, by unanimity of the solution and the assumption that $x^{\theta'} \neq y$, individual i does not receive her peak amount either in state θ at x^{θ} or in state θ' at y; that is, $x_i^{\theta} \neq p_i(\theta)$ and $y_i \neq p_i(\theta')$.

Step 1: $x_i^{\theta} < M$.

Assume, to the contrary, that $x_i^{\theta} = M$. Then, it follows from $x_i^{\theta} \neq p_i(\theta)$ that $p_i(\theta) < x_i^{\theta} = M$. We consider three cases, according to whether the sum of peaks in state θ is greater than M or not.

Suppose that the sum of peaks in state θ is equal to M. Then, individual i receives her peak amount in state θ at x^{θ}, which is not the case.

Suppose that the sum of peaks in state θ is greater than M. Given that the equal-distance allocation x^{θ} is efficient in the state θ, every individual receives a fraction of M that is not greater than her peak amount. It follows from this that, in particular, $x_i^{\theta} \leq p_i(\theta)$, which is incompatible with the assumption that $p_i(\theta) < x_i^{\theta} = M$.

Suppose that the sum of peaks in state θ is lower than M. Thus, the equal-distance allocation assigns a positive fraction of M to every individual in state θ, which contradicts the assumption that individual i gets M at x^{θ}. This concludes the proof of *step 1*.

Step 2: If the fraction $r_i\left(x_i^{\theta}\right)$ of M exists, then either $y_i = x_i^{\theta}$ or $y_i = r_i\left(x_i^{\theta}\right)$. If the fraction $r_i\left(x_i^{\theta}\right)$ of M does not exists, then $y_i = x_i^{\theta}$.

To obtain a contradiction, we suppose that $x_i^{\theta} \neq y_i$ and that also $y_i \neq r_i\left(x_i^{\theta}\right)$ if the fraction $r_i\left(x_i^{\theta}\right)$ exists. Thus, from the fact that the allocation y is an element of $C_i\left(\theta,x^{\theta}\right)$, it follows that individual i judges x_i^θ better than y_i in state θ, that is, $x_i^\theta P_i(\theta) y_i$. So, the fraction y_i that individual i gets at y is an element of the strict lower contour set of $R_i(\theta)$ at x_i^θ, which can be of two types: (1) if $x_i^\theta < p_i(\theta)$, then the strict lower contour set of $R_i(\theta)$ at x_i^θ is the interval $\left[0, x_i^\theta\right[\cup \left]r_i\left(x_i^\theta\right), M\right]$ if the fraction $r_i\left(x_i^\theta\right)$ of M exists, or else the interval $\left[0, x_i^\theta\right[$; and (2) if $x_i^\theta > p_i(\theta)$, then the strict lower contour set of $R_i(\theta)$ at x_i^θ is the interval $\left[0, r_i\left(x_i^\theta\right)\right[\cup \left]x_i^\theta, M\right]$ if the fraction $r_i\left(x_i^\theta\right)$ of M exists, or else the interval $\left]x_i^\theta, M\right]$.

Suppose that the strict lower contour set of $R_i(\theta)$ at x_i^θ is of type (1). This implies that the set $C_i\left(\theta, x^\theta\right)$ contains allocations of $X(M)$ which assign to individual i any element of the interval $\left[0, x_i^\theta\right[$ and, moreover, any element of the interval $\left]r_i\left(x_i^\theta\right), M\right]$ provided that $r_i\left(x_i^\theta\right)$ exists.

We distinguish two cases, according to whether the fraction y_i is an element of the interval $\left[0, x_i^\theta\right[$ or not. Suppose that y_i is an element of $\left[0, x_i^\theta\right[$. We proceed by cases, according to whether $y_i < p_i(\theta')$ or $y_i > p_i(\theta')$. Thus, let us suppose that $y_i < p_i(\theta')$. Then, the lower contour set of $R_i(\theta')$ at y_i is the interval $[0, y_i] \cup [r_i(y_i), M]$ if the fraction $r_i(y_i)$ of M exists, or else the interval $[0, y_i]$, implying that the lower contour set of $R_i(\theta')$ at y consists of allocations of $X(M)$ which assign to individual i any element of the interval $[0, y_i]$ and, moreover, any element of the interval $[r_i(y_i), M]$ if $r_i(y_i)$ exists. Consequently, and irrespective of whether the fraction $r_i(y_i)$ exists, given that y_i is an element of the interval $\left[0, x_i^\theta\right[$, we can find in $C_i\left(\theta, x^\theta\right)$ an allocation which assigns to individual i an element of the interval $\left]y_i, x_i^\theta\right[$ and which is not an element of the lower contour set of $R_i(\theta')$ at y, and this is incompatible with the premises of part (2)(a) of Condition μ^*(ii).

Suppose that $y_i > p_i(\theta')$. Then, y assigns a positive fraction of M to individual i; that is, $y_i > 0$. Moreover, the lower contour set of $R_i(\theta')$ at y_i is the interval $[0, r_i(y_i)] \cup [y_i, M]$ if the fraction $r_i(y_i)$ of M exists, or else the interval $[y_i, M]$. This means that the lower contour set of $R_i(\theta')$ at y consists of allocations of $X(M)$ which assign to individual i any element of the interval $[y_i, M]$ and, moreover, any element of the interval $[0, r_i(y_i)]$ provided that this $r_i(y_i)$ exists. Therefore, and irrespective of whether the fraction $r_i(y_i)$ exists, given that y_i is an element of the interval $\left]0, x_i^\theta\right[$, one can check that there is in $C_i\left(\theta, x^\theta\right)$ an allocation which assigns to individual i an element of the interval $[0, y_i[$ and which is not an element of the lower contour set of $R_i(\theta')$ at y, yielding a contradiction. This concludes the proof of the case that y_i is an element of $\left[0, x_i^\theta\right[$.

We next turn to deal with the case that y_i is an element of the interval $\left]r_i\left(x_i^\theta\right), M\right]$. Again, we proceed by cases, according to whether $y_i < p_i(\theta')$ or $y_i > p_i(\theta')$.

Suppose that $y_i < p_i(\theta')$. Then, individual i does not receive the whole commodity M at y; that is, $y_i < M$. In addition, as already noted above, the lower contour set of $R_i(\theta')$ at y_i is the interval $[0, y_i] \cup [r(y_i), M]$ if the fraction $r_i(y_i)$ of M exists, or else the interval $[0, y_i]$, which implies that the lower contour set of $R_i(\theta')$ at y consists of allocations of $X(M)$ which assign to individual i any element of the interval $[0, y_i]$ and, moreover, any element of the interval $[r_i(y_i), M]$ provided that this $r_i(y_i)$ exists. Irrespective of whether $r_i(y_i)$ exists, we deduce from the assumption that y_i is an element of the interval $\left]r_i\left(x_i^\theta\right), M\right]$ that there is in $C_i\left(\theta, x^\theta\right)$ an allocation which assigns to individual i an element of the interval $]y_i, M[$ and which is not an element of the lower contour set of $R_i(\theta')$ at y, which yields a contradiction.

Suppose that $y_i > p_i(\theta')$. Then, y assigns a positive fraction of M to individual i; that is, $y_i > 0$. Furthermore, as already noted, in this case the lower contour set of $R_i(\theta')$ at y_i is the interval $[0, r_i(y_i)] \cup [y_i, M]$ if the fraction $r_i(y_i)$ of M exists, or else the interval $[y_i, M]$, which implies that the lower contour set of $R_i(\theta')$ at y consists of allocations of $X(M)$ which assign to individual i any element of the interval $[y_i, M]$ and, moreover, any element of the interval $[0, r_i(y_i)]$ provided that $r_i(y_i)$ exists. Irrespective of whether $r_i(y_i)$ exists, it follows from the fact that y_i is an element of the interval $\left]r_i\left(x_i^\theta\right), M\right]$ that one can find in $C_i\left(\theta, x^\theta\right)$ an allocation which assigns to individual i an element of the interval $]0, y_i[$ and which is not an element of the lower contour set of $R_i(\theta')$ at y, which is a contradiction. This concludes the proof of the case where the strict lower contour set of $R_i(\theta)$ at x_i^θ is of type (1).

We conclude the proof of *step 2* by mentioning that, suitably modified, the above proof applies to the case where the strict lower contour set of $R_i(\theta)$ at x_i^θ is of type (2).

Step 3: The set $S_i\left(\theta'; x^\theta, \theta\right)$ is non-empty if $x_i^\theta = y_i$.

Suppose that $x_i^\theta = y_i$. By *step 1*, we have that $x_i^\theta = y_i < M$. Then, the allocation y assigns a positive fraction of M to some individual $q \neq i$, and this follows from the fact that $y \in X(M)$. Thus, individual q's assignment at y is $y_q = p_q(\theta') > 0$.

Now, take any individual p who differs from both individual i and individual q. Then, the allocation y assigns the fraction $y_p = p_p(\theta')$ to individual p. Note that y_p is lower than M given that $y \in X(M)$ and that individual q gets a positive fraction of M at this y. For a positive number $\epsilon > 0$ sufficiently small, define z by setting $z_p = p_p(\theta') + \epsilon$, $z_q = p_q(\theta') - \epsilon$ and $z_k = y_k$ for every other individual k. Then, this z is an element of $X(M)$ which assigns y_i to individual i, $z_p \neq p_p(\theta')$ to individual p and $z_q \neq p_q(\theta')$ to individual q, resulting in a element of the set $S_i\left(\theta'; x^\theta, \theta\right)$. This concludes the proof of *step 3*.

Step 4: The set $S_i\left(\theta'; x^\theta, \theta\right)$ is non-empty if $y_i \neq x_i^\theta$ and the fraction $r_i\left(x_i^\theta\right)$ exists.

It follows from *step 3* that $y_i = r_i\left(x_i^\theta\right)$. Moreover, individual i judges the fractions x_i^θ and y_i as equally good in state θ, resulting in $xI_i(\theta)\,y$. We distinguish two cases, according to whether $y_i < M$ or not.

Suppose that $y_i < M$. Then, the allocation y assigns a positive fraction of M to some individual $q \neq i$, and this follows from the fact that $y \in X(M)$. This implies that individual q's assignment at y is $y_q = p_q(\theta') > 0$. As in *step 3*, take any individual p who differs from both individual i and individual q. Individual p's assignment at y is lower than M, and this follows from $y_q > 0$ and the assumption that y is an element of $X(M)$. For a positive number $\epsilon > 0$ sufficiently small, define z by setting $z_p = p_p(\theta') + \epsilon$, $z_q = p_q(\theta') - \epsilon$ and $z_k = y_k$ for every other individual k. Then, this z is an element of $X(M)$ which assigns $y_i = r_i\left(x_i^\theta\right)$ to individual i, $z_p \neq p_p(\theta')$ to individual p and $z_q \neq p_q(\theta')$ to individual q, resulting in a element of the set $S_i\left(\theta'; x^\theta, \theta\right)$, as was to be proved.

Finally, we consider the case where $y_i = r_i\left(x_i^\theta\right) = M$ and show that it is incompatible with the assumption that $y \in C_i\left(\theta, x^\theta\right)$. Note that the allocation $y \in X(M)$ assigns the entire commodity M to individual i and nothing to everybody else; that is, $y = \left(r_i\left(x_i^\theta\right), 0_{-i}\right)$. Also, $x^\theta < M$, by *step 1*. Consequently, given that the fraction $r_i\left(x_i^\theta\right)$ exists, it must be the case that $x_i^\theta < p_i(\theta) < r_i\left(x_i^\theta\right) = M$. It follows from the definition of the set $C_i\left(\theta, x^\theta\right)$ and from the fact that $xI_i(\theta)\,y$ that y is not an element of $C_i\left(\theta, x^\theta\right)$, as was to be shown. This concludes the proof of *step 4*. ■

A natural question, then, is whether the equal-distance solution can also be a two-person partially-honestly Nash implementable SCR when the family $\mathcal{H}$ has as elements all non-empty subsets of the society N. The answer is no.[16]

3.4 Applications to bargaining games

While the previous subsection considered problems of allocating an infinitely divisible commodity among a group of individuals with single-peaked preferences, in this subsection we consider problems of allocating utilities among a group of individuals with von Neumann-Morgenstern preferences who bargain over the division of one unit of a perfectly divisible commodity. We show that the *Nash* solution is partially-honestly Nash implementable when the prevailing state is known by the individuals but is unknown by the mechanism designer.

Then, we will assume that the set of possible divisions - *allocations* - of one unit of a perfectly divisible commodity among the n individuals is given by $A \equiv \{a \in \mathbb{R}^n_+ \mid \sum_{i=1}^n a_i \leq 1\}$, with a as a typical allocation and with a_i as a typical fraction obtained by individual i at a. This set A is kept fixed throughout. In addition, we take the complete waste of the commodity as the *disagreement point* $d = 0$, which will also be the origin of the individual utilities.

A *bargaining game* is a triplet (N, Δ, θ) such that:

- N is a finite set of individuals, with $n \geq 2$.
- Δ is the set of *outcomes*, which consists of all probability measures on the Borel σ-algebra of the space A, with p as a typical element.
- θ is a state in Θ, at which every individual j's preferences over $[0,1]$ are identified by a continuous and monotonic von Neumann-Morgenstern ordering.[17] Thus, individual j's preferences in state θ can be represented by a continuous, increasing and von Neumann-Morgenstern utility function $u_j(\cdot;\theta) : [0,1] \to \mathbb{R}$ such that individual j's expected utility of a probability measure p in Δ is:

$$U_j(p;\theta) \equiv \int_A u_j(a_j;\theta)\, \mathrm{d}p(a), \quad \text{for every } p \in \Delta.$$

[16] To see this, suppose that there are $n = 2$ individuals and two states θ and θ' such that individual i's peak amount is equal to M in both states, whereas individual j's peak amount is equal to M in state θ and equal to half of M in state θ'. For the rationing problem with single-peaked preferences $(N, X(M), \theta)$, one can check that the equal-distance solution assigns half of M to every individual; that is, $x_i = x_j = \frac{M}{2}$. Moreover, one can also check that for the problem $(N, X(M), \theta')$ the equal-distance allocation is such that individual i obtains a fraction equal to three-fourths of M and individual j a fraction equal to one-fourth. Note that, by construction, $L_i(\theta, x) = L_i(\theta', x)$ and that $X(M) \subseteq L_j(\theta', x)$. Since the singleton $\{i\}$ is an element of the family $\mathcal{H}$, one can now easily check that the equal-distance solution violates part (2)(a) of Condition μ^*(ii) under the specification that $Y = X$. The reason is that there cannot exist any allocation $z \neq x$ of $X(M)$ in the set $L_i(\theta, x)$ such that individual i is indifferent between this z and x according to her ordering $R_i(\theta')$. This is so because the commodity M is not disposable.

The preceding arguments also apply entirely to the so-called *equal-sacrifice* solution, which divides the commodity M so that all upper contour sets are of the same size subject to non-negativity. In this case, however, we also need that in state θ' individual j judges one-third of M and two-thirds of M as equally good. Under this specification, the equal-sacrifice solution assigns to individual i half of M in state θ and two-thirds of M in state θ'. For a recent survey of the literature on rationing problems with single-peaked preferences in which the equal-sacrifice solution is discussed see Thomson (2014).

[17] An ordering $R_j(\theta)$ on $[0,1]$ is *monotonic* if $a_j \geq b_j \implies a_j R_j(\theta)\, b_j$, for every $a_j, b_j \in [0,1]$.

In addition, this utility function is uniquely determined up to a positive affine transformation.[18] Therefore, for the sake of simplicity, we also assume that $u_j(0;\theta) = 0$ and that $u_j(1;\theta) = 1$ in state θ.

Write (N, Δ, Θ) for the class of bargaining games, with (N, Δ, θ) as a typical element, where the set Θ consists of all representations of continuous and monotonic orderings over $[0, 1]$ that are consistent with the von Neumann-Morgenstern axioms; that is, the domain Θ is *unrestricted*. To save writing, write $U(p;\theta)$ for the utility allocation $(U_1(p;\theta), \cdots, U_1(p;\theta))$ generated by the outcome p in state θ.

Let (N, Δ, θ) be a bargaining game. Define the utility possibility set associated with this bargaining game as:

$$U(\Delta;\theta) \equiv \left\{ (U_j(p;\theta))_{j\in N} \,|p \in \Delta \right\},$$

which is a non-empty, compact and convex set in $\mathbb{R}^n$.[19] In addition, since the utility functions representing individuals' preferences are increasing, this set $U(\Delta;\theta)$ is also comprehensive, which amounts to free disposal of utility.[20]

As already noted in Vartiainen (2007b), for every non-empty, convex and compact subset S of $\mathbb{R}^n_+$ there is a bargaining game (N, Δ, θ) in the family (N, Δ, Θ) for which the utility possibility set $U(\Delta;\theta)$ is S; that is, $U(\Delta;\theta) = S$. Therefore, in the actual setting, every element of the class of standard bargaining problems in $\mathbb{R}^n_+$ is the image of some element of the family $\{U(\Delta;\theta)\}_{\theta\in\Theta}$ of utility possibility sets generated by the class (N, Δ, Θ) of bargaining games; that is, there is an onto function from the family $\{U(\Delta;\theta)\}_{\theta\in\Theta}$ of utility possibility sets to the class of standard bargaining problems in $\mathbb{R}^n_+$. Indeed, from the welfaristic viewpoint, that is, from the point of view where only utility allocations matter, these two classes are basically equivalent.

Definition 11 The *Nash* solution of a bargaining game (N, Δ, θ), denoted by ν, is the collection of all outcomes p and q of Δ that generate the same utility allocations $U(p;\theta) = U(q;\theta)$ and that maximize the product of utilities over the utility possibility set $U(\Delta, \theta)$,

$$\nu(\theta) \equiv \left\{ \begin{array}{c} q|p, q \in \arg\max_{m\in\Delta} \left\{ \prod_{j\in N} U_j(m;\theta) \,|U(m;\theta) \in U(\Delta;\theta) \right\} \\ \text{and } U(q;\theta) = U(p;\theta) \end{array} \right\}.$$

Thus, this solution is derived under the so-called welfaristic assumption: The solution depends only on the Nash property of the utility allocations.

[18] A function $v : [0, 1] \to \mathbb{R}$ is a *positive affine transformation* of $u_j(\cdot;\theta)$ if there exists a positive real number $\beta > 0$ and a real number γ such that $v(a_j) = \beta u_j(a_j;\theta) + \gamma$, for every $a_j \in [0, 1]$.

[19] Its convexity follows from the Lyapunov's theorem for nonatomic vector measures, whereas its compactness follows from the fact the set $U(\Delta;\theta)$ is the image of the compact set Δ under the profile of continuous functions $U(\cdot;\theta) \equiv (U_j(\cdot;\theta))_{j\in N}$ (in the topology of weak convergence).

[20] In symbols, a nonempty set $S \subseteq \mathbb{R}^n$ is said to be *comprehensive* if $x \in S$ and $0 \leq y \leq x$ together imply $y \in S$, where it is understood that for every two n-dimensional Euclidean vectors a and b, $a \geq b$ means that $a_i \geq b_i$ for every individual i, $a > b$ means that $a \neq b$ and $a_i \geq b_i$ for every individual i.

Since the Nash solution is a risk sensitive bargaining solution, it follows from Vartiainen (2007b; Corollary 1, p. 343) that this solution fails Maskin monotonicity.[21] The following claim establishes that the Nash solution does not satisfy the no veto-power condition either: In the abstract Arrovian domain, the condition of *no veto-power* says that if an outcome is at the top of the preferences of all individuals but possibly one, then it should be chosen irrespective of the preferences of the remaining individual: that individual cannot veto it.

Claim 3 Let $n = 3$. Then, the Nash solution does not satisfy the condition of no veto-power.

Proof. Since this solution is unanimous, we can set $X = \Delta$ as per Sjöström (1991). Assume, to the contrary, that the Nash solution satisfies the condition of no veto-power.

Suppose that there are three individuals and a state θ, at which each individual j's ordering over the interval $[0, 1]$ is represented by a linear utility function; that is, $u_j(a_j; \theta) = a_j$ for every $a_j \in [0, 1]$. Therefore, the triplet (N, Δ, θ) is a bargaining game with a utility possibility set $U(\Delta; \theta)$, which is equal to the convex three-dimensional polyhedron with vertices at the following elements of the space A:

$$a^0 \equiv (0,0,0)\,,\ a^1 \equiv (0.5,0,0)\,,\ a^2 \equiv (0.5,0.5,0)\,,\ a^3 \equiv (0,0.5,0) \text{ and } a^4 \equiv (0,0,1)\,.$$

By abuse of notation, write a for the degenerate probability measure in Δ that picks the allocation a in A with certainty.

In the bargaining game (N, Δ, θ), the utility allocation generated by the probability measure a^2 in state θ is $U(a^2; \theta) \equiv (0.5, 0.5, 0)$, which is an element of $U(\Delta; \theta)$. Since the probability measure a^2 is an outcome for which $U_j(\cdot; \theta)$ attains its largest value over the set X for individual $j = 1, 2$, no veto-power implies that this outcome is an element of the Nash solution at θ, in violation of the definition of the Nash solution. ■

In contrast with the above negative results, the Nash solution is partially-honestly Nash implementable when there are $n \geq 3$ individuals:

Proposition 3 Let (N, Δ, Θ) be a class of bargaining games with $n \geq 3$. Let Assumption 1 and Assumption 2 be given. Then, the Nash solution is partially-honestly Nash implementable.

Proof. Let the premises hold. In the context of bargaining games, the set X coincides with the space Δ. We show that the Nash solution satisfies Condition μ^*(ii) with respect to $Y = X$. A typical Nash-optimal outcome at state θ is denoted by p^θ.

Since this solution is unanimous, we can set $Y = X$ as per Sjöström (1991), and so Y contains the range of ν. In addition, let

$$C_i\left(\theta, p^\theta\right) \equiv L_i\left(\theta, p^\theta\right) \text{ and } S_i\left(\theta; p^\theta, \theta\right) \equiv \nu(\theta)\,, \quad \text{for every pair } (i, \theta) \in N \times \Theta.$$

One can check that for every state θ, it holds that $p^\theta \in C_i\left(\theta, p^\theta\right) \subseteq L_i\left(\theta, p^\theta\right) \subseteq Y$ for every individual i. Moreover, one can also check that $p^\theta \in S_i\left(\theta; p^\theta, \theta\right)$, establishing part (1)

[21] A bargaining solution is risk sensitive when an increase in one's opponent's risk aversion is advantageous to other bargainers. For a recent study on the effects on bargaining solutions when bargainers become more risk averse and when they become more uncertainty averse see Driesen et al. (2015).

of Condition μ^*(ii) when $\theta' = \theta$. Next, let us show that the Nash solution satisfies part (2) of Condition μ^*(ii) when $\theta' = \theta$. We do it by showing that the outcome q is a Nash-optimal outcome at state θ provided that this $q \in L_i\left(\theta, p^\theta\right)$ is an outcome for which $U_i\left(\cdot;\theta\right)$ attains its largest value on the set $L_i\left(\theta, p^\theta\right)$ for some individual i and that this q is also an outcome for which $U_j\left(\cdot;\theta\right)$ attains its largest value on the set Y for every other individual j. To see this, note that $U_i\left(p^\theta;\theta\right) = U_i\left(q;\theta\right)$ and that $U_j\left(q;\theta\right) \geq U_j\left(p^\theta;\theta\right)$ for every individual $j \neq i$. By the efficiency of the Nash solution, it must be the case that $U_j\left(q;\theta\right) = U_j\left(p^\theta;\theta\right)$ for every individual $j \neq i$. Thus, by the definition of the Nash solution it follows that q is an element of $\nu\left(\theta\right)$, as was to be shown. In summary, the Nash solution satisfies Condition μ^*(ii) when $\theta' = \theta$.[22]

We next turn to deal with the case where $\theta \neq \theta'$. Let us then first provide a construction of the set $S_i\left(\theta' p^\theta, \theta\right)$ for every individual i when $\theta \neq \theta'$. To this end, for every triplet (i, θ, θ') with $\theta \neq \theta'$, define the set $S_i\left(\theta'; p^\theta, \theta\right)$ as follows:

- For all $q \in Y$, if $q \in C_i\left(\theta, p^\theta\right) \subseteq L_i\left(\theta', q\right)$ and $Y \subseteq L_j\left(\theta', q\right)$ for every other individual j and if $q \notin \nu\left(\theta'\right)$, then:

$$S_i\left(\theta'; p^\theta, \theta\right) \equiv \left\{r \in C_i\left(\theta, p^\theta\right) | U_i\left(r;\theta'\right) = U_i\left(q;\theta'\right) \text{ and } U_j\left(r;\theta'\right) = 0 \text{ for every } j \neq i\right\}.$$

- In all other cases, $S_i\left(\theta'; p^\theta, \theta\right) \equiv C_i\left(\theta, p^\theta\right)$.

Firstly, suppose that the premises of part (2)(a) of Condition μ^*(ii) never apply to outcomes in $C_i\left(\theta, p^\theta\right)$. Then, $S_i\left(\theta'; p^\theta, \theta\right)$ coincides with the non-empty set $C_i\left(\theta, x^\theta\right)$, which shows that part (1)(a) as well as part (2)(a) of Condition μ^*(ii) are satisfied for this i.

Secondly, suppose that the premises of part (2)(a) of the condition apply to at least one outcome $q \in C_i\left(\theta, p^\theta\right)$. Then, to satisfy part (2)(a) of Condition μ^*(ii) we need to have that this q is not an element of $S_i\left(\theta'; p^\theta, \theta\right)$ and, moreover, that the intersection $S_i\left(\theta'; p^\theta, \theta\right) \cap I_i\left(\theta', q, Y\right)$ is not empty. This is the case by construction of the set $S_i\left(\theta'; p^\theta, \theta\right)$ provided that this set is not empty. Indeed, if the set $S_i\left(\theta'; p^\theta, \theta\right)$ is not empty, then Condition μ^*(ii) is satisfied because there would exist an outcome r in $S_i\left(\theta'; p^\theta, \theta\right)$ such that the expected utility of individual i at r and at q in state θ' is the same, that is, $U_i\left(r;\theta'\right) = U_i\left(q;\theta'\right)$, establishing that the intersection $S_i\left(\theta'; p^\theta, \theta\right) \cap I_i\left(\theta', q, Y\right)$ is not empty, as well as because every element of $S_i\left(\theta'; p^\theta, \theta\right)$ is an outcome of $C_i\left(\theta, p^\theta\right)$ which results in a zero expected utility in state θ' for every individual $j \neq i$, establishing that the outcome q cannot be an element of this $S_i\left(\theta'; p^\theta, \theta\right)$.

Thus, to show that the set $S_i\left(\theta'; p^\theta, \theta\right)$ is not empty, it suffices to show that this set is not empty for every triplet (i, θ, θ') for which the premises of part (2)(a) of Condition μ^*(ii) apply to some $q \in C_i\left(\theta, p^\theta\right)$. To this end, take any of these triplets and denote it by (i, θ, θ').

Given that the utility allocation which assigns $U_i\left(q;\theta'\right)$ to individual i and zero to every other individual j is an element of the utility possibility set $U\left(\Delta;\theta'\right)$, it follows from this that there is a probability measure s in Δ which generates this utility allocation. From the

[22] The *Pareto optimal* set of Δ at θ is:

$$P\left(\theta\right) \equiv \left\{q \in \Delta | \text{ there is no } p \in \Delta : U\left(p;\theta\right) > U\left(q;\theta\right)\right\}, \quad \text{for every } \theta \in \Theta.$$

The Nash solution is efficient since $\nu\left(\theta\right) \subseteq P\left(\theta\right)$ for every $\theta \in \Theta$.

available ones, let r denote the one for which it also holds that $U_i(q;\theta) = U_i(r;\theta)$. This r exists because the space of outcomes Δ consists of all probability measures on the Borel σ-algebra of the space A. Thus, this r is an element of $C_i(\theta, p^\theta)$ for which it holds that $U_i(q;\theta') = U_i(r;\theta')$ and that $U_j(r;\theta') = 0$ for every individual $j \neq i$, establishing that the set $S_i(\theta'; p^\theta, \theta)$ is not empty. ■

4. The characterization theorem

In this section, we also consider non-unanimous SCRs and discuss a complete characterization of partially-honest Nash implementation. In the standard Nash implementation theory, as Moore and Repullo's (1990) Condition μ(iii) states, a SCR F must be unanimous with respect to a subset Y of X, with $F(\Theta) \subseteq Y$, if it is Nash implementable. Unfortunately, this condition is not a necessary one for partially-honest Nash implementation. Thus, we establish a new necessary condition, called Condition μ^*(iii). This condition, when combined with Condition μ^*(ii) and with another necessary condition, called Condition μ^*(i), provide a full characterization of the class of SCRs that are Nash implementable with partially-honest individuals. Condition μ^*(i) is a weak variant of Maskin monotonicity.

There are several key considerations underpinning our condition. These are presented from the viewpoint of necessity. To this end, take any SCR F satisfying Condition μ^*(ii). Suppose that it is partially-honestly Nash implementable by Γ.

4.1 Condition μ^* (i)

Let us consider two states θ and θ' and let H be the set of partially honest individuals. Suppose that $x = g(m)$ is F-optimal at θ but is not F-optimal at θ' and that for each individual i, outcome x is maximal for i over the set $C_i(\theta, x) = g(M_i, m_{-i})$ under the state θ'. Recall that the set $S_i(\theta'; x, \theta) = g(T_i^\Gamma(\theta'), m_{-i})$ represents the set of outcomes that this individual can attain by playing truthful strategy choices for θ' when the state moves from θ to θ', keeping the other individuals' equilibrium strategy choices fixed at m_{-i}. Thus, in order to break the Nash equilibrium at (θ', H) via a unilateral deviation there must exist a partially-honest individual h who can find it profitable unilaterally to deviate from the strategy profile m. This means that the strategy choice m_h is not a truthful one for θ' (that is, $m_h \notin T_h^\Gamma(\theta')$) and that there is a truthful strategy choice $m'_h \in T_h^\Gamma(\theta')$ such that this h judges the outcomes $g(m'_h, m_{-h}) = z$ and $g(m) = x$ as equally good according to her preference $R_h(\theta')$. In other words, at least one partially-honest individual h needs to find a truthful outcome $z \in S_h(\theta'; x, \theta)$ that is equally good to x according to her ordering $R_h(\theta')$ in order to have a unilateral non-material profitable deviation from the profile m. Therefore, the outcome z is an element of $S_h(\theta'; x, \theta) \cap I_h(\theta', x)$. Let us formalize this discussion into the following condition:

Definition 12 The SCR $F : \Theta \to X$ satisfies *Condition* μ^*(i) provided that for all θ, θ' and H, if $x \in F(\theta) \backslash F(\theta')$ and $x \in C_i(\theta, x) \subseteq L_i(\theta', x)$ for all $i \in N$, then there exists $h \in H$ such that $S_h(\theta'; x, \theta) \cap I_h(\theta', x, Y) \neq \varnothing$.

It is worth emphasizing that the above condition, *per se*, does not impose any restriction on the class of SCRs that are partially-honestly Nash implementable. This is due to the fact that one can always construct individual i's set of truthful outcomes, $S_i(\theta'; x, \theta)$, by satisfying the requirement that $x \in S_i(\theta'; x, \theta)$ when the premises of the condition are met. In other words, one can always make the implication of the condition trivially true. This construction is also consistent with Theorem 1 of Dutta and Sen (2012), according to which the partially-honest Nash implementability is assured by no veto-power when to be honest means to report the true preferences of individuals. However, this construction is not allowed when our objective is to provide a full characterization of SCRs that are partially-honestly Nash implementable. The reason is that the construction of the set $S_i(\theta'; x, \theta)$ needs to be compatible with Condition μ^*(ii) as well as with other feasibility constraints that are introduced below. These constraints are indeed needed because the canonical mechanism for Nash implementation (Repullo, 1987, p. 40) fails to partially-honestly Nash implements F. However, if F is a unanimous SCR and, moreover, it satisfies Condition μ^*(ii), then it is possible to construct the set $S_i(\theta'; x, \theta)$ in a way that both Condition μ^*(i) and Condition μ^*(ii) are both satisfied.[23]

4.2 Condition μ^*(iii)(A)

In order to formalize the above-mentioned feasibility constraints, we need to introduce additional notation. The class of all subsets of N is denoted by $\mathcal{P}(N)$. Any non-empty element T of the family $\mathcal{P}(N)$ is associated with a profile of states $\left(\bar{\theta}^j\right)_{j \in T}$, denoted by $\bar{\theta}^T$. The state $\bar{\theta}^j \in \Theta$ can be thought of as the *state announced by individual* $j \in T$. The profile $\bar{\theta}^T_{-i}$ is obtained from $\bar{\theta}^T$ by omitting the state announced by individual i, that is, $\bar{\theta}^T_{-i} \equiv \left(\bar{\theta}^j\right)_{j \in T \setminus \{i\}}$.

The maximal set of outcomes associated with the triplet (Y', θ, i) for any $Y' \subseteq X$ is $M_i(Y', \theta) \equiv \{x \in Y' | x R_i(\theta) y \text{ for all } y \in Y'\}$. We write $M(Y', \theta)$ for the set of outcomes that are maximal with respect to $R_i(\theta)$ over the set Y', for every individual i; that is, $M(Y', \theta) \equiv \bigcap_{i \in N} M_i(Y', \theta)$.

Suppose that F is partially-honestly Nash implemented by Γ. Fix any individual i and any state $\bar{\theta}^i \in \Theta$. Thus, the first feasibility constraint, denoted by $Y_i\left(\bar{\theta}^i\right)$, is defined by

$$Y_i\left(\bar{\theta}^i\right) \equiv g\left(T_i^\Gamma\left(\bar{\theta}^i\right), M_{-i}\right).$$

[23] To see this point, let us suppose that F satisfies unanimity as well as Condition μ^*(ii). Let us consider any two states θ and θ' and any outcome x such that x is F-optimal at θ but is not F-optimal at θ'. Moreover, assume that $C_\ell(\theta, x) \subseteq L_\ell\left(\theta', x\right)$ for all $\ell \in N$. Then, Condition μ^*(ii) implies that there exists a non-empty set $S_\ell\left(\theta'; x, \theta\right)$ for each ℓ. Fix any agent ℓ. We can construct a new set $S^*_\ell\left(\theta'; x, \theta\right)$ from $S_\ell\left(\theta'; x, \theta\right)$ as follows: (a) $S^*_\ell\left(\theta'; x, \theta\right) \equiv S_\ell\left(\theta'; x, \theta\right) \cup \{x\}$ if there is an agent $k \neq \ell$ for whom this x is not a maximal element of Y under θ'; (b) otherwise, $S^*_\ell\left(\theta'; x, \theta\right) \equiv S_\ell\left(\theta'; x, \theta\right)$. By construction, one can see that the intersection $S^*_\ell\left(\theta'; x, \theta\right) \cap I_\ell\left(\theta', x, Y\right)$ is not empty. Moreover, one can see that Condition μ^*(i) is (trivially) satisfied. Finally, one can easily check that this construction is consistent with Conditions μ^*(ii).

Since the set of truthful strategy choices for $\bar{\theta}^i$, $T_i^\Gamma\left(\bar{\theta}^i\right)$, is non-empty, it is plain that $Y_i\left(\bar{\theta}^i\right)$ is non-empty. Similar to the set $S_i\left(\theta';x,\theta\right)$ of Condition μ^*(ii), $Y_i\left(\bar{\theta}^i\right)$ is the set of outcomes that this i can attain by playing truthful strategy choices for $\bar{\theta}^i$.

For what follows, let us fix any non-empty subset $T\subseteq N$ and let us associate it with the profile $\bar{\theta}^T$. The second feasibility constraint, denoted by $Y_T\left(\bar{\theta}^T\right)$, is defined by

$$Y_T\left(\bar{\theta}^T\right)\equiv g\left(\left(T_i^\Gamma\left(\bar{\theta}^i\right)\right)_{i\in T},\prod_{i\in N\setminus T}M_i\right)\text{ if }T\neq N, \tag{1}$$

and by

$$Y_T\left(\bar{\theta}^T\right)\equiv g\left(\left(T_i^\Gamma\left(\bar{\theta}^i\right)\right)_{i\in T}\right)\text{ if }T=N. \tag{2}$$

Again, this set is not empty since set of truthful strategy choices for $\bar{\theta}^i$, $T_i^\Gamma\left(\bar{\theta}^i\right)$, is non-empty, for each $i\in T$.[24] Similar to $Y_i\left(\bar{\theta}^i\right)$, we can view this $Y_T\left(\bar{\theta}^T\right)$ as the set of outcomes that $i\in T$ can attain by playing truthful strategy choices for $\bar{\theta}^i$ while every other individual $j\in T$ is playing a truthful strategy for $\bar{\theta}^j$. Let us summarize these feasibility constraints as follows: Let Y be the set of outcomes specified by Condition μ^*(ii).

(A)-(0). For every $\left(x,\theta,H,T,\bar{\theta}^T\right)\in Y\times\Theta\times\mathcal{H}\times\mathcal{P}\left(N\right)\times\Theta^{|T|}$, there exist a collection of non-empty sets $\left(Y_i\left(\bar{\theta}^i\right)\right)_{i\in N}$ and a non-empty subset $Y_T\left(\bar{\theta}^T\right)$, with $Y_T\left(\bar{\theta}^T\right)\equiv Y$ if $T=\varnothing$, such that $Y_T\left(\bar{\theta}^T\right)=Y_j\left(\bar{\theta}^j\right)$ if $T=\{j\}$.

Two additional properties of the sets $Y_T\left(\theta^T\right)$ and $Y_i\left(\bar{\theta}^i\right)$ will be introduced below, which emerge as direct implications of the fact that Γ partially-honestly Nash implements F. The common feature they share is that in cases an outcome x is not F-opitmal to one state θ, then there must exist a deviant i who can find an alternative outcome that is as least as good as x under θ and that is an element of the set $Y_{\bar{T}}\left(\bar{\theta}^{\bar{T}}\right)$ for some $\bar{T}\in\mathcal{P}\left(N\right)$, where $\bar{T}\equiv T\setminus\{i\}$ or $\bar{T}\equiv T\cup\{i\}$.

First, let us consider the case where x is unanimously top-ranked but is not F-optimal at θ. Then, x cannot be an equilibrium outcome for $\left(\Gamma,\succsim^{\Gamma,\theta,H}\right)$. Under Assumption 2, this implies that for any element $\{i\}\in\mathcal{H}$ and any strategy profile m such that $g\left(m\right)=x$, m cannot be an equilibrium profile for $\left(\Gamma,\succsim^{\Gamma,\theta,\{i\}}\right)$. As x is unanimously top-ranked at state θ but is not F-optimal at θ, it must be the case that m_i is not a truthful strategy for θ, that is, $m_i\notin T_i^\Gamma\left(\theta\right)$. Moreover, there should be a truthful strategy $m_i'\in T_i^\Gamma\left(\theta\right)$ such that i judges $g\left(m_i',m_{-i}\right)$ and x to be as equally good and that $g\left(m_i',m_{-i}\right)$ is an i's truthful outcome for θ; that is, $g\left(m_i',m_{-i}\right)\in Y_i\left(\theta\right)\cap I_i\left(\theta,x,Y\right)$. Finally, note that x cannot be supported by any

[24] If $T=\varnothing$, then we define $Y_T\left(\bar{\theta}^T\right)$ by $Y_T\left(\bar{\theta}^T\right)=Y\equiv g\left(M\right)$.

strategy profile in which i plays a truthful strategy for θ - otherwise, this strategy profile is an equilibrium for $\left(\Gamma, \succcurlyeq^{\Gamma,\theta,\{i\}}\right)$, which is a contradiction. By defining $Y_{T\cup\{h\}}\left(\bar{\theta}^T_{-h}, \theta^h\right)$ in a way similar to $Y_T\left(\bar{\theta}^T\right)$, with $\theta^h = \theta$, one can also see that $x \notin Y_{T\cup\{h\}}\left(\bar{\theta}^T_{-h}, \theta^h\right)$.[25] Let us formalize this discussion into the following condition:

(A)-(a). If $x \in M(Y,\theta) \backslash F(\theta)$ and $H = \{h\}$, then $x \notin Y_{T\cup\{h\}}\left(\bar{\theta}^T_{-h}, \theta^h\right)$, with $\theta^h = \theta$, and $Y_h\left(\theta^h\right) \cap I_h\left(\theta^h, x, Y\right) \neq \varnothing$.

Second, let us consider the case where an outcome x is an element of $Y_T\left(\bar{\theta}^T\right)$ but x is not F-optimal at θ. Since condition (A)-(a) holds, it cannot be that this x is maximal for every i over the set Y at θ and that $\bar{\theta}^i = \theta$; otherwise, this condition and Assumption 2 would imply that x is F-optimal at θ, which is a contradiction.

A case that is not covered by condition (A)-(a) is the case where x is maximal for every individual i over $Y_T\left(\bar{\theta}^T\right)$ at θ and where $Y_T\left(\bar{\theta}^T\right) \neq Y$. In this case, given that x is not F-optimal at θ, this x cannot be an equilibrium outcome for $\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$. This implies that for any set H and any strategy profile m such that $g(m) = x$, m cannot be an equilibrium for $\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$. In other words, there exists an individual $i \in N$ and a feasible outcome $g(m'_i, m_{-i}) \in Y$ such that $g(m'_i, m_{-i})$ is at least as good as $g(m)$ under θ. We distinguish two mutually exclusive cases.

- Suppose that there is an i who judges $g(m'_i, m_{-i})$ to be better than $g(m)$ under θ, that is, $g(m'_i, m_{-i})\, P_i(\theta)\, g(m)$. This i needs to be an element of T. To see this, first note that $i \in T$ if $T = N$. Second, if $T \neq N$, then $Y_T\left(\bar{\theta}^T\right)$ is as defined in (1). Since $g(m)$ is maximal over $Y_T\left(\bar{\theta}^T\right)$ under θ, for every $j \in N$, it follows from definition of $Y_T\left(\bar{\theta}^T\right)$ that $g(m)$ is maximal over $g(M_j, m_{-j})$ under θ, for every $j \in N\backslash T$. This means that i cannot be an element of $N\backslash T$ given our supposition that $g(m'_i, m_{-i})\, P_i(\theta)\, g(m)$. Now, by defining $Y_{T\backslash\{i\}}\left(\bar{\theta}^T_{-i}\right)$ in a way similar to $Y_T\left(\bar{\theta}^T\right)$, one can see that $g(m'_i, m_{-i})$ is an element of $Y_{T\backslash\{i\}}\left(\bar{\theta}^T_{-i}\right)$.[26] In summary, we can conclude for this case that there exist $i \in T$ and $g(m'_i, m_{-i}) \in Y_{T\backslash\{i\}}\left(\bar{\theta}^T_{-i}\right)$ such that $g(m'_i, m_{-i})\, P_i(\theta)\, g(m)$.

- Otherwise, for each individual i, there is no outcome in $g(M_i, m_{-i})$ which is better than $g(m)$ under θ. Thus, since m is not an equilibrium for $\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$, there must exist a partially-honest individual $h \in H$ who can find it profitable unilaterally to deviate from m. This means that the strategy choice m_h is not a truthful one for θ (that is, $m_h \notin T^{\Gamma}_h(\theta)$) and that there is a truthful strategy choice $m'_h \in T^{\Gamma}_h(\theta)$ such that this h judges $g(m'_h, m_{-h})$ and $g(m)$ as equally good under θ; that is, $g(m'_h, m_{-h}) \in I_h(\theta, x, Y)$.

[25] Recall that $\bar{\theta}^T_{-h} \equiv \left(\bar{\theta}^T_j\right)_{j\in T\backslash\{h\}}$.

[26] Recall that by Condition (A)-(0), it follows that $Y_{T\backslash\{i\}}\left(\bar{\theta}^T_{-i}\right) \equiv Y$ if $T\backslash\{i\} = \varnothing$.

Observe that if $h \in T$, then it must be the case that $\bar{\theta}^h \neq \theta$; otherwise, this h cannot find any profitable unilateral deviation. Now, by defining $Y_{T\cup\{h\}}\left(\bar{\theta}_{-h}^T, \theta^h\right)$ in a way similar to $Y_T\left(\bar{\theta}^T\right)$, one can see that $g\left(m_i', m_{-i}\right)$ is also an element of $Y_{T\cup\{h\}}\left(\bar{\theta}_{-h}^T, \theta^h\right)$. In summary, we can conclude that for this case there exists an $h \in H$ for whom the intersection $Y_{T\cup\{h\}}\left(\bar{\theta}_{-h}^T, \theta^h\right) \cap I_h\left(\theta^h, x, Y\right)$ is not empty, where $\theta^h = \theta$, and for whom $\bar{\theta}^h \neq \theta$ if $h \in T$. Hence:

(A)-(b). If $T \neq \varnothing$ and $x \in M\left(Y_T\left(\bar{\theta}^T\right), \theta\right) \backslash F(\theta)$, then there exist $i \in T$ and $y \in Y_{T\backslash\{i\}}\left(\bar{\theta}_{-i}^T\right)$ such that $yP_i(\theta)\,x$, otherwise, there exists $h \in H$, with $\bar{\theta}^h \neq \theta$ if $h \in T$, for whom $Y_{T\cup\{h\}}\left(\bar{\theta}_{-h}^T, \theta^h\right) \cap I_h\left(\theta^h, x, Y\right) \neq \varnothing$, where $\theta^h = \theta$.

In summary, if F is partially-honestly Nash implementable, then the following condition must be satisfied:

Definition 13 The SCR $F : \Theta \to X$ satisfies *Condition* μ^*(iii)(A) provided that it satisfies Condition (A)-(0), Condition (A)-(a), and Condition (A)-(b).

Remark 4 If F is a unanimous SCR, then we can set $Y = X = Y_i(\theta) = Y_T\left(\bar{\theta}^T\right)$ for all $T \in \mathcal{P}(N)$, all $\bar{\theta}^T$ and all θ. Under these definitions, one can see that Condition μ^*(iii)(A) is satisfied.

4.3 Condition μ^*(iii)(B)

In this subsection, we present the last condition that emerges as a direct implication of the fact that Γ partially-honestly Nash implements F. We name this condition as Condition μ^*(iii)(B). To introduce this condition, we need to lay down its premises. To this end, suppose that F is partially-honestly Nash implemented by Γ. Thus, it satisfies Condition (A)-(0).

First, fix any individual i, any H and any θ and θ'. Suppose that x is F-optimal at θ. Also, assume that when the state moves from θ to θ' an outcome $z \in C_i(\theta, x)$ is maximal for i over $C_i(\theta, x)$ under θ' but this z is not F-optimal at θ'. Note that since N is an element of $\mathcal{H}$, by Assumption 2, there exists a profile m such that m is an equilibrium for $\left(\Gamma, \succsim^{\Gamma,\theta,N}\right)$ and that $g(m) = x$. As we have already discussed in section 3, $C_i(\theta, x) = g(M_i, m_{-i})$ represents the set of outcomes that i can generate by varying her own strategy, keeping the other individuals' equilibrium strategy choices fixed at m_{-i}. Then, given that m is an equilibrium for $\left(\Gamma, \succsim^{\Gamma,\theta,N}\right)$ such that $g(m) = x$ and that $z \in C_i(\theta, x)$, it follows that $g\left(m_i', m_{-i}\right) = z$ for some $m_i' \in M_i$. To economize on notation, we write m' for $\left(m_i', m_{-i}\right)$. Observe that m' is not an equilibrium for $\left(\Gamma, \succsim^{\Gamma,\theta',H}\right)$ given that z is not F-optimal at θ'.

Before proceeding with the discussion of Condition μ^*(iii)(B), recall that set $S_i(\theta'; x, \theta) = g\left(T_i^{\Gamma}(\theta'), m_{-i}\right)$ represents the set of outcomes that i can attain by playing truthful strategy choices for θ' when the state moves from θ to θ', keeping the other individuals' equilibrium strategy choices fixed at m_{-i}.

Second, fix any non-empty set $T \in \mathcal{P}(N)$ and any profile $\bar{\theta}^T$ such that T satisfies at least one of the following properties:

(I) $i \in T \implies z \in S_i\left(\bar{\theta}^i; x, \theta\right)$; and

(II) $T\backslash\{i\} \neq \varnothing \implies x \in S_k(\theta; x, \theta)$ with $\theta = \bar{\theta}^k$ for any $k \in T\backslash\{i\}$.

Given our interpretation of the pair $\left(T, \bar{\theta}^T\right)$, it is clear that if i is an element of T, then i is playing a truthful strategy choice for $\bar{\theta}^i$, and so z is an i's truthful outcome for $\bar{\theta}^i$. Similarly, if k is an element of $T\backslash\{i\}$, then property (II) requires that this k is playing a truthful strategy choice for $\theta = \bar{\theta}^k$, and so x is a k's truthful outcome for θ. By defining the set $Y_T\left(\bar{\theta}^T\right)$ of Condition (A)-(0) as in (1) or as in (2), one can see that $z \in Y_T\left(\bar{\theta}^T\right)$.[27]

Third, let us suppose that at least one of the following cases holds:

(1) $S_i(\theta'; x, \theta) \subseteq SL_i(\theta', z)$ and $T \neq \varnothing$;
(2) $i \notin H$; and
(3) $i \in T$ with $\bar{\theta}^i = \theta'$.

Suppose that case (1) applies. This implies that z is not an i's truthful outcome for θ' and that i cannot find any profitable unilateral deviation from m' given that $C_i(\theta, x) \subseteq L_i(\theta', z)$. However, given that m' is not an equilibrium for $\left(\Gamma, \succsim^{\Gamma,\theta',H}\right)$, there exist an $\ell \neq i$ and an outcome $g\left(m''_\ell, m'_{-\ell}\right)$ such that $g\left(m''_\ell, m'_{-\ell}\right) R_\ell(\theta') g(m')$. We distinguish two mutually exclusive cases.

(α) Suppose that for each individual $\ell \neq i$, there is no outcome in $g\left(M_\ell, m'_{-\ell}\right)$ which is better than $g(m') = z$ under θ'. This implies that $g\left(M_\ell, m'_{-\ell}\right) \subseteq L_\ell(\theta', z)$, for every $\ell \neq i$. Given that i cannot find any profitable unilateral deviation from m', it must be the case that the deviant ℓ needs to be a partially-honest individual in $H\backslash\{i\}$ such that $g\left(m''_\ell, m'_{-\ell}\right) I_\ell(\theta') g(m')$, that $m''_\ell \in T^\Gamma_\ell(\theta')$ and that $m'_\ell \notin T^\Gamma_\ell(\theta')$. Then, $g\left(m''_\ell, m'_{-\ell}\right) \in I_\ell(\theta', z, Y)$. Observe that if $\ell \in T$, then property (II) implies that $\bar{\theta}^\ell = \theta$, and so it must be the case that $\theta \neq \theta'$; otherwise, this ℓ cannot find any profitable unilateral deviation. By defining $Y_{T\cup\{\ell\}}\left(\bar{\theta}^T_{-\ell}, \theta^\ell\right)$ in a way similar to $Y_T\left(\bar{\theta}^T\right)$, one can see that $g\left(m''_\ell, m'_{-\ell}\right)$ is also an element of $Y_{T\cup\{\ell\}}\left(\bar{\theta}^T_{-h}, \theta^\ell\right)$. In summary, we can conclude that for this case there exists an $h \in H$ for whom the intersection $Y_{T\cup\{h\}}\left(\bar{\theta}^T_{-h}, \theta^h\right) \cap I_h\left(\theta^h, x, Y\right)$ is not empty, where $\theta^h = \theta'$, and for whom $\bar{\theta}^h \neq \theta'$ if $h \in T$.

(β) Suppose that $g\left(m''_\ell, m'_{-\ell}\right) P_\ell(\theta') g(m')$ for some $\ell \neq i$. Then, by defining $Y_{T\backslash\{\ell\}}\left(\bar{\theta}^T_{-\ell}\right)$ in a way similar to $Y_T\left(\bar{\theta}^T\right)$, one can see that $g\left(m''_\ell, m'_{-\ell}\right)$ is an element of $Y_{T\backslash\{\ell\}}\left(\bar{\theta}^T_{-\ell}\right)$.[28]

[27] In the case where the set T were an empty set, one can also see that $z \in Y_T\left(\bar{\theta}^T\right) = Y$, by Condition (A)-(0). Also, note that properties (I)-(II) are both satisfied in the case where $T = \varnothing$.

[28] Recall that by Condition (A)-(0), it follows that $Y_{T\backslash\{\ell\}}\left(\bar{\theta}^T_{-\ell}\right) \equiv Y$ if $T\backslash\{\ell\} = \varnothing$.

In summary, we can conclude for this case that there exist an individual $\ell \neq i$ and an outcome $g\left(m''_{\ell}, m'_{-\ell}\right) \in Y_{T \setminus \{\ell\}}\left(\bar{\theta}^{T}_{-\ell}\right)$ such that $g\left(m''_{\ell}, m'_{-\ell}\right) P_{\ell}(\theta)\, g(m')$.

By a similar reasoning and by invoking property (I) in the case that $i \in T$, one can show that either (α) or (β) holds in case (2) as well as in case (3). Therefore:

Definition 14 The SCR $F : \Theta \to X$ satisfies *Condition μ^* (iii)(B)* provided for all $i \in N$, all θ and θ', all T and $\bar{\theta}^{T}$, and all H, if $x \in F(\theta)$, $z \in C_i(\theta, x) \subseteq L_i(\theta', z)$ and $z \notin F(\theta')$ and if, moreover, T satisfies at least one of the following properties:
(I) $i \in T \implies z \in S_i\left(\bar{\theta}^{i}; x, \theta\right)$;
(II) $T \setminus \{i\} \neq \varnothing \implies x \in S_k(\theta; x, \theta)$ with $\theta = \bar{\theta}^{k}$ for each $k \in T \setminus \{i\}$,
then $z \in Y_T\left(\bar{\theta}^{T}\right)$. Moreover, if at least one of the following three requirements are satisfied: (1) $S_i(\theta'; x, \theta) \subseteq SL_i(\theta', z)$ and $T \neq \varnothing$; (2) $i \notin H$; (3) $i \in T$ with $\bar{\theta}^{i} = \theta'$, then one of the following two statements holds:
(α) there exists $h \in H \setminus \{i\}$, with $\theta' \neq \theta$ if $h \in T$, for whom $Y_{T \cup \{h\}}\left(\bar{\theta}^{T}_{-h}, \theta^{h}\right) \cap I_h\left(\theta^{h}, z, Y\right) \neq \varnothing$, where $\theta^{h} = \theta'$;
(β) there exist $\ell \in N \setminus \{i\}$ and $z^{(\ell)} \in Y_{T \setminus \{\ell\}}\left(\bar{\theta}^{T}_{-\ell}\right)$ such that $z^{(\ell)} P_{\ell}(\theta')\, z$.

Remark 5 If F is unanimous, then we can set $Y = X = Y_T\left(\bar{\theta}^{T}\right)$ for all $\left(T, \bar{\theta}^{T}\right)$. Condition μ^*(iii)-(B) is (trivially) satisfied, since for every individual ℓ we can always take an element of X so as to meet either (α) or (β).

Remark 6 Condition μ^*(iii)-(B) implies Condition μ^*(ii)-(2)-(b).

In what follows, we say a SCR satisfies *Condition μ^* (iii)* if it satisfies Condition μ^*(iii)(A) and Condition μ^*(iii)(B). Moreover, we say that F satisfies *Condition μ^** if it satisfies Condition μ^*(i), Condition μ^*(ii) and Condition μ^*(iii). Now, we are ready to state our characterization of SCRs that are partially-honestly Nash implementable.

Theorem 3 Let $n \geq 3$. Suppose that Assumption 1 and that Assumption 2 hold. The SCR $F : \Theta \twoheadrightarrow X$ is partially-honestly Nash implementable if and only if it satisfies Condition μ^*.

Proof. Suppose that F is partially-honestly Nash implementable. The proof of Theorem 1 in Appendix A shows that F satisfies Condition μ^*(ii). The proof that F satisfies Condition μ^*(i) as well as Condition μ^*(iii) is relegated in Appendix C. Suppose that F satisfies Condition μ^*(i), Condition μ^*(ii) and Condition μ^*(iii). The proof that F is partially-honestly Nash implementable can be found in Appendix B. ■

4.4 On the implementability of the egalitarian bargaining allocation rule

It is well-known that the egalitarian solution, defined over bargaining problems, is not a unanimous solution. In what follows, we show that this solution satisfies Condition μ^*, and so it is partially-honestly Nash implementable according to Theorem 3.

Like in section 3.4, here we assume allocation problems of infinitely divisible commodities among a group of individuals. However, unlike in section 3.4, we will not assume that each individual's preference is represented by a von Neumann-Morgenstern (vNM) utility function.[29] Here, we simply assume that individual i's preference over her consumption space is continuous, monotonic, and convex. Then, it admits a numerical represetantion. Moreover, following Roemer (1986, 1988) and Yoshihara (2003), a bargaining solution is defined as an allocation rule that associates a subset of feasible allocations to a profile of utility functions.

Let us consider allocation problems in pure exchange economies, as in Roemer (1986, 1988). Let $X \equiv \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^{nm}_{+} \mid \sum_{i \in N} x_i \leq \omega\right\}$ be the set of feasible allocations, with $\omega \in \mathbb{R}^{m}_{++}$ as the fixed social endowment of m commodities. Let $\mathbb{R}^{m}_{+}$ be the consumption space common to all individuals. For each $i \in N$ and each state $\theta \in \Theta$, let $u_i(\cdot; \theta)$ be a continuous, increasing, and concave utility function in state θ, which is defined by $u_i(\cdot; \theta) : \mathbb{R}^{m}_{+} \to \mathbb{R}_{+}$ with $u_i(\mathbf{0}; \theta) = 0$ and $u_i(x_i; \theta) > 0$ for all $x_i \in \mathbb{R}^{m}_{++}$. Then, given $\theta \in \Theta$, the utility possibility set is given by $U(\theta) \equiv \left\{\boldsymbol{u}(x) \equiv (u_i(x_i; \theta))_{i \in N} \in \mathbb{R}^{n}_{+} \mid x \in X\right\}$ with $\mathbf{0} \in U(\theta)$. Let the disagreement point be fixed by $d = \mathbf{0}$ throughout this section. Then, the class of bargaining problems is defined by $\mathcal{U} \equiv \{U(\theta) \mid \theta \in \Theta\}$. The *egalitarian solution* is a mapping $E : \mathcal{U} \to \mathbb{R}^{n}_{+}$ such that for each $U(\theta) \in \mathcal{U}$, $E(U(\theta)) \in \mathbb{R}^{n}_{++}$ satisfies the following properties: (a) $E_i(U(\theta)) = E_j(U(\theta))$ holds for any $i, j \in N$, and (b) there is no $\boldsymbol{u}(x) \in U(\theta)$ such that $\boldsymbol{u}(x) \gg E(U(\theta))$. Moreover, a *bargaining allocation rule* is a correspondence $\varphi : \Theta \twoheadrightarrow X$ such that (i) for each $\theta \in \Theta$, $\varphi(\theta) \neq \varnothing$, (ii) for any $x, x' \in \varphi(\theta)$, $\boldsymbol{u}(x) = \boldsymbol{u}(x')$ holds, and (iii) for any $x'' \in X$ with $\boldsymbol{u}(x'') = \boldsymbol{u}(x)$ for some $x \in \varphi(\theta)$, $x'' \in \varphi(\theta)$ holds. A bargaining allocation rule φ is an *egalitarian* bargaining solution, denoted by φ^E, if and only if for each $\theta \in \Theta$, $\boldsymbol{u}\left(\varphi^E(\theta)\right) = E(U(\theta))$ holds.

It is well-known that φ^E is not Nash implementable, since it does not satisfy Maskin monotonicity. This is because the class of bargaining problems contains (not necessarily strict) comprehensive and convex problems derived from the class of continuous, (not necessarily strongly) increasing, and concave utility functions, and thus a non-egalitarian, unanimously top-ranked utility allocation can exist for some bargaining problems. Thus, we cannot test the partially-honest implementability of φ^E by applying the characterization result of Theorem 1. By applying the characterization result of Theorem 3, however, we can show that φ^E is partially-honestly Nash implementable.

Proposition 4 Let (N, X, Θ) be a class of pure exchange economies with $n \geq 3$. Let Assumption 1 and Assumption 2 be given. Then, the egalitarian bargaining solution φ^E is partially-honestly Nash implementable.

Proof. We show that φ^E satisfies Condition μ^* with respect to X.

(i) First, let us show that φ^E satisfies Condition μ^*(ii). Take any $\theta \in \Theta$ and any $x \in \varphi^E(\theta)$. Then, let $C_i(\theta, x) \equiv L_i(\theta, x) \equiv \{y \in X \mid u_i(y_i; \theta) \leq u_i(x_i; \theta)\}$ for each $i \in N$. Given $x \in X$ and $i \in N$, let $x_i^{\mathbf{0}} \equiv (x_i, \mathbf{0}_{-i})$ where $\mathbf{0}_{-i}$ means that any individual other than i receives the zero consumption bundle. Given a possibility that there exists $y^* \in$

[29] We do not need to assume vNM utility preferences in underlying economic environments of bargaining problems, since the egalitarian solution cannot satisfy the so-called Scale Invariance axiom.

$\left[\cap_{N\setminus\{i\}} M_j(X,\theta)\right]\setminus\varphi^E(\theta)$, let us define the following subset of individuals:

$$S=\left\{j\in N \mid u_j\left(y_j^*;\theta\right)=u_j\left(x_j;\theta\right) \text{ for some } y^*\in\left[\cap_{N\setminus\{i\}} M_j(X,\theta)\right]\setminus\varphi^E(\theta) \text{ and some } i\in N\right\}.$$

Note that if there exists $y^*\in\left[\cap_{N\setminus\{i\}} M_j(X,\theta)\right]\setminus\varphi^E(\theta)$ and $y^*\in C_i(\theta,x)\subseteq L_i(\theta,y^*)$ holds, then $i\in S$.

Then, for each $j\in N$, let us define the set $S_j(\theta';x,\theta)$ as follows:
(a) if $j\in S$, then

$$\begin{aligned} S_j(\theta';x,\theta) &\equiv \cup_{y\in\arg\max_{z\in C_j(\theta,x)} u_j(z_j;\theta')}\left\{y_j^{\mathbf{0}}\right\}\cup\{x\} \text{ if and only if } \theta'=\theta; \\ S_j(\theta';x,\theta) &\equiv \cup_{y\in\arg\max_{z\in C_j(\theta,x)} u_j(z_j;\theta')}\left\{y_j^{\mathbf{0}}\right\} \text{ if and only if } \theta'\neq\theta. \end{aligned}$$

(b) if $j\notin S$, then

$$\begin{aligned} S_j(\theta';x,\theta) &\equiv SL_j(\theta,x) \text{ if and only if } \theta'=\theta; \\ S_j(\theta';x,\theta) &\equiv \cup_{y\in\arg\max_{z\in C_j(\theta,x)} u_j(z_j;\theta')}\left\{y_j^{\mathbf{0}}\right\} \text{ if and only if } \theta'\neq\theta. \end{aligned}$$

Such a definition of $S_j(\theta';x,\theta)$ is well-defined, since $\mathbf{0}\in S_j(\theta;x,\theta)$ and $\max_{z\in C_j(\theta,x)} u_j(z_j;\theta')$ is well-defined by the compactness of $C_j(\theta,x)=L_j(\theta,x)$ and the continuity of $u_j(\cdot;\theta')$. Moreover, Condition μ^*(ii) is satisfied with this definition. First, by the definition, $S_i(\theta';x,\theta)\subseteq L_i(\theta,x)=C_i(\theta,x)$ for all $i\in N$. Thus, part (1)-(a) of Condition μ^*(ii) is satisfied. Second, for every $i\in N$, if $\theta'=\theta$, then $x\in S_i(\theta';x,\theta)$ or $S_i(\theta';x,\theta)=SL_i(\theta,x)$ holds. Therefore, part (1)-(b) of Condition μ^*(ii) is satisfied. Third, let $y\notin\varphi^E(\theta')$ satisfy $y\in C_i(\theta,x)\subseteq L_i(\theta',y)$ and $Y\subseteq L_j(\theta',y)$ for all $j\neq i$. Suppose $\theta'\neq\theta$. Then by the definition, $y_i^{\mathbf{0}}\in S_i(\theta';x,\theta)\cap I_i(\theta',y,X)$ holds, and $y\neq y_i^{\mathbf{0}}$ holds by $y\in C_i(\theta,x)$, $u_j(y_j;\theta')>u_j(\mathbf{0};\theta')$, and $Y\subseteq L_j(\theta',y)$ for all $j\neq i$. Therefore, $y\notin S_i(\theta';x,\theta)$. Thus, part (2)-(a) of Condition μ^*(ii) is satisfied. Suppose $\theta'=\theta$. Then, the set S is non-empty and $i\in S$. Then by the definition, $y_i^{\mathbf{0}}\in S_i(\theta';x,\theta)\cap I_i(\theta',y,X)$ holds, and $y\neq y_i^{\mathbf{0}}$ holds by $y\in C_i(\theta,x)$, $u_j(y_j;\theta')>u_j(\mathbf{0};\theta')$, and $Y\subseteq L_j(\theta',y)$ for all $j\neq i$. Therefore, $y\notin S_i(\theta';x,\theta)$. Thus, part (2)-(a) of Condition μ^*(ii) is satisfied. Thus, in summary, part (2)-(a) of Condition μ^*(ii) is satisfied.

(ii) Second, let us show that φ^E satisfies Condition μ^*(iii)-(A).

Let

$$X_i^{\mathbf{0}}\equiv\left\{x_i^{\mathbf{0}}\in\mathbb{R}_+^{nm}\mid\exists x\in X: x_i^{\mathbf{0}}=(x_i,\mathbf{0}_{-i})\right\},$$

and $X^{\mathbf{0}}\equiv\cup_{i\in N}X_i^{\mathbf{0}}$. Take any $\left(x,\theta,H,T,\bar{\theta}^T\right)\in X\times\Theta\times\mathcal{H}\times\mathcal{P}(N)\times\Theta^{|T|}$, and let $Y_T\left(\bar{\theta}^T\right)\equiv X\setminus\left[\cup_{i\in T}M\left(X,\bar{\theta}^i\right)\right]$ if $T\neq\varnothing$; and $Y_T\left(\bar{\theta}^T\right)\equiv X$ if $T=\varnothing$. For any $i\in N$, let $Y_i(\theta)\equiv X\setminus M(X,\theta)$, $Y_i\left(\bar{\theta}^i\right)\equiv X\setminus M\left(X,\bar{\theta}^i\right)$, and $Y_{T\setminus\{i\}}\left(\bar{\theta}_{-i}^T\right)\equiv X\setminus\left[\cup_{j\in T\setminus\{i\}}M\left(X,\bar{\theta}^j\right)\right]$. Moreover, for any $h\in H$, let $Y_{T\cup\{h\}}\left(\bar{\theta}_{-h}^T,\theta^h\right)\equiv X\setminus\left[\cup_{i\in T\setminus\{h\}}M\left(X,\bar{\theta}^i\right)\cup M\left(X,\theta^h\right)\right]$ for any $\theta^h\in\Theta$. These specifications are consistent with the requirements of Condition μ^*(iii).

To see φ^E satisfies Condition μ^*(iii)-(A)-(a), let $x\in M(X,\theta)$, $x\notin\varphi^E(\theta)$, and $H=\{h\}$. Note that $Y_{T\cup\{h\}}\left(\bar{\theta}_{-h}^T,\theta^h\right)=X\setminus\left[\cup_{i\in T\setminus\{h\}}M\left(X,\bar{\theta}^i\right)\cup M\left(X,\theta^h\right)\right]$ for $\theta^h=\theta$. Therefore,

$x \in M(X,\theta)$ implies $x \notin Y_{T\cup\{h\}}\left(\bar{\theta}^T_{-h},\theta^h\right)$ holds for $\theta^h=\theta$. Moreover, let $z^{(h,\theta)} \equiv (x_h,\mathbf{0}_{-h}) = x^{\mathbf{0}}_h$. Then, $z^{(h,\theta)} \in Y_h(\theta)\cap I_h(\theta,x,X)$ holds, as $Y_h(\theta)=X\backslash M(X,\theta)\supseteq X^{\mathbf{0}}$, by our specification. Thus, φ^E satisfies Condition μ^*(iii)-(A)-(a).

To see φ^E satisfies Condition μ^*(iii)-(A)-(b), for any $T\neq\varnothing$, any $\theta\in\Theta$, and any $x\notin\varphi^E(\theta)$, let $x\in M\left(Y_T\left(\bar{\theta}^T\right),\theta\right)$. Note that $\boldsymbol{u}(M(X,\theta);\theta)$ is the utility allocation corresponding to the unanimous allocations $M(X,\theta)$ at θ, which is a unique point in the utility possibility set $U(\theta)$. Therefore, $U(\theta)\setminus\{\boldsymbol{u}(M(X,\theta);\theta)\}$ is not closed in $\mathbb{R}^n_+$. In this case, there is no unanimously maximal utility allocation within $U(\theta)\setminus\{\boldsymbol{u}(M(X,\theta);\theta)\}$ at θ. Likewise, $U(\theta)\setminus\left[\cup_{i\in T}\left\{\boldsymbol{u}\left(M\left(X,\bar{\theta}^i\right);\theta\right)\right\}\cup\{\boldsymbol{u}(M(X,\theta);\theta)\}\right]$ is not closed, within which there is no unanimously maximal utility allocation at θ. Then, correspondingly, $M(X\backslash M(X,\theta),\theta)=\varnothing$ holds as well as $M\left(X\backslash\left[\cup_{i\in T}M\left(X,\bar{\theta}^i\right)\cup M(X,\theta)\right],\theta\right)=\varnothing$ holds. Therefore, whenever there exists $k\in T$ such that $\bar{\theta}^k=\theta$, $M\left(Y_T\left(\bar{\theta}^T\right),\theta\right)=M\left(X\backslash\left[\cup_{i\in T}M\left(X,\bar{\theta}^i\right)\cup M(X,\theta)\right],\theta\right)=\varnothing$ holds. Note that even if $M(X,\theta)=\varnothing$, $M(X\backslash M(X,\theta),\theta)=M(X,\theta)=\varnothing$ holds. Then, since $M\left(X\backslash\left[\cup_{i\in T}M\left(X,\bar{\theta}^i\right)\right],\theta\right)=M(X,\theta)$ whenever $M(X,\theta)\cap\left[\cup_{i\in T}M\left(X,\bar{\theta}^i\right)\right]=\varnothing$, $M\left(X\backslash\left[\cup_{i\in T}M\left(X,\bar{\theta}^i\right)\right],\theta\right)=M(X,\theta)=\varnothing$ holds. Thus, if $\bar{\theta}^k=\theta$ for some $k\in T$, then $x\in M\left(Y_T\left(\bar{\theta}^T\right),\theta\right)$ does not hold. Thus, in this case, φ^E vacuously satisfies Condition μ^*(iii)-(A)-(b). Consider the case that for any $k\in T$, $\bar{\theta}^k\neq\theta$. Then, if $x\in M\left(Y_T\left(\bar{\theta}^T\right),\theta\right)$, then let $w^{(h,\theta)}\equiv(x_h,\mathbf{0}_{-h})=x^{\mathbf{0}}_h$. Then, $w^{(h,\theta)}\in Y_{T\cup\{h\}}\left(\bar{\theta}^T_{-h},\theta\right)\cap I_h(\theta,x,X)$ holds, as $Y_{T\cup\{h\}}\left(\bar{\theta}^T_{-h},\theta\right)\supseteq X^{\mathbf{0}}$, by our specification. Thus, φ^E satisfies Condition μ^*(iii)-(A)-(b). In summary, φ^E satisfies Condition μ^*(iii)-(A).

(iii) Next, let us show that φ^E satisfies Condition μ^*(iii)-(B). Let $x\in\varphi^E(\theta)$, and for any given $i\in N$ and any given $\theta'\in\Theta$, let $z\in C_i(\theta,x)\subseteq L_i(\theta',z)$ and $z\notin\varphi^E(\theta')$. By the construction of $S_j(\theta';x,\theta)$ for each $j\in N\backslash S$, $x\notin S_j(\theta';x,\theta)$ whenever $\theta'=\theta$. Therefore, the premise (II) of Condition μ^*(iii)-(B) is never met whenever $S=\{i\}$.

Let $\theta'=\theta$. In this case, $i\in S$. Suppose the premise (I) of Condition μ^*(iii)-(B) holds for $\bar{\theta}^i=\theta$. Then, the construction of $S_i(\theta;x,\theta)$ implies $z=z^{\mathbf{0}}_i$ holds, and so $z\in X^{\mathbf{0}}\subseteq Y_T\left(\bar{\theta}^T\right)$. Moreover, there exist $\ell\in N\backslash\{i\}$ and an outcome $z^{\mathbf{0}}_\ell\equiv\left(z^{(\ell)}_\ell,\mathbf{0}_{-\ell}\right)\in X^{\mathbf{0}}\subseteq Y_{T\backslash\{\ell\}}\left(\bar{\theta}^T_{-\ell}\right)$ such that $z^{\mathbf{0}}_\ell P_\ell(\theta')\,z$. Therfore, the claim (β) of Condition μ^*(iii)-(B) holds. Suppose the premise (I) of Condition μ^*(iii)-(B) holds for $\bar{\theta}^i\neq\theta$. Again, the construction of $S_i\left(\bar{\theta}^i;x,\theta\right)$ implies $z=z^{\mathbf{0}}_i$ holds, and so $z\in X^{\mathbf{0}}\subseteq Y_T\left(\bar{\theta}^T\right)$. Therefore, as in the case of $\bar{\theta}^i=\theta$, the claim (β) of Condition μ^*(iii)-(B) holds.

Suppose that the premise (I) of Condition μ^*(iii)-(B) does not hold, but the premise (II) of Condition μ^*(iii)-(B) holds. This implies that for every $k\in T$, $k\in S$. Then, given the construction of $S_i(\theta;x,\theta)$ and the present supposition, neither the premise (1) nor the premise (3) of Condition μ^*(iii)-(B) holds. Therefore, let $i\notin H$. Then, for each $j\in N\backslash S$, there exists $z^{\mathbf{0}}_j\in X^{\mathbf{0}}\subseteq Y_T\left(\bar{\theta}^T,\theta^j\right)$ for $\theta^j=\theta$. By definition of $z^{\mathbf{0}}_j$, $z^{\mathbf{0}}_j\in I_j(\theta,z,X)$ holds.

Since $i \notin N \backslash S \neq \varnothing$, the claim (α) of Condition μ^*(iii)-(B) holds whenever $H \cap (N \backslash S) \neq \varnothing$. If $H \subseteq S \backslash \{i\}$, then it implies that for any $h \in H$, $z \in C_h(\theta, x) \subseteq L_h(\theta, z)$ holds by $u_h(z_h; \theta) = u_h(x_h; \theta)$. Since φ^E satisfies part (2)-(a) of Condition μ^*(ii), either $z \notin \cap_{N \backslash \{i\}} M_j(X, \theta)$ or $z \in S_i(\theta; x, \theta)$ holds. Note that $H \subseteq S \backslash \{i\}$ and $i \in S$ imply that $\# S > 1$. This implies there is at least one individual k other than i such that $u_k(z_k; \theta) = u_k(x_k; \theta)$. Therefore, $z \neq z_i^{\mathbf{0}}$, which implies $z \notin S_i(\theta; x, \theta)$, and so $z \notin \cap_{N \backslash \{i\}} M_j(X, \theta)$ holds. Thus, $z \in Y_T(\theta) = X \backslash M(X, \theta)$, and there exist $\ell \in N \backslash S$ and outcome $z^{(\ell)} \in X$ such that $z^{(\ell)} P_\ell(\theta') z$. Therefore, the claim (β) of Condition μ^*(iii)-(B) holds whenever $H \cap (N \backslash S) = \varnothing$.

Suppose that $T = \varnothing$. Then, again only the premise (2) that $i \notin H$ is available. In this case, $z \in X = Y_T\left(\overline{\theta}^T\right)$ holds, and the claim (α) of Condition μ^*(iii)-(B) always holds by $z_j^{\mathbf{0}} \in I_j(\theta, z, X)$ for any $j \neq i$. In summary, φ^E satisfies Condition μ^*(iii)-(B).

(iv) To see φ^E satisfies Condition μ^*(i), let $x \in \varphi^E(\theta) \backslash \varphi^E(\theta')$ and $C_\ell(\theta, x) \subseteq L_\ell(\theta', x)$ holds for all $\ell \in N$. This implies $\theta' \neq \theta$. Then, for each $h \in H$, it follows that $x_h^{\mathbf{0}} \in S_h(\theta'; x, \theta) \cap I_h(\theta', x, X)$, by the definition of $S_h(\theta'; x, \theta)$. Thus, φ^E satisfies Condition μ^*(i). ■

5. Conclusion

The main practical aim of adopting an axiomatic approach to implementation theory is to distinguish between implementable and non-implementable SCRs. Drawing from the recent literature on implementation with partially-honest individuals, this paper identifies necessary and sufficient conditions for the Nash implementation in a many-person setting with partially-honest individuals. Existing results on the subject have thus far offered only sufficient conditions in a variety of environments.

In an environment in which knowledge is dispersed, how individuals will interact with the mechanism designer is a natural starting point when it comes to Nash implement a SCR. A particular kind of communication is, as we have done in this paper, to ask participants to report the entire state of the world. There is, however, no reason to restrict attention to such schemes.

On this issue, Lombardi and Yoshihara (2016) have recently identified conditions for Nash implementation with partially-honest individuals which, if satisfied, send us back to the limitations imposed by Maskin's theorem. In terms of mechanisms, these conditions basically result in the impossibility to structure the communication in a way that does not allow the mechanism designer to elicit enough information of individuals' characteristics from the partially-honest participants. For instance, the limitations of Maskin's theorem remain valid when participants are asked to report only their own characteristics.

However, this does not mean that there are not mechanisms that resemble real-life mechanisms and that, at the same time, allow us to escape the limitations imposed by Maskin monotonicity in a setting with partially-honest individuals. One of these mechanisms is represented by the price-quantity mechanism (studied, for example, in Dutta et al. 1995; Sjöström, 1996; and Saijo et al., 1996), in which each individual chooses prices of commodities as well as a consumption bundle as her strategy choice. This is so because the announcement of prices serves the purpose to acquire some local information about individuals' indifference curves, such as the common marginal rate of substitution at an efficient allocation. Indeed, we now know that the Walrasian set solution is Nash implementable in a many-person setting with partially-honest individuals by this type of market mechanism (see Lombardi and Yoshihara, 2017).[30]

Postlewaite and Schmeidler (1986), Palfrey and Srivastava (1989) and Jackson (1991) have shown that Maskin's theorem can be generalized to Bayesian environments. A necessary condition for Bayesian Nash implementation is Bayesian monotonicity. In a Bayesian environment involving at least three individuals, Bayesian monotonicity combined with no veto-power is sufficient for Bayesian Nash implementation provided that a necessary condition called closure and the Bayesian incentive compatibility condition are satisfied (Jackson, 1991). Korpela (2014) studies Bayesian Nash implementation and provides sufficient conditions for implementation in a setting with partially-honest participants. This characterization result shows that Bayesian monotonicity becomes redundant in this environment, and so there are far fewer limitations for Bayesian Nash implementation when individuals have a taste for honesty. As yet, where the exact boundaries of those limitations lay for Bayesian environments is far from known. This subject is left for future research.

References

D. Abreu, A. Sen, Subgame perfect implementation: a necessary and almost sufficient condition, J. Econ. Theory 50 (1990) 285-299.

D. Abreu, A. Sen, Virtual implementation in Nash equilibrium, Econometrica 59 (1991) 997–1021.

L.C. Corchón, C. Herrero, A decent proposal, Spanish Econ. Rev. 6 (2004) 107–125.

P. Dasgupta, P. Hammond, E. Maskin, The implementation of social choice rules: Some general results on incentive compatibility, Rev. Econ. Stud. 46 (1979) 185-216.

B. Driesen, M. Lombardi, H. Peters, Feasible sets, comparative risk aversion, and comparative uncertainty aversion in bargaining, available at Maastricht University, Research Memoranda 15 (2015).

B. Dutta, A. Sen, A necessary and sufficient condition for two-person Nash implementation, Rev. Econ. Stud. 58 (1991) 121-128.

B. Dutta, A. Sen, R. Vohra, Nash implementation through elementary mechanisms in economic environments, Econ. Des. 1 (1995) 173-204.

B. Dutta, A. Sen, Nash implementation with partially honest individuals, Games Econ Behav 74 (2012) 154-169.

D. Gale, L.S. Shapley, College Admissions and the Stability of Marriage, Amer. Math. Monthly 69 (1962) 9-14.

J.V. Howard, A social choice rule and its implementation in perfect equilibrium, J. Econ. Theory 56 (1992) 142-159.

[30] The provided characterization does not rely on any sort of "tail-chasing" construction.

M.O. Jackson, Bayesian implementation, Econometrica 59 (1991) 461-477.

M.O. Jackson, Implementation in Undominated Strategies: A Look at Bounded Mechanisms, Rev. Econ. Stud. 59 (1992) 757-775.

M.O. Jackson, A crash course in implementation theory, Soc. Choice Welfare 18 (2001) 655-708.

E. Kalai, J.O. Ledyard, Repeated Implementation, J. Econ. Theory 83 (1998) 308-317.

T. Kara, T. Sönmez, Nash implementation of matching rules, J. Econ. Theory 68 (1996) 425-439.

N. Kartik, O. Tercieux, Implementation with evidence, Theoretical Econ. 7 (2012) 323-355.

N. Kartik, O. Tercieux, R. Holden, Simple mechanism and preference for honesty, Games Econ. Behav. 83 (2014) 284-290.

D.E. Knuth, Marriages stables, Les Presses de l'Université de Montréal, Montréal, 1976.

V. Korpela, Bayesian implementation with partially honest individuals, Soc. Choice Welfare 43 (2014) 647-658.

J. Lee, H. Sabourian, Efficient Repeated Implementation, Econometrica 79 (2011) 1967-1994.

M. Lombardi, N. Yoshihara, A full characterization of Nash implementation with strategy space reduction, Econ. Theory 54 (2013) 131-151.

M. Lombardi, N. Yoshihara, Natural implementation with semi-responsible agents in pure exchange economies, (2017), forthcoming in Int. J. Game Theory.

M. Lombardi, N. Yoshihara, Treading a Fine Line: (Im)possibilities for Nash Implementation with Partially-honest Individuals," Discussion Paper Series 651, Institute of Economic Research, Hitotsubashi University (2016).

E. Maskin, Nash equilibrium and welfare optimality, Rev. Econ. Stud. 66 (1999) 23-38.

E. Maskin, T. Sjöström, Implementation theory, in: K. Arrow, A.K. Sen, K. Suzumura (Eds), Handbook of Social Choice and Welfare, Elsevier Science, Amsterdam, 2002, pp. 237–288.

H. Matsushima, A new approach to the implementation problem, J. Econ. Theory 45 (1988) 128–144.

H. Matsushima, Behavioral aspects of implementation theory, Econ. Letters 100 (2008a) 161-164.

H. Matsushima, Role of honesty in full implementation, J. Econ. Theory 139 (2008b) 353-359.

C. Mezzetti, L. Renou, Repeated Nash Implementation (June 29, 2012). Available at SSRN: http://ssrn.com/abstract=2096184.

E. Miyagawa, Subgame-perfect implementation of bargaining solutions, Games Econ. Behav. 41 (2002) 292–308.

J. Moore, R. Repullo, Subgame perfect implementation, Econometrica 56 (1988) 1191-1220.

J. Moore, R. Repullo, Nash implementation: A full characterization, Econometrica 58 (1990) 1083-1100.

H. Moulin, Implementing the Kalai–Smorodinsky bargaining solution, J. Econ. Theory 33 (1984) 32–45.

S. Mukherjee, N. Muto, E. Ramaekers, Implementation in undominated strategies with partially honest agents, Core DP 2017/11.

J. Naeve, Nash implementation of the Nash bargaining solution using intuitive message spaces, Econ. Lett. 62 (1999) 23-28.

J.F. Nash, The bargaining problem, Econometrica 18 (1950) 155-162.

J.F. Nash, Two person cooperative games, Econometrica 21 (1953) 128-140.

J. Ortner, Direct implementation with partially honest individuals, Games Econ. Behav. 90 (2015) 1-16.

T. Palfrey, S. Srivastava, Implementation with incomplete information in exchange economies, Econometrica 57 (1989) 115-134.

T. Palfrey, S. Srivastava, Nash-implementation using undominated strategies, Econometrica 59 (1991) 479-501.

A. Postlewaite, D. Schmeidler, Implementation in differential information economies, J. Econ. Theory 39 (1986) 14-33.

R. Repullo, A simple proof of Maskin's theorem on Nash implementation. Soc. Choice Welfare 4 (1987), 39-41

J. E. Roemer, Equality of Resources Implies Equality of Welfare, Quart. J. Econ. 101 (1986), 751-784.

J. E. Roemer, Axiomatic Bargaining Theory on Economic Environments, J. Econ. Theory 45 (1988), 1-31.

A. E. Roth, M. A. Oliveira Sotomayor, Two-sided matching, Econometric Society Monographs No. 18, Cambridge University Press, Cambridge, 1990.

T. Saijo, Y. Tatamitani, T. Yamato, Toward natural implementation, Int. Econ. Rev. 37 (1996) 949-980.

A. Saporiti, Securely implementable social choice rules with partially honest agents, J. Econ. Theory 154 (2014) 216-228.

R. Serrano, A comment on the Nash program and the theory of implementation, Econ. Lett. 55 (1997) 203–208.

T. Sjöström, On the necessary and sufficient conditions for Nash implementation, Soc. Choice Welfare 8 (1991) 333-340.

T. Sjöström, Implementation by demand mechanisms Econ. Des. 1 (1996) 343-354.

Y. Sprumont, The division problem with single-peaked preferences: A characterization of the uniform allocation rule, Econometrica 89 (1991) 509-519.

K. Tadenuma, M. Toda, Implementable stable solutions to pure matching problems, Math. Soc. Sci. 35 (1998) 121-132.

W. Thomson, Consistent solutions to the problem of fair division when preferences are single-peaked, J. Econ. Theory 63 (1994a), 219-245.

W. Thomson, Cooperative models of bargainings, in: Aumann, R.J., Hart, S. (Eds.), Handbook of Game Theory with Economic Applications, vol. 2, Elsevier Science, Amsterdam, 1994b, pp. 1238-1284

W. Thomson, Fully allocating a commodity among agents with single-peaked preferences (May 2014). Available at: http://www.iser.osaka-u.ac.jp/collabo/20140524/School_Choice_Problems.pdf

H. Vartiainen, Subgame perfect implementation: a full characterization, J. Econ. Theory 133 (2007a) 111-126.

H. Vartiainen, Nash implementation and bargaining problem, Soc. Choice Welfare 29 (2007b) 333-351.

N. Yoshihara, Characterizations of Bargaining Solutions in Production Economies with Unequal Skills, J. Econ. Theory 108, (2003), 256-285.

6. Appendices

6.1 Appendix A: Proof of Theorem 1

Let the premises hold. Suppose that SCR $F : \Theta \twoheadrightarrow X$ satisfies unanimity.

Let us first show that F satisfies Condition μ^*(ii) with respect to $Y \subseteq X$ if it is partially-honestly Nash implemented by the mechanism $\Gamma = (M, g)$. Let Γ be the mechanism that partially-honestly Nash implements F. Then, $T_i^{\Gamma}\left(\bar{\theta}\right) \neq \varnothing$ for every pair $\left(i, \bar{\theta}\right) \in N \times \Theta$ and, moreover, it holds that

$$F\left(\bar{\theta}\right) = NA\left(\Gamma, \succcurlyeq^{\Gamma, \bar{\theta}, \bar{H}}\right), \quad \text{for every pair } \left(\bar{\theta}, \bar{H}\right) \in \Theta \times \mathcal{H}.$$

Let

$$Y = \{z \in X | g(m) = z \text{ for some } m \in M\}.$$

Thus, Y contains the range of F.

For what follows, fix any pair $(x, \theta) \in Y \times \Theta$ with $x \in F(\theta)$.

Given that $N \in \mathcal{H}$ by Assumption 2, there exists m such that $g(m) = x$ and that $m \in NE\left(\Gamma, \succcurlyeq^{\Gamma, \theta, N}\right)$. Thus, for every $i \in N$, let

$$C_i(\theta, x) = g(M_i, m_{-i}). \tag{A1}$$

Clearly, $x \in C_i(\theta, x) \subseteq L_i(\theta, x)$ and $C_i(\theta, x) \subseteq Y$. For what follows, fix also any pair $(\theta', H) \in \Theta \times \mathcal{H}$.

Given that $g(m) = x$ and $m \in NE\left(\Gamma, \succcurlyeq^{\Gamma, \theta, N}\right)$, define $S_i(\theta'; x, \theta)$ as follows:

$$S_i(\theta'; x, \theta) = g\left(T_i^{\Gamma}(\theta'), m_{-i}\right). \tag{A2}$$

Clearly, $S_i(\theta'; x, \theta) \neq \varnothing$ and, moreover, $S_i(\theta'; x, \theta) \subseteq C_i(\theta, x)$, establishing part (1)(a) of Condition μ^*(ii).

Next, we show that F satisfies part (1)(b) of Condition μ^*(ii). Take any $h \in H$ and suppose that $\theta' = \theta$. Also, suppose that $x \notin S_h(\theta'; x, \theta)$. It follows that $m_h \notin T_h^\Gamma(\theta)$. Suppose that there exists $z \in S_h(\theta'; x, \theta)$ such that $zR_h(\theta')x$. Given that $z \in S_h(\theta'; x, \theta)$, it follows that there exists $m'_h \in T^\Gamma(\theta)$ such that $g(m'_h, m_{-h}) = z$. Thus, $(m'_h, m_{-h}) \succ_h^{\Gamma,\theta} m$, which contradicts that $m \in NE(\Gamma, \succcurlyeq^{\Gamma,\theta,N})$. We thus conclude that $xP_h(\theta)z$ for all $z \in S_h(\theta; x, \theta)$.

Finally, we show that F satisfies part (2) of Condition μ^*(ii). Fix any pair $(i, y) \in N \times C_i(\theta, x)$. Then, given that $g(m) = x$ and $m \in NE(\Gamma, \succcurlyeq^{\Gamma,\theta,N})$, it follows that $(m'_i, m_{-i}) = y$ for some $m'_i \in M_i$. To economize on notation, we write m' for (m'_i, m_{-i}).

Suppose that $C_i(\theta, x) \subseteq L_i(\theta', y)$ and that $Y \subseteq L_i(\theta', y)$. Moreover, suppose that $y \notin F(\theta')$. By the partially-honest Nash implementability of F, we have that $m' \notin NE(\Gamma, \succcurlyeq^{\Gamma,\theta',H})$. Given that $g(M_k, m'_k) \subseteq L_k(\theta', y)$ for every $k \in N$, there is a deviant partially-honest individual $h \in H$ who can find it profitable unilaterally to deviate from m'. Thus, it is the case that $m'_h \notin T_h^\Gamma(\theta')$ and that there is $m''_h \in T_h^\Gamma(\theta')$ such that $g(T_h^\Gamma(\theta'), m'_{-h}) \cap I_h(\theta', y, Y) \neq \varnothing$. This shows that the intersection

$$S_i(\theta'; x, \theta) \bigcap I_i(\theta', y, Y) \tag{A3}$$

is not empty if $H = \{i\}$. Suppose that $H = \{i\}$. If $g(m''_i, m'_{-i}) = y$, then $y \in NA(\Gamma, \succcurlyeq^{\Gamma,\theta',H})$, which contradicts that $y \notin F(\theta')$. Thus, when $H = \{i\}$, we have that $y \notin S_i(\theta'; x, \theta)$ and that the intersection in (A3) is not empty, establishing part (2)(a) of Condition μ^*(ii).

Finally, let us show that F satisfies part (2)(b) of Condition μ^*(ii) as well. Thus, suppose that the deviant h identified above is different from i given that $i \notin H$ and, moreover, that $\theta' = \theta$. Recall that for this deviant individual it holds that $m'_h = m_h \notin T_h^\Gamma(\theta')$. Since $g(m) = x$ and $m \in NE(\Gamma, \succcurlyeq^{\Gamma,\theta,H})$, it holds that the deviant individual h cannot break it via any unilateral deviation.[31]

Assume, to the contrary, that $x \in S_j(\theta'; x, \theta)$ for all $j \in H$. Then, the deviant individual $h \in H$ identified above can find a strategy choice $\hat{m}_h \in T_h^\Gamma(\theta')$ such that $g(\hat{m}_h, m_{-h}) = x$. Since $m'_h = m_h \notin T_h^\Gamma(\theta')$, it follows that $\hat{m}_h \neq m_h$ and so the deviant individual h can break the strategy profile m from being a Nash equilibrium of $(\Gamma, \succcurlyeq^{\Gamma,\theta,H})$, which is a contradiction. Thus, it is the case that $x \notin S_j(\theta'; x, , \theta)$ for some individual $j \in H$, establishing part (2)(b) of Condition μ^*(ii).

In what follows, we show that F is partially-honestly Nash implementable if it satisfies Condition μ^*(ii) with respect to $Y \subseteq X$. To this end, suppose that F satisfies Condition μ^*(ii) with respect to $Y \subseteq X$.

Agent i's strategy choice space is defined by

$$M_i = (\Theta \cup \Omega) \times X \times N,$$

where Ω is a non-empty set such that its intersection with Θ is empty and that there is a bijection ϕ from Θ to Ω. Thus, individual i's strategy consists of an outcome in X, an

[31] Observe that $m \in NE(\Gamma, \succcurlyeq^{\Gamma,\theta,H})$ given that $m \in NE(\Gamma, \succcurlyeq^{\Gamma,\theta,N})$ and that $H \subseteq N$.

element of the set $\Theta \cup \Omega$ and an individual index $k \in N$. A typical strategy played by individual i is denoted by $m_i = (m_1^i, x^i, k^i)$ with m_1^i as a typical element of $\Theta \cup \Omega$. The strategy choice space of individuals is the product space $M = \prod_{i \in N} M_i$, with m as a typical strategy profile.

For every pair $(\bar{\theta}, x) \in \Theta \times Y$ with $x \in F(\bar{\theta})$, define individual p's set $\sigma_p(\bar{\theta}, x)$ as follows:

$$\sigma_p\left(\bar{\theta}, x\right) = \begin{cases} \left\{\phi\left(\bar{\theta}\right)\right\} \times \{x\} \times N & \text{if for one and only one individual } q \\ & \text{it holds that } x \in S_q\left(\bar{\theta}; x, \bar{\theta}\right) \text{ and } q \neq p; \\ \left\{\bar{\theta}\right\} \times \{x\} \times N & \text{otherwise.} \end{cases}$$

Write $\sigma(\bar{\theta}, x)$ for a typical profile of sets, that is, $\sigma(\bar{\theta}, x) = \left(\sigma_p(\bar{\theta}, x)\right)_{p \in N}$; and write $\sigma_1^p(\bar{\theta}, x)$ for a typical first coordinate of the set $\sigma_p(\bar{\theta}, x)$.

Definition 15 For every pair $(\bar{\theta}, x) \in \Theta \times Y$ with $x \in F(\bar{\theta})$ and every strategy profile $m \in M$,
(a) m is *consistent with* $\sigma(\bar{\theta}, x)$ if $m_i \in \sigma_i(\bar{\theta}, x)$ for every individual $i \in N$.
(b) m is *quasi-consistent with* $\sigma(\bar{\theta}, x)$ if $m_i \notin \sigma_i(\bar{\theta}, x)$ for one and only one individual $i \in N$.

The outcome function g is defined by the following three rules:

Rule 1: If m is consistent with $\sigma(\bar{\theta}, x)$, then $g(m) = x$.

Rule 2: If m is quasi-consistent with $\sigma(\bar{\theta}, x)$ and $m_i \notin \sigma_i(\theta, x)$ for some $i \in N$, then we can have three cases:

1. If $m_1^i = \theta^i = \bar{\theta}$ or $m_1^i = \phi(\theta^i) = \phi(\bar{\theta})$, then $g(m) = x$.

2. If $m_1^i = \theta^i \neq \bar{\theta}$ or $m_1^i = \phi(\theta^i) \neq \phi(\bar{\theta})$, then given that $\theta^i = (\phi^{-1} \circ \phi)(\theta^i)$: (a) $g(m) = x^i$ if $x^i \in S_i(\theta^i; x, \bar{\theta})$; (b) $g(m) = x^i$ if $x^i \in C_i(\theta^i, x) \backslash S_i(\theta^i; x, \bar{\theta})$ and $S_i(\theta^i; x, \bar{\theta}) \subseteq SL_i(\theta^i, x^i)$; (c) $g(m) = y$ if $x^i \in C_i(\bar{\theta}, x) \backslash S_i(\theta^i; x, \bar{\theta})$ and $y \in S_i(\theta^i; x, \bar{\theta}) \cap I_i(\theta^i, x^i, Y)$; (d) otherwise, $g(m) = z$ for some $z \in S_i(\theta^i; x, \bar{\theta})$.

3. If $m_1^i = \theta^i = \bar{\theta} \neq \sigma_1^i(\bar{\theta}, x)$, then: (a) $g(m) = x^i$ if $x^i \in S_i(\theta^i; x, \bar{\theta})$; (b) otherwise, $g(m) = z$ for some $z \in S_i(\theta^i; x, \bar{\theta})$.

Rule 3: Otherwise, a modulo game is played: divide the sum $\sum_{i \in N} k^i$ by n and identify the remainder, which can be either 0, 1, $\cdots$, or $n - 1$. The individual having the same index of the remainder is declared the winner of the game and the alternative implemented is the one she selects, with the convention that the winner is individual n if the remainder is 0.

By the above definitions, it follows that $\Gamma = (M, g)$ is a mechanism. We show that this Γ partially-honestly implements F.

For every individual i, define the truth-telling correspondence as follows:

$$T_i^\Gamma\left(\bar{\theta}\right) = \left\{\bar{\theta}\right\} \times X \times N, \quad \text{for every state } \bar{\theta} \in \Theta.$$

It is clear that the truth-telling correspondence is not empty, as required by Definition 4.

Thus, we are left to show that

$$F\left(\bar{\theta}\right)=NA\left(\Gamma,\succsim^{\Gamma,\bar{\theta},\bar{H}}\right),\quad \text{for every pair}\ \left(\bar{\theta},\bar{H}\right)\in\Theta\times\mathcal{H}.$$

To this end, fix any pair $(\theta,H)\in\Theta\times\mathcal{H}$.

Let us first show that if $x\in F(\theta)$, then there is a strategy profile $m\in NE\left(\Gamma,\succsim^{\Gamma,\theta,H}\right)$ with $g(m)=x$.

Suppose that $x\in F(\theta)$. Given that the profile $\sigma(\theta,x)$ is well-defined, take any strategy profile m that is consistent with $\sigma(\theta,x)$. Thus, m falls into *Rule 1* and $x=g(m)$. We claim that $m\in NE\left(\Gamma,\succsim^{\Gamma,\theta,H}\right)$.

To see this, first observe that any deviation of j will get her to an outcome in $C_j(\theta,x)$ by *Rule 2*, and so $g(M_j,m_{-j})\subseteq C_j(\theta,x)$. Since $C_j(\theta,x)\subseteq L_j(\theta,x)$, such deviations are not profitable if $j\notin H$. To see that such deviations are also not profitable for $j\in H$, we proceed according to whether $\sigma_1^j(\theta,x)=\{\theta\}$ or not.

Suppose that $\sigma_1^j(\theta,x)=\{\theta\}$. Then, given that $m_j\in T_j^\Gamma(\theta)$, there is no unilateral profitable deviation for this $j\in H$. Suppose that $\sigma_1^j(\theta,x)=\{\phi(\theta)\}$. Then, $m_j\notin T_j^\Gamma(\theta)$, and, moreover, $x\notin S_j(\theta;x,\theta)$, by definition of $\sigma_j(\theta,x)$. Note that by definition of *Rule 2.3* any deviation to a truthful strategy choice for θ by this j will result in outcomes of $S_j(\theta;x,\theta)$. Since part (1)(b) of Condition μ^*(ii) implies that $S_j(\theta;x,\theta)\subseteq SL_j(\theta,x)$, there is no unilateral profitable deviation for this j.

In summary, j's deviations from m are not profitable, and so $m\in NE\left(\Gamma,\succsim^{\Gamma,\theta,H}\right)$, as we sought

For the converse, suppose that $m\in NE\left(\Gamma,\succsim^{\Gamma,\theta,H}\right)$. We show that $g(m)\in F(\theta)$. To obtain a contradiction, we suppose that $g(m)\notin F(\theta)$. We proceed by cases.

Case 1: m falls into *Rule 3*

Given the richness of the strategy space we see that $X\subseteq g(M_j,m_{-j})$ for every j. Since $m\in NE\left(\Gamma,\succsim^{\Gamma,\theta,H}\right)$, it follows that $X\subseteq L_j(\theta,g(m))$ for every j. Given that the SCR F is unanimous, $g(m)\in F(\theta)$, which is a contradiction.

Case 2: m falls into *Rule 1*

Then, $g(m)=x$. If $\theta=\bar{\theta}$, there is an immediate contradiction. We thus suppose that $\theta\neq\bar{\theta}$. It follows that $m_h\notin T_h^\Gamma(\theta)$ for every $h\in H$. Fix any $h\in H$. This h can change m_h into $m_h'=\left(\theta,x,k^h\right)\in T_h^\Gamma(\theta)$ so as to induce *Rule 2.2* and to obtain an outcome $g(m_h',m_{-h})$ such that $g(m_h',m_{-h})\,I_h(\theta)\,x$. Therefore, $(m_h',m_{-h})\succ_h^{\Gamma,\theta}m$, which contradicts that $m\in NE\left(\Gamma,\succsim^{\Gamma,\theta,H}\right)$.

Case 3: m falls into *Rule 2.1*

Then, $g(m)=x$. Again, if $\theta=\bar{\theta}$, there is an immediate contradiction. We thus suppose that $\theta\neq\bar{\theta}$. Therefore, we have that $m_h\notin T_h^\Gamma(\theta)$ for every $h\in H$. Fix any $h\in H$. Suppose that $h=i$. This i can change m_i into $m_i'=(\theta,x,k^i)\in T_i^\Gamma(R)$ so as to induce *Rule 2.2* and to obtain $g(m_i',m_{-i})$ such that $g(m_i',m_{-i})\,I_i(\theta)\,x$. Therefore, $(m_i',m_{-i})\succ_i^{\Gamma,\theta}m$, which contradicts that $m\in NE\left(\Gamma,\succsim^{\Gamma,\theta,H}\right)$. Thus, suppose that $h\neq i$. This h can change m_h into

$m'_h = \left(\theta, x, k^h\right) \in T_h^{\Gamma}(R)$ so as to induce *Rule 3.* To attain x, h has only to adjust k^h so as to win the modulo game. Thus, $(m'_h, m_{-h}) \succ_h^{\Gamma,\theta} m$, which contradicts that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$.

Case 4: m falls into *Rule 2.3*

Then, $g(m) \in S_i\left(\theta^i; x, \bar{\theta}\right)$. Given that in this case it must hold that $m_i \notin \sigma_i\left(\bar{\theta}, x\right)$ and that $m_1^i = \theta^i = \bar{\theta} \neq \sigma_1^i\left(\bar{\theta}, x\right)$, it follows from the definition of the profile $\sigma\left(\bar{\theta}, x\right)$ and the fact that m falls into *Rule 2.3* that $x \in S_q\left(\bar{\theta}; x, \bar{\theta}\right)$ for one and only one individual $q \neq i$,[32] and so $g(m) \neq x$. We proceed according to whether $\theta = \bar{\theta}$ or not.

Sub-case 4.1: $\theta = \theta^i = \bar{\theta}$

Observe that $x \notin S_i(\theta; x, \theta)$ given that $\bar{\theta} \neq \sigma_1^i\left(\bar{\theta}, x\right)$. Suppose that $i \in H$. Given that i can attain x by inducing *Rule 1*, we have that $x \in g(M_i, m_{-i})$. Given that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$, it also holds that $g(m) R_i(\theta) x$. However, since $x \in C_i(\theta, x) \subseteq L_i(\theta, x)$ and, moreover, $g(m) \in S_i(\theta; x, \theta) \subseteq C_i(\theta, x)$, it follows that $x I_i(\theta) g(m)$, which contradicts part (1)(b) of Condition μ^*(ii). Therefore, it must be the case that $i \notin H$.

Suppose that $H \setminus \{q\} \neq \varnothing$. Then, take any $h \in H \setminus \{q\}$. Note that $m_h \notin T_h^{\Gamma}(\theta)$ given that $\sigma_1^h(\theta, x) = \{\phi(\theta)\}$. This h can change m_h into $m'_h = \left(\theta, g(m), k^h\right) \in T_h^{\Gamma}(\theta)$ so as to induce *Rule 3.* To attain $g(m)$, h has only to adjust k^h so as to win the modulo game. Thus, $(m'_h, m_{-h}) \succ_h^{\Gamma,\theta} m$, which contradicts that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$. Otherwise, let $H \setminus \{q\} = \varnothing$. Given that $H \neq \varnothing$ and that $i \notin H$, we are left to consider the case where $H = \{q\}$. Recall that $x \in S_q(\theta; x, \theta)$ for one and only one individual $q \neq i$.

Let us show that $Y \subseteq L_j(\theta, g(m))$ for every $j \neq i$. To this end, take any $z \in Y \setminus \{x\}$ and any $j \neq i$. By changing m_j into $m'_j = (\phi(\theta), z, k^j)$, j can induce *Rule 3.* To attain z, this j has only to adjust k^j so as to win the modulo game. To attain x, j has only to adjust k^j so as to allow $k \in N \setminus \{i, j\}$ to win the modulo game. Thus, we have that $Y \subseteq g(M_j, m_{-j})$ and so $Y \subseteq L_j(\theta, g(m))$ given that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$, as was to be shown.

Next, let us show that $C_i(\theta, x) \subseteq L_i(\theta, g(m))$. To attain x, i can change m_i into any strategy choice in $\sigma_i(\theta, x)$ and induce *Rule 1.* Thus, $x \in g(M_i, m_{-i})$. Since $g(m) \in C_i(\theta, x)$ and $C_i(\theta, x) \subseteq L_i(\theta, x)$ and, moreover, since $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$, we see that $x I_i(\theta) g(m)$. By transitivity, it follows from $C_i(\theta, x) \subseteq L_i(\theta, x)$ that $C_i(\theta, x) \subseteq L_i(\theta, g(m))$.

Since $Y \subseteq L_j(\theta, g(m))$ for every $j \neq i$ and $C_i(\theta, x) \subseteq L_i(\theta, g(m))$ and since, moreover, $H = \{q\}$ and $g(m) \notin F(\theta)$, part (2)(b) of Condition μ^*(ii) implies that $x \notin S_q(\theta; x, \theta)$, which is a contradiction.

Sub-case 4.2: $\theta \neq \theta^i = \bar{\theta}$

Note that $m_h \notin T_h^{\Gamma}(\theta)$ for every $h \in H$. Fix any $h \in H$. Suppose that $h = i$. This i can change m_i into $m'_i = (\theta, g(m), k^i) \in T_i^{\Gamma}(\theta)$ so as to induce *Rule 2.2* and to obtain $g(m'_i, m_{-i})$ such that $g(m'_i, m_{-i}) I_i(\theta) g(m)$.[33] Therefore, $(m'_i, m_{-i}) \succ_i^{\Gamma,\theta} m$, which contradicts that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$. Thus, suppose that $h \neq i$. This h can change m_h into $m'_h = \left(\theta, g(m), k^h\right) \in T_h^{\Gamma}(\theta)$ so as to induce *Rule 3.* To attain $g(m)$, h has only to adjust k^h so as to win the modulo game. Thus, $(m'_h, m_{-h}) \succ_h^{\Gamma,\theta} m$, which contradicts that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$.

[32] If $q = i$, then $\sigma_i^1\left(\bar{\theta}, x\right) = \left\{\bar{\theta}\right\}$, which is not the case.

[33] Note that if $g(m) \notin S_i\left(\theta; x, \bar{\theta}\right)$ and there does not exist any outcome $y \in C_i\left(\bar{\theta}, x\right)$ such that $y \in S_i\left(\theta; x, \bar{\theta}\right) \cap I_i(\theta, g(m), Y)$, then by part (b) of *Rule 2.2* it follows that $g(m'_i, m_{-i}) = g(m)$.

Case 5: m falls into *Rule 2.2*

Let us show that $Y \subseteq L_j(\theta, g(m))$ for every $j \neq i$. To this end, take any $z \in Y \backslash \{x\}$ and any $j \neq i$. By changing m_j into $m'_j = (\phi(\theta), z, k^j)$, j can induce *Rule 3*. To attain z, this j has only to adjust k^j so as to win the modulo game. To attain x, j has only to adjust k^j so as to allow $k \in N \backslash \{i, j\}$ to win the modulo game. Thus, we have that $Y \subseteq g(M_j, m_{-j})$ and so $Y \subseteq L_j(\theta, g(m))$ given that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$. Since the choice of j is arbitrary, we have that $Y \subseteq L_j(\theta, g(m))$ for each $j \neq i$.

Next, let us show that $C_i\left(\bar{\theta}, x\right) \subseteq L_i(\theta, g(m))$. To attain x, i can change m_i into $m'_i \in \sigma_i\left(\bar{\theta}, x\right)$ and induce *Rule 1*. Thus, $x \in g(M_i, m_{-i})$. Let us proceed according to whether $\theta = \bar{\theta}$ or not.

Suppose that $\theta = \bar{\theta}$. Since $g(m) \in C_i\left(\bar{\theta}, x\right)$ and $C_i\left(\bar{\theta}, x\right) \subseteq L_i\left(\bar{\theta}, x\right)$ and, moreover, since $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$, we see that $x I_i(\theta) g(m)$. By transitivity, it follows from $C_i(\theta, x) \subseteq L_i(\theta, x)$ that $C_i(\theta, x) \subseteq L_i(\theta, g(m))$.

Suppose thus that $\theta \neq \bar{\theta}$. Note that $g(M_i, m_{-i}) \subseteq L_i(\theta, g(m))$ given that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$. By changing m_i into $m'_i = (\phi(\theta), z^i, k^i)$ with $z^i \in S_i\left(\theta; x, \bar{\theta}\right)$, i can induce *Rule 2.2* and obtain this z^i via part (a) of the outcome function. It follows that i can also attain every outcome in $S_i\left(\theta; x, \bar{\theta}\right)$, establishing that $S_i\left(\theta; x, \bar{\theta}\right) \cup \{x\} \subseteq L_i(\theta, g(m))$. Assume, to the contrary, that there exists $w \in C_i\left(\bar{\theta}, x\right) \backslash S_i\left(\theta; x, \bar{\theta}\right)$ such that $w P_i(\theta) g(m)$. By transitivity, we see that $S_i\left(\theta; x, \bar{\theta}\right) \cup \{x\} \subseteq SL_i(\theta, w)$. Individual i can change m_i into $m'_i = (\phi(\theta), w, k^i)$ so as to obtain $g(m'_i, m_{-i}) = w$ by part (b) of *Rule 2.2*, which contradicts that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$. Thus, we conclude that $C_i\left(\bar{\theta}, x\right) \subseteq L_i(\theta, g(m))$.

(+) Suppose that $\sigma_1^h\left(\bar{\theta}, x\right) = \left\{\phi\left(\bar{\theta}\right)\right\}$ for some $h \in H \backslash \{i\}$ if the set $H \neq \{i\}$. Then, $m_h \notin T_h^\Gamma(\theta)$. By changing m_h into $m'_h = \left(\theta, g(m), k^h\right) \in T_h^\Gamma(\theta)$, h can induce *Rule 3*. To attain $g(m)$, this h has only to adjust k^h so as to win the modulo game. It follows that $(m'_h, m_{-h}) \succ_h^{\Gamma,\theta} m$, which contradicts that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$. Therefore, it must be the case that $\sigma_1^h(\theta, x) = \{\theta\}$ for every $h \in H \backslash \{i\}$ if the set $H \neq \{i\}$.

We distinguish the following cases: (1) $i \notin H$, (2) $H = \{i\}$ and (3) $i \in H$ and $H \cap (N \backslash \{i\}) \neq \varnothing$.

Sub-case 5.1: $i \notin H$

Suppose that $\bar{\theta} \neq \theta$. Then, $m_h \notin T_h^\Gamma(\theta)$ for each $h \in H$. Fix any h. The contradiction that $m \notin NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$ follows from the argument used for (+). Thus, in what follows, we assume that $\bar{\theta} = \theta$. We distinguish whether $x \in S_h(\theta; x, \theta)$ for some $h \in H$ or not.

Suppose that $x \in S_h(\theta; x, \theta)$ for some $h \in H$. Since $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$ and since, moreover, Assumption 2 holds, it follows that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,\{h\}}\right)$. Since $g(m) \in C_i(\theta, x) \subseteq L_i(\theta, g(m))$ and $Y \subseteq L_j(\theta, g(m))$ for every $j \neq i$ and since, moreover, $x \in S_h(\theta; x, \theta)$ and $\{h\} \in \mathcal{H}$, part (2)(b) of Condition μ^*(ii) implies that $g(m) \in F(\theta)$, which is a contradiction.

Suppose that $x \notin S_h(\theta; x, \theta)$ for every $h \in H$. Since $H \neq \{i\}$, it follows from (+) that $\sigma_1^h(\theta, x) = \{\theta\}$ for every $h \in H$. This implies that $m_h \in T_h^\Gamma(\theta)$ for every $h \in H$. Suppose that $x \in S_p(\theta; x, \theta)$ for some $p \in N \backslash H$ with $p \neq i$. Since $g(m) \in C_i(\theta, x) \subseteq L_i(\theta, g(m))$ and $Y \subseteq L_j(\theta, g(m))$ for every $j \neq i$ and since, moreover, $x \in S_p(\theta; x, \theta)$ and $\{p\} \in \mathcal{H}$, part (2)(b) of Condition μ^*(ii) implies that $g(m) \in F(\theta)$, which is a contradiction. Therefore, we have established that $x \notin S_j(\theta; x, \theta)$ for every $j \neq i$. Furthermore, given that $H \neq \varnothing$ and that

$i \notin H$ and given that $m_h \in T_h^\Gamma(\theta)$ for every $h \in H$, it cannot be that $x \in S_q(\theta; x, \theta)$ for one and only one individual $q = i$. It follows that $x \notin S_i(\theta; x, \theta)$. By Assumption 2, it also holds that $\{i\} \in \mathcal{H}$. Since $\bar{\theta} = \theta$, part (1)(b) of Condition μ^*(ii) implies that $S_i(\theta; x, \theta) \subseteq SL_i(\theta, x)$ for i when $\{i\} \in \mathcal{H}$ is considered. Now, since $g(m) \in C_i(\theta, x) \subseteq L_i(\theta, x)$ and since $x \in C_i(\theta, x) \subseteq L_i(\theta, g(m))$, it follows that $xI_i(\theta)\, g(m)$, and so $S_i(\theta; x, \theta) \subseteq SL_i(\theta, g(m))$, by transitivity. However, since $g(m) \in C_i(\theta, x) \subseteq L_i(\theta, g(m))$ and $Y \subseteq L_j(\theta, g(m))$ for every $j \neq i$ and since, moreover, $\{i\} \in \mathcal{H}$ and $g(m) \notin F(\theta)$, part (2)(a) of Condition μ^*(ii) implies that $S_i(\theta; x, \theta) \cap I_i(\theta, g(m), Y) \neq \varnothing$, which contradicts that $S_i(\theta; x, \theta) \subseteq SL_i(\theta, g(m))$.

Sub-case 5.2: $H = \{i\}$

Then, $m_i \in T_i^\Gamma(\theta)$. To see this, assume, to the contrary, that $m_i \notin T_i^\Gamma(\theta)$. We proceed according to whether $\sigma_1^i(\bar{\theta}, x) = \{\theta\}$ or not.

Suppose that $\sigma_1^i(\bar{\theta}, x) = \{\theta\}$. Then, $\theta = \bar{\theta}$. To attain x, i can change m_i into $m_i' = (\theta, x, k^i) \in T_i^\Gamma(\theta)$ and induce *Rule 1*. Since $H = \{i\}$, it follows that $(m_i', m_{-i}) \succ_i^{\Gamma,\theta} m$, which contradicts that $m \in NE(\Gamma, \succcurlyeq^{\Gamma,\theta,H})$.

Suppose that $\sigma_1^i(\bar{\theta}, x) \neq \{\theta\}$. We proceed according to whether $\theta = \bar{\theta}$ or not. Suppose that $\theta \neq \bar{\theta}$. Then, by changing m_i into $m_i' = (\theta, g(m), k^i) \in T_i^\Gamma(\theta)$, i can induce *Rule 2.2*. Since $g(m_i', m_{-i})\, I_i(\theta)\, g(m)$, it follows that $(m_i', m_{-i}) \succ_i^{\Gamma,\theta} m$, which contradicts that $m \in NE(\Gamma, \succcurlyeq^{\Gamma,\theta,H})$. Suppose that $\theta = \bar{\theta}$. Then, $\sigma_1^i(\bar{\theta}, x) = \{\phi(\bar{\theta})\}$ given that $\sigma_1^i(\bar{\theta}, x) \neq \{\theta\}$, and so it must be the case that $x \in S_q(\bar{\theta}; x, \theta)$ for one and only one individual $q \neq i$ and, consequently, that $\sigma_1^p(\bar{\theta}, x) = \{\phi(\bar{\theta})\}$ for every $p \neq q$.[34] Since $g(m) \in C_i(\theta, x) \subseteq L_i(\theta, g(m))$ and $Y \subseteq L_j(\theta, g(m))$ for every $j \neq i$ and since, moreover, $g(m) \notin F(\theta)$, part (2)(a) of Condition μ^*(ii) implies for $\{i\} = H$ that $g(m) \notin S_i(\theta; x, \theta)$ and that there is $z \in S_i(\theta; x, \theta) \cap I_i(\theta, g(m), Y)$. Thus, by changing m_i into $m_i' = (\theta, z, k^i) \in T_i^\Gamma(\theta)$, i can induce *Rule 2.3* and obtain $g(m_i', m_{-i}) = z$, which contradicts that $m \in NE(\Gamma, \succcurlyeq^{\Gamma,\theta,H})$. We conclude that $m_i \in T_i^\Gamma(\theta)$.

Since $g(m) \in C_i(\bar{\theta}, x) \subseteq L_i(\theta, g(m))$ and $Y \subseteq L_j(\theta, g(m))$ for every $j \neq i$ and since, moreover, either $g(m) \in S_i(\theta; x, \bar{\theta})$ or $S_i(\theta; x, \bar{\theta}) \subseteq SL_i(\theta, g(m))$, part (2)(a) of Condition μ^*(ii) implies that $g(m) \in F(\theta)$, which is a contradiction.

Sub-case 5.3: $i \in H$ and $H \cap (N \backslash \{i\}) \neq \varnothing$

Then, from the same arguments used for *Sub-case 5.1*, one can see that $\bar{\theta} = \theta$. It also follows from (+) that $\sigma_1^h(\theta, x) = \theta$ for every $h \in H \backslash \{i\}$, and so $m_h \in T_h^\Gamma(\theta)$ for every $h \in H \backslash \{i\}$. Note that $m_i \notin T_i^\Gamma(\theta)$ given that $m_1^i = \theta^i \neq \theta$ or $m_1^i = \phi(\theta^i) \neq \phi(\theta)$. Also, note that given that $g(m) \in C_i(\theta, x) \subseteq L_i(\theta, x)$ and that $C_i(\theta, x) \subseteq L_i(\theta, g(m))$ we have that $g(m)\, I_i(\theta)\, x$. We proceed according to whether $\sigma_1^i(\theta, x) = \theta$ or not.

Suppose that $\sigma_1^i(\theta, x) = \theta$. Then, by changing m_i into $m_i' = (\theta, x, k^i) \in T_i^\Gamma(\theta)$, i can induce *Rule 1* and obtain $g(m_i', m_{-i}) = x$. Given that $g(m)\, I_i(\theta)\, x$, it follows that $(m_i', m_{-i}) \succ_i^{\Gamma,\theta} m$, which contradicts that $m \in NE(\Gamma, \succcurlyeq^{\Gamma,\theta,H})$.

Suppose that $\sigma_1^i(\theta, x) \neq \theta$. Thus, $\sigma_1^i(\theta, x) = \phi(\theta)$, and so there exists exactly one $q \neq i$ such that $x \in S_q(\theta; x, \theta)$ and, consequently, $\sigma_1^p(\theta, x) = \phi(\theta)$ for every $p \neq q$. Given that $i \in H$ and $H \cap (N \backslash \{i\}) \neq \varnothing$ and given that $m_h \in T_h^\Gamma(\theta)$ for every $h \in H \backslash \{i\}$, it needs to be the case that $H = \{q, i\}$.

[34] Again, if $q = i$, then $\sigma_i^1(\bar{\theta}, x) = \{\bar{\theta}\}$, which is not the case.

Part (1)(b) of Condition μ^*(ii) implies that $S_i(\theta; x, \theta) \subseteq SL_i(\theta, x)$. Furthermore, given that $g(m) I_i(\theta) x$, it also follows from transitivity that $S_i(\theta; x, \theta) \subseteq SL_i(\theta, g(m))$. Since $m \in NE\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$, and since, moreover, Assumption 2 holds, $m \in NE\left(\Gamma, \succcurlyeq^{\Gamma,\theta,\{i\}}\right)$. Since the premises of part (2)(a) of Condition μ^*(ii) are met, we have that $S_i(\theta; x, \theta) \cap I_i(\theta, g(m), Y) \neq \varnothing$, which is a contradiction. ■

6.2 Appendix B: Proof of Theorem 3, "if" part

Let the premises hold. Let $F : \Theta \twoheadrightarrow X$ be a SCR. In what follows, we show that F is partially-honestly Nash implementable if it satisfies Condition μ^*. To this end, suppose that F satisfies Condition μ^* with respect to $Y \subseteq X$ with $F(\Theta) \subseteq Y$.

Let us define a mechanism $\Gamma = (M, g)$. First, individual i's strategy choice space is defined by:

$$M_i = (\Theta \cup \Omega) \times Y \times N \times N,$$

where Ω is a non-empty set such that its intersection with Θ is empty and that there is a bijection ϕ from Θ to Ω. Thus, individual i's strategy consists of an outcome in X, an element of the set $\Theta \cup \Omega$ and a pair of individual indices in $N \times N$. A typical strategy played by individual i is denoted by $m_i = (m_1^i, x^i, k^i, z^i)$ with m_1^i as a typical element of $\Theta \cup \Omega$. The strategy choice space of individuals is the product space $M = \prod_{i \in N} M_i$, with m as a typical strategy profile.

For every individual i, define the truth-telling correspondence as follows:

$$T_i^{\Gamma}\left(\bar{\theta}\right) = \left\{\bar{\theta}\right\} \times Y \times N \times N, \quad \text{for every state } \bar{\theta} \in \Theta.$$

It is clear that the truth-telling correspondence is not empty for every individual i, as required by Definition 4.

For every strategy choice profile $m \in M$, define the sets of individuals $T(m)$ as follows:

$$T(m) = \left\{i \in N | m_i \in T_i^{\Gamma}\left(\theta^i\right) \text{ for some } \theta^i \in \Theta\right\} \tag{A4}$$

Thus, if the strategy choice m_i of individual i is truthful for some state θ^i, then this individual is an element of the set $T(m)$.

For every pair $\left(\bar{\theta}, x\right) \in \Theta \times Y$ with $x \in F\left(\bar{\theta}\right)$, define individual p's set $\sigma_p\left(\bar{\theta}, x\right)$ as follows:

$$\sigma_p\left(\bar{\theta}, x\right) = \begin{cases} \left\{\bar{\theta}\right\} \times \{x\} \times \{1\} \times \{1\} & \text{if } x \in S_p\left(\bar{\theta}; x, \bar{\theta}\right); \\ \left\{\phi\left(\bar{\theta}\right)\right\} \times \{x\} \times \{1\} \times \{1\} & \text{otherwise.} \end{cases}$$

Write $\sigma\left(\bar{\theta}, x\right)$ for a typical profile of sets, that is, $\sigma\left(\bar{\theta}, x\right) = \left(\sigma_p\left(\bar{\theta}, x\right)\right)_{p \in N}$; and write $\sigma_1^p\left(\bar{\theta}, x\right)$ for a typical first coordinate of the set $\sigma_p\left(\bar{\theta}, x\right)$.

Definition 16 For every pair $\left(\bar{\theta}, x\right) \in \Theta \times Y$ with $x \in F\left(\bar{\theta}\right)$ and every strategy profile $m \in M$,
(a) m is *consistent with* $\sigma\left(\bar{\theta}, x\right)$ if $m_i \in \sigma_i\left(\bar{\theta}, x\right)$ for every individual $i \in N$.
(b) m is *quasi-consistent with* $\sigma\left(\bar{\theta}, x\right)$ if $m_i \notin \sigma_i\left(\bar{\theta}, x\right)$ for one and only one individual $i \in N$.

The outcome function g is defined by the following three rules:

Rule 1: If m is consistent with $\sigma\left(\bar{\theta}, x\right)$, then $g\left(m\right) = x$.

Rule 2: If m is quasi-consistent with $\sigma\left(\bar{\theta}, x\right)$ and $m_i \notin \sigma_i\left(\bar{\theta}, x\right)$ for some $i \in N$, then we can have three cases:

1. If $m_1^i = \theta^i = \bar{\theta}$ or $m_1^i = \phi\left(\theta^i\right) = \phi\left(\bar{\theta}\right)$, then $g\left(m\right) = x$.

2. If $m_1^i = \theta^i \neq \bar{\theta}$ or $m_1^i = \phi\left(\theta^i\right) \neq \phi\left(\bar{\theta}\right)$, then

$$g\left(m\right) = \begin{cases} x^i & \text{if } x^i \in S_i\left(\theta^i; x, \overline{\theta}\right); \\ x^i & \begin{array}{c}\text{if } x^i \in C_i\left(\overline{\theta}, x\right) \backslash S_i\left(\theta^i; x, \overline{\theta}\right),\ S_i\left(\theta^i; x, \overline{\theta}\right) \subseteq SL_i\left(\theta^i, x^i\right), \\ \text{and } m_i \in \left\{\phi\left(\theta^i\right)\right\} \times \left\{x^i\right\} \times N \times N;\end{array} \\ y & \begin{array}{c}\text{if } x^i \in C_i\left(\overline{\theta}, x\right) \backslash S_i\left(\theta^i; x, \overline{\theta}\right), \\ \text{and } \exists y \in S_i\left(\theta^i; x, \overline{\theta}\right) \cap I_i\left(\theta^i, x^i, Y\right);\end{array} \\ z \in S_i\left(\theta^i; x, \overline{\theta}\right) & \text{otherwise.} \end{cases}$$

3. If $m_1^i = \theta^i = \bar{\theta} \neq \sigma_1^i\left(\bar{\theta}, x\right)$, then: (a) $g\left(m\right) = x^i$ if $x^i \in S_i\left(\theta^i; x, \overline{\theta}\right)$; (b) otherwise, $g\left(m\right) = z$ for some $z \in S_i\left(\theta^i; x, \overline{\theta}\right)$.

Rule 3: Otherwise, a modulo game is played: divide the sum $\sum_{i \in N} k^i$ by n and identify the remainder, which can be either 0, 1, $\cdots$, or $n-1$.[35] The individual having the same index of the remainder is declared the winner of the game such that n is the winner when the remainder is 0. If individual i wins the modulo game, then we can have two cases:

1. If the set $T\left(m\right)$ is not empty, then: (a) $g\left(m\right) = x^i$ if $x^i \in Y_{T(m)}\left(\theta^{T(m)}\right)$; (b) otherwise, $g\left(m\right) = z$ for some $z \in Y_{T(m)}\left(\theta^{T(m)}\right)$.

2. If the set $T\left(m\right)$ is empty, then $g\left(m\right) = x^i$.

By the above definitions, it follows that $\Gamma = \left(M, g\right)$ is a mechanism. We show that this Γ partially-honestly implements F. Thus, we are left to show that

$$F\left(\bar{\theta}\right) = NA\left(\Gamma, \succcurlyeq^{\Gamma, \bar{\theta}, \bar{H}}\right) \quad \text{for every pair } \left(\bar{\theta}, \bar{H}\right) \in \Theta \times \mathcal{H}.$$

To this end, fix any pair $\left(\theta, H\right) \in \Theta \times \mathcal{H}$.

[35] Note that only the first entry of the pair $\left(k^i, z^i\right) \in N \times N$ is considered in order to compute the sum $\sum_{i \in N} k^i$.

The proof of the assertion that if $x \in F(\theta)$, then there exists a strategy profile m such that $g(m) = x$ and that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$ can be found in the proof of Theorem 1 - Appendix A.

For the converse, suppose that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$. We show that $g(m) \in F(\theta)$. To obtain a contradiction, we suppose that $g(m) \notin F(\theta)$. We proceed by cases.

Case 1: m falls into *Rule 1*

Then, $g(m) = x$. Note that $\theta \neq \bar{\theta}$ given that $x \in F\left(\bar{\theta}\right) \backslash F(\theta)$. So, $m_h \notin T_h^{\Gamma}(\theta)$ for all $h \in H$. Fix any $i \in N$. We first show that $C_i\left(\bar{\theta}, x\right) \subseteq L_i(\theta, g(m))$. Take any $x^i \in S_i\left(\theta; x, \bar{\theta}\right)$ such that $x^i \neq x$. This i can change m_i into $m_i' = (\theta, x^i, 1, 1)$ so as to obtain $g(m_i', m_{-i}) = x^i$, by *Rule 2.2*. Therefore, $S_i\left(\theta; x, \bar{\theta}\right) \cup \{x\} \subseteq g(M_i, m_{-i})$, and so $S_i\left(\theta; x, \bar{\theta}\right) \cup \{x\} \subseteq L_i(\theta, g(m))$, given that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$. Suppose that there exists $x^i \in C_i\left(\bar{\theta}, x\right) \backslash S_i\left(\theta; x, \bar{\theta}\right)$ such that $x^i P_i(\theta)\, g(m)$. By transitivity, $S_i\left(\theta; x, \bar{\theta}\right) \cup \{x\} \subseteq SL_i(\theta, x^i)$. Then, i can change m_i into $m_i' = (\phi(\theta), x^i, 1, 1)$ so as to obtain $g(m_i', m_{-i}) = x^i$, by *Rule 2.2*, which contradicts that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$. Since the choice of i is arbitrary, we have that $C_i\left(\bar{\theta}, x\right) \subseteq L_i(\theta, g(m))$ for all i.

Since $g(m) = x \notin F(\theta)$, Condition μ^*(i) implies that there exist $h \in H$ and $z^{(h)} \in S_h\left(\theta; x, \bar{\theta}\right) \cap I_h(\theta, x, Y)$. Then, by *Rule 2.2* of the mechanism, $g(m_h', m_{-h}) = z^{(h)}$ holds when this h changes m_h into $m_h' = \left(\theta, z^{(h)}, 1, 1\right) \in T_h^{\Gamma}(\theta)$, which contradicts the assumption that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$.

Case 2: m falls into *Rule 2.1*

Then, $g(m) = x$. Note that $\theta \neq \bar{\theta}$ given that $x \in F\left(\bar{\theta}\right) \backslash F(\theta)$. So, $m_h \notin T_h^{\Gamma}(\theta)$ for all $h \in H$. By the same reasoning used for *Case 1*, one can see that $C_i\left(\bar{\theta}, x\right) \subseteq L_i(\theta, g(m))$.

Note that $x \in S_k\left(\bar{\theta}; x, \bar{\theta}\right)$ for each $k \in T(m) \setminus \{i\}$, by definition of $\sigma_k\left(\bar{\theta}, x\right)$ and by definition of $T(m)$ given in (A4), if $T(m) \setminus \{i\} \neq \varnothing$. Thus, property (II) of Condition μ^*(iii)(B) is met if $T(m) \setminus \{i\} \neq \varnothing$. Also, note that this property is vacuously satisfied when $T(m) \setminus \{i\} = \varnothing$. Therefore, in what follows, we will assume that this property of Condition μ^*(iii)(B) is met.

Note that if $i \in T(m)$, then $m_1^i = \theta^i = \bar{\theta}$, by definition of *Rule 2.1* and by definition of $T(m)$ given in (A4). Also, note that $\sigma_1^i\left(\bar{\theta}, x\right) = \bar{\theta}$ if $i \in T(m)$; otherwise, if $\sigma_1^i\left(\bar{\theta}, x\right) = \phi\left(\bar{\theta}\right)$ and $i \in T(m)$, then m would fall into *Rule 2.3*, which is not the case. Thus, $x = g(m) \in S_i\left(\bar{\theta}; x, \bar{\theta}\right)$ if $i \in T(m)$. This implies that property (I) of Condition μ^*(iii)(B) is met if $i \in T(m)$. Also, note that this property is vacuously satisfied when $i \notin T(m)$. Therefore, in what follows, we will assume that this property of Condition μ^*(iii)(B) is met as well.

Given that both property (I) and property (II) are satisfied, Condition μ^*(iii)(B) implies that $g(m) \in Y_{T(m)}\left(\theta^{T(m)}\right)$. Recall that $Y_{T(m)}\left(\theta^{T(m)}\right) = Y$ if $T(m) = \varnothing$, by Condition μ^*(iii)(A)(0).

We proceed according to whether $i \in H$ or not.

Sub-case 2.1: $i \in H$

Given that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$ and that i can induce *Rule 2.2* by changing $m_i \notin T_i^{\Gamma}(\theta)$ into $m_i' \in T_i^{\Gamma}(\theta)$, because $\theta \neq \bar{\theta}$, it does not exist any $x^i \in S_i\left(\theta; x, \bar{\theta}\right)$ such that

$x^i R_i(\theta)\, g(m)$. Thus, $S_i\left(\theta; x, \overline{\theta}\right) \subseteq SL_i(\theta, x)$. We proceed according to whether $m_1^i = \theta^i = \bar{\theta}$ or $m_1^i = \phi\left(\theta^i\right) = \phi\left(\bar{\theta}\right)$.

Sub–sub-case 2.1.1: $m_1^i = \phi\left(\theta^i\right) = \phi\left(\bar{\theta}\right)$

Then, $i \notin T(m)$, by definition of $T(m)$ given in (A4). Suppose that $T(m) \setminus \{i\} \neq \varnothing$. Since $S_i\left(\theta; x, \overline{\theta}\right) \subseteq SL_i(\theta, x)$ and $T(m) \setminus \{i\} \neq \varnothing$, Condition μ^*(iii)(B) implies that either statement (α) or statement (β) holds.

Case α: Statement (α) of Condition μ^*(iii)(B) holds.

Then, there exists an $h \in H \setminus \{i\}$ who can change $m_h \notin T_h^{\Gamma}(\theta)$ into $m_h' = \left(\theta, z^{(h)}, k^h, 2\right) \in T_h^{\Gamma}(\theta)$ so as to induce *Rule 3*. To obtain $z^{(h)} \in Y_{T\left(m_{-h}, m_h'\right)}\left(\theta^{T\left(m_{-h}, m_h'\right)}\right) \cap I_h(\theta, x, Y)$, h has only to adjust k^h by which she becomes the winner of the modulo game, which contradicts that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$.

Case β: Statement (β) of Condition μ^*(iii)(B) holds.

Then, there exists an $\ell \in N \setminus \{i\}$ who can change m_ℓ into $m_\ell' = \left(\phi(\theta), z^{(\ell)}, k^\ell, 2\right)$ so as to induce *Rule 3*. To obtain $z^{(\ell)} \in Y_{T\left(m_{-h}, m_h'\right)}\left(\theta^{T\left(m_{-h}, m_h'\right)}\right)$, with $\ell \notin T(m_{-h}, m_h')$ by definition in (A4), such that $z^{(\ell)} P_\ell(\theta)\, g(m)$, ℓ has only to adjust k^ℓ by which she becomes the winner of the modulo game, which contradicts that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$.

Suppose that $T(m) \setminus \{i\} = \varnothing$. Thus, $T(m) = \varnothing$.

(*) Since $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$ and $i \in H$ and since, moreover, $\{i\} \in \mathcal{H}$, by Assumption 2, then $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,\{i\}}\right)$. Note that every $\ell \neq i$ can induce *Rule 3.2* and win the modulo game. Since $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,\{i\}}\right)$, we see that $Y \subseteq L_\ell(\theta, g(m))$ for each $\ell \neq i$. Then, since $S_i\left(\theta; x, \overline{\theta}\right) \subseteq SL_i(\theta, g(m))$ and $\{i\} \in \mathcal{H}$, and since, moreover, $C_i\left(\overline{\theta}, x\right) \subseteq L_i(\theta, g(m))$, Condition μ^*(ii)(2)(a) implies that $g(m) \in F(\theta)$, which is a contradiction.

Sub–sub-case 2.1.2: $m_1^i = \theta^i = \bar{\theta}$

Then, $i \in T(m)$. Since $S_i\left(\theta; x, \overline{\theta}\right) \subseteq SL_i(\theta, x)$ and $T(m) \neq \varnothing$, Condition μ^*(iii)(B) implies that either statement (α) or statement (β) holds. The contradiction that $m \notin NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$ follows either from the argument used for *Case* α or from the argument used for *Case* β.

Sub-case 2.2: $i \notin H$

Condition μ^*(iii)(B) implies that either statement (α) or statement (β) holds. Let us proceed according to whether $T(m) \neq \varnothing$ or not. Suppose that $T(m) \neq \varnothing$. The contradiction that $m \notin NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$ follows either from the argument used for *Case* α or from the argument used for *Case* β. Therefore, let us suppose that $T(m) = \varnothing$.

(**) Then, every $\ell \neq i$ can induce *Rule 3.2* and win the modulo game. Since $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$, we see that $Y \subseteq L_\ell(\theta, g(m))$ for each $\ell \neq i$. It follows that only statement (α) of Condition μ^*(iii)(B) can hold. Thus, the same type of argument used for *Case* α shows that $m \notin NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$, which is a contradiction.

Case 3: m corresponds to *Rule 2.2*

We first show that $C_i\left(\overline{\theta},x\right)\subseteq L_i\left(\theta,g\left(m\right)\right)$. Note that i can induce *Rule 1* and obtain x. Also, note that $g\left(m\right)R_i\left(\theta\right)x$ given that $g\left(m\right)\in NA\left(\Gamma,\succsim^{\Gamma,\theta,H}\right)$. We proceed according to whether $\theta=\overline{\theta}$ or not. Suppose that $\theta=\overline{\theta}$. Then, given that $x\in F\left(\theta\right)$, Condition μ^{*}(ii) implies that $C_i\left(\overline{\theta},x\right)\subseteq L_i\left(\overline{\theta},x\right)$, and so $xI_i\left(\overline{\theta}\right)g\left(m\right)$, given that $g\left(m\right)\in C_i\left(\overline{\theta},x\right)$. Assume, to the contrary, that there exists $x^{i}\in C_i\left(\overline{\theta},x\right)$ such that $x^{i}P_i\left(\overline{\theta}\right)g\left(m\right)$. By transitivity, $x^{i}P_i\left(\overline{\theta}\right)x$, which contradicts that $C_i\left(\overline{\theta},x\right)\subseteq L_i\left(\overline{\theta},x\right)$. Suppose that $\theta\neq\overline{\theta}$. By same the reasoning used in *Case 1*, one can obtain that $C_i\left(\overline{\theta},x\right)\subseteq L_i\left(\theta,g\left(m\right)\right)$.

By the same type of argument used for *Case 2*, one can see that property (II) of Condition μ^{*}(iii)(B) is satisfied. In what follows, we assume that this property is met. By definition of g for the case of *Rule 2.2*, one can see that $g\left(m\right)\in S_i\left(\theta^{i};x,\bar{\theta}\right)$ if $i\in T\left(m\right)$.[36] Thus, in what follows, we assume that property (I) of Condition μ^{*}(iii)(B) is satisfied as well. Given that both property (I) and property (II) are satisfied, Condition μ^{*}(iii)(B) implies that $g\left(m\right)\in Y_{T(m)}\left(\theta^{T(m)}\right)$. Recall that $Y_{T(m)}\left(\theta^{T(m)}\right)=Y$ if $T\left(m\right)=\varnothing$, by Condition μ^{*}(iii)(A)(0).

We proceed according to whether $i\in H$ or not.

Sub-case 3.1: $i\notin H$

Condition μ^{*}(iii)(B) implies that either statement (α) or statement (β) holds. Note that if statement (α) should hold in the case where $\bar{\theta}=\theta$, then the deviant $h\in H\backslash\left\{i\right\}$ of Case α is such that $h\notin T\left(m\right)$; otherwise, Condition μ^{*}(iii)(B) is violated. Consequently, in the case where $\bar{\theta}=\theta$, it holds that $m_h\notin T_h^{\Gamma}\left(\theta\right)$ for the deviant $h\in H\backslash\left\{i\right\}$ of Case α. Also, note that $m_h\notin T_h^{\Gamma}\left(\theta\right)$ for all $h\in H\backslash\left\{i\right\}$ if $\theta\neq\overline{\theta}$. Thus, in either case, $m_h\notin T_h^{\Gamma}\left(\theta\right)$ for the deviant $h\in H\backslash\left\{i\right\}$ of Case α.

Let us proceed according to whether $T\left(m\right)=\varnothing$ or not. The contradiction that $m\notin NE\left(\Gamma,\succsim^{\Gamma,\theta,H}\right)$ follows from the argument used for *Sub-case 2.2*.

Sub-case 3.2: $i\in H$

We proceed according to whether $\theta\neq\overline{\theta}$ or not.

Sub-sub-case 3.2.1: $\theta\neq\overline{\theta}$

First, observe that $m_h\notin T_h^{\Gamma}\left(\theta\right)$ for all $h\in H\backslash\left\{i\right\}$. Second, observe that $m_i\in T_i^{\Gamma}\left(\theta\right)$ or $S_i\left(\theta;x,\bar{\theta}\right)\cap I_i\left(\theta,g\left(m\right),Y\right)=\varnothing$. To see it, suppose that $m_i\notin T_i^{\Gamma}\left(\theta\right)$ and that there exists $z\in S_i\left(\theta;x,\bar{\theta}\right)\cap I_i\left(\theta,g\left(m\right),Y\right)$. Then, i can change $m_i\notin T_i^{\Gamma}\left(\theta\right)$ into $m_i'=\left(\theta,z,2,2\right)\in T_i^{\Gamma}\left(\theta\right)$. Since $\theta\neq\overline{\theta}$, and so $m_i'\notin\sigma_i\left(\bar{\theta},x\right)$, the profile $\left(m_{-i},m_i'\right)$ falls into *Rule 2.2*. Since $z\in S_i\left(\theta;x,\bar{\theta}\right)$, *Rule 2.2* implies that $g\left(m_{-i},m_i'\right)=z$. It follows that $\left(m_i',m_{-i}\right)\succ_i^{\Gamma,\theta}m$, which contradicts that $m\in NE\left(\Gamma,\succsim^{\Gamma,\theta,H}\right)$.

Suppose that $m_i\in T_i^{\Gamma}\left(\theta\right)$, and so $\theta^{i}=\theta$ and $i\in T\left(m\right)$. Condition μ^{*}(iii)(B) implies that either statement (α) or statement (β) holds. The contradiction that $m\notin NE\left(\Gamma,\succsim^{\Gamma,\theta,H}\right)$ follows either from the argument used for *Case* α or from the argument used for *Case* β.

Suppose that $S_i\left(\theta;x,\bar{\theta}\right)\cap I_i\left(\theta,g\left(m\right),Y\right)=\varnothing$ and that $m_i\notin T_i^{\Gamma}\left(\theta\right)$. This implies that $S_i\left(\theta;x,\bar{\theta}\right)\subseteq SL_i\left(\theta,g\left(m\right)\right)$, given that $S_i\left(\theta;x,\bar{\theta}\right)\subseteq C_i\left(\bar{\theta},x\right)$, by part (1)(a) of Condition μ^{*}(ii), and that $C_i\left(\bar{\theta},x\right)\subseteq L_i\left(\theta,g\left(m\right)\right)$.

[36] By definition of $T\left(m\right)$ given in (A4), $m_1^{i}=\theta^{i}$ if $i\in T\left(m\right)$.

Let us proceed according to whether $m_1^i = \theta^i \neq \bar{\theta}$ or $m_1^i = \phi\left(\theta^i\right) \neq \phi\left(\bar{\theta}\right)$.

Case ψ: $m_1^i = \phi\left(\theta^i\right) \neq \phi\left(\bar{\theta}\right)$

Then, $i \notin T(m)$. Suppose that $T(m) \neq \varnothing$. Since $S_i\left(\theta; x, \bar{\theta}\right) \subseteq SL_i(\theta, g(m))$ and $T(m) \neq \varnothing$, Condition μ^*(iii)(B) implies that either statement (α) or statement (β) holds. The contradiction that $m \notin NE\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$ follows either from the argument used for *Case* α or from the argument used for *Case* β. Thus, suppose $T(m) = \varnothing$. Then, the same type of argument used in (*) shows that $g(m) \in F(\theta)$, which is a contradiction.

Case δ: $m_1^i = \theta^i \neq \bar{\theta}$.

So, $i \in T(m)$. Since $S_i\left(\theta; x, \bar{\theta}\right) \subseteq SL_i(\theta, g(m))$ and $T(m) \neq \varnothing$, Condition μ^*(iii)(B) implies that either statement (α) or statement (β) holds. The contradiction follows as before, considering only the case $T(m) \neq \varnothing$ in the argument of *Case* ψ.

Sub-sub-case 3.2.2: $\theta = \bar{\theta}$

Then, given that either $m_1^i = \theta^i \neq \bar{\theta}$ or $m_1^i = \phi\left(\theta^i\right) \neq \phi\left(\bar{\theta}\right)$, by definition of *Rule 2.2*, $m_i \notin T_i^\Gamma(\theta)$. Moreover, given that $x \in C_i(\theta, x) \subseteq L_i(\theta, x)$, by Condition μ^*(ii), given that $g(m) \in C_i(\theta, x)$, by definition of *Rule 2.2*, and given that $C_i(\theta, x) \subseteq L_i(\theta, g(m))$, it follows that $g(m)\, I_i(\theta)\, x$. Suppose that $x \in S_i(\theta; x, \theta)$, so that $\sigma_1^i(\theta, x) = \theta$. Given that $m_i \notin T_i^\Gamma(\theta)$ and given that $i \in H$, i can change $m_i \notin T_i^\Gamma(\theta)$ into $m_i' = (\theta, x, 1, 1) \in T_i^\Gamma(\theta)$. The profile (m_{-i}, m_i') falls into *Rule 1*, and so $g(m_{-i}, m_i') = x$. Thus, $(m_i', m_{-i}) \succ_i^{\Gamma,\theta} m$, which contradicts that $m \in NE\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$. We conclude that $x \notin S_i(\theta; x, \theta)$. Since $i \in H$, part (1)(b) of Condition μ^*(ii) implies that $S_i(\theta; x, \theta) \subseteq SL_i(\theta, x)$. Since $g(m)\, I_i(\theta)\, x$ and since, moreover, $R_i(\theta)$ is transitive, it follows that $S_i(\theta; x, \theta) \subseteq SL_i(\theta, g(m))$. We proceed according to whether $m_1^i = \theta^i \neq \theta$ or $m_1^i = \phi\left(\theta^i\right) \neq \phi(\theta)$.

Suppose that $m_1^i = \phi\left(\theta^i\right) \neq \phi(\theta)$. The contradiction follows from arguments similar to those used for *Case* ψ. Otherwise, suppose that $m_1^i = \theta^i \neq \theta$. The contradiction follows from arguments similar to those used for *Case* δ. Note that if statement (α) should hold, then the deviant $h \in H \setminus \{i\}$ of Case α is such that $h \notin T(m)$; otherwise, Condition μ^*(iii)(B) is violated. Consequently, it holds that $m_h \notin T_h^\Gamma(\theta)$ for the deviant $h \in H \setminus \{i\}$ of Case α.

Case 4: m corresponds to *Rule 2.3*

Then, $g(m) \in S_i\left(\theta^i; x, \bar{\theta}\right)$. Given that $m_1^i = \theta^i = \bar{\theta} \neq \sigma_1^i\left(\bar{\theta}, x\right)$, it follows that $x \notin S_i\left(\bar{\theta}; x, \bar{\theta}\right)$, by definition of $\sigma_i\left(\bar{\theta}, x\right)$, and so $g(m) \neq x$. Furthermore, $C_i\left(\bar{\theta}, x\right) \subseteq L_i(\theta, g(m))$ follows from the argument used for *Case 3*. Since $x \in C_i\left(\bar{\theta}, x\right)$, by Condition μ^*(ii), it follows that $g(m)\, R_i(\theta)\, x$. We proceed according to whether $\theta = \bar{\theta} = \theta^i$ or $\theta \neq \bar{\theta} = \theta^i$.

Suppose that $\theta = \bar{\theta} = \theta^i$. Suppose that $i \in H$. Note that $x \in g(M_i, m_{-i})$ given that i can attain it by inducing *Rule 1*. Also, note that $x R_i(\theta)\, g(m)$ given that $C_i(\theta, x) \subseteq L_i(\theta, x)$, by Condition μ^*(ii), given that $S_i(\theta; x, \theta) \subseteq C_i(\theta, x)$, by part (1)(a) of Condition μ^*(ii), and given that $g(m) \in S_i(\theta; x, \theta)$. Thus, since $g(m)\, R_i(\theta)\, x$, we have that $g(m)\, I_i(\theta)\, x$, which contradicts part (1)(b) of Condition μ^*(ii), given that $i \in H$ and that $x \notin S_i(\theta; x, \theta)$. We conclude that $i \notin H$, and so Condition μ^*(iii)(B) implies that either statement (α) or statement (β) holds. Let us proceed according to whether $T(m) \neq \varnothing$ or not. The contradiction that $m \notin NE\left(\Gamma, \succcurlyeq^{\Gamma,\theta,H}\right)$ follows from arguments similar to those used for

Sub-case 2.2, noting that statement (α) implies that the deviant $h \in H\backslash\{i\}$ is such that $h \notin T(m)$, given that $\theta = \bar{\theta}$, and so $m_h \notin T_h^{\Gamma}(\theta)$.

Suppose that $\theta \neq \theta^i = \bar{\theta}$. Note that $i \in T(m)$ and that $m_h \notin T_h^{\Gamma}(\theta)$ for all $h \in H$. We proceed according to whether $i \in H$ or not.

Sub-case 4.1: $i \in H$

Let us show that $S_i\left(\theta; x, \bar{\theta}\right) \cap I_i(\theta, g(m), Y) = \varnothing$. Assume, to the contrary, that there exists $z \in S_i\left(\theta; x, \bar{\theta}\right) \cap I_i(\theta, g(m), Y)$. Then, i can change $m_i \notin T_i^{\Gamma}(\theta)$ into $m_i' = (\theta, z, 2, 2) \in T_i^{\Gamma}(\theta)$. Since $\theta \neq \bar{\theta}$, and so $m_i' \notin \sigma_i\left(\bar{\theta}, x\right)$, the profile (m_{-i}, m_i') falls into *Rule 2.2*. Moreover, since $z \in S_i\left(\theta; x, \bar{\theta}\right)$, *Rule 2.2* implies that $g(m_{-i}, m_i') = z$. It follows that $(m_i', m_{-i}) \succ_i^{\Gamma,\theta} m$, which contradicts that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$. We conclude that $S_i\left(\theta; x, \bar{\theta}\right) \cap I_i(\theta, g(m), Y) = \varnothing$.

This implies that $S_i\left(\theta; x, \bar{\theta}\right) \subseteq SL_i(\theta, g(m))$, given that $S_i\left(\theta; x, \bar{\theta}\right) \subseteq C_i\left(\bar{\theta}, x\right)$, by part (1)(a) of Condition μ^*(ii), and that $C_i\left(\bar{\theta}, x\right) \subseteq L_i(\theta, g(m))$. Since $S_i\left(\theta; x, \bar{\theta}\right) \subseteq SL_i(\theta, g(m))$ and $i \in T(m)$, Condition μ^*(iii)(B) implies that either statement (α) or statement (β) holds. The contradiction that $m \notin NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$ follows either from the argument used for *Case* α or from the argument used for *Case* β.

Sub-case 4.2: $i \notin H$

Condition μ^*(iii)(B) implies that either statement (α) or statement (β) holds. Let us proceed according to whether $T(m) \neq \varnothing$ or not. The contradiction that $m \notin NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$ follows from the argument used for *Sub-case 2.2*.

Case 5: m falls into *Rule 3.2*

Then, $T(m) = \varnothing$, and so $m_h \notin T_h^{\Gamma}(\theta)$ for every $h \in H$. Take any $i \in N$ and any $x^i \in Y$. Let us consider $m_i' = (\phi(\theta), x^i, k^i, 2)$. We proceed according to whether $x^i \in F(\theta)$ or not.

Suppose that $x^i \in F(\theta)$. Then, $m_i' \notin \sigma_i(\theta, x^i)$, by definition of $\sigma_i(\theta, x^i)$. Moreover, given that m falls into *Rule 3*, $m_j \notin \sigma_j(\theta, x^i)$ for some $j \neq i$, and so (m_i', m_{-i}) falls into *Rule 3.2*. To obtain x^i, i has only to adjust k^i by which she becomes the winner of the modulo game. Otherwise, suppose $x^i \notin F(\theta)$. Then, (m_i', m_{-i}) falls into *Rule 3.2*. To obtain x^i, i has only to adjust k^i by which she becomes the winner of the modulo game.

Since the choice of $x^i \in Y$ is arbitrary, one can see that $Y \subseteq g(M_i, m_{-i})$. Moreover, given that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$, it holds that $Y \subseteq L_i(\theta, g(m))$. Since the choice of i is arbitrary, one can see that $g(m) \in M(Y, \theta)\backslash F(\theta)$.

Given that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$, it follows from Assumption 2 that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,\{h\}}\right)$ for every $h \in H$. Thus, in what follows, fix any $h \in H$. Since $g(m) \in M(Y, \theta)\backslash F(\theta)$, Condition μ^*(iii)-(A)-(a) implies for $\{h\}$ that $g(m) \notin Y_h(\theta)$, given that $T(m) = \varnothing$, and that there exists $z^{(h,\theta)} \in Y_h(\theta) \cap I_h(\theta, g(m), Y)$. By changing $m_h \notin T_h^{\Gamma}(\theta)$ into $m_h' = \left(\theta, z^{(h,\theta)}, k^h, 2\right) \in T_h^{\Gamma}(\theta)$, h can induce *Rule 3.1* with $T(m_h', m_{-h}) = \{h\}$. To see this, note in the case where $z^{(h,\theta)} \in F(\theta)$ it holds that $m_h' \notin \sigma_h\left(\theta, z^{(h,\theta)}\right)$ and that $m_j \notin \sigma_j\left(\theta, z^{(h,\theta)}\right)$ for some $j \neq h$. To obtain $z^{(h,\theta)}$, h has only to adjust k^h by which she becomes the winner of the modulo game, which contradicts $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,\{h\}}\right)$.

Case 6: m falls into *Rule 3.1*

Then, $T(m) \neq \varnothing$ and $g(m) \in Y_{T(m)}\left(\theta^{T(m)}\right)$. Given that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$, it follows from Assumption 2 that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,\{h\}}\right)$ for every $h \in H$. Thus, in what follows, fix any $h \in H$.

Let first show that $g(m) \in M\left(Y_{T(m)}\left(\theta^{T(m)}\right), \theta\right)$. To this end, fix any $i \in N$ and any $x^i \in Y_{T(m)}\left(\theta^{T(m)}\right)$. By changing m_i into $m_i' = (\phi(\theta), x^i, k^i, 2)$, i can induce *Rule 3* with $T(m_i', m_{-i}) = T(m) \setminus \{i\}$. To see this, note that in the case where $x^i \in F(\theta)$ it holds that $m_i' \notin \sigma_i(\theta, x^i)$ and that $m_j \notin \sigma_j(\theta, x^i)$ for some $j \neq i$. To obtain x^i, i has only to adjust k^i by which she becomes the winner of the modulo game. Since the choice of $x^i \in Y_{T(m)}\left(\theta^{T(m)}\right)$ is arbitrary, one can see that $Y_{T(m)}\left(\theta^{T(m)}\right) \subseteq g(M_i, m_{-i})$. Moreover, given that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,\{h\}}\right)$, it holds that $Y_{T(m)}\left(\theta^{T(m)}\right) \subseteq L_i(\theta, g(m))$. Since the choice of i is arbitrary, one can also see that $g(m) \in M\left(Y_{T(m)}\left(\theta^{T(m)}\right), \theta\right)$.

Since $g(m) \in M\left(Y_{T(m)}\left(\theta^{T(m)}\right), \theta\right) \setminus F(\theta)$ and since $T(m) \neq \varnothing$, Condition μ^*(iii)-(A)-(b) implies that either

Case p: There exist $i \in T(m)$ and $y \in Y_{T(m)\setminus\{i\}}\left(\theta_{-i}^{T(m)}\right)$ such that $yP_i(\theta)\, g(m)$; or

Case q: There exist $h \in H$, with $\theta^h \neq \theta$ if $h \in T(m)$, and $w^{(h,\theta)} \in Y_{T(m)\cup\{h\}}\left(\theta_{-h}^{T(m)}, \theta\right) \cap I_h(\theta, g(m), Y)$.

Suppose that *Case p* holds. Let us consider $m_i' = (\phi(\theta), y, k^i, 2)$. Note that $m_i' \notin \sigma_i(\theta, y)$ if $y \in F(\theta)$. Also, note that $m_j \notin \sigma_j(\theta, y)$ for some $j \neq i$ if $y \in F(\theta)$, given that m falls into *Rule 3.1*. Then, (m_i', m_{-i}) falls into *Rule 3*. If $T(m_i', m_{-i}) = \varnothing$, then (m_i', m_{-i}) falls into *Rule 3.2*. To obtain y, i has only to adjust k^i by which she becomes the winner of the modulo game. Therefore, $(m_i', m_{-i}) \succ_i^{\Gamma,\theta} m$, which contradicts that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,\{h\}}\right)$. Suppose that $T(m_i', m_{-i}) \neq \varnothing$. Then, (m_i', m_{-i}) falls into *Rule 3.1* and $T(m_i', m_{-i}) = T(m) \setminus \{i\}$. To obtain y, i has only to adjust k^i by which she becomes the winner of the modulo game. Therefore, $(m_i', m_{-i}) \succ_i^{\Gamma,\theta} m$, which contradicts that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,\{h\}}\right)$.

Suppose that *Case q* holds. Note that for the deviant h it holds that $m_h \notin T_h^{\Gamma}(\theta)$ if $h \in T(m)$. Also, note that $m_h \notin T_h^{\Gamma}(\theta)$ if $h \notin T(m)$. Let us consider $m_h' = \left(\theta, w^{(h,\theta)}, k^h, 2\right) \in T_h^{\Gamma}(\theta)$. Note that $m_h' \notin \sigma_h\left(\theta, w^{(h,\theta)}\right)$ if $w^{(h,\theta)} \in F(\theta)$. Also, note that $m_j \notin \sigma_j\left(\theta, w^{(h,\theta)}\right)$ for some $j \neq h$ if $w^{(h,\theta)} \in F(\theta)$, given that m falls into *Rule 3.1*. Then, (m_h', m_{-h}) falls into *Rule 3.1* with $T(m_h', m_{-h}) = T(m) \cup \{h\}$. To obtain $w^{(h,\theta)}$, h has only to adjust k^h by which she becomes the winner of the modulo game. Therefore, $(m_h', m_{-h}) \succ_h^{\Gamma,\theta} m$, which contradicts that $m \in NE\left(\Gamma, \succsim^{\Gamma,\theta,\{h\}}\right)$. ■

6.3 *Appendix C: Proof of Theorem 3, "only if" part - not for publication*

Let us show that F satisfies Condition μ^*(i) and Condition μ^*(iii) with respect to $Y \subseteq X$ if it is partially-honest Nash implementable. Let $\Gamma = (M, g)$ be the mechanism that partially-honestly Nash implements F. Then, $T_i^{\Gamma}\left(\bar{\theta}\right) \neq \varnothing$ for every pair $\left(i, \bar{\theta}\right) \in N \times \Theta$ and, moreover, $F\left(\bar{\theta}\right) = NA\left(\Gamma, \succsim^{\Gamma,\bar{\theta},\bar{H}}\right)$, for every pair $\left(\bar{\theta}, \bar{H}\right) \in \Theta \times \mathcal{H}$.

The fact that F satisfies Condition μ^*(ii) with respect to $Y \subseteq X$ follows from Theorem 1. In what follows, define Y as in the proof of Theorem 1; that is, $Y \equiv g(M)$.

6.3.1 Proof of Condition μ^*(iii)-(A)

Fix any $\left(x, \theta, H, T, \bar{\theta}^T\right) \in Y \times \Theta \times \mathcal{H} \times \mathcal{P}(N) \times \Theta^{|T|}$ with $\bar{\theta}^T \equiv \left(\bar{\theta}^j\right)_{j \in T}$.

To show that F satisfies Condition (A)-(0), fix any $i \in N$ and any $\bar{\theta}^i \in \Theta$. Define $Y_i\left(\bar{\theta}^i\right)$ as follows:

$$Y_i\left(\bar{\theta}^i\right) \equiv g\left(T_i^\Gamma\left(\bar{\theta}^i\right), \prod_{j \in N \setminus \{i\}} M_j\right). \tag{A5}$$

Given that $T_i^\Gamma\left(\bar{\theta}^i\right) \neq \varnothing$, by implementability, and that $Y \equiv g(M)$, it follows that $\varnothing \neq Y_i\left(\bar{\theta}^i\right) \subseteq Y$.

Define $Y_T\left(\bar{\theta}^T\right)$ as follows:

$$Y_T\left(\bar{\theta}^T\right) \equiv Y \text{ if } T = \varnothing; \tag{A6}$$

$$Y_T\left(\bar{\theta}^T\right) \equiv g\left(\prod_{j \in T} T_j^\Gamma\left(\bar{\theta}^j\right), \prod_{i \in N \setminus T} M_i\right) \text{ if } T \neq N; \tag{A7}$$

and

$$Y_T\left(\bar{\theta}^T\right) \equiv g\left(\prod_{j \in T} T_j^\Gamma\left(\bar{\theta}^j\right)\right) \text{ if } T = N. \tag{A8}$$

Given that $T_j^\Gamma\left(\bar{\theta}^j\right) \neq \varnothing$ for every $j \in T$, one can see that $Y_T\left(\bar{\theta}^T\right) \neq \varnothing$. Also, by definition of Y, it follows that $Y_T\left(\bar{\theta}^T\right) \subseteq Y$. By definition of $Y_i\left(\bar{\theta}^i\right)$ and by definition of $Y_T\left(\bar{\theta}^T\right)$, one can see that $Y_i\left(\bar{\theta}^i\right) = Y_T\left(\bar{\theta}^T\right)$ if $T = \{i\}$. Thus, F satisfied Condition (A)-(0).

Next, let us show that F meets Condition (A)-(a). Assume that $x \in M(Y, \theta) \setminus F(\theta)$ and that $H = \{h\}$. Fix any $m \in M$ such that $g(m) = x$. Since $x \notin NA\left(\Gamma, \succsim^{\Gamma,\theta,H}\right) = F(\theta)$ and $x = g(m) \in M(Y, \theta)$, it must be the case that $m_h \notin T_h^\Gamma(\theta)$ and that there exists $m_h' \in T_h^\Gamma(\theta)$ such that $g(m_h', m_{-h})\, I_h(\theta)\, g(m)$. Let $g(m_h', m_{-h}) \equiv z^{(h,\theta)}$ and $\theta^h \equiv \theta$. Since $Y_h\left(\theta^h\right)$ is defined as in (A5), the outcome $z^{(h,\theta)} \equiv g(m_h', m_{-h}) \in Y_h\left(\theta^h\right) \cap I_h\left(\theta^h, x, Y\right)$. We are left to show that $x \notin Y_{T \cup \{h\}}\left(\bar{\theta}_{-h}^T, \theta^h\right)$, with $Y_{T \cup \{h\}}\left(\bar{\theta}_{-h}^T, \theta^h\right) = Y_h\left(\theta^h\right)$ if $T = \varnothing$. Let $\bar{T} \equiv T \cup \{h\}$ and $\bar{\theta}^{\bar{T}} \equiv \left(\bar{\theta}_{-h}^T, \theta^h\right)$. Assume, to the contrary, that $x \in Y_{\bar{T}}\left(\bar{\theta}^{\bar{T}}\right)$. Then, there exists $m^1 \in M$ such that $g(m^1) = x \in Y_{\bar{T}}\left(\bar{\theta}^{\bar{T}}\right)$. It follows from the definition of $Y_{\bar{T}}\left(\bar{\theta}^{\bar{T}}\right)$ in (A7) if $\bar{T} \neq N$, or in (A8) if $\bar{T} = N$, that $m_h^1 \in T_h^\Gamma\left(\theta^h\right)$. Since $H = \{h\}$, it follows that

$g(m^1) \in NA\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$, which is a contradiction. We conclude that F satisfies Condition (A)-(a).

Let us show that F satisfies Condition (A)-(b). To this end, suppose that $T \neq \varnothing$ and that $x \in M\left(Y_T\left(\bar{\theta}^T\right), \theta\right) \setminus F(\theta)$. Since $x \in Y_T\left(\bar{\theta}^T\right)$, it follows from the definition of $Y_T\left(\bar{\theta}^T\right)$ that there exists $m \in M$ such that $g(m) = x$ and that $m_j \in T_j^\Gamma\left(\bar{\theta}^j\right)$ for every $j \in T$. Moreover, since $g(m) \notin F(\theta) = NA\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$, there exist $i \in N$ and $m_i' \in M_i$ such that $g(m_i', m_{-i})\, R_i(\theta)\, g(m)$.

- Suppose that $g(m_i', m_{-i})\, P_i(\theta)\, g(m)$ for some $m_i' \in M_i$ and some $i \in N$. Note that $i \in T$ if $T = N$. Also note that $i \in T$ if $T \neq N$. To see it, observe that since $Y_T\left(\bar{\theta}^T\right)$ is defined as in (A7) if $T \neq N$ and since, moreover, $g(m) \in Y_T\left(\bar{\theta}^T\right) \cap M\left(Y_T\left(\bar{\theta}^T\right), \theta\right)$, it holds that $g(M_k, m_{-k}) \subseteq L_k(\theta, g(m))$ for every $k \in N \setminus T$, and so $i \in T$. Thus, in either case, $i \in T$. Let $\bar{T} \equiv T \setminus \{i\}$ and $\bar{\theta}^{\bar{T}} \equiv \left(\bar{\theta}^T_{-i}\right) \equiv \left(\bar{\theta}^j\right)_{j \in \bar{T}}$. Since $m_j \in T_j^\Gamma\left(\bar{\theta}^j\right)$ for every $j \in T \setminus \{i\}$, it follows from the definition of $Y_{\bar{T}}\left(\bar{\theta}^{\bar{T}}\right)$ in (A7) if $\bar{T} \neq N$, or in (A6) if $\bar{T} = \varnothing$, that $g(m_i', m_{-i}) \in Y_{\bar{T}}\left(\bar{\theta}^{\bar{T}}\right)$. Let $g(m_i', m_{-i}) \equiv y$. Thus, we conclude that $y \in Y_{T \setminus \{i\}}\left(\bar{\theta}^T_{-i}\right)$ and $i \in T$.
- Suppose that $g(M_i, m_{-i}) \subseteq L_i(\theta, g(m))$ for every $i \in N$. Since $g(m) \notin F(\theta) = NA\left(\Gamma, \succsim^{\Gamma,\theta,H}\right)$, then for some $h \in H$ it holds that $m_h \notin T_h^\Gamma(\theta)$ and that there exists $m_h' \in T_h^\Gamma(\theta)$ such that $g(m_h', m_{-h})\, I_h(\theta)\, g(m)$. Let $\theta \equiv \theta^h$ and $w^{(h,\theta)} \equiv g(m_h', m_{-h})$. We proceed according to whether $h \in T$ or not.
  - Suppose that $h \in T$. Let $\bar{T} \equiv T$ and $\bar{\theta}^{\bar{T}} \equiv \left(\bar{\theta}^T_{-h}, \theta^h\right) \equiv \left(\left(\bar{\theta}^j\right)_{j \in \bar{T} \setminus \{h\}}, \theta^h\right)$. Note that $\bar{\theta}^h \neq \theta$ given that $m_h \notin T_h^\Gamma(\theta)$. Thus, $\bar{\theta}^h \neq \theta$ if $h \in T$. Define $Y_{\bar{T}}\left(\bar{\theta}^{\bar{T}}\right)$ as in (A7) if $\bar{T} \neq N$, or as in (A8) if $\bar{T} = N$. It follows that $w^{(h,\theta)} \in Y_{\bar{T}}\left(\bar{\theta}^{\bar{T}}\right) \cap I_h\left(\theta^h, x, Y\right)$.
  - Suppose that $h \notin T$. Let $\bar{T} \equiv T \cup \{h\}$ and $\bar{\theta}^{\bar{T}} \equiv \left(\bar{\theta}^T_{-h}, \theta^h\right) \equiv \left(\left(\bar{\theta}^j\right)_{j \in T}, \theta^h\right)$. Note that $\bar{T} \in \mathcal{P}(N)$. Define the set $Y_{\bar{T}}\left(\bar{\theta}^{\bar{T}}\right)$ as in (A7) if $\bar{T} \neq N$, or as in (A8) if $\bar{T} = N$. Again, we have that $w^{(h,\theta)} \in Y_{\bar{T}}\left(\bar{\theta}^{\bar{T}}\right) \cap I_h\left(\theta^h, x, Y\right)$.

In summary, if the premises of Condition (A)-(b) are met, then either there exist $i \in T$ and $y \in Y_{T \setminus \{i\}}\left(\bar{\theta}^T_{-i}\right)$ such that $y P_i(\theta) x$; or there exist $h \in H$, with $\bar{\theta}^h \neq \theta$ if $h \in T$, and $w^{(h,\theta)} \in Y_{T \cup \{h\}}\left(\bar{\theta}^T_{-h}, \theta^h\right) \cap I_h\left(\theta^h, x, Y\right)$, where $\theta^h = \theta$. Thus, F satisfies Condition (A)-(b).

6.3.2 Proof of Condition μ^*(iii)-(B)

Fix any $\left(x,\theta,H,T,\bar{\theta}^T\right) \in Y \times \Theta \times \mathcal{H} \times \mathcal{P}(N) \times \Theta^{|T|}$ with $\bar{\theta}^T \equiv \left(\bar{\theta}^j\right)_{j\in T}$ such that $x \in F(\theta)$. Moreover, fix any $(i,\theta') \in N \times \Theta$. Suppose that $z \in C_i(\theta,x) \subseteq L_i(\theta',z)$ and $z \notin F(\theta')$. Let T be such that:

(I) $i \in T \implies z \in S_i\left(\bar{\theta}^i;x,\theta\right)$; and

(II) $T\backslash\{i\} \neq \varnothing \implies \bar{\theta}^k = \theta$ and $x \in S_k\left(\bar{\theta}^k;x,\theta\right)$ for any $k \in T\backslash\{i\}$.

As shown in the proof of Theorem 1, given that $x \in F(\theta)$ and given that $N \in \mathcal{H}$ by Assumption 2, there exists $m \in NE\left(\Gamma,\succcurlyeq^{\Gamma,\theta,N}\right)$ such that $g(m) = x$ and $C_i(\theta,x) \equiv g(M_i,m_{-i})$. Moreover, $z \in C_i(\theta,x) \subseteq L_i(\theta',z)$ implies that there exists $m' \equiv (m_i',m_{-i})$ such that $g(m') = z$, and so $z \in C_i(\theta,x) = g\left(M_i,m_{-i}'\right)$. For each $\ell \in N$, define $S_i(\theta';x,\theta)$ as in (A2). Note that if $\bar{\theta}^k = \theta$ and $x \in S_k\left(\bar{\theta}^k;x,\theta\right)$ for any $k \in T\backslash\{i\}$, then $m \in NE\left(\Gamma,\succcurlyeq^{\Gamma,\theta,N}\right)$ is such that $(m_k)_{k\in T\backslash\{i\}} \in \prod_{k\in T\backslash\{i\}} T_k^\Gamma(\theta)$. Also, note that if $i \in T$ and $z = g(m') \in S_i\left(\bar{\theta}^i;x,\theta\right)$, then $m_i' \in T_i^\Gamma\left(\bar{\theta}^i\right)$, where $S_i\left(\bar{\theta}^i;x,\theta\right) \equiv g\left(T_i^\Gamma\left(\bar{\theta}^i\right),m_{-i}\right)$. Thus, by definition of $Y_T\left(\bar{\theta}^T\right)$ in (A7) if $\varnothing \neq T \neq N$, or in (A8) if $T = N$, or in (A6) if $T = \varnothing$, one can see that $g(m') = z \in Y_T\left(\bar{\theta}^T\right)$.

Next, let one of the following three requirements hold:

(1) $S_i(\theta';x,\theta) \subseteq SL_i(\theta',z)$ and $T \neq \varnothing$;
(2) $i \notin H$;
(3) $i \in T$ with $\bar{\theta}^i = \theta'$.

First, let requirement (2) hold. Then, since $z = g(m') \in C_i(\theta,x) \equiv g\left(M_i,m_{-i}'\right) \subseteq L_i(\theta',z)$ but $m' \notin NE\left(\Gamma,\succcurlyeq^{\Gamma,\theta',H}\right)$ since $z \notin F(\theta') = NA\left(\Gamma,\succcurlyeq^{\Gamma,\theta',H}\right)$, it follows that either there exists $h \in H\backslash\{i\}$ for whom it holds that $m_h' \notin T_h^\Gamma(\theta')$ and that $g\left(m_h'',m_{-h}'\right) I_h(\theta') z$ for some $m_h'' \in T_h^\Gamma(\theta')$, or there exists $j \in N\backslash\{i\}$ such that $g\left(m_j'',m_{-j}'\right) P_j(\theta') z$ for some $m_j'' \in M_j$.

- Suppose that there exists $j \in N\backslash\{i\}$ such that $g\left(m_j'',m_{-j}'\right) P_j(\theta') z$ for some $m_j'' \in M_j$. Let $g\left(m_j'',m_{-j}'\right) \equiv z^{(j)}$, $\bar{T} = T\backslash\{j\}$ and $\bar{\theta}^{\bar{T}} \equiv \bar{\theta}^T_{-j} \equiv \left(\bar{\theta}^k\right)_{k\in\bar{T}}$. It follows from the definition of $Y_{\bar{T}}\left(\bar{\theta}^{\bar{T}}\right)$ in (A7) if $\bar{T} \neq N$, or in (A6) if $\bar{T} = \varnothing$, that $z^{(j)} \in Y_{\bar{T}}\left(\bar{\theta}^{\bar{T}}\right)$, and so statement (β) of Condition (B) holds if requirement (2) is met.

- Suppose that there exists $h \in H\backslash\{i\}$ for whom it holds that $m_h' \notin T_h^\Gamma(\theta')$ and that $g\left(m_h'',m_{-h}'\right) I_h(\theta') z$ for some $m_h'' \in T_h^\Gamma(\theta')$. Note that if $h \in T$, then property (II) implies that $\bar{\theta}^h = \theta$. Then, if $h \in T$, then $\theta' \neq \theta$; otherwise, $m_h' \in T_h^\Gamma(\theta')$, which is a contradiction. Let $z^{(h)} \equiv g\left(m_h'',m_{-h}'\right)$ and $\theta^h \equiv \theta'$, and so $z^{(h)} I_h(\theta') z$. Moreover, let $\bar{T} \equiv T\backslash\{h\} \cup \{h\}$ and $\bar{\theta}^{\bar{T}} \equiv \left(\bar{\theta}^T_{-h},\theta^h\right) \equiv \left(\left(\bar{\theta}^k\right)_{k\in T\backslash\{h\}},\theta^h\right)$. it follows from the definition of $Y_{\bar{T}}\left(\bar{\theta}^{\bar{T}}\right)$ in (A7) if $\bar{T}\neq N$, or as in (A8) if $\bar{T}=N$, that $z^{(h)}\in Y_{\bar{T}}\left(\bar{\theta}^{\bar{T}}\right)\cap I_h\left(\theta^h,z,Y\right)$, and so statement ($\alpha$) of Condition (B) holds if requirement (2) is met.

Let requirement (1) hold. Since $S_i\left(\theta';x,\theta\right)\subseteq SL_i\left(\theta',z\right)$, it cannot be that $m_i'\in T_i^{\Gamma}\left(\theta'\right)$. Thus, let $m_i'\notin T_i^{\Gamma}\left(\theta'\right)$. Furthermore, since $S_i\left(\theta';x,\theta\right)\subseteq SL_i\left(\theta',z\right)$, it follows from definition of $S_i\left(\theta';x,\theta\right)$ in (A2) that $g\left(m'\right)P_i\left(\theta'\right)g\left(m_i'',m_{-i}'\right)$ for all $m_i''\in T_i^{\Gamma}\left(\theta'\right)$, and so i cannot find a profitable unilateral deviation from m' if $i\in H$. Of course, i cannot find a profitable unilateral deviation from m' if $i\notin H$ given that $C_i\left(\theta,x\right)\subseteq L_i\left(\theta',z\right)$, by assumption. Again, since $z=g\left(m'\right)\in C_i\left(\theta,x\right)\equiv g\left(M_i,m_{-i}'\right)\subseteq L_i\left(\theta',z\right)$ but $m'\notin NE\left(\Gamma,\succsim^{\Gamma,\theta',H}\right)$ since $z\notin F\left(\theta'\right)=NA\left(\Gamma,\succsim^{\Gamma,\theta',H}\right)$, it follows that either there exists $h\in H\backslash\left\{i\right\}$ for whom it holds that $m_h'\notin T_h^{\Gamma}\left(\theta'\right)$ and that $g\left(m_h'',m_{-h}'\right)I_h\left(\theta'\right)z$ for some $m_h''\in T_h^{\Gamma}\left(\theta'\right)$, or there exists $j\in N\backslash\left\{i\right\}$ such that $g\left(m_j'',m_{-j}'\right)P_j\left(\theta'\right)z$ for some $m_j''\in M_j$. By the same argument used for the case where requirement (2) holds, we see that either statement (α) or statement (β) of Condition (B) holds if requirement (1) is met.

Let requirement (3) hold. This means that $m_i'\in T_i^{\Gamma}\left(\theta'\right)=T_i^{\Gamma}\left(\bar{\theta}^i\right)$. Again, since $z=g\left(m'\right)\in C_i\left(\theta,x\right)\equiv g\left(M_i,m_{-i}'\right)\subseteq L_i\left(\theta',z\right)$ but $m'\notin NE\left(\Gamma,\succsim^{\Gamma,\theta',H}\right)$ since $z\notin F\left(\theta'\right)=NA\left(\Gamma,\succsim^{\Gamma,\theta',H}\right)$, it follows that either there exists $h\in H\backslash\left\{i\right\}$ for whom it holds that $m_h'\notin T_h^{\Gamma}\left(\theta'\right)$ and that $g\left(m_h'',m_{-h}'\right)I_h\left(\theta'\right)z$ for some $m_h''\in T_h^{\Gamma}\left(\theta'\right)$, or there exists $j\in N\backslash\left\{i\right\}$ such that $g\left(m_j'',m_{-j}'\right)P_j\left(\theta'\right)z$ for some $m_j''\in M_j$. By the same argument used for the case where requirement (2) holds, we see that either statement (α) or statement (β) of Condition (B) holds if requirement (3) is met.

6.3.3 Proof of Condition μ^*(i)

To show that F satisfies Condition μ^*(i), take any $\theta,\theta'\in\Theta$ such that $x\in F\left(\theta\right)\backslash F\left(\theta'\right)$ and $C_\ell\left(\theta,x\right)\subseteq L_\ell\left(\theta',x\right)$ for all $\ell\in N$. Given that $N\in\mathcal{H}$, by Assumption 2, there exists $m\in NE\left(\Gamma,\succsim^{\Gamma,\theta,N}\right)$ such that $g\left(m\right)=x$. Define $C_i\left(\theta,x\right)$ as in (A1) and $S_i\left(\theta';x,\theta\right)$ as in (A2). Fix any H. Since $g\left(m\right)\notin NA\left(\Gamma,\succsim^{\Gamma,\theta',H}\right)$ and since $C_\ell\left(\theta,x\right)\equiv g\left(M_\ell,m_{-\ell}\right)\subseteq L_\ell\left(\theta',x\right)$ for all $\ell\in N$, it follows there exists $h\in H$ for whom $m_h\notin T_h^{\Gamma}\left(\theta'\right)$ and there exists $m_h'\in T_h^{\Gamma}\left(\theta'\right)$ such that $g\left(m_h',m_{-h}\right)I_h\left(\theta'\right)g\left(m\right)$. Then, by the definition of $S_h\left(\theta';x,\theta\right)\equiv g\left(T_h^{\Gamma}\left(\theta'\right),m_{-h}\right)$, $g\left(m_h',m_{-h}\right)\in S_h\left(\theta';x,\theta\right)\cap I_h\left(\theta',x,Y\right)$ holds. Thus, F satisfies Condition μ^*(i). ■