Essays on Mean-Variance Portfolio Selections and Utility Maximizations in Mathematical Finance

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Contents

Chapter 1

Overview

1.1 Introduction

1.1.1 Mean-variance portfolio selections

Mean-variance portfolio selection is a problem of the allocation of wealth among various securities so as to attain the optimal trade-off between the expected return of the portfolio and its risk measured by the variance of the portfolio. This problem was first proposed and solved in the single-period setting by Markowitz (1952). Markowitz formulated the problem of minimizing a portfolio's variance subject to the constraint that its expected return equals a constant level. This analysis has long been recognized as the basis of modern portfolio theory.

Being widely used in both academia and industry, this mean-variance paradigm has also inspired the development of the multiperiod mean-variance portfolio selections. As examples of studies in discrete-time multiperiod mean-variance portfolio selections, we have Hakansson (1971), Pliska (1997) and Li and Ng (2000). Hakansson (1971) has investigated relations between the optimal growth portfolio (i.e., the portfolio which is chosen so as to maximize its expected logarithmic utility) and the mean-variance efficiency. In the textbook Pliska (1997), a multiperiod mean-variance portfolio selection in a finite model (i.e., a security model where the probability space is a finite set) is treated. The most related article to our study is Li and Ng (2000) and they have solved a mean-variance portfolio selection in a general discrete-time model where the growth rates of the security prices at each period are assumed to be independent random variables. They used the framework of multiobjective optimization and introduced an embedding technique which embeds the original problem in quadratic utility optimization problems so that dynamic programming can be used to obtain explicit solutions.

Continuous-time mean-variance portfolio selections have been studied by various approaches. As examples of those studies, we have Zhou and Li (2000), Lim (2004), Framstad et al. (2004), Basak and Chabakauri (2010) and Bielecki et al. (2005). The embedding technique has also been employed by Zhou and Li (2000) to solve a continuous-time mean-variance portfolio selection using the stochastic linear-quadratic (LQ) control theory in a diffusion model with deterministic coefficients. Lim (2004) has also dealt with the problem by the stochastic LQ control in a diffusion model when the coefficients are random. Framstad et al. (2004) treated a continuous-time mean-variance portfolio selection in a jump-diffusion model as an application of their main result about the stochastic maximum principle in the model. Basak and Chabakauri (2010) tackled the problem in a continuous-time Markovian model driven by two Brownian motions directly applying dynamic programming without the embedding technique and they derived the time-consistent solution to the problem. Bielecki et al. (2005) has studied a continuous-time mean-variance portfolio selection with a condition which prohibits the portfolio taking negative value so that the theory can be more suitable for the investment situation in the real world.

The multiperiod mean-variance criteria can be applied to practical investment problems because of their solvability and explicit results. For example, both Delong and Gerrard (2007) and Wang et al. (2007) solved optimal investment problems for insurance companies which are formulated as continuous-time mean-variance portfolio selections. Moreover Basak and Chabakauri (2010) stated that the multiperiod mean-variance criteria can be used as a benchmark for evaluation of investment. These articles motivate us to struggle to obtain more simple and elementary solution to multiperiod mean-variance selections. In Chapter 2 of this thesis, we will demonstrate an alternative and perhaps simpler approach to the problems both in discrete time and in continuous time.

1.1.2 Utility maximizations

Utility maximization is also a basic problem in mathematical finance. This is the problem of an economic agent who invests in a financial market so as to maximize the expected utility of her terminal wealth as well as intertemporal consumption from her wealth. Optimal consumption-portfolio policies have traditionally been derived by stochastic dynamic programming. Samuelson (1969), Merton (1969) and Merton (1971) are the pioneering papers in this field. In particular, in the framework of a continuous-time model, Merton (1969) derived a Hamilton-Jacobi-Bellman (HJB) equation for the value function of the optimization problem for the first time. He also provided the closed-form solution of this equation when the utility function is a power function, the logarithmic or an exponential function. Their work have been greatly extended to various settings, but we make no attempt to survey in detail. Rogers (2013) includes a lot of variations of the original problem presented by Merton (1969) and Merton (1971) in simple diffusion models and readers are referred to references therein for more information on this topic.

Pham (2009) offers an introduction to further developments in dynamic programming for utility maximizations and other optimization problems in mathematical finance. In particular, Chapter 4 of Pham (2009) is devoted to explanation of application of viscosity solutions. The viscosity solution is a kind of weak solutions of some partial differential equations. One definition of a viscosity solution of the HJB variational inequality which is employed to deal with singular stochastic optimization problems (i.e., the problems whose Hamiltonians of the HJB equations diverge) is stated in Definition 4.6 in Chapter 4 of this thesis. In Chapter 4 of Pham (2009), it has been proved that the value function of a utility maximization problem which is possibly singular is the unique viscosity solution of the corresponding HJB variational inequality. Obtaining similar results in various settings is often taken to be the main purpose of research in the literature related to viscosity solutions in mathematical finance. In Chapter 4 of this thesis, we will attempt conversely to derive a viscosity solution of the HJB variational inequality for a maximization problem of the utility from the terminal wealth. The meaning of this experiment is that if we know that a particular function is a viscosity solution of the HJB variational inequality, then we can confirm that the function coincides with the value function in the model where the value function is the unique viscosity solution of the HJB variational inequality.

Another important method of solving utility maximizations is the martingale representation approach. In martingale approach, one solves a utility maximization problem by separating it into two parts. First, she transforms the problem into a static utility maximization problem and maximizes the objective function over the set of all outcomes that are attainable in the given financial market. Then, she applies the martingale representation theorem to determine the portfolio trading strategy which generates the optimal outcome obtained in the first step. The martingale approach in complete markets is originally developed by Pliska (1986), Cox and Huang (1989), Cox and Huang (1991) and Karatzas et al. (1987). In particular, Cox and Huang (1989) has characterized the optimal policy with a linear partial differential equation (PDE), which is much easier to solve than the nonlinear PDE obtained in dynamic programming. In an incomplete market setting, He and Pearson (1991) extended the result of Cox and Huang (1989) and related the optimal policy to the solution of a quasi-linear PDE. They introduced the notion of minimax local martingale measure which is characterized by proving a duality theorem relating the original utility maximization problem to a dual problem. The minimax local martingale measure is the solution of the dual problem and used both in proving the existence of a solution to the original problem and in characterizing the solution. However, the author believes that the duality argument which is somewhat technical is not essential when we consider only characterization of the solution. In Chapter 3 of this thesis, we will try to provide an alternative approach to derive the PDE without using the duality principle.

1.2 Overview

The subsequent chapters are summarized as follows.

1.2.1 Chapter 2: On explicit solutions to mean-variance portfolio selections via mean-variance hedgings

In Chapter 2, multiperiod mean-variance portfolio selections are solved both in a discrete-time model and in a continuous-time model. A multiperiod mean-variance portfolio selection concerns minimizing the variance of the terminal value of the portfolio while keeping the expected value of the terminal value at a constant level. Unlike the literature referred in the previous section, we will deal with the problem applying the results of mean-variance hedging problems.

We will begin with deriving an explicit solution to a multiperiod mean-variance portfolio selection in a discrete-time model. We will employ the ordinary Lagrange multiplier method and see that the problem of minimizing the Lagrangian with respect to the investment strategies can be regarded as a simple mean-variance hedging. Then we can solve the mean-variance hedging by applying the result given by Gugushvili (2003), which investigated the mean-variance hedging in a general discrete time model by dynamic programming.

Next, we will find an explicit solution to a continuous-time mean-variance portfolio selection by a similar approach which is employed in discrete time. We will solve the mean-variance hedging in continuous time appeared in the process of solving using the result given by Rheinländer and Schweizer (1997), which analyzed the mean-variance hedging by projections. Our main result will show that the optimal strategy of the original problem is obtained as a multiple of the optimal strategy of the mean-variance hedging. This result and the method of solving the problem may be somewhat simpler than those in the earlier studies mentioned in the previous section.

Furthermore our approach may be valid for continuous-time mean-variance portfolio selections in general semimartingale models if we employ the result given by Jeanblanc et al. (2012) to handle the mean-variance hedgings.

1.2.2 Chapter 3: Remarks on optimal strategies to utility maximizations in continuous time incomplete markets

In an incomplete continuous-time diffusion model, He and Pearson (1991) related the optimal strategy of the utility maximization problem to the solution of a quasi-linear PDE by analyzing the dual problem. In Chapter 3 of this thesis, we will attempt to propose an alternative approach to derive the PDE without using the duality principle. We will see that the optimal solution can be characterized by an equivalent martingale measure from a simple necessary condition of optimality. Then we will identify the equivalent martingale measure and derive the PDE by Itô's formula. The optimal strategy can be obtained from the solution of the PDE.

Moreover we apply our method to a utility maximization problem in a model where the price process is assumed to be a compound Poisson process. In contrast to the literature which considered utility maximizations when the price processes are not continuous such as Aase (1984), Bellamy (2001) and so on, the jump sizes of the price process are unpredictable. First, we will specify the whole set of equivalent martingale measures in the model. Then, assuming that the optimal solution exists, an equation which relates to the optimal solution of the utility maximization will be derived by the same approach as in the diffusion case.

1.2.3 Chapter 4: On discrete Itô formulas and discrete Hamilton-Jacobi-Bellman equations

In Chapter 4, we will prove a discrete Itô formula for discrete jump-diffusion processes and derive a discrete Hamilton-Jacobi-Bellman (dHJB) equation for expected utility maximization problems in a discrete jump-diffusion model. Moreover we will analyze a relation between a Hamilton-Jacobi-Bellman (HJB) variational inequality in a continuous-time geometric Brownian model and a dHJB equation in a random walk model derived by Ishimura and Mita (2009).

A discrete Itô formula was originally obtained by Fujita and Kawanishi (2008) for random walks. The proof can be done by simply checking that the formula holds in either case where the increment of the random walk is 1 or -1. We will extend this idea to discrete jump-diffusion processes. Here, a discrete jump-diffusion process is a discrete stochastic process which have a discrete Poisson process term in addition to a random walk term.

Applying the discrete Itô formula for random walks, Ishimura and Mita (2009) derived a dHJB equation and prove the verification theorem. Since the derivation of the dHJB equation and the proof of the verification theorem have been essentially carried out by the discrete Itô formula, we can generalize the result in a discrete jump-diffusion model. Deriving the optimality equation for the utility maximization in the model, we will apply the discrete Itô formula derived in advance to the optimality equation and obtain the dHJB equation in the discrete jump-diffusion model.

After that, we will also provide an application of the dHJB equation in a random walk model. As we have mentioned in the previous section, we will show that a proper limit of a solution of the dHJB equation becomes a viscosity solution of the corresponding HJB variational inequality in continuous time. If the conditions in the theorem are satisfied, this result enables us to specify the value function of the optimization problem in continuous time and consequently find the optimal solution when we know that the value function is the unique viscosity solution of the HJB variational inequality.

Chapter 2

On explicit solutions to mean-variance portfolio selections via mean-variance hedgings

2.1 Introduction

In this chapter, we propose a method of solving multiperiod mean-variance portfolio selections both in a discrete-time model and in a continuous-time model. As mentioned in Chapter 1, this problem has been studied through various approaches. Unlike the literature , we will deal with the problem applying the results of mean-variance hedging problems. Note that a mean-variance hedging problem is the problem of determining the value of a financial option by minimizing the expected value of the quadratic hedging error.

First, we solve a multiperiod mean-variance portfolio selection in a discrete-time model. Li and Ng (2000) referred in Chapter 1 is the most related study to ours and our model is more general in a sense that the growth rates of the security price in our model are not independent random variables but only assumed that they satisfy the deterministic mean-variance tradeoff condition explained below. We employ the ordinary Lagrange multiplier method and see that the problem of minimizing the Lagrangian with respect to the investment strategies can be regarded as a simple mean-variance hedging. Then we can construct the explicit solution to the mean-variance hedging by applying the result given by Gugushvili (2003), which investigated the mean-variance hedging in a general discrete time model by dynamic programming.

Next, we treat a continuous-time mean-variance portfolio selection in a continuous semimartingale model by a similar approach which is employed in discrete time. We transform the original problem into a mean-variance hedging in continuous time and solve it using the result given by Rheinländer and Schweizer (1997), which analyzed the mean-variance hedging by projections. Our main result shows that the optimal strategy of the original problem is obtained as a multiple of the optimal strategy of the mean-variance hedging. This result and the method of solving the problem may be somewhat simpler than those in the earlier studies mentioned in the previous chapter.

The rest of the chapter is organized as follows. In Section 2.2, a multi-period mean-variance portfolio selection in discrete time is solved after recalling results of a mean-variance hedging. A mean-variance portfolio selection in a continuous semimartingale model is solved in Section 2.3 using a solution of a mean-variance hedging in continuous time. Section 2.4 concludes the chapter.

2.2 In discrete time

In this section, a multiperiod mean-variance portfolio selection in discrete time is solved using results of a mean-variance hedging obtained by Gugushvili (2003).

2.2.1 A mean-variance hedging in discrete time

In this subsection, results of a mean-variance hedging problem in discrete time are recalled.

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t=0,1,2,\cdots,T})$ be a filtered probability space where the filtration is assumed to satisfy $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$.

We set a discrete-time market model with a finite terminal time $T > 0$. In the market, there is a risk-free asset. In this section, we assume that the growth rate of this risk-free asset is permanently zero. There is also a risky asset whose price at time t is denoted by S_t, $t = 0, 1, 2, \cdots, T$ where $S_0 > 0$ is a constant and each S_t is $\mathcal{F}_t$-measurable. We use a notation $\Delta S_t := S_t - S_{t-1}$, $t = 1, 2, \cdots, T$.

We summarize results of a mean-variance hedging problem in discrete time. A mean-variance hedging is the problem to determine a value of a financial option by minimizing the expected value of the quadratic hedging error. Let an $\mathcal{F}_T$-measurable random variable H be the payoff of a financial option of which we want to know the price at initial time $t = 0$. A constant $c > 0$ denotes the agent's initial wealth. The whole set of his strategy is defined by

$$\Pi := \left\{ z = (z_t)_{t=0,1,2,\cdots,T-1} \,\middle|\, z \text{ is an adapted process} \right.$$
$$\left. \text{such that } \sum_{s=0}^{T-1} z_s \Delta S_{s+1} \text{ is square integrable.} \right\}.$$

The terminal value of the agent's self-financed portfolio corresponding to a strategy z can be written as $c + \sum_{s=0}^{T-1} z_s \Delta S_{s+1}$. Then the mean-variance hedging problem is defined by

$$\inf_{z \in \Pi} E\left[\left(c + \sum_{s=0}^{T-1} z_s \Delta S_{s+1} - H\right)^2\right]. \tag{2.1}$$

This problem is solved in Gugushvili (2003) by the dynamic programming in discrete time and the optimal solution and the optimal strategy are given as follows.

Theorem 2.1 (Theorem 1 in Gugushvili (2003))**.** *Define the value function $v(t,x)$ corresponding to the mean-variance hedging problem (2.1) as*

$$\begin{aligned} v(t,x) &= \inf_{z \in \Pi} E\left[\left(x + \sum_{s=t}^{T-1} z_s \Delta S_{s+1} - H\right)^2 \middle| \mathcal{F}_t\right], \\ v(T,x) &= (x-H)^2. \end{aligned}$$

Then $v(t,x)$ is a square trinomial in x,

$$v(t,x) = a_t x^2 + 2b_t x + c_t$$

where a, b and c are adapted processes which are determined recurrently by

$$\begin{aligned} a_t &= E[a_{t+1}|\mathcal{F}_t] - \frac{(E[a_{t+1}\Delta S_{t+1}|\mathcal{F}_t])^2}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]}, \\ b_t &= E[b_{t+1}|\mathcal{F}_t] - \frac{E[a_{t+1}\Delta S_{t+1}|\mathcal{F}_t]E[b_{t+1}\Delta S_{t+1}|\mathcal{F}_t]}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]}, \\ c_t &= E[c_{t+1}|\mathcal{F}_t] - \frac{(E[b_{t+1}\Delta S_{t+1}|\mathcal{F}_t])^2}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]}, \\ a_T &= 1, \quad b_T = -H, \quad c_T = H. \end{aligned} \tag{2.2}$$

Moreover, the optimal strategy z^* is given by

$$z_t^* = -\frac{E[b_{t+1}\Delta S_{t+1}|\mathcal{F}_t]}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]} - \left(c + \sum_{s=0}^{t-1} z_s^* \Delta S_{s+1}\right) \frac{E[a_{t+1}\Delta S_{t+1}|\mathcal{F}_t]}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]}. \tag{2.3}$$

Proof. See Theorem 1 in Gugushvili (2003). ∎

2.2.2 A mean-variance portfolio selection in discrete time

In this subsection, a multiperiod mean-variance portfolio selection problem in discrete time is solved explicitly.

The mean-variance portfolio selection seeks the strategy which minimizes the variance of the terminal value of the portfolio while keeping the expectation of the terminal value of the portfolio at a constant level. Imposing an assumption which simplifies the model on the growth rate of S, we derive an explicit solution to the mean-variance portfolio selection problem. In particular, the form of the optimal strategy is fully explicit and not even recurrent.

We consider the following problem:

$$\begin{aligned} &\inf_{z\in\Pi} \ Var\left(c+\sum_{s=0}^{T-1} z_s \Delta S_{s+1}\right), \\ \text{subject to } \ & E\left[c+\sum_{s=0}^{T-1} z_s \Delta S_{s+1}\right] \geq A \end{aligned} \tag{2.4}$$

where Var means the variance of random variables and A is a sufficiently large constant such that $A > c$. Here, we examine the following specific case. Put

$$R_{t+1} := \frac{S_{t+1}-S_t}{S_t} \ \text{ and } \ M_t := \frac{(E[R_{t+1}|\mathcal{F}_t])^2}{E[R_{t+1}^2|\mathcal{F}_t]}, \quad 0 \leq t \leq T-1.$$

We assume here that M_t are deterministic for all t. In this case, we obtain the next result.

Theorem 2.2. *Put $R_{t+1} := \Delta S_{t+1}/S_t$ and $M_t := (E[R_{t+1}|\mathcal{F}_t])^2/E[R_{t+1}^2|\mathcal{F}_t]$, $0 \leq t \leq T-1$. If M_t are deterministic, then the optimal strategy z^* to the* *problem (2.4) is given by*

$$z_t^* S_t = \left\{ \frac{A - c\prod_{s=0}^{T-1}(1-M_s)}{1-\prod_{s=0}^{T-1}(1-M_s)} - \left(c + \sum_{s=0}^{t-1} z_s^* \Delta S_{s+1} \right) \right\} \frac{E[R_{t+1}|\mathcal{F}_t]}{E[R_{t+1}^2|\mathcal{F}_t]} \tag{2.5}$$

or

$$z_t^* S_t = \frac{A - c}{1-\prod_{s=0}^{T-1}(1-M_s)} \prod_{s=0}^{t-1} \left(1 - \frac{R_{s+1} E[R_{s+1}|\mathcal{F}_s]}{E[R_{s+1}^2|\mathcal{F}_s]} \right) \frac{E[R_{t+1}|\mathcal{F}_t]}{E[R_{t+1}^2|\mathcal{F}_t]} \tag{2.6}$$

and the optimal solution is obtained by

$$Var\left(c + \sum_{s=0}^{T-1} z_s^* \Delta S_{s+1} \right) = \frac{(A-c)^2 \prod_{s=0}^{T-1}(1-M_s)}{1-\prod_{s=0}^{T-1}(1-M_s)}. \tag{2.7}$$

Proof. We begin with solving the following problem:

$$\begin{aligned} \inf_{z\in\Pi} \ & Var\left(c + \sum_{s=0}^{T-1} z_s \Delta S_{s+1} \right), \\ \text{subject to } \ & E\left[c + \sum_{s=0}^{T-1} z_s \Delta S_{s+1} \right] = B \end{aligned} \tag{2.8}$$

where B is a constant such that $B \geq A$. It is obvious that the solution to the original problem (2.4) can be obtained by minimizing the solution to the above problem (2.8) in terms of B. The Lagrangian corresponding to this problem (2.8) is obtained by

$$\mathcal{L}(z,\lambda) = E\left[\left(c + \sum_{s=0}^{T-1} z_s \Delta S_{s+1} - B \right)^2 + \lambda \left(B - E\left[c + \sum_{s=0}^{T-1} z_s \Delta S_{s+1} \right] \right) \right]$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier. By Theorem 2 in Section 8.4 in Luenberger (1969), the optimal solution $z^B \in \Pi$ and $\lambda^B \in \mathbb{R}$ to the problem (2.8) is given by a saddle point of $\mathcal{L}$, i.e., z^B and λ^B which satisfy

$$\mathcal{L}(z^B, \lambda) \leq \mathcal{L}(z^B, \lambda^B) \leq \mathcal{L}(z, \lambda^B)$$

for all $z \in \Pi$ and $\lambda \in \mathbb{R}$. We start with minimizing $\mathcal{L}(z,\lambda)$ in terms of z for given $\lambda \in \mathbb{R}$.

As the example described in Section 3.3 of Øksendal and Sulem (2005), since

$$E\left[\left(c+\sum_{s=0}^{T-1} z_s\Delta S_{s+1}-B\right)^2+\lambda\left(B-E\left[c+\sum_{s=0}^{T-1} z_s\Delta S_{s+1}\right]\right)\right]$$
$$=E\left[\left(c+\sum_{s=0}^{T-1} z_s\Delta S_{s+1}-\left(B+\frac{1}{2}\lambda\right)\right)^2\right]-\frac{1}{4}\lambda^2,$$

we should solve the following problem in advance:

$$\inf_{z\in\Pi} E\left[\left(c+\sum_{s=0}^{T-1} z_s\Delta S_{s+1}-\left(B+\frac{1}{2}\lambda\right)\right)^2\right]. \tag{2.9}$$

This is a mean-variance hedging and can be solved by Theorem 2.1 above. We define the value function of this problem (2.9) by

$$v^{B+\frac{1}{2}\lambda}(t,x):=\operatorname*{essinf}_{z\in\Pi}\, E\left[\left(x+\sum_{s=t}^{T-1} z_s\Delta S_{s+1}-\left(B+\frac{1}{2}\lambda\right)\right)^2\middle|\mathcal{F}_t\right]$$

Then, by Theorem 2.1, $v^{B+\frac{1}{2}\lambda}$ can be written by

$$v^{B+\frac{1}{2}\lambda}(t,x)=a_t x^2+2b_t x+c_t$$

where a,b and c are adapted processes which are determined recurrently by

$$\begin{aligned}
a_t &= E[a_{t+1}|\mathcal{F}_t]-\frac{(E[a_{t+1}\Delta S_{t+1}|\mathcal{F}_t])^2}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]},\\
b_t &= E[b_{t+1}|\mathcal{F}_t]-\frac{E[a_{t+1}\Delta S_{t+1}|\mathcal{F}_t]E[b_{t+1}\Delta S_{t+1}|\mathcal{F}_t]}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]},\\
c_t &= E[c_{t+1}|\mathcal{F}_t]-\frac{(E[b_{t+1}\Delta S_{t+1}|\mathcal{F}_t])^2}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]},\\
a_T &= 1,\quad b_T=-\left(B+\frac{1}{2}\lambda\right),\quad c_T=\left(B+\frac{1}{2}\lambda\right)^2.
\end{aligned} \tag{2.10}$$

Moreover, the optimal strategy z^B is determined by

$$z_t^B=-\frac{E[b_{t+1}\Delta S_{t+1}|\mathcal{F}_t]}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]}-\left(c+\sum_{s=0}^{t-1} z_s^B\Delta S_{s+1}\right)\frac{E[a_{t+1}\Delta S_{t+1}|\mathcal{F}_t]}{E[a_{t+1}(\Delta S_{t+1})^2|\mathcal{F}_t]}. \tag{2.11}$$

In the present setting, since $M_t = (E[R_{t+1}|\mathcal{F}_t])^2/E[R_{t+1}^2|\mathcal{F}_t]$, $0 \le t \le T-1$ are deterministic where $R_{t+1} = \Delta S_{t+1}/S_t$, (2.10) is, inductively, partially rewritten by

$$\begin{aligned} &a_t = a_{t+1}(1-M_t), \quad b_t = b_{t+1}(1-M_t), \\ &a_T = 1, \quad b_T = -\Big(B + \frac{1}{2}\lambda\Big), \end{aligned}$$

so that

$$a_t = \prod_{s=t}^{T-1}(1-M_s), \quad b_t = -\Big(B+\frac{1}{2}\lambda\Big)\prod_{s=t}^{T-1}(1-M_s). \tag{2.12}$$

We can also deduce from (2.10) that

$$c_t = \Big(B+\frac{1}{2}\lambda\Big)^2\prod_{s=t}^{T-1}(1-M_s).$$

Then the value function becomes

$$v^{B+\frac{1}{2}\lambda}(t,x) = \bigg(x - \Big(B+\frac{1}{2}\lambda\Big)\bigg)^2\prod_{s=t}^{T-1}(1-M_s).$$

Now we can calculate λ^B. Substitute $z = z^B$ in the Lagrangian. Then the Lagrangian can be written as

$$\begin{aligned} \mathcal{L}(z^B,\lambda) &= v^{B+\frac{1}{2}\lambda}(0,c) - \frac{1}{4}\lambda^2 \\ &= \bigg(c - \Big(B+\frac{1}{2}\lambda\Big)\bigg)^2\prod_{s=0}^{T-1}(1-M_s) - \frac{1}{4}\lambda^2. \end{aligned}$$

Therefore $\lambda^B \in \mathbb{R}$ which maximizes this $\mathcal{L}(z^B,\lambda)$ is given by

$$\lambda^B = \frac{2(B-c)\prod_{s=0}^{T-1}(1-M_s)}{1-\prod_{s=0}^{T-1}(1-M_s)}.$$

Substituting (2.12) and $\lambda = \lambda^B$ into (2.11), we get

$$\begin{aligned} z_t^B S_t &= \left\{B + \frac{1}{2}\lambda^B - \bigg(c + \sum_{s=0}^{t-1} z_s^B \Delta S_{s+1}\bigg)\right\}\frac{E[R_{t+1}|\mathcal{F}_t]}{E[R_{t+1}^2|\mathcal{F}_t]} \\ &= \left\{\frac{B - c\prod_{s=0}^{T-1}(1-M_s)}{1-\prod_{s=0}^{T-1}(1-M_s)} - \bigg(c + \sum_{s=0}^{t-1} z_s^B \Delta S_{s+1}\bigg)\right\}\frac{E[R_{t+1}|\mathcal{F}_t]}{E[R_{t+1}^2|\mathcal{F}_t]} \end{aligned} \tag{2.13}$$

This can be also written by

$$z_t^B S_t = \left(\frac{B-c}{1-\prod_{s=0}^{T-1}(1-M_s)} - \sum_{s=0}^{t-1} z_s^B \Delta S_{s+1} \right) \frac{E[R_{t+1}|\mathcal{F}_t]}{E[R_{t+1}^2|\mathcal{F}_t]}$$

or

$$z_t^B S_t R_{t+1} = \left(\frac{B-c}{1-\prod_{s=0}^{T-1}(1-M_s)} - \sum_{s=0}^{t-1} z_s^B S_s R_{s+1} \right) \frac{R_{t+1} E[R_{t+1}|\mathcal{F}_t]}{E[R_{t+1}^2|\mathcal{F}_t]}.$$

By solving this equation, we obtain

$$z_t^B S_t = \frac{B-c}{1-\prod_{s=0}^{T-1}(1-M_s)} \prod_{s=0}^{t-1} \left(1 - \frac{R_{s+1} E[R_{s+1}|\mathcal{F}_s]}{E[R_{s+1}^2|\mathcal{F}_s]} \right) \frac{E[R_{t+1}|\mathcal{F}_t]}{E[R_{t+1}^2|\mathcal{F}_t]}. \tag{2.14}$$

Finally, the solution to (2.4) is given as follows. The solution to (2.8) is obtained as

$$Var\left(c + \sum_{s=0}^{T-1} z_s^B \Delta S_{s+1} \right) = \mathcal{L}(z^B, \lambda^B) = \frac{(B-c)^2 \prod_{s=0}^{T-1}(1-M_s)}{1-\prod_{s=0}^{T-1}(1-M_s)} \tag{2.15}$$

for each $B \geq A$. Obviously, this is minimized when $B = A$. Therefore the solution to the original problem (2.4) can be obtained by substituting $B = A$ into (2.13), (2.14) and (2.15) yields (2.5), (2.6) and (2.7), respectively. This concludes the proof. ■

Remark 2.3. We get some implication of the optimal strategy from the representation in (2.5). In (2.5), $z_t^* S_t$ means the amount of the money which the agent should spend on the risky asset and $c + \sum_{s=0}^{t-1} z_s^* \Delta S_{s+1}$ is the value of his portfolio at time t. Then (2.5) implies that the quantity that the agent should invest in the risky asset is the difference between a constant $(A - c\prod_{s=0}^{T-1}(1-M_s))/(1-\prod_{s=0}^{T-1}(1-M_s))$ and the value of the portfolio adjusted by $E[R_{t+1}|\mathcal{F}_t]/E[R_{t+1}^2|\mathcal{F}_t]$.

Remark 2.4. We can confirm that $E[c + \sum_{s=0}^{T-1} z_s^* \Delta S_{s+1}] = A$. Indeed, from

(2.6), we have

$$E[z_t^* S_t R_{t+1}] = \frac{A-c}{1-\prod_{s=0}^{T-1}(1-M_s)} \prod_{s=0}^{t-1}(1-M_s)M_t$$
$$= \frac{A-c}{1-\prod_{s=0}^{T-1}(1-M_s)} \left(\prod_{s=0}^{t-1}(1-M_s) - \prod_{s=0}^{t}(1-M_s)\right)$$

for $t \geq 1$ and

$$E[z_0^* S_0 R_1] = \frac{A-c}{1-\prod_{s=0}^{T-1}(1-M_s)} M_0 = \frac{A-c}{1-\prod_{s=0}^{T-1}(1-M_s)}(1-(1-M_0)).$$

Therefore we conclude that

$$E[\sum_{s=0}^{T-1} z_s^* \Delta S_{s+1}] = \sum_{s=0}^{T-1} E[z_s^* S_s Rs + 1]$$
$$= \frac{A-c}{1-\prod_{s=0}^{T-1}(1-M_s)} \left\{1-(1-M_0) + \sum_{t=1}^{T-1}\left(\prod_{s=0}^{t-1}(1-M_s) - \prod_{s=0}^{t}(1-M_s)\right)\right\}$$
$$= \frac{A-c}{1-\prod_{s=0}^{T-1}(1-M_s)} \left(1 - \prod_{s=0}^{T-1}(1-M_s)\right)$$
$$= A-c.$$

Remark 2.5. The condition that M_t are deterministic is called the *deterministic mean-variance tradeoff* condition which is provided by Schweizer (1995). It is known that when this condition is satisfied, the variance-optimal martingale measure coincides with the minimal martingale measure, which can be obtained explicitly with ease (Corollary 4.2 in Schweizer (1995)). This is one reason why we can get an explicit optimal solution to the mean-variance portfolio selection. We also note that it can be easily checked that $1/\prod_{s=0}^{T-1}(1-M_s)$ is equal to the square mean of the density of the variance-optimal martingale measure (or the minimal martingale measure) in the model (see (2.21) in Schweizer (1995) for the expression of the density of the minimal martingale measure). This assure us that the solution (2.7) is consistent with the solution in the continuous-time in the next section.

2.3 In continuous time

In this section, a continuous-time mean-variance portfolio selection in a continuous semimartingale modes is solved applying results of a mean-variance hedging obtained by Rheinländer and Schweizer (1997).

2.3.1 A mean-variance hedging in continuous time

In this subsection, we recall results of a mean-variance hedging in continuous time.

We set a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0\le t\le T})$ with a filtration $(\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual condition. For simplicity, we assume that $\mathcal{F}_0$ is trivial and $\mathcal{F} = \mathcal{F}_T$. Let X be a continuous $\mathbb{R}^d$-valued semimartingale which represents the price of d risky assets. In this section, we assume that the growth rate of the risk-free asset is zero. We define the whole set of trading strategies by

$$\Theta := \left\{ \theta = (\theta_s)_{0\le s\le T} \middle| \theta \text{ is an } \mathbb{R}^d\text{-valued predictable process such that} \right.$$
$$\left. \int_0^T \theta_s^\top dX_s \text{ is a square-integrable semimartingale.} \right\}.$$

In this section, we also define $G_T(\Theta) := \{\int_0^T \theta_s^\top dX_s | \theta \in \Theta\}$ and assume that $G_T(\Theta)$ is closed in $L^2(P)$.

Next, we define the variance-optimal martingale measure in this model. The following three definitions are taken from Rheinländer and Schweizer (1997).

Definition 2.6. *Let W denotes the linear subspace of $L^\infty(\Omega, \mathcal{F}, P)$ spanned by the simple stochastic integrals of the form $Y = h^\top(X_{T_2} - X_{T_1})$ where $T_1 \le T_2 \le T$ are stopping times such that $X_{t\wedge T_2}$ is bounded and h is any bounded $\mathbb{R}^d$-valued $\mathcal{F}_{T_1}$-measurable random variable.*

Definition 2.7. *We define a set $\mathcal{M}^s(P)$ as the space of all signed measures $Q \ll P$ with $Q(\Omega) = 1$ and*

$$E\left[\frac{dQ}{dP}Y\right] = 0, \quad \forall Y \in W.$$

Moreover we define the following set of densities:

$$\mathcal{D}^s := \left\{ \frac{dQ}{dP} \middle| Q \in \mathcal{M}^s(P) \right\}.$$

We assume that $\mathcal{D}^s \cap L^2(P) \neq \emptyset$ hereafter.

Definition 2.8. *The variance-optimal martingale measure $\tilde{P}$ is the element of $\mathcal{M}^s(P)$ such that $\tilde{D} = d\tilde{P}/dP$ is in $L^2(P)$ and minimizes $E[D^2]$ over all $D \in \mathcal{D}^s \cap L^2(P)$.*

The following lemma from Schweizer (1996) shows that $\tilde{D}$ can be represented by a stochastic integral with respect to X. This representation has an important role in the proof of our main result in the subsequent section.

Lemma 2.9 (Lemma 1 (b) in Schweizer (1996))**.** *$\tilde{P}$ is given by*

$$\tilde{D} = E[\tilde{D}^2] + \int_0^T \tilde{\zeta}_s^\top dX_s \tag{2.16}$$

for some $\tilde{\zeta} \in \Theta$.

Proof. See Lemma 1 (b) in Schweizer (1996). ∎

Here, we can describe the solution of a mean-variance hedging in continuous time. The solution of a mean-variance hedging when the payoff of the derivative is constant is given as follows.

Lemma 2.10 (Lemma 4 in Rheinländer and Schweizer (1997))**.** *Assume that $G_T(\Theta)$ is closed in $L^2(P)$ and $\mathcal{D}^s \cap L^2(P) \neq \emptyset$. Then the solution $\tilde{\theta}$ of a mean-variance hedging problem*

$$\inf_{\theta \in \Theta} E\left[\left(1 - \int_0^T \theta_s^\top dX_s\right)^2\right]$$

is given by

$$\tilde{\theta}_s = -\frac{\tilde{\zeta}_s}{E[\tilde{D}^2]}$$

for $0 \leq s \leq T$ *where* $\tilde{\zeta}$ *is the integrand in (2.16).*

Proof. See Lemma 4 in Rheinländer and Schweizer (1997). ■

2.3.2 A mean-variance portfolio selection in continuous time

In this subsection, a mean-variance portfolio selection in continuous time is solved explicitly. Let a constant $c > 0$ denote the agent's initial wealth. Then the terminal value of the agent's self-financed portfolio corresponding to a strategy θ can be written as $c + \int_0^T \theta_s^\top dX_s$. We consider the following problem:

$$\begin{aligned} &\inf_{\theta \in \Theta} \; Var\left(c + \int_0^T \theta_s^\top dX_s\right) \\ &\text{subject to } \; E\left[c + \int_0^T \theta_s^\top dX_s\right] \geq A \end{aligned} \tag{2.17}$$

where A is a sufficiently large constant such that $A > c$. Hereafter, we suppose that $E[\tilde{D}^2] > 1$. Otherwise, the original probability measure P and the variance-optimal martingale measure $\tilde{P}$ almost surely coincide and it results in $E[c + \int_0^T \theta_s^\top dX_s] = c$ for any $\theta \in \Theta$. Then, in the current situation where $A > c$, it is obvious that the mean-variance portfolio selection (2.17) does not admit any optimal solution.

A solution to this problem (2.17) can be obtained as follows.

Theorem 2.11. *For the mean-variance portfolio selection problem (2.17), an optimal strategy* θ^* *is given by*

$$\theta_s^* = -\frac{A - c}{E[\tilde{D}^2] - 1}\tilde{\zeta}_s$$

for $0 \le s \le T$ *where* $\tilde{\zeta}$ *is the integrand in (2.16) and the optimal solution is obtained by*

$$Var\left(c + \int_0^T \theta_s^{*\top} dX_s\right) = \frac{(A-c)^2}{E[\tilde{D}^2] - 1}. \tag{2.18}$$

Proof. As the proof of Theorem 2.2, we first solve the following problem:

$$\begin{aligned} &\inf_{\theta \in \Theta} \ Var\left(c + \int_0^T \theta_s^\top dX_s\right) \\ &\text{subject to } \ E\left[c + \int_0^T \theta_s^\top dX_s\right] = B \end{aligned} \tag{2.19}$$

where B is a constant such that $B \ge A$. The Lagrangian corresponding to this problem (2.19) is obtained by

$$\mathcal{L}(\theta, \lambda) = E\left[\left(c + \int_0^T \theta_s^\top dX_s - B\right)^2 + \lambda\left(B - E\left[c + \int_0^T \theta_s^\top dX_s\right]\right)\right]$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier. As the proof of Theorem 2.2, by Theorem 2 in Section 8.4 in Luenberger (1969), the optimal solution $\theta^B \in \Theta$ and $\lambda^B \in \mathbb{R}$ to the problem (2.19) is given by a saddle point of $\mathcal{L}$. The Lagrangian can be rewritten as

$$\mathcal{L}(\theta, \lambda) = E\left[\left(c + \int_0^T \theta_s^\top dX_s - \left(B + \frac{1}{2}\lambda\right)\right)^2\right] - \frac{1}{4}\lambda^2. \tag{2.20}$$

The problem of minimizing this with respect to θ can be regarded as a mean-variance hedging when the payoff of a derivative is $B - c + \lambda/2$. Therefore, by Lemma 2.10, the strategy θ^B which minimizes (2.20) is given by

$$\theta_s^B = -\frac{B - c + \lambda/2}{E[\tilde{D}^2]}\tilde{\zeta}_s \tag{2.21}$$

for $0 \le s \le T$. Then, by (2.16),

$$\begin{aligned} \mathcal{L}(\theta^B, \lambda) &= E\left[\left(c - \frac{B - c + \lambda/2}{E[\tilde{D}^2]}\int_0^T \tilde{\zeta}_s^\top dX_s - \left(B + \frac{1}{2}\lambda\right)\right)^2\right] - \frac{1}{4}\lambda^2 \\ &= E\left[\left(c - \frac{B - c + \lambda/2}{E[\tilde{D}^2]}(\tilde{D} - E[\tilde{D}^2]) - \left(B + \frac{1}{2}\lambda\right)\right)^2\right] - \frac{1}{4}\lambda^2 \\ &= \frac{(B - c + \lambda/2)^2}{E[\tilde{D}^2]} - \frac{1}{4}\lambda^2. \end{aligned}$$

This is maximized when λ equals

$$\lambda^B = \frac{2(B-c)}{E[\tilde{D}^2]-1}.$$

Substituting $\lambda = \lambda^B$ into (2.21), θ^B is revealed to be

$$\theta_s^B = -\frac{B-c}{E[\tilde{D}^2]-1}\tilde{\zeta}_s \tag{2.22}$$

for $0 \le s \le T$.

Then the optimal solution to (2.19) is obtained by

$$\begin{aligned} Var\bigg(c + \int_0^T \theta_s^{B\top} dX_s\bigg) &= E\bigg[\bigg(c - \frac{B-c}{E[\tilde{D}^2]-1}\int_0^T \tilde{\zeta}_s^\top dX_s - B\bigg)^2\bigg] \\ &= (B-c)^2 E\bigg[\bigg(1 + \frac{\tilde{D} - E[\tilde{D}^2]}{E[\tilde{D}^s]-1}\bigg)^2\bigg] \\ &= (B-c)^2 \frac{E[(\tilde{D}-1)^2]}{(E[\tilde{D}^2]-1)^2} \\ &= \frac{(B-c)^2}{E[\tilde{D}^2]-1} \end{aligned} \tag{2.23}$$

for each $B \ge A$. Since this is minimized when $B = A$, the solution to the original problem (2.17) can be obtained by substituting $B = A$ into (2.22) and (2.23) and this concludes the proof. ■

Remark 2.12. We can check that $E[c + \int_0^T \theta_s^{*\top} dX_s] = A$ as follows:

$$\begin{aligned} E\bigg[c - \frac{A-c}{E[\tilde{D}^2]-1}\int_0^T \tilde{\zeta}_s^\top dX_s\bigg] &= c - \frac{A-c}{E[\tilde{D}^2]-1}E[\tilde{D} - E[\tilde{D}^2]] \\ &= c - \frac{A-c}{E[\tilde{D}^2]-1}(1 - E[\tilde{D}^2]) \\ &= A. \end{aligned}$$

Remark 2.13. Note that our result (2.18) is consistent with (6.9) in Zhou and Li (2000) if our $E[\tilde{D}^2]$, which is the square mean of the density of the variance-optimal measure is equal to $\exp\{\int_0^T \rho(t)dt\}$ in Zhou and Li (2000) and the risk-free rate is zero. Obviously, the result (2.18) is also consistent with the solution (2.7) in the previous section.

Remark 2.14. Our approach may be valid for continuous-time mean-variance portfolio selections in general semimartingale models if we employ the result given by Jeanblanc et al. (2012) to handle the mean-variance hedging. To our knowledge, there is no study which solved such a problem. This may show the novelty of our approach.

2.4 Conclusion

In this chapter, explicit solutions to multiperiod mean-variance portfolio selection problems in a discrete-time model and a continuous semimartingale model are provided. We have dealt with the problem applying the results of mean-variance hedging problems. In discrete time, for obtaining an explicit solution, we have assumed that the discounted price process of the security satisfies the deterministic mean-variance tradeoff condition. Then we have derived the explicit solution and realized a relation between the optimal solution and the variance-optimal martingale measure (or the minimal martingale measure) in the model. In the continuous-time model, we have solved a mean-variance portfolio selection problem by the same approach as the discrete-time case. The result shows that the optimal strategy of the original problem is obtained as a multiple of the optimal strategy of the mean-variance hedging. Our approach may be valid for continuous-time mean-variance portfolio selections in general semimartingale models if we employ the result given by Jeanblanc et al. (2012) to handle the mean-variance hedging. However, we have always assumed that the growth rate of the risk-free asset is zero and a way to remove this limitation is not obvious.

We have also not concerned the condition which prohibits the value of the portfolio becoming negative as Bielecki et al. (2005). It is left for the future to overcome these shortcomings.

Chapter 3

Remarks on optimal strategies to utility maximizations in continuous time incomplete markets

3.1 Introduction

In this chapter, we treat utility maximization problems through the martingale approach introduced in Chapter 1. In particular, we are interested in the PDE which characterizes the optimal policy of the problem. As mentioned in Chapter 1, the PDE is derived for the first time by Cox and Huang (1989) in a complete market. He and Pearson (1991) extended the result and derive the PDE in an incomplete market through analysis of the dual problem. In this chapter, we attempt to propose an alternative approach to derive the PDE without using the duality principle in incomplete markets. We may observe that the derivation of the PDE can be somewhat simplified when we consider only characterization of the solution.

We consider a utility maximization problem in a market model which includes M risky assets whose discounted prices are driven by the N-dimensional Brownian motion where $M \leq N$. First, we recall the whole set of equivalent martingale measures in the model. Next, we think of the objective function as a functional defined on the set of trading strategies. Then we see that the optimal solution of the utility maximization problem can be characterized by an equivalent martingale measure from a simple necessary condition of optimality of the maximization problem of the functional. The equivalent martingale measure and the PDE which characterizes the optimal strategy are identified by Itô's formula and the optimal strategy can be obtained from the solution of the PDE.

Moreover we apply our method to a utility maximization problem in a model where the price process is assumed to be a compound Poisson process. We study the case where the jump sizes of the price process are unpredictable. First, we specify the whole set of equivalent martingale measures in the model. Then, assuming that the optimal solution exists, an equation which relates to the optimal solution of the utility maximization is derived by the same approach as in the diffusion case.

The rest of the chapter is organized as follows. In Section 3.2, a security market model by Brownian motions will be specified, the all equivalent martingale measures in the model will be characterized and a utility maximization problem will be recalled. Under the assumption that the utility maximization problem admits an optimal strategy, Section 3.3 will derive the PDE which provides the optimal strategy as a necessary condition of optimality. A case of a simple jump model is treated in Section 3.4. Section 3.5 concludes the chapter.

3.2 Model settings and a utility maximization problem

In this section, a utility maximization problem is recalled.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $T > 0$ be the finite terminal time. Let $W(t) = (W^1(t), W^2(t), \cdots, W^N(t))^\top, 0 \leq t \leq T$ denote an N-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ and we suppose that we have a filtration $\mathcal{F}_t, 0 \leq t \leq T$ which is an augmented filtration generated by W. Note that, for a matrix A, $A^\top$ denotes the transpose of A.

We consider a single investor's utility maximization problem. We assume that there are $M(\leq N)$ risky assets that the investor can trade in a market. The discounted price process of those risky assets, denoted by $S(t) = (S^1(t), S^2(t), \cdots, S^M(t))^\top, 0 \leq t \leq T$, is supposed to be defined by

$$\begin{aligned} dS(t) &= a(t)dt + b(t)dW(t), \quad 0 \leq t \leq T, \\ S(0) &= S_0 \end{aligned}$$

where a is an $M \times 1$ vector valued progressively measurable process, b is an $M \times N$ matrix valued progressively measurable process and $S_0 > 0$ is a constant. We suppose further that

$$\int_0^T |a(t)|dt < \infty \quad a.s.$$

and $\mathrm{rank}(b(t)) = M$ for all t where, for a matrix A, we assume $|A| = \mathrm{tr}(AA^\top)^{\frac{1}{2}}$.

By Proposition 2.2 in Pagè (1987) or Proposition 1 in He and Pearson (1991), the Radon-Nikodym derivatives of equivalent martingale measures in the current model are characterized as follows. As He and Pearson (1991), defining $\kappa(t) = -b(t)^\top(b(t)b(t)^\top)^{-1}a(t),\ 0 \le t \le T$, we assume

$$E\left[\exp\left\{\frac{1}{2}\int_0^T |\kappa(t)|^2 dt\right\}\right] < \infty$$
$$E\left[\int_0^T |b(t)|^4 dt\right] < \infty$$

for ensuring the existence of an equivalent martingale measure. If Q is an equivalent martingale measure with respect to P, then $dQ/dP = \xi_\nu(T)$ for some $\nu \in \mathrm{Ker}(b)$ such that $\int_0^T |\nu(t)|^2 dt < \infty$ $a.s.$ where

$$\xi_\nu(t) = \exp\left\{\int_0^t (\kappa(s)+\nu(s))^\top dW(s) - \frac{1}{2}\int_0^t (|\kappa(s)|^2 + |\nu(s)|^2) ds\right\}, \quad 0 \le t \le T \tag{3.1}$$

with

$$\mathrm{Ker}(b) = \{\nu | \nu \text{ is progressively measurable,}$$
$$\nu(t) \in \mathbb{R}^N \text{ and } b(t)\nu(t) = 0,\ 0 \le t \le T\}.$$

The whole set Θ of the investor's strategies is defined by

$$\Theta := \left\{ z(t) = (z^1(t), z^2(t), \cdots, z^M(t))^\top, 0 \le t \le T \,\middle|\, z \text{ is adapted,} \right.$$
$$\left. \int_0^T |z(t)^\top a(t)| dt < \infty \ \ a.s. \ \text{ and } \ \int_0^T |z(t)^\top b(t)|^2 dt < \infty \ \ a.s. \right\}$$

where $z^i(t), i = 1, 2, \cdots, M$ is the number of the i th asset that the investor holds at time $t, 0 \le t \le T$. Obviously, Θ forms a vector space on $\mathbb{R}$. The investor's gain processes are defined by

$$G_t(z) := \int_0^t z(s)^\top dS(s), \qquad z \in \Theta, \qquad 0 \le t \le T.$$

Let $x > 0$ be the investor's initial endowment. Then the value of a self-financed portfolio corresponding to a strategy $z \in \Theta$ at time t can be obtained by $x + G_t(z)$. Suppose that the investor has a utility $u(y)$ defined on

$(k, \infty), -\infty \leq k < \infty$. Suppose further that u is a continuously differentiable, strictly increasing and strictly concave function. In this chapter, we also suppose that u is bounded from below and it is the severest restriction in comparison with the model in He and Pearson (1991). We also assume that the inverse function of the derivative of the utility function $I(y) := (u')^{-1}(y)$ defined on $\left(\lim_{l \to k+0} u'(l), \infty\right)$ is continuously differentiable.

Finally, we can define the following utility maximization problem:

$$\sup_{z \in \Theta} E[u(x + G_T(z))]. \tag{3.2}$$

3.3 Optimal strategies in Brownian models

In this section, the PDE which yields the optimal strategy to the problem (3.2) is obtained.

Before providing the result, we need some preparations. We assume that the problem (3.2) admits an optimal strategy which is denoted by $z_0(t),\ 0 \leq t \leq T$. We define a functional f on Θ by

$$f(z) := E[u(x + G_T(z))], \quad z \in \Theta.$$

Theorem 1 of Section 7.4 in Luenberger (1969) suggests that if z_0 is an optimal strategy, the Gateaux differential of f at z_0 equals zero, i.e.,

$$\frac{\partial}{\partial \alpha} f(z_0 + \alpha z)|_{\alpha=0} = E[u'(x + G_T(z_0))G_T(z)] = 0, \quad \forall z \in \Theta$$

where $\alpha \in \mathbb{R}$. The interchange of the expectation and the differentiation is confirmed from the cndition that u is bounded from below. Then, since $u' > 0$, we can claim that $u'(x+G_T(z_0))/E[u'(x+G_T(z_0))]$ is the Radon-Nikodym density of an equivalent martingale measure. Therefore, there exists a process

$\nu_0 \in \mathrm{Ker}(b)$ such that

$$\frac{u'(x+G_T(z_0))}{E[u'(x+G_T(z_0))]} = \xi_{\nu_0}(T) \tag{3.3}$$

where $\xi_{\nu_0}(t),\ 0 \le t \le T$ is the process defined by (3.1) corresponding to ν_0.

The following theorem provides an explicit expression of z_0 when (ξ_{ν_0}, S) are Markov processes. Below we use the abbreviation that $F_t = \partial F(t,\xi,S)/\partial t$, $F_\xi = \partial F(t,\xi,S)/\partial \xi$, $F_{\xi\xi} = \partial^2 F(t,\xi,S)/\partial \xi^2$, F_S is a $M \times 1$ vector valued process $\partial F(t,\xi,S)/\partial S$, $F_{S\xi}$ is a $M \times 1$ vector valued process $\partial^2 F(t,\xi,S)/\partial S \partial \xi$ and F_{SS} is an $M \times M$ matrix valued process $\partial^2 F(t,\xi,S)/\partial S^2$.

Theorem 3.1. *Suppose that (ξ_{ν_0}, S) forms a Markov process where ξ_{ν_0} is defined by (3.3) and z_0 is an optimal strategy to (3.2). Define the function $F(t,\xi,S)$ by*

$$F(t,\xi_{\nu_0}(t),S(t)) := \frac{1}{\xi_{\nu_0}(t)} E[I(E[u'(x+G_T(z_0))]\xi_{\nu_0}(T))\xi_{\nu_0}(T)|\xi_{\nu_0}(t),S(t)]$$

where $I = (u')^{-1}$. Suppose further that F is a $C^{1,2,2}$ function and $F_\xi \neq 0$. Then F satisfies the partial differential equation

$$F_t + \frac{1}{2}|\kappa|^2 \xi_{\nu_0}^2 F_{\xi\xi} + |\kappa|^2 \xi_{\nu_0} F_\xi + \frac{1}{2}\mathrm{tr}(F_{SS} bb^\top) - F_{S\xi}^\top a = 0$$

with the boundary conditions

$$\begin{aligned} F(T,\xi_{\nu_0}(T),S(T)) &= I(E[u'(x+G_T(z_0))\xi_{\nu_0}(T)]), \\ F(0,1,S_0) &= x \end{aligned}$$

where $\nu_0 = 0$. Moreover, the optimal strategy z_0 is given from F by

$$z_0^\top = \xi_{\nu_0} F_\xi \kappa^\top b^\top (bb^\top)^{-1} + F_S^\top.$$

Proof. Since $u'(x+G_T(z_0)) = E[u'(x+G_T(z_0))]\xi_{\nu_0}(T)$,

$$x + G_T(z_0) = I(E[u'(x+G_T(z_0))]\xi_{\nu_0}(T)).$$

By multiplying both side of this equation by $\xi_{\nu_0}(T)$, we obtain

$$(x + G_T(z_0))\xi_{\nu_0}(T) = I(E[u'(x + G_T(z_0))]\xi_{\nu_0}(T))\xi_{\nu_0}(T).$$

Since the left hand side of this equation is a P-martingale, by taking conditional expectations,

$$(x + G_t(z_0))\xi_{\nu_0}(t) = E[I(E[u'(x + G_T(z_0))]\xi_{\nu_0}(T))\xi_{\nu_0}(T)|\mathcal{F}_t].$$

Then we get

$$\begin{aligned}
x + G_t(z_0) &= \frac{1}{\xi_{\nu_0}(t)}E[I(E[u'(x + G_T(z_0))]\xi_{\nu_0}(T))\xi_{\nu_0}(T)|\mathcal{F}_t] \\
&= \frac{1}{\xi_{\nu_0}(t)}E[I(E[u'(x + G_T(z_0))]\xi_{\nu_0}(T))\xi_{\nu_0}(T)|\xi_{\nu_0}(t), S(t)] \\
&= F(t, \xi_{\nu_0}(t), S(t)).
\end{aligned}$$

By Itô's formula,

$$\begin{aligned}
z_0^\top a dt + z_0^\top b dW =& \left(F_t + \frac{1}{2}(|\kappa|^2 + |\nu_0|^2)\xi_{\nu_0}^2 F_{\xi\xi} + F_S^\top a + \frac{1}{2}\mathrm{tr}(F_{SS}bb^\top) + F_{S\xi}^\top b\kappa\right)dt \\
&+ (\xi_{\nu_0}F_\xi(\kappa + \nu_0)^\top + F_S^\top b)dW.
\end{aligned}$$

Then we obtain the following simultaneous equations:

$$z_0^\top a = F_t + \frac{1}{2}(|\kappa|^2 + |\nu_0|^2)\xi_{\nu_0}^2 F_{\xi\xi} + F_S^\top a + \frac{1}{2}\mathrm{tr}(F_{SS}bb^\top) + F_{S\xi}^\top b\kappa, \tag{3.4}$$

$$z_0^\top b = \xi_{\nu_0}F_\xi(\kappa + \nu_0)^\top + F_S^\top b. \tag{3.5}$$

By multiplying both sides of (3.5) by $b^\top(bb^\top)^{-1}$, $z_0^\top$ is determined as

$$z_0^\top = \xi_{\nu_0}F_\xi\kappa^\top b^\top(bb^\top)^{-1} + F_S^\top. \tag{3.6}$$

Substituting this into (3.5), we have

$$\xi_{\nu_0}F_\xi\kappa^\top = \xi_{\nu_0}F_\xi(\kappa + \nu_0)^\top$$

by the definition of κ and we get $F_\xi \nu_0^\top = 0$. This implies $\nu_0 = 0$ by the assumption $F_\xi \neq 0$. From (3.6) and $\nu_0 = 0$, (3.4) becomes

$$F_t + \frac{1}{2}|\kappa|^2 \xi_{\nu_0}^2 F_{\xi\xi} + |\kappa|^2 \xi_{\nu_0} F_\xi + \frac{1}{2}\mathrm{tr}(F_{SS} bb^\top) - F_{S\xi}^\top a = 0.$$

■

3.4 An optimal strategy in a jump model

In this section, we apply the method of deriving the equation which characterizes the solution given in the previous section to a utility maximization problem in a compound Poisson model.

Let $N(t),\ 0 \leq t \leq T$ be a Poisson process defined on $(\Omega, \mathcal{F}, P)$ with the intensity $\lambda > 0$ and $T_i,\ i = 1, 2, \ldots$ denote times of i-th jump of N. Let $C_i,\ i = 1, 2, \ldots$ be independent and identically distributed (i.i.d.) random variables which are supposed to have common distribution μ. Suppose further that each C_i and N are independent, $i = 1, 2, \ldots$. It is also assumed that we have a filtration $\mathcal{F}_t^{N,C},\ 0 \leq t \leq T$ which is defined by $\mathcal{F}_t^{N,C} := \sigma(N(s), C_{N(s)}, s \leq t)$.

Suppose that there are one risky asset and one risk-free asset in the market. We suppose further that the discounted asset price process $Y(t)$ is given by

$$\begin{aligned} dY(t) &= Y(t-)(\alpha dt + C_{N(t)} dN(t)), \qquad 0 \leq t \leq T, \\ Y(0) &= Y_0 \end{aligned} \tag{3.7}$$

where α and Y_0 are constants and $\alpha \neq 0$.

We consider a problem which is similar to the problem (3.2) in the model

(3.7). We define the whole set of strategies by

$$\Theta^Y := \left\{ z(t),\ 0 \le t \le T \,\middle|\, z \text{ is predictable}, \int_0^T |\alpha z(t) Y(t-)| dt < \infty\ a.s. \right.$$
$$\left. \text{and} \int_0^T |C_{N(t)} z(t) Y(t-)| dt < \infty,\ a.s. \right\}$$

and the gain processes by

$$H_t(z) := \int_0^t z(s) dY(s), \quad z \in \Theta^Y, \quad 0 \le t \le T$$

and we consider the following utility maximization problem:

$$\sup_{z \in \Theta^Y} E[u(x + H_T(z))] \tag{3.8}$$

where u is a utility function which satisfies the same conditions provided in Section 3.2. We assume that the problem (3.8) admits an optimal strategy and the strategy is denoted by $z^Y(t),\ 0 \le t \le T$. In the subsequent argument, we see that the similar result to Theorem 3.1 may hold true in this situation.

First, the whole of the equivalent martingale measures for the model (3.7) is characterized. Let Q be an equivalent martingale measure with respect to P. Then, according to Theorem 4.2 in Bardhan and Chao (1996), there exists a predictable process $\zeta(t, y)$ which satisfies

$$\zeta > 0 \quad \text{and} \quad -\frac{\alpha}{\int_{\mathbb{R}} y \zeta(t, y) d\mu(y)} > 0, \quad \forall t$$

and dQ/dP is written by $dQ/dP = R_\zeta(T)$ where

$$R_\zeta(t) := \exp\left\{ \lambda t + \int_0^t \frac{\alpha}{\int_{\mathbb{R}} y \zeta(s, y) d\mu(y)} ds + \int_0^t \log\left(\frac{-\alpha \zeta(s, C_{N(s)})}{\lambda \int_{\mathbb{R}} y \zeta(s, y) d\mu(y)} \right) dN(s) \right\}, \tag{3.9}$$

for $0 \le t \le T$. To see this, since we have not used Poisson random measures, we should show the correspondence of notations between Bardhan and Chao (1996) and ours. The correspondence is as follows.

- Our model is the case where $d = 0$ (i.e. no Brownian motions), $n = 1$, $\sigma = 1$ and $r = 0$ in Bardhan and Chao (1996),
- a characteristic of a martingale measure, $\varphi_1(t, z) = \zeta(t, z)$,
- the rate of jumps, $\lambda_1(t) = \lambda$,
- the distribution of jump sizes, $\phi_1(t, z) = \mu(z)$,
- expectations of jump sizes at t, $\alpha(t) = E[C_1]$,
- expectations of jump sizes at t under the measure changed by φ (or ζ), $\tilde{\alpha}(t) = \int_{\mathbb{R}} z\zeta(t, z)d\mu(z)$ and
- the drift term, $b(t) = \alpha + \lambda E[C_1]$ (see equation (2.1) in Bardhan and Chao (1996)),

where the left hand sides of above equations are notations in Bardhan and Chao (1996) and right hand sides are ours. Then ϑ_1 in Bardhan and Chao (1996) is determined by equation (4.4) in Bardhan and Chao (1996). In our model, equation (4.4) in Bardhan and Chao (1996) is

$$\lambda\Bigg(E[C_1] - \vartheta_1(t)\int_{\mathbb{R}} z\zeta(t, z)d\mu(z)\Bigg) = \alpha + \lambda E[C_1].$$
$$\therefore \quad \vartheta_1(t) = -\frac{\alpha}{\lambda \int_{\mathbb{R}} z\zeta(t, z)d\mu(z)}.$$

Then equation (4.5) in Bardhan and Chao (1996) can be expressed as follows. The term in the second line in equation (4.5) in Bardhan and Chao (1996) is

$$\begin{aligned}\prod_{n=1}^{N(t)}\left(-\frac{\alpha\zeta(T_n, C_{N_{T_n}}))}{\lambda \int_{\mathbb{R}} z\zeta(T_n, z)d\mu(z)}\right) &= \exp\left\{\sum_{n=1}^{N(t)}\log\left(-\frac{\alpha\zeta(T_n, C_{N_{T_n}}))}{\lambda \int_{\mathbb{R}} z\zeta(T_n, z)d\mu(z)}\right)\right\} \\ &= \exp\left\{\int_0^t \log\left(-\frac{\alpha\zeta(s, C_{N_s})}{\lambda \int_{\mathbb{R}} z\zeta(s, z)d\mu(z)}\right)dN(s)\right\}\end{aligned}$$

and the argument of the exponential function in the third line in equation (4.5) in Bardhan and Chao (1996) is

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} \left(1 + \frac{\alpha\zeta(s,z)}{\lambda \int_{\mathbb{R}} y\zeta(s,y)d\mu(y)}\right)\lambda d\mu(z)ds &= \lambda t + \int_0^t \int_{\mathbb{R}} \frac{\alpha\zeta(s,z)}{\int_{\mathbb{R}} y\zeta(s,y)d\mu(y)} d\mu(z)ds \\ &= \lambda t + \int_0^t \frac{\alpha}{\int_{\mathbb{R}} y\zeta(s,y)d\mu(y)} ds\end{aligned}$$

where the last equation is a consequence of equation (4.3) in Bardhan and Chao (1996) which is $\int_{\mathbb{R}} \zeta(s,z)d\mu(z) = 1$ in our model. This concludes that every martingale measure has a density R_ζ by Theorem 4.2 in Bardhan and Chao (1996). Suppose ζ_0 is the process corresponding to z^Y, i.e.,

$$\frac{u'(x + H_T(z^Y))}{E[u'(x + H_T(z^Y))]} = R_{\zeta_0}(T) \tag{3.10}$$

and we assume that R_{ζ_0} is a Markov process.

Next, by the same argument as the proof of Theorem 3.1 in the previous section, when we define a function $J(t,R)$ by

$$J(t, R_{\zeta_0}(t)) := \frac{1}{R_{\zeta_0}(t)} E[I(E[u'(x + H_T(z^Y))]R_{\zeta_0}(T))R_{\zeta_0}(T)|R_{\zeta_0}(t)],$$

we can assert that by Itô's formula,

$$\begin{aligned}&\alpha z^Y(t)Y(t-)dt + C_{N(t)}z^Y(t)Y(t-)dN(t) = dJ(t, R_{\zeta_0}(t)) \\ \therefore\ &\alpha z^Y(t)Y(t-)dt + C_{N(t)}z^Y(t)Y(t-)dN(t) \\ &\quad = \left\{ J_t(t, R_{\zeta_0}(t-)) + \left(\lambda + \frac{\alpha}{\int_{\mathbb{R}} y\zeta_0(t-,y)d\mu(y)}\right) R_{\zeta_0}(t-)J_R(t, R_{\zeta_0}(t-)) \right\} dt \\ &\qquad + \left[J\left(t, -\frac{\alpha\zeta_0(t, C_{N(t)})}{\lambda \int_{\mathbb{R}} y\zeta_0(t,y)d\mu(y)} R_{\zeta_0}(t-)\right) - J(t, R_{\zeta_0}(t-)) \right] dN(t),\end{aligned}$$

where the subscripts of the function J denote the partial differentials of J with respect to each argument. Then we obtain the following simultaneous equations:

$$
\begin{aligned}
\alpha z^Y(t)Y(t-) &= J_t(t, R_{\zeta_0}(t-)) + \left(\lambda + \frac{\alpha}{\int_{\mathbb{R}} y\zeta_0(t-, y)d\mu(y)}\right) R_{\zeta_0}(t-) J_R(t, R_{\zeta_0}(t-)),\\
C_{N(t)} z^Y(t)Y(t-) &= J\left(t, -\frac{\alpha\zeta_0(t, C_{N(t)})}{\lambda\int_{\mathbb{R}} y\zeta_0(t, y)d\mu(y)} R_{\zeta_0}(t-)\right) - J(t, R_{\zeta_0}(t-)).
\end{aligned}
\tag{3.11}
$$

From these, we conclude that J satisfies the following equation:

$$
\begin{aligned}
&\frac{C_{N(t)}}{\alpha}\left[J_t(t, R_{\zeta_0}(t-)) + \left(\lambda + \frac{\alpha}{\int_{\mathbb{R}} y\zeta_0(t-, y)d\mu(y)}\right) R_{\zeta_0}(t-) J_R(t, R_{\zeta_0}(t-))\right]\\
&\qquad = J\left(t, -\frac{\alpha\zeta_0(t, C_{N(t)})}{\lambda\int_{\mathbb{R}} y\zeta_0(t, y)d\mu(y)} R_{\zeta_0}(t-)\right) - J(t, R_{\zeta_0}(t-)).
\end{aligned}
$$

Therefore, we have obtained the following proposition.

Proposition 3.2. *Suppose that z^Y is an optimal strategy to (3.8) and R_{ζ_0} defined by (3.10) is a Markov process where R_ζ is defined by (3.9). Define the function $J(t, R)$ by*

$$
J(t, R_{\zeta_0}(t)) := \frac{1}{R_{\zeta_0}(t)} E[I(E[u'(x + H_T(z^Y))] R_{\zeta_0}(T)) R_{\zeta_0}(T) | R_{\zeta_0}(t)],
$$

where $I = (u')^{-1}$. Suppose further that J is a $C^{1,1}$ function. Then J satisfies the equation

$$
\begin{aligned}
&\frac{C_{N(t)}}{\alpha}\left[J_t(t, R_{\zeta_0}(t-)) + \left(\lambda + \frac{\alpha}{\int_{\mathbb{R}} y\zeta_0(t-, y)d\mu(y)}\right) R_{\zeta_0}(t-) J_R(t, R_{\zeta_0}(t-))\right]\\
&\qquad = J\left(t, -\frac{\alpha\zeta_0(t, C_{N(t)})}{\lambda\int_{\mathbb{R}} y\zeta_0(t, y)d\mu(y)} R_{\zeta_0}(t-)\right) - J(t, R_{\zeta_0}(t-)).
\end{aligned}
\tag{3.12}
$$

Remark 3.3. The optimal strategy z^Y can be calculated from (3.11) as soon as the equation (3.12) is solved with proper boundary conditions. However, unlike the Brownian case in the previous section, we have not been able to determine the characteristic of the martingale measure ζ_0 yet. For further characterization, we must find a relation between ζ_0 and the function J from the second equation in (3.11).

3.5 Conclusion

In this chapter, we have proposed an alternative approach to derive the PDE which relates to the optimal solution of the utility maximization problem in a diffusion model without using the duality principle. We characterized the optimal solution with an equivalent martingale measure from a simple necessary condition of optimality. Moreover we applied our method to a utility maximization problem in a compound Poisson model with unpredictable jump sizes. However, in the result in the compound Poisson model, a strong assumption that the Radon-Nikodym derivative process corresponding to the optimal solution is a Markov process is imposed. Moreover the way to solve the obtained equation is not obvious. It is left for future subjects to resolve these difficulties.

Chapter 4

On discrete Itô formulas and discrete Hamilton-Jacobi-Bellman equations

4.1 Introduction

In this chapter, we prove a discrete Itô formula for discrete jump-diffusion processes and derive a discrete Hamilton-Jacobi-Bellman (dHJB) equation for expected utility maximization problems in a discrete jump-diffusion model. Moreover we will analyze a relation between a Hamilton-Jacobi-Bellman (HJB) variational inequality in a continuous-time geometric Brownian model and a dHJB equation in a random walk model derived by Ishimura and Mita (2009).

A discrete Itô formula was originally obtained by Fujita and Kawanishi (2008) for random walks. Since the increment of the random walk is 1 or

-1, the proof can be done by simply checking that the formula holds in each case. We extend this idea to discrete jump-diffusion processes. Here, a discrete jump-diffusion process is a discrete stochastic process which have a discrete Poisson process term as well as a random walk term. Although the formula includes some additional term in comparison with the formula for randam walks, the proof remains simple.

Ishimura and Mita (2009) applied the discrete Itô formula for random walks to the analysis of a utility maximization problem in discrete time. They derived a dHJB equation and proved the verification theorem, which shows that the solution of the dHJB equation actually provides the optimal solution of the utility maximization problem. Since the derivation of the dHJB equation and the proof of the verification theorem have been essentially carried out by the discrete Itô formula, we can generalize the result in a discrete jump-diffusion model. We set a financial model and a utility maximization problem as discrete analogues of ones in Aase (1984) which studied utility maximizations in a simple jump-diffusion model. Then, deriving the optimality equation for the utility maximization in the model, we apply the discrete Itô formula derived in advance to the optimality equation and obtain the dHJB equaiton in the discrete jump-diffusion model.

After that, we will also provide an application of the dHJB equation in a random walk model. As we have mentioned in Chapter 1, we provide the condition which assures that a proper limit of a solution of the dHJB equation becomes a viscosity solution of the corresponding HJB variational inequality in continuous time. The viscosity solution is a kind of weak solutions of some partial differential equations. The HJB variational inequality is often employed to deal with singular stochastic optimization problems (i.e., the problems whose Hamiltonians of the HJB equations diverge). In Chapter 4 of Pham (2009), it has been proved that the value function of a utility maximization problem which is possibly singular is the unique viscosity solution of the corresponding HJB variational inequality. Our result conversely derives a viscosity solution of the HJB variational inequality from the solutions of the dHJB equations in the random walk model derived by Ishimura and Mita (2009). If the conditions in the theorem are satisfied, this result enables us to specify the value function of the optimization problem and consequently find the optimal solution.

The rest of the chapter is organized as follows. Section 4.2 will derive a discrete Itô formula and a dHJB equation in a discrete jump-diffusion model. In Section 4.3 , it will be revealed that a solution of a dHJB equation in a random walk model converges to a viscosity solution of an HJB variational inequality. Section 4.4 concludes the chapter.

4.2 A discrete Itô formula and a dHJB equation in discrete jump-diffusion models

In this section, we prove a discrete Itô formula for discrete jump-diffusion processes and derive a dHJB equation in a discrete jump-diffusion model.

4.2.1 A discrete Itô formula for discrete jump-diffusion processes

In this subsection, we show that a discrete Itô formula holds for discrete jump-diffusion processes.

Let $(B_t)_{t=0,1,2,\ldots}$ be a symmetric random walk and $(N_t)_{t=0,1,2,\ldots}$ be a discrete Poisson process. To be precise, $(B_t)_{t=0,1,2,\ldots}$ and $(N_t)_{t=0,1,2,\ldots}$ are supposed to be discrete stochastic processes such that

$$
\begin{aligned}
&P(B_{t+1} - B_t = 1) = P(B_{t+1} - B_t = -1) = \frac{1}{2}, \\
&P(N_{t+1} - N_t = 1) = q,\ P(N_{t+1} - N_t = 0) = 1 - q
\end{aligned}
$$

for $t = 0, 1, 2, \ldots$ and $B_0 = N_0 = 0$ where $0 < q < 1$.

Lemma in Fujita and Kawanishi (2008) and Proposition 1 in Ishimura and Mita (2009) showed that a discrete Itô formula holds for random walks. Fujita and Kawanishi (2008) used the formula to prove Itô's formula for Brownian motions in continuous time and Ishimura and Mita (2009) derived a dHJB equation in a random walk model and prove the verification theorem by the formula. We exhibit the formula which is a little more general, though the proof is unchanged.

Theorem 4.1 (Lemma in Fujita and Kawanishi (2008), Proposition 1 in Ishimura and Mita (2009))**.** *Let* $(Y_t)_{t=0,1,2,\ldots}$ *be a discrete stochastic process which satisfies*

$$
Y_{t+1} - Y_t = m_t + s_t(B_{t+1} - B_t)
$$

for some $\mathbb{R}$*-valued random variables* m_t *and* s_t*,* $t = 0, 1, 2, \ldots$*. Then, for any function* $f : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$*, the following equation holds:*

$$
\begin{aligned}
&f(Y_{t+1}, t+1) - f(Y_t, t) \\
&= f(Y_t + m_t, t+1) - f(Y_t, t+1) \\
&+ \frac{1}{2}\Big(f(Y_t + m_t + s_t, t+1) - f(Y_t + m_t - s_t, t+1)\Big)(B_{t+1} - B_t) \\
&+ \frac{1}{2}\Big(f(Y_t + m_t + s_t, t+1) - 2f(Y_t + m_t, t+1) + f(Y_t + m_t - s_t, t+1)\Big) \\
&+ f(Y_t, t+1) - f(Y_t, t)
\end{aligned}
$$

for $t = 0, 1, 2, \ldots$*.*

Proof. If $B_{t+1} - B_t = 1$, then the right hand side agrees with the left hand side of the equation. If $B_{t+1} - B_t = -1$, both sides of the equation agrees. This finishes the proof. ■

We can generalize this result for discrete jump-diffusion processes which have an additional term driven by the discrete Poisson process as well as the random walk.

Theorem 4.2. *Let $(X_t)_{t=0,1,2,\ldots}$ be a discrete stochastic process which satisfies*

$$X_{t+1} - X_t = m_t + s_t(B_{t+1} - B_t) + l_t(N_{t+1} - N_t)$$

for some $\mathbb{R}$-valued random variables m_t, s_t and l_t, $t = 0, 1, 2, \ldots$. Then, for any function $f : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$, the following equation holds:

$$\begin{aligned}
&f(X_{t+1}, t+1) - f(X_t, t)\\
&= f(X_t + m_t, t+1) - f(X_t, t+1)\\
&+ \frac{1}{2}\Big(f(X_t + m_t + s_t, t+1) - f(X_t + m_t - s_t, t+1)\Big)(B_{t+1} - B_t)\\
&+ \frac{1}{2}\Big(f(X_t + m_t + s_t, t+1) - 2f(X_t + m_t, t+1) + f(X_t + m_t - s_t, t+1)\Big)\\
&+ f(X_t, t+1) - f(X_t, t)\\
&+ \frac{1}{2}\Big(f(X_t + m_t + s_t + l_t, t+1) - f(X_t + m_t + s_t, t+1)\\
&\quad - f(X_t + m_t - s_t + l_t, t+1) + f(X_t + m_t - s_t, t+1)\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot (B_{t+1} - B_t)(N_{t+1} - N_t)\\
&+ \frac{1}{2}\Big(f(X_t + m_t + s_t + l_t, t+1) - f(X_t + m_t + s_t, t+1)\\
&\quad + f(X_t + m_t - s_t + l_t, t+1) - f(X_t + m_t - s_t, t+1)\Big)(N_{t+1} - N_t)
\end{aligned}$$

for $t = 0, 1, 2, \ldots$.

Proof. In a similar way to the proof of Theorem 4.1, we only have to show the both sides of the equation agrees when $(B_{t+1} - B_t, N_{t+1} - N_t) = (1, 1)$,

$(1,0)$, $(-1,1)$ and $(-1,0)$, respectively. Since it is straightforward, we omit details. ■

4.2.2 A dHJB equation in a discrete jump-diffusion model

In this subsection, we derive a dHJB equation in a discrete jump-diffusion model. This is a direct extension of Theorem 2 in Ishimura and Mita (2009).

We set an expected utility maximization problem in a discrete jump-diffusion model. Suppose that we have a filtration $(\mathcal{F}_t)_{t=0,1,2,\ldots}$ which is defined by $\mathcal{F}_t = \sigma(B_0, B_1, \ldots, B_t, N_0, N_1, \ldots, N_t)$ for $t = 0, 1, 2, \ldots, T$. We define the whole set of trading strategies by

$$\Theta := \{u = (u_t)_{0\leq t\leq T-1} | u_t \text{ is a } \mathcal{F}_t\text{-measureable random variable} \\ \text{for } t = 0, 1, 2, \ldots, T-1.\}.$$

We also define an auxiliary set of trading strategies by

$$\Theta(t, T-1) := \{u = (u_s)_{t\leq s\leq T-1} | u_s \text{ is a } \mathcal{F}_s\text{-measureable random variable} \\ \text{for } s = t, t+1, \ldots, T-1.\}.$$

We consider the following problem:

$$\sup_{u\in\Theta} E[U(X_T^{u,0,y})] \tag{4.1}$$

where U is a utility function and, for $u \in \Theta(t, T-1)$, $0 \leq t \leq T$ and $y \in \mathbb{R}$, $X^{u,t,y}$ satisfies

$$\begin{aligned} X_{s+1}^{u,t,y} - X_s^{u,t,y} =& ((\mu - r)u_s + r)X_s^{u,t,y} \\ &+ \sigma u_s X_s^{u,t,y}(B_{s+1} - B_s) + c u_s X_s^{u,t,y}(N_{s+1} - N_s), \\ X_t^{u,t,y} =& y \end{aligned}$$

for $t \leq s \leq T-1$ where μ, r, σ and c are constants such that $\mu > r$. In words, $X_s^{u,t,y}$ is the value of the portfolio which is started with value y at time t and traded by the strategy u until time s. This is a discrete analogue of the problem studied in Aase (1984). For convenience, we use the following notations:

$$\alpha^u := (\mu - r)u + r, \quad \beta^u := \sigma u, \quad \gamma^u := cu$$

for $u \in \mathbb{R}$. We define a value function associated with the problem (4.1) by

$$\begin{aligned} v(x,t) &:= \sup_{u \in \Theta(t,T-1)} E[U(X_T^{u,t,x})|\mathcal{F}_t], \qquad (4.2) \\ v(x,T) &= U(x) \end{aligned}$$

for $x \in \mathbb{R}$ and $t = 0, 1, 2, \ldots, T-1$. In this section, we assume that $v(x,t) < \infty$ for all x and t.

From the value function v, we can obtain a dHJB equation as follows. Since the dynamic programming in discrete time is unfamiliar, we give a proof in detail for completeness. For the argument of the dynamic programming in the following proof, we have referred to proofs of Proposition A.1 and Proposition 2.1 in Gugushvili (2003) which show the same fact in the context of mean-variance hedgings in discrete time.

Theorem 4.3. *For $t = 0, 1, 2, \ldots, T-1$, the value function v defined by (4.2) satisfies a dHJB equation*

$$\begin{aligned} &\sup_{u \in \mathbb{R}} \Big\{ V(x + \alpha^u x, t+1) - V(x, t+1) \\ &+ \frac{1}{2}\Big(V(x + \alpha^u x + \beta^u x, t+1) - 2V(x + \alpha^u x, t+1) \\ &+ V(x + \alpha^u x - \beta^u x, t+1) \Big) + V(x, t+1) - V(x,t) \\ &+ \frac{q}{2}\Big(V(x + \alpha^u x + \beta^u x + \gamma^u x, t+1) - V(x + \alpha^u x + \beta^u x, t+1) \\ &+ V(x + \alpha^u x - \beta^u x + \gamma^u x, t+1) - V(x + \alpha^u x - \beta^u x, t+1) \Big) \Big\} = 0. \qquad (4.3) \end{aligned}$$

Proof. We divide the proof into four steps.

(i) First, we show that for any fixed $\hat{u} \in \Theta$,

$$
\begin{aligned}
\sup_{u\in\Theta(t+1,T-1)} & E\left[U\left(X_T^{u,t+1,X_{t+1}^{\hat{u},0,y}}\right)\middle|\mathcal{F}_t\right] \\
&=E\left[\sup_{u\in\Theta(t+1,T-1)} E\left[U\left(X_T^{u,t+1,X_{t+1}^{\hat{u},0,y}}\right)\middle|\mathcal{F}_{t+1}\right]\middle|\mathcal{F}_t\right]. \qquad (4.4)
\end{aligned}
$$

Indeed, for any fixed $\hat{u} \in \Theta$,

$$
\begin{aligned}
E\left[\sup_{u\in\Theta(t+1,T-1)} E\left[U\left(X_T^{u,t+1,X_{t+1}^{\hat{u},0,y}}\right)\middle|\mathcal{F}_{t+1}\right]\middle|\mathcal{F}_t\right]
&\geq E\left[E\left[U\left(X_T^{u,t+1,X_{t+1}^{\hat{u},0,y}}\right)\middle|\mathcal{F}_{t+1}\right]\middle|\mathcal{F}_t\right] \\
&= E\left[U\left(X_T^{u,t+1,X_{t+1}^{\hat{u},0,y}}\right)\middle|\mathcal{F}_t\right].
\end{aligned}
$$

Then, taking supremum, we have

$$
\begin{aligned}
E&\left[\sup_{u\in\Theta(t+1,T-1)} E\left[U\left(X_T^{u,t+1,X_{t+1}^{\hat{u},0,y}}\right)\middle|\mathcal{F}_{t+1}\right]\middle|\mathcal{F}_t\right] \\
&\geq \sup_{u\in\Theta(t+1,T-1)} E\left[U\left(X_T^{u,t+1,X_{t+1}^{\hat{u},0,y}}\right)\middle|\mathcal{F}_t\right]. \qquad (4.5)
\end{aligned}
$$

On the other hand, there exists a sequence $u^n \in \Theta(t+1, T-1)$, $n = 1, 2, \ldots$ such that

$$
E\left[U\left(X_T^{u^n,t+1,X_{t+1}^{\hat{u},0,y}}\right)\middle|\mathcal{F}_{t+1}\right]
$$

converges to

$$
\sup_{u\in\Theta(t+1,T-1)} E\left[U\left(X_T^{u,t+1,X_{t+1}^{\hat{u},0,y}}\right)\middle|\mathcal{F}_{t+1}\right]
$$

monotonously from below as $n \to \infty$. Then, by the monotone convergence theorem, we have

$$
\begin{aligned}
E\left[U\left(X_T^{n,t+1,X_{t+1}^{\hat{u},0,y}}\right)\middle|\mathcal{F}_t\right] &= E\left[E\left[U\left(X_T^{u^n,t+1,X_{t+1}^{\hat{u},0,y}}\right)\middle|\mathcal{F}_{t+1}\right]\middle|\mathcal{F}_t\right] \\
&\to E\left[\sup_{u\in\Theta(t+1,T-1)} E\left[U\left(X_T^{u,t+1,X_{t+1}^{\hat{u},0,y}}\right)\middle|\mathcal{F}_{t+1}\right]\middle|\mathcal{F}_t\right].
\end{aligned}
$$

as $n \to \infty$. Since

$$\sup_{u\in\Theta(t+1,T-1)} E\Big[U\Big(X_T^{u,t+1,X_{t+1}^{\hat{u},0,y}}\Big)\Big|\mathcal{F}_t\Big] \geq E\Big[U\Big(X_T^{u^n,t+1,X_{t+1}^{\hat{u},0,y}}\Big)\Big|\mathcal{F}_t\Big],$$

taking the limit, we get

$$\begin{aligned}\sup_{u\in\Theta(t+1,T-1)} & E\Big[U\Big(X_T^{u,t+1,X_{t+1}^{\hat{u},0,y}}\Big)\Big|\mathcal{F}_t\Big] \\ &\geq E\Big[\sup_{u\in\Theta(t+1,T-1)} E\Big[U\Big(X_T^{u,t+1,X_{t+1}^{\hat{u},0,y}}\Big)\Big|\mathcal{F}_{t+1}\Big]\Big|\mathcal{F}_t\Big]. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we obtain

$$\begin{aligned}\sup_{u\in\Theta(t+1,T-1)} & E\Big[U\Big(X_T^{u,t+1,X_{t+1}^{\hat{u},0,y}}\Big)\Big|\mathcal{F}_t\Big] \\ &= E\Big[\sup_{u\in\Theta(t+1,T-1)} E\Big[U\Big(X_T^{u,t+1,X_{t+1}^{\hat{u},0,y}}\Big)\Big|\mathcal{F}_{t+1}\Big]\Big|\mathcal{F}_t\Big]. \end{aligned}$$

(ii) Second, we assert that for all $\hat{u} \in \Theta$ and $y \in \mathbb{R}$, $v(X_t^{\hat{u},0,y}, t)$ is a supermartingale, i.e.,

$$v(X_t^{\hat{u},0,y}, t) \geq E[v(X_{t+1}^{\hat{u},0,y}, t+1)|\mathcal{F}_t] \tag{4.7}$$

for $0 \leq t \leq T-1$. Indeed, from (4.4), we have

$$\begin{aligned} & E[v(X_{t+1}^{\hat{u},0,y}, t+1)|\mathcal{F}_t] \\ &\quad = E\Big[\sup_{u\in\Theta(t+1,T-1)} E\Big[U\Big(X_T^{u,t+1,X_{t+1}^{\hat{u},0,y}}\Big)\Big|\mathcal{F}_{t+1}\Big]\Big|\mathcal{F}_t\Big] \\ &\quad = \sup_{u\in\Theta(t+1,T-1)} E\Big[U\Big(X_T^{u,t+1,X_{t+1}^{\hat{u},0,y}}\Big)\Big|\mathcal{F}_t\Big] \\ &\quad \leq \sup_{u\in\Theta(t,T-1)} E\Big[U\Big(X_T^{u,t,X_t^{\hat{u},0,y}}\Big)\Big|\mathcal{F}_t\Big] \\ &\quad = v(X_t^{\hat{u},0,y}, t) \end{aligned}$$

for $0 \leq t \leq T-1$.

(iii) Third, we show that

$$v(x,t) = \sup_{u\in\Theta(t,T-1)} E[v(X_{t+1}^{u,t,x}, t+1)|\mathcal{F}_t] \tag{4.8}$$

for $x \in \mathbb{R}$ and $0 \le t \le T-1$. Indeed, setting $\hat{u}_s = 0$ for $s \le t-1$ in (4.7), we have

$$v(y(1+r)^t, t) \ge E[v(X_{t+1}^{\hat{u},t,y(1+r)^t}, t+1)|\mathcal{F}_t].$$

Since $x \in \mathbb{R}$ and $\hat{u}_t$ are arbitrary, we get

$$v(x,t) \ge \sup_{u\in\Theta(t,T-1)} E[v(X_{t+1}^{u,t,x}, t+1)|\mathcal{F}_t].$$

On the other hand, for any $\hat{u} \in \Theta(t, T-1)$,

$$\begin{aligned}
&E[v(X_{t+1}^{\hat{u},t,x}, t+1)|\mathcal{F}_t] \\
&= E\left[\sup_{u\in\Theta(t+1,T-1)} E\left[U\left(X_T^{u,t+1,X_{t+1}^{\hat{u},t,x}}\right)\middle|\mathcal{F}_{t+1}\right]\middle|\mathcal{F}_t\right] \\
&\ge E\left[U\left(X_T^{\hat{u},t+1,X_{t+1}^{\hat{u},t,x}}\right)\middle|\mathcal{F}_t\right] \\
&= E\left[U\left(X_T^{\hat{u},t,x}\right)\middle|\mathcal{F}_t\right].
\end{aligned}$$

Then

$$\begin{aligned}
\sup_{u\in\Theta(t,T-1)} E[v(X_{t+1}^{u,t,x}, t+1)|\mathcal{F}_t] &\ge \sup_{u\in\Theta(t,T-1)} E\left[U\left(X_T^{u,t,x}\right)\middle|\mathcal{F}_t\right] \\
&= v(x,t).
\end{aligned}$$

Therefore we have

$$v(x,t) = \sup_{u\in\Theta(t,T-1)} E[v(X_{t+1}^{u,t,x}, t+1)|\mathcal{F}_t].$$

(iv) Finally, we can show the claim of the theorem. From (4.8), we have

$$\begin{aligned}
&v(x,t) \\
&= \sup_{u\in\Theta(t,T-1)} E[v(x + \alpha^{u_t}x + \beta^{u_t}x(B_{t+1} - B_t) + \gamma^{u_t}x(N_{t+1} - N_t), t+1)|\mathcal{F}_t].
\end{aligned}$$

Applying Theorem 4.2, we get

$$
\begin{aligned}
&\sup_{u\in\Theta(t,T-1)} E\Big[v(x+\alpha^{u_t}x,t+1)-v(x,t+1)\\
&\quad+\frac{1}{2}\Big(v(x+\alpha^{u_t}x+\beta^{u_t}x,t+1)-2v(x+\alpha^{u_t}x,t+1)\\
&\quad+v(x+\alpha^{u_t}x-\beta^{u_t}x,t+1)\Big)+v(x,t+1)-v(x,t)\\
&\quad+\frac{q}{2}\Big(v(x+\alpha^{u_t}x+\beta^{u_t}x+\gamma^{u_t}x,t+1)-v(x+\alpha^{u_t}x+\beta^{u_t}x,t+1)\\
&\qquad+v(x+\alpha^{u_t}x-\beta^{u_t}x+\gamma^{u_t}x,t+1)-v(x+\alpha^{u_t}x-\beta^{u_t}x,t+1)\Big)\Big|\mathcal{F}_t\Big]=0.
\end{aligned}
$$

Then, for $x\in\mathbb{R}$, v satisfies

$$
\begin{aligned}
&\sup_{u\in\mathbb{R}}\Big\{v(x+\alpha^{u}x,t+1)-v(x,t+1)\\
&+\frac{1}{2}\Big(v(x+\alpha^{u}x+\beta^{u}x,t+1)-2v(x+\alpha^{u}x,t+1)\\
&+v(x+\alpha^{u}x-\beta^{u}x,t+1)\Big)+v(x,t+1)-v(x,t)\\
&+\frac{q}{2}\Big(v(x+\alpha^{u}x+\beta^{u}x+\gamma^{u}x,t+1)-v(x+\alpha^{u}x+\beta^{u}x,t+1)\\
&+v(x+\alpha^{u}x-\beta^{u}x+\gamma^{u}x,t+1)-v(x+\alpha^{u}x-\beta^{u}x,t+1)\Big)\Big\}=0.
\end{aligned}
$$

This concludes the proof. ■

Remark 4.4. Theorem 4.3 is a generalization of Theorem 2 in Ishimura and Mita (2009) when $U_1\equiv 0$ in their article. Indeed, we can derive dHJB equation derived in Ishimura and Mita (2009),

$$
\begin{aligned}
&\sup_{u\in\mathbb{R}}\Big\{V(x+\alpha^{u}x,t+1)-V(x,t+1)\\
&+\frac{1}{2}\Big(V(x+\alpha^{u}x+\beta^{u}x,t+1)-2V(x+\alpha^{u}x,t+1)\\
&+V(x+\alpha^{u}x-\beta^{u}x,t+1)\Big)+V(x,t+1)-V(x,t)\Big\}=0,
\end{aligned}
$$

by letting $q=0$ in (4.3)

Remark 4.5. Related to Theorem 4.3, a verification theorem also holds. However, since its claim and proof are almost the same as Theorem 3 in Ishimura and Mita (2009), it is not described in this dissertation.

4.3 On a relation between solutions of a HJB equation and a dHJB equation

In this section, we analyze a relation between an HJB variational inequality in a continuous-time geometric Brownian model and a dHJB equation in a random walk model derived by Ishimura and Mita (2009). Our main result asserts that the limit of a solution of the dHJB equation is a solution of the corresponding HJB variational inequality.

In this section, we use the following notations:

$$
\begin{aligned}
L^u J(x,t) :=& \alpha^u x \frac{\partial}{\partial x} J(x,t) + \frac{1}{2}(\beta^u x)^2 \frac{\partial^2}{\partial x^2} J(x,t) + \frac{\partial}{\partial t} J(x,t) \\
\mathcal{L}^{u,\Delta t} V(x,t) :=& V(x + \alpha^u x \Delta t, t + \Delta t) - V(x, t + \Delta t) \\
&+ \frac{1}{2}\Big[V(x + \alpha^u x \Delta t + \beta^u x \sqrt{\Delta t}, t + \Delta t) - 2V(x + \alpha^u x \Delta t, t + \Delta t) \\
&+ V(x + \alpha^u x \Delta t - \beta^u x \sqrt{\Delta t}, t + \Delta t)\Big] + V(x, t + \Delta t) - V(x,t) \\
=& \frac{1}{2}\Big[V(x + \alpha^u x \Delta t + \beta^u x \sqrt{\Delta t}, t + \Delta t) \\
&+ V(x + \alpha^u x \Delta t - \beta^u x \sqrt{\Delta t}, t + \Delta t)\Big] - V(x,t).
\end{aligned}
$$

Obviously, the following relation holds:

$$
L^u J(x,t) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \mathcal{L}^{u,\Delta t} J(x,t). \tag{4.9}
$$

With these notations, a dHJB equation derived in Ishimura and Mita (2009) when the time intervals are changed to be Δt,

$$\sup_{u\in\mathbb{R}}\Big\{V(x+\alpha^u x\Delta t, t+\Delta t)-V(x,t+\Delta t)$$
$$+\frac{1}{2}\Big[V(x+\alpha^u x\Delta t+\beta^u x\sqrt{\Delta t}, t+\Delta t)-2V(x+\alpha^u x\Delta t, t+\Delta t)$$
$$+V(x+\alpha^u x\Delta t-\beta^u x\sqrt{\Delta t}, t+\Delta t)\Big]+V(x,t+\Delta t)-V(x,t)\Big\}=0$$

can be written by

$$\sup_{u\in\mathbb{R}}\mathcal{L}^{u,\Delta t}V(x,t)=0.$$

Furthermore an HJB variational inequality for a singular stochastic control problem can be defined as follows: Suppose that there exists a continuous function $G(x,p,m,t)$ defined on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times[0,T)$ such that

$$\sup_{u\in A}\Big\{\alpha^u xp+\frac{1}{2}(\beta^u x)^2 m\Big\}<\infty \ \text{ if and only if } \ G(x,p,m,t)\geq 0,$$
$$\forall(x,p,m,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times[0,T)$$

and $G(x,p,m,t)\geq G(x,p,\hat{m},t)$ when $m\leq\hat{m}$ where $A\subset\mathbb{R}$ is a set which is chosen so that the trading strategy can be admissible in each problem. Then the equation

$$\min\Big\{-\sup_{u\in A}L^u J(x,t), G\Big(x,\frac{\partial J}{\partial x}(x,t),\frac{\partial^2 J}{\partial x^2}(x,t),t\Big)\Big\}=0 \tag{4.10}$$

is called a HJB variational inequality. Singular stochastic control problems often arise in cases such as $A\subset\mathbb{R}$ is not bounded and so on and HJB variational inequalities are used to handle those problems. See Section 4.3 in Chapter 4 of Pham (2009) for more details about HJB variational inequalities.

Next, we state the definition of viscosity solutions of (4.10). We have referred to Definition 4.2.1 in Pham (2009) to describe the following definition of viscosity solutions of the HJB variational inequality.

Definition 4.6. A locally bounded function $H(x,t)$, $(x,t) \in \mathbb{R} \times [0,T)$ is called

(i) a viscosity subsolution of (4.10) on $\mathbb{R} \times [0,T)$ if

$$\min\left\{ -\sup_{u\in A} L^u J(\bar{x},\bar{t}), G\left(\bar{x}, \frac{\partial J}{\partial x}(\bar{x},\bar{t}), \frac{\partial^2 J}{\partial x^2}(\bar{x},\bar{t}), \bar{t}\right)\right\} \leq 0$$

for all $(\bar{x},\bar{t}) \in \mathbb{R} \times [0,T)$ and for any $C^{2,1}$ function $J(x,t)$ such that $H(x,t) \leq J(x,t),\ \forall x,t$ and $H(\bar{x},\bar{t}) = J(\bar{x},\bar{t})$,

(ii) a viscosity supersolution of (4.10) on $\mathbb{R} \times [0,T)$ if

$$\min\left\{ -\sup_{u\in A} L^u J(\bar{x},\bar{t}), G\left(\bar{x}, \frac{\partial J}{\partial x}(\bar{x},\bar{t}), \frac{\partial^2 J}{\partial x^2}(\bar{x},\bar{t}), \bar{t}\right)\right\} \geq 0$$

for all $(\bar{x},\bar{t}) \in \mathbb{R} \times [0,T)$ and for any $C^{2,1}$ function $J(x,t)$ such that $H(x,t) \geq J(x,t),\ \forall x,t$ and $H(\bar{x},\bar{t}) = J(\bar{x},\bar{t})$ and

(iii) a viscosity solution of (4.10) if it is both a subsolution and supersolution of (4.10).

We provide our main result in this section. It claims that the limit of a solution of the dHJB equation becomes a solution of the HJB variational inequality when an optimal solution of the dHJB equation exists and some conditions are satisfied.

Theorem 4.7. *Suppose that there exists a continuous function $G(x,p,m,t)$ defined on $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,T)$ such that*

$$\sup_{u\in A}\left\{\alpha^u xp + \frac{1}{2}(\beta^u x)^2 m\right\} < \infty \quad \textit{if and only if} \quad G(x,p,m,t) \geq 0,$$
$$\forall (x,p,m,t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,T)$$

and $G(x,p,m,t) \geq G(x,p,\hat{m},t)$ when $m \leq \hat{m}$ for given $A \subset \mathbb{R}$. Suppose also that the dHJB equation $\sup_{u\in\mathbb{R}} \mathcal{L}^{u,\Delta t} V(x,t) = 0$ admits an optimal strategy $\tilde{u}^{\Delta t}$

and an optimal solution $H^{\Delta t}(x,t)$ for each $0 < \Delta t \le 1$ such that the limit $\tilde{u} = \lim_{\Delta t \downarrow 0} \tilde{u}^{\Delta t}$ exists and $H^{\Delta t}(x,t)$ uniformly converges to a locally bounded function $H(x,t)$. Suppose further that $\tilde{u}$ has a bounded and closed neighborhood $B \subset A$ and there exists a real number $\xi_0 > 0$ such that $\tilde{u}^{\xi} \in B$ for any $0 < \xi \le \xi_0$. Then $H(x,t)$ is a viscosity solution of the HJB variational inequality

$$\min\left\{ -\sup_{u\in A} L^u J(x,t),\quad G\left(x, \frac{\partial J}{\partial x}(x,t), \frac{\partial^2 J}{\partial x^2}(x,t), t \right) \right\} = 0.$$

Proof. We divide the proof into two steps.

(i) First, we show that the viscosity subsolution property. Suppose that a $C^{2,1}$ function $J(x,t)$ and a point $(\bar{x},\bar{t})$ satisfy $H(x,t) \le J(x,t),\ \forall (x,t)$ and $H(\bar{x},\bar{t}) = J(\bar{x},\bar{t})$. Assume on the contrary that

$$\min\left\{ -\sup_{u\in A} L^u J(\bar{x},\bar{t}),\quad G\left(\bar{x}, \frac{\partial J}{\partial x}(\bar{x},\bar{t}), \frac{\partial^2 J}{\partial x^2}(\bar{x},\bar{t}), \bar{t} \right) \right\} > 0.$$

Then we have $-\sup_{u\in A} L^u J(\bar{x},\bar{t}) > 0$. From this, we get $L^u J(\bar{x},\bar{t}) < 0$ for all $u \in A$. Since, from (4.9), we have

$$\lim_{\epsilon\downarrow 0} \frac{1}{\epsilon} \mathcal{L}^{u,\epsilon} J(\bar{x},\bar{t}) = L^u J(\bar{x},\bar{t}) < 0$$

for each $u \in A$, there exists a number $\epsilon(u) > 0$ such that $\mathcal{L}^{u,\epsilon} J(\bar{x},\bar{t}) < 0$ for all $0 < \epsilon \le \epsilon(u)$. In particular, if we put $\epsilon_0 = \min_{u\in B} \epsilon(u)$, then we have $\mathcal{L}^{u,\epsilon} J(\bar{x},\bar{t}) < 0$ for all $u \in B$ and $0 < \epsilon \le \epsilon_0$. Moreover since

$$\begin{aligned}
&\mathcal{L}^{u,\epsilon} J(\bar{x},\bar{t}) \\
&= \frac{1}{2}\Big\{ J(\bar{x} + \alpha^u \bar{x}\epsilon + \beta^u \bar{x}\sqrt{\epsilon}, \bar{t}+\epsilon) + J(\bar{x} + \alpha^u \bar{x}\epsilon - \beta^u \bar{x}\sqrt{\epsilon}, \bar{t}+\epsilon) \Big\} - J(\bar{x},\bar{t}) \\
&\ge \frac{1}{2}\Big\{ H(\bar{x} + \alpha^u \bar{x}\epsilon + \beta^u \bar{x}\sqrt{\epsilon}, \bar{t}+\epsilon) + H(\bar{x} + \alpha^u \bar{x}\epsilon - \beta^u \bar{x}\sqrt{\epsilon}, \bar{t}+\epsilon) \Big\} - H(\bar{x},\bar{t}) \\
&= \lim_{\delta\downarrow 0} \bigg[\frac{1}{2}\Big\{ H^\delta(\bar{x} + \alpha^u \bar{x}\epsilon + \beta^u \bar{x}\sqrt{\epsilon}, \bar{t}+\epsilon) \\
&\qquad\qquad + H^\delta(\bar{x} + \alpha^u \bar{x}\epsilon - \beta^u \bar{x}\sqrt{\epsilon}, \bar{t}+\epsilon) \Big\} - H^\delta(\bar{x},\bar{t}) \bigg] \\
&= \lim_{\delta\downarrow 0} \mathcal{L}^{u,\epsilon} H^\delta(\bar{x},\bar{t}),
\end{aligned}$$

there exists a number $\delta_0 > 0$ such that $\mathcal{L}^{u,\epsilon} H^\delta(\bar{x}, \bar{t}) < 0$ for all $u \in B$, $0 < \epsilon \leq \epsilon_0$ and $0 < \delta \leq \delta_0$. However, when we take a number $0 < \xi \leq \min\{\epsilon_0, \delta_0, \xi_0\}$ and $u = \tilde{u}^\xi$, since $\tilde{u}^\xi$ and H^ξ solve the dHJB equation $\sup_{u \in \mathbb{R}} \mathcal{L}^{u,\xi} V(x,t) = 0$, we must have $\mathcal{L}^{\tilde{u}^\xi,\xi} H^\xi(\bar{x}, \bar{t}) = 0$ and it yields a contradiction. Therefore

$$\min\left\{ -\sup_{u \in A} L^u J(\bar{x}, \bar{t}), G\left(\bar{x}, \frac{\partial J}{\partial x}(\bar{x}, \bar{t}), \frac{\partial^2 J}{\partial x^2}(\bar{x}, \bar{t}), \bar{t}\right)\right\} \leq 0$$

must be true.

(ii) Second, we show that the viscosity supersolution property. Suppose that a $C^{2,1}$ function $J(x,t)$ and $(\bar{x}, \bar{t})$ satisfy $H(x,t) \geq J(x,t)$, $\forall (x,t)$ and $H(\bar{x}, \bar{t}) = J(\bar{x}, \bar{t})$. Assume on the contrary that

$$\min\left\{ -\sup_{u \in A} L^u J(\bar{x}, \bar{t}), G\left(\bar{x}, \frac{\partial J}{\partial x}(\bar{x}, \bar{t}), \frac{\partial^2 J}{\partial x^2}(\bar{x}, \bar{t}), \bar{t}\right)\right\} < 0.$$

Then we have either

$$-\sup_{u \in A} L^u J(\bar{x}, \bar{t}) < 0 \quad \text{or} \quad G\left(\bar{x}, \frac{\partial J}{\partial x}(\bar{x}, \bar{t}), \frac{\partial^2 J}{\partial x^2}(\bar{x}, \bar{t}), \bar{t}\right) < 0.$$

However, since $G(\bar{x}, \partial J(\bar{x}, \bar{t})/\partial x, \partial^2 J(\bar{x}, \bar{t})/\partial x^2, \bar{t}) < 0$ means $-\sup_{u \in A} L^u J(\bar{x}, \bar{t}) = -\infty$, we only have to consider the case where $-\sup_{u \in A} L^u J(\bar{x}, \bar{t}) < 0$. When $\sup_{u \in A} L^u J(\bar{x}, \bar{t}) > 0$, there exists some $\bar{u} \in A$ such that $L^{\bar{u}} J(\bar{x}, \bar{t}) > 0$. Furthermore since we have

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathcal{L}^{\bar{u},\epsilon} J(\bar{x}, \bar{t}) = L^{\bar{u}} J(\bar{x}, \bar{t}) > 0,$$

there exists a number $\epsilon_1 > 0$ such that $\mathcal{L}^{\bar{u},\epsilon} J(\bar{x}, \bar{t}) > 0$ for all $0 < \epsilon \leq \epsilon_1$. We can also obtain

$$
\begin{aligned}
&\mathcal{L}^{\bar{u},\epsilon} J(\bar{x},\bar{t}) \\
&= \frac{1}{2}\Big\{ J(\bar{x}+\alpha^{\bar{u}}\bar{x}\epsilon+\beta^{\bar{u}}\bar{x}\sqrt{\epsilon},\bar{t}+\epsilon) + J(\bar{x}+\alpha^{\bar{u}}\bar{x}\epsilon-\beta^{\bar{u}}\bar{x}\sqrt{\epsilon},\bar{t}+\epsilon)\Big\} - J(\bar{x},\bar{t}) \\
&\le \frac{1}{2}\Big\{ H(\bar{x}+\alpha^{\bar{u}}\bar{x}\epsilon+\beta^{\bar{u}}\bar{x}\sqrt{\epsilon},\bar{t}+\epsilon) + H(\bar{x}+\alpha^{\bar{u}}\bar{x}\epsilon-\beta^{\bar{u}}\bar{x}\sqrt{\epsilon},\bar{t}+\epsilon)\Big\} - H(\bar{x},\bar{t}) \\
&= \lim_{\delta\downarrow 0}\Bigg[\frac{1}{2}\Big\{ H^{\delta}(\bar{x}+\alpha^{\bar{u}}\bar{x}\epsilon+\beta^{\bar{u}}\bar{x}\sqrt{\epsilon},\bar{t}+\epsilon) \\
&\qquad\qquad + H^{\delta}(\bar{x}+\alpha^{\bar{u}}\bar{x}\epsilon-\beta^{\bar{u}}\bar{x}\sqrt{\epsilon},\bar{t}+\epsilon)\Big\} - H^{\delta}(\bar{x},\bar{t})\Bigg] \\
&= \lim_{\delta\downarrow 0} \mathcal{L}^{\bar{u},\epsilon} H^{\delta}(\bar{x},\bar{t}).
\end{aligned}
$$

Then there exists a number $\delta_1 > 0$ such that $\mathcal{L}^{\bar{u},\epsilon} H^{\delta}(\bar{x},\bar{t}) > 0$ for all $0 < \epsilon \le \epsilon_1$ and $0 < \delta \le \delta_1$. In particular, for any $\eta \le \min\{\epsilon_1, \delta_1\}$, we can deduce that $\mathcal{L}^{\bar{u},\eta} H^{\eta}(\bar{x},\bar{t}) > 0$. However, since $\bar{u} \in A$ and $\tilde{u}^{\eta}$ is a optimal strategy of the dHJB equation, we have

$$
\mathcal{L}^{\bar{u},\eta} H^{\eta}(\bar{x},\bar{t}) \le \mathcal{L}^{\tilde{u}^{\eta},\eta} H^{\eta}(\bar{x},\bar{t}) = 0
$$

and it yields a contradiction. Therefore

$$
\min\Bigg\{ -\sup_{u\in A} L^{u} J(\bar{x},\bar{t}), G\Bigg(\bar{x}, \frac{\partial J}{\partial x}(\bar{x},\bar{t}), \frac{\partial^2 J}{\partial x^2}(\bar{x},\bar{t}), \bar{t}\Bigg)\Bigg\} \ge 0
$$

must be true.

Finally, from (i) and (ii), we get the assertion. ■

Remark 4.8. By Theorem 4.7, we may use dHJB equations to analyze HJB variational inequalities in continuous time as long as dHJB equations admit optimal solutions and some conditions are satisfied. However, in most cases where the Hamiltonians of HJB equations are singular, the corresponding dHJB equations may also diverge. This flaw must be overcome for practical use of this result.

4.4 Conclusion

In this chapter, we have proved a discrete Itô formula for discrete jump-diffusion processes and derived a dHJB equation for expected utility maximization problems in a discrete jump-diffusion model. We have also provided an application of the dHJB equation in a random walk model. It has been shown that a proper limit of a solution of the dHJB equation becomes a viscosity solution of the corresponding HJB variational inequality in continuous time under some conditions. However, in most cases where the Hamiltonians of HJB equations are singular and HJB variational inequalities should be used, the corresponding optimization problem in discrete time may also be singular. Then the condition that the dHJB equation admits a solution may not be realistic. It is left for the future to overcome this weakness.

Chapter 5

Conclusions

This dissertation focused on two basic optimization problems in mathematical finance, the multiperiod mean-variance portfolio selection and the utility maximization problem, and provided some new attempts for each field of study.

In Chapter 2, explicit solutions to multiperiod mean-variance portfolio selection problems in a discrete-time model and a continuous semimartingale model have been provided. Unlike the literature referred in Chapter 1, we have dealt with the problem applying the results of mean-variance hedging problems. In the discrete-time model, we have employed the ordinary Lagrange multiplier method to tackle the problem in a simple way. We have seen that the problem of minimizing the Lagrangian with respect to the investment strategies can be regarded as a mean-variance hedging problem which was solved by Gugushvili (2003) using dynamic programming. For obtaining an explicit solution, we have assumed that the discounted price process of the security satisfies the deterministic mean-variance tradeoff condition. Then we have derived the explicit solution and realized a relation between the optimal solution and the variance-optimal martingale measure (or the minimal martingale measure) in the model. In the continuous-time model, we have solved a mean-variance portfolio selection problem by the same approach as the discrete-time case with the result of the mean-variance hedging obtained by Rheinländer and Schweizer (1997). The result shows that the optimal strategy of the original problem is obtained as a multiple of the optimal strategy of the mean-variance hedging. Our approach may be valid for continuous-time mean-variance portfolio selections in general semimartingale model if we employ the result given by Jeanblanc et al. (2012) to handle the mean-variance hedging. However, we have always assumed that the growth rate of the risk-free asset is zero and a way to remove this limitation is not obvious. We have also not concerned the condition which prohibits the value of the portfolio becoming negative as Bielecki et al. (2005). It is left for the future to overcome these shortcomings.

In Chapter 3, we have proposed an alternative approach to derive the PDE which relates to the optimal solution of the utility maximization problem in a diffusion model without using the duality principle. We characterized the optimal solution with an equivalent martingale measure from a simple necessary condition of optimality. Then we have identified the equivalent martingale measure and derived the PDE by Itô's formula. Moreover we applied our method to a utility maximization problem in a compound Poisson model with unpredictable jump sizes. Assuming that the optimal solution exists, an equation which relates to the optimal solution of the utility maximization was derived. However, in the result in the compound Poisson model, a strong assumption that the Radon-Nikodym derivative process corresponding to the optimal solution is a Markov process is imposed. Moreover the way to solve the obtained equation is not obvious. It is left for a future subject to resolve these difficulties.

In Chapter 4, we have proved a discrete Itô formula for discrete jump-diffusion processes and derived a dHJB equation for expected utility maximization problems in a discrete jump-diffusion model. After deriving the optimality equation for the utility maximization in the model, we applied the discrete Itô formula derived in advance to the optimality equation and obtained the dHJB equation in the model. We have also provided an application of the dHJB equation in a random walk model. It has been shown that a proper limit of a solution of the dHJB equation becomes a viscosity solution of the corresponding HJB variational inequality in continuous time. If the conditions in the theorem are satisfied, this result enables us to specify the value function of the optimization problem in continuous time and consequently find the optimal solution when we know that the value function is the unique viscosity solution of the HJB variational inequality. However, in most cases where the Hamiltonians of HJB equations are singular and HJB variational inequalities should be used, the corresponding optimization problem in discrete time may also be singular. Then the condition that the dHJB equation admits a solution may not be realistic. It is left for the future to overcome this weakness.

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