博士学位請求論文要旨

Essays on Mean-Variance Portfolio Selections and Utility Maximizations in Mathematical Finance

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Introduction

This dissertation focuses on two basic optimization problems in mathematical finance, the multiperiod mean-variance portfolio selection and the utility maximization problem, and provides some new attempts for each field of study.

Mean-variance portfolio selection is a problem of the allocation of wealth among various securities so as to attain the optimal trade-off between the expected return of the portfolio and its risk measured by the variance of the portfolio. This problem was first proposed and solved in the single-period setting by Markowitz (1952). Being widely used in both academia and industry, this mean-variance paradigm has also inspired the development of the multiperiod mean-variance portfolio selections both in discrete time and in continuous time through various approaches (for example, Li and Ng, 2000; Zhou and Li, 2000; Lim, 2004; Framstad et al., 2004; Basak and Chabakauri, 2010; Bielecki et al., 2005).

The multiperiod mean-variance criteria can be applied to practical investment problems because of their solvability and explicit results. For example, both Delong and Gerrard (2007) and Wang et al. (2007) solved optimal investment problems for insurance companies which are formulated as continuous-time mean-variance portfolio selections. Moreover Basak and Chabakauri (2010) stated that the multiperiod mean-variance criteria can be used as a benchmark for evaluation of investment. These articles motivate us to struggle to obtain a simpler and more elementary solution to multiperiod mean-variance selections. In Chapter 2 of this thesis, we demonstrate an alternative and perhaps simpler approach to the problems both in discrete time and in continuous time.

Utility maximization is also a basic problem in mathematical finance. This is the problem of an economic agent who invests in a financial market so as to maximize the expected utility of her terminal wealth as well as intertemporal consumption from her wealth. Optimal consumption-portfolio policies have traditionally been derived by stochastic dynamic programming. In particular, in the framework of a continuous-time model, Merton (1969) derived a Hamilton-Jacobi-Bellman (HJB) equation for the value function of the optimization problem for the first time. He also provided the closed-form solution of this equation when the utility function is a power function, the logarithmic or an exponential function.

As an example of further developments in dynamic programming for utility maximizations, Chapter 4 of Pham (2009) offers an introduction to the application of viscosity solutions. The viscosity solution is a kind of weak solutions of some partial differential equations. In Chapter 4 of Pham (2009), it has been proved that the value function of a utility maximization problem which is possibly singular is the unique viscosity solution of the corresponding HJB variational inequality. In Chapter 4 of this thesis, we attempt conversely to derive a viscosity solution of the HJB variational inequality for a maximization problem of the utility from the terminal wealth through the use of the discrete HJB (dHJB) equation in a random walk model. The meaning of this experiment is that if we know that a particular function is a viscosity solution of the HJB variational inequality, then we can confirm that the function coincides with the value function in the model where the value function is the unique viscosity solution of the HJB variational inequality. Furthermore we also discuss the derivation of the dHJB equation in discrete jump-diffusion models.

Another important method of solving utility maximizations is the martingale representation approach. In particular, through the martingale approach in a Markovian setting, Cox and Huang (1989) has characterized the optimal policy with a linear partial differential equation (PDE), which is much easier to solve than the nonlinear PDE obtained in dynamic programming. In an incomplete market setting, He and Pearson (1991) extended the result of Cox and Huang (1989) and related the optimal policy to the solution of a quasi-linear PDE using the duality principle both in proving the existence of a solution and in characterizing the solution. However, the author believes that the duality argument which is somewhat technical is not essential when we consider only characterization of the solution. In Chapter 3 of this thesis, we try to provide an alternative approach to derive the PDE without using the duality principle in incomplete markets.

Summary of Chapter 2

In Chapter 2, explicit solutions to multiperiod mean-variance portfolio selection problems in a discrete-time model and a continuous semimartingale model are provided. We deal with the problem applying the results of mean-variance hedging problems. In the discrete-time model, we employ the ordinary Lagrange multiplier method to tackle the problem in a simple way. We see that the problem of minimizing the Lagrangian with respect to the investment strategies can be regarded as a mean-variance hedging problem which was solved by Gugushvili (2003) using dynamic programming. For obtaining an explicit solution, we assume that the discounted price process of the security satisfies a condition, which is called the deterministic mean-variance tradeoff condition. Then we derive the explicit solution and realize a relation between the optimal solution and the variance-optimal martingale measure (or the minimal martingale measure) in the model. In the continuous-time model, we solve a mean-variance portfolio selection problem by the same approach as the discrete-time case with the result of the mean-variance hedging obtained by Rheinländer and Schweizer (1997). The result shows that the optimal strategy of the original problem is obtained as a multiple of the optimal strategy of the mean-variance hedging.

Summary of Chapter 3

In Chapter 3, we propose an alternative approach to derive the PDE which relates to the optimal solution of the utility maximization problem in an incomplete diffusion model without using the duality principle. We characterize the optimal solution with an equivalent martingale measure from a simple necessary condition of optimality. Then we identify the equivalent martingale measure and derive the PDE by Itô's formula. Moreover we apply our method to a utility maximization problem in a compound Poisson model with unpredictable jump sizes. Assuming that the optimal solution exists, an equation which relates to the optimal solution of the utility maximization is derived.

Summary of Chapter 4

In Chapter 4, we prove a discrete Itô formula for discrete jump-diffusion processes and derive a dHJB equation for expected utility maximization problems in a discrete jump-diffusion model generalizing ones in random walk models. After deriving the optimality equation for the utility maximization in the model, we apply the discrete Itô formula derived in advance to the optimality equation and yield the dHJB equation in the model. We also provide an application of the dHJB equation in a random walk model. It is shown that a proper limit of a solution of the dHJB equation becomes a viscosity solution of the corresponding HJB variational inequality in continuous time. If the conditions in the theorem are satisfied, this result enables us to specify the value function of the optimization problem in continuous time and consequently find the optimal solution when we know that the value function is the unique viscosity solution of the HJB variational inequality.

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