Bayesian Nash Equilibrium and Variational Inequalities*

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Abstract

This paper provides a sufficient condition for the existence and uniqueness of a Bayesian Nash equilibrium by regarding it as a solution of a variational inequality. The payoff gradient of a game is defined as a vector whose component is a partial derivative of each player's payoff function with respect to the player's own action. If the Jacobian matrix of the payoff gradient is negative definite for each state, then a Bayesian Nash equilibrium is unique. This result unifies and generalizes the uniqueness of an equilibrium in a complete information game by Rosen (Econometrica 33: 520, 1965) and that in a team by Radner (Ann. Math. Stat. 33: 857, 1962). In a Bayesian game played on a network, the Jacobian matrix of the payoff gradient coincides with the weighted adjacency matrix of the underlying graph.

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1 Introduction

This paper explores a sufficient condition for the existence and uniqueness of a Bayesian Nash equilibrium in a class of Bayesian games where action sets are closed intervals and each player's payoff function is concave and continuously differentiable with respect to the player's own action. This class of Bayesian games has many applications such as Cournot and Bertrand competition, private provision of public goods, rent seeking, and strategic interaction on networks. A special case of our sufficient condition includes strict concavity of a potential function in a Bayesian potential games (Radner, 1962; Ui, 2009).

We formulate a Bayesian Nash equilibrium as a solution of a variational inequality in an infinite-dimensional space (Kinderlehrer and Stampacchia, 1980), which is one representation of the first-order condition for an equilibrium. This representation not only gives us an elementary proof for the uniqueness but also allows us to use the existence theorem for solutions of variational inequalities (Browder, 1965; Hartman and Stampacchia, 1966). It is well known that a Nash equilibrium of a complete information game is a solution of a variational inequality in a finite-dimensional space (Lions and Stampacchia, 1967; Bensoussan, 1974). Thus, it is hardly surprising that a Bayesian Nash equilibrium is a solution of a variational inequality in an infinite-dimensional space. To the best of the author's knowledge, however, the resulting implications are not necessarily well-documented. This paper fills this gap in the literature and shed new light on the variational inequality approach to game theory.

In the main results, we construct a vector whose component is a partial derivative of each player's payoff function with respect to the player's own action. This vector is referred to as the payoff gradient of the game. The payoff gradient is said to be strictly monotone if its Jacobian matrix is negative definite for each state.[1] It is said to be strongly monotone if it is strictly monotone and the maximum eigenvalue of the Jacobian matrix has a strictly negative supremum over the actions and the states. We show that if the payoff gradient is strictly monotone, then there exists at most one equilibrium, and if the payoff gradient is strongly monotone or if it is strictly monotone and the payoff functions are quadratic, then there exists a unique equilibrium. In particular, we consider a linear

[1]To be more precise, negative definiteness of the Jacobian matrix is a sufficient condition for strict monotonicity of the payoff gradient.

quadratic Gaussian (LQG) game, whose payoff functions are quadratic and private signals are normally distributed, and obtains the unique equilibrium in a closed form, which is linear in private signals.

Our condition is an extension of the sufficient condition for the uniqueness of a Nash equilibrium by Rosen (1965), who shows that a Nash equilibrium is unique if the payoff gradient of a complete information game is strictly monotone. As shown by Ui (2008), the unique Nash equilibrium is also a unique correlated equilibrium. We can show the uniqueness of a correlated equilibrium as a special case of our results because a Bayesian game is reduced to a complete information game with a correlation device when payoff functions are independent of the state.

Our condition is also an extension of the sufficient condition for the uniqueness of a Bayesian Nash equilibrium by Radner (1962). Radner (1962) studies a team, an identical interest Bayesian game with a common payoff function,[2] and shows that if the common payoff function is strictly concave in an action profile, a Bayesian Nash equilibrium is a unique maximizer of the expected value of the common payoff function. As a special case, Radner (1962) considers an LQG team and obtains the unique equilibrium in a closed form. Radner's results are used to study Bayesian potential games (Monderer and Shapley, 1996; van Heumen et al., 1996). A Bayesian potential game has the same best-response correspondence as that of a team,[3] the common payoff function of which is referred to as a potential function. If the potential function is strictly concave, a Bayesian Nash equilibrium is a unique maximizer of the expected value of the potential function, as shown by Ui (2009).

Our results generalize Radner's results and the applications to Bayesian potential games in the following sense. A Bayesian game is a Bayesian potential game if and only if the Jacobian matrix of the payoff gradient is symmetric (Monderer and Shapley, 1996), in which case the Jacobian matrix coincides with the Hessian matrix of a potential function. Moreover, the potential function is strictly concave if and only if the payoff gradient is strictly monotone (Ui, 2008). Thus, we can restate the results of Radner (1962) and Ui (2009) as follows: a Bayesian Nash equilibrium is unique if the Jacobian matrix is both symmetric and negative definite. Our results show that the symmetry requirement is not

[2]The theory of teams precedes Harsanyi (1967–1968).

[3]Ui (2009) studies a game satisfying this condition and calls it a best-response Bayesian potential game.

necessary.

For example, most studies on LQG games assume that the Jacobian matrix is symmetric and negative definite, i.e., an LQG game is a Bayesian potential game with a strictly concave potential function.[4] In order to analyze communication in a network, however, Calvó-Armengol et al. (2015) consider an LQG game in which the Jacobian matrix is asymmetric and show the existence and uniqueness of a linear Bayesian Nash equilibrium, while it has been an open question under what condition the linear equilibrium is a unique equilibrium. Our results show that this linear equilibrium is a unique equilibrium if the Jacobian matrix is negative definite.

As an application, we consider aggregative games (Selten, 1970), in which each player's payoff depends on the player's own action and the aggregate of all players' actions. We give a simple sufficient condition for the uniqueness of a Bayesian Nash equilibrium and apply it to a Cournot game and a rent-seeking game. We also consider games played on networks (Ballester et al., 2006; Bramoullé et al., 2014), or network games for short.[5] A Bayesian game with quadratic payoff functions is mathematically equivalent to a Bayesian network game, where the Jacobian matrix of the payoff gradient equals the negative of a weighted adjacency matrix of the underlying graph. Thus, a Bayesian network game has a unique equilibrium if the weighted adjacency matrix is positive definite. We can use this result to study Bayesian network games with random adjacency matrices, whereas most previous studies on Bayesian network games assume a constant adjacency matrix with a special structure (Blume et al., 2015; de Martì and Zenou, 2015; Calvó-Armengol et al., 2015).

The organization of the paper is as follows. Preliminary definitions and results are summarized in Section 2. Section 3 discusses the concept of strictly monotone payoff gradients. Section 4 reports the main results. Section 5 is devoted to applications.

[4]Basar and Ho (1974) was the first to use Radner's results to study LQG games that are not teams, followed by many studies on information sharing (Clark, 1983; Vives, 1984; Gal-Or, 1985), information acquisition (Li et al., 1987; Vives, 1988), and social value of information (Morris and Shin, 2002; Angeletos and Pavan, 2007; Ui and Yoshizawa, 2015), among others. LQG games in these studies are Bayesian potential games.

[5]See Jackson and Zenou (2015) for a survey.

2 Preliminaries

Consider a Bayesian game with a set of players $N = \{1, \ldots, n\}$. Player $i \in N$ has a set of actions $X_i \subseteq \mathbb{R}$, which is a closed interval. We write $X = \prod_{i \in N} X_i$ and $X_{-i} = \prod_{j \neq i} X_j$. Player i's payoff function is a measurable function $u_i : X \times \Omega \to \mathbb{R}$, where $(\Omega, \mathcal{F}, P)$ is a probability space. Player i's information is given by a measurable mapping $\eta_i : \Omega \to Y_i$, where $(Y_i, \mathcal{Y}_i)$ is a measurable space. Player i's strategy is a measurable mapping $\sigma_i : Y_i \to X_i$ with $E[\sigma_i(\eta_i)^2] < \infty$. We regard two strategies σ_i^1, σ_i^2 as the same strategy if $\sigma_i^1(\eta_i(\omega)) = \sigma_i^2(\eta_i(\omega))$ almost everywhere. Let Σ_i denote player i's set of strategies. We write $\Sigma = \prod_{i \in N} \Sigma_i$ and $\Sigma_{-i} = \prod_{j \neq i} \Sigma_j$. We assume that $E[u_i(\sigma, \omega)]$ exists for all $\sigma \in \Sigma$.

We fix N, X, and $(\Omega, \mathcal{F}, P)$ throughout this paper and simply denote a Bayesian game by $(\mathbf{u}, \eta)$, where $\mathbf{u} = (u_i)_{i \in N}$ and $\eta = (\eta_i)_{i \in N}$. We say that $(\mathbf{u}, \eta)$ is *smooth* if $u_i((\cdot, x_{-i}), \omega) : X_i \to \mathbb{R}$ is continuously differentiable for each $x_{-i} \in X_{-i}$, $i \in N$, and a.e. $\omega \in \Omega$, and $E[(\partial u_i(\sigma, \omega)/\partial x_i)^2] < \infty$ for each $\sigma \in \Sigma$ and $i \in N$. We write $\nabla \mathbf{u}(x, \omega) \equiv (\partial u_i(x, \omega)/\partial x_i)_{i \in N}$ and call it the payoff gradient of $\mathbf{u}$. We say that $(\mathbf{u}, \eta)$ is *concave* if $u_i((\cdot, x_{-i}), \omega) : X_i \to \mathbb{R}$ is concave for each $x_{-i} \in X_{-i}$, $i \in N$, and a.e. $\omega \in \Omega$.

A strategy profile $\sigma \in \Sigma$ is a Bayesian Nash equilibrium if, for a.e. $\omega \in \Omega$,

$$E\left[u_i\left(\sigma(\eta), \omega\right) \mid \eta_i\right] \geq E\left[u_i\left((x_i, \sigma_{-i}(\eta_{-i})), \omega\right) \mid \eta_i\right] \tag{1}$$

for each $x_i \in X_i$ and $i \in N$, where $\sigma(\eta) = (\sigma_i(\eta_i))_{i \in N}$, $\sigma_{-i}(\eta_{-i}) = (\sigma_j(\eta_j))_{j \neq i}$, and $E[\,\cdot \mid \eta_i]$ is the conditional expectation operator given $\eta_i(\omega)$.

In this paper, we study an equilibrium of a smooth concave Bayesian game by converting the first-order condition as follows.[6] First, we exchange the order of integration and differentiation to obtain the following representation.[7]

Lemma 1. *Let $(\mathbf{u}, \eta)$ be a smooth concave Bayesian game. Then, $\sigma \in \Sigma$ is a Bayesian Nash equilibrium if and only if, for a.e. $\omega \in \Omega$,*

$$E\Big[\frac{\partial}{\partial x_i} u_i(\sigma(\eta), \omega)(x_i - \sigma_i(\eta_i)) \Big| \eta_i\Big] \leq 0 \textit{ for each } x_i \in X_i \textit{ and } i \in N. \tag{2}$$

[6]Even if a concave Bayesian game is not smooth, we can obtain a similar first-order condition in terms of subderivatives and extend our main results using multi-valued variational inequalities, which is beyond the scope of this paper.

[7]Of course, the same first-order condition is valid under a suitable condition without concavity.

Proof. See Appendix A. □

If $\sigma \in \Sigma$ is a Bayesian Nash equilibrium, then, for a.e. $\omega \in \Omega$,

$$E\Big[\frac{\partial}{\partial x_i} u_i(\sigma(\eta), \omega)(\sigma_i'(\eta_i) - \sigma_i(\eta_i)) \,\Big|\, \eta_i\Big] \le 0 \text{ for each } \sigma_i' \in \Sigma_i \text{ and } i \in N$$

by Lemma 1. By taking the expectation with respect to η_i, we have

$$E\Big[\frac{\partial}{\partial x_i} u_i(\sigma(\eta), \omega)(\sigma_i'(\eta_i) - \sigma_i(\eta_i))\Big] \le 0 \text{ for each } \sigma_i' \in \Sigma_i \text{ and } i \in N.$$

By adding up the above over $i \in N$, we obtain

$$E\Big[\nabla \mathbf{u}(\sigma(\eta), \omega)^\top (\sigma'(\eta) - \sigma(\eta))\Big] \le 0 \text{ for each } \sigma' \in \Sigma, \tag{3}$$

where we regard $\nabla \mathbf{u}$, $\sigma(\eta)$, and $\sigma'(\eta)$ as column vectors and $x^\top$ denotes the transpose of a vector or a matrix x. The next lemma shows that this condition is not only necessary but also sufficient for a Bayesian Nash equilibrium.

Lemma 2. *Let* $(\mathbf{u}, \eta)$ *be a smooth concave Bayesian game. Then,* $\sigma \in \Sigma$ *is a Bayesian Nash equilibrium if and only if* (3) *holds.*

Proof. See Appendix A. □

For example, consider a Bayesian game with quadratic payoff functions:[8]

$$u_i(x, \omega) = -q_{ii}(\omega) x_i^2 - 2\sum_{j \neq i} q_{ij}(\omega) x_i x_j + 2\theta_i(\omega) x_i + h_i(x_{-i}, \omega), \tag{4}$$

where $q_{ij} : \Omega \to \mathbb{R}$ with $q_{ii}(\omega) > 0$, $\theta_i : \Omega \to \mathbb{R}$, and $h_i : X_{-i} \times \Omega \to \mathbb{R}$ for $i, j \in N$. We write $Q = [q_{ij}(\omega)]_{n \times n}$ and $\theta = (\theta_1(\omega), \ldots, \theta(\omega))^\top$. Then, (3) is reduced to

$$E\left[(Q\sigma(\eta) - \theta)^\top (\sigma'(\eta) - \sigma(\eta))\right] \ge 0 \text{ for each } \sigma' \in \Sigma. \tag{5}$$

Now suppose that Q is positive definite for a.e. $\omega \in \Omega$.[9] Then, (5) implies the uniqueness of an equilibrium.[10] In fact, if $\sigma^1, \sigma^2 \in \Sigma$ are equilibria,

$$E\left[(Q\sigma^1(\eta) - \theta)^\top (\sigma^2(\eta) - \sigma^1(\eta))\right] \ge 0 \text{ and } E\left[(Q\sigma^2(\eta) - \theta)^\top (\sigma^1(\eta) - \sigma^2(\eta))\right] \ge 0,$$

[8] Examples include Cournot and Bertrand games with linear demand functions.

[9] We say that a square matrix M is positive definite if $M + M^\top$ is positive definite. Note that $x^\top M x = x^\top (M + M^\top) x/2$.

[10] Positive definiteness of Q is not directly related to strategic complementarities or substitutabilities. Note that this game exhibits strategic complementarities (substitutabilities) if all non-diagonal elements are positive (negative).

which implies that

$$E\left[(\sigma^2(\eta) - \sigma^1(\eta))^\top Q(\sigma^2(\eta) - \sigma^1(\eta))\right] \leq 0.$$

Because Q is positive definite for a.e. $\omega \in \Omega$, it follows that $\sigma^1(\eta) = \sigma^2(\eta)$ almost everywhere.

In the subsequent sections, we consider more general smooth concave Bayesian games and discuss not only the uniqueness but also the existence of Bayesian Nash equilibria on the basis of (3), which is shown to be a variational inequality.

3 Strict monotonicity

Let $S \subseteq \mathbb{R}^n$ be a convex set. A mapping $F : S \to \mathbb{R}^n$ is strictly monotone if $(F(x) - F(y))^\top(x - y) > 0$ for each $x, y \in S$ with $x \neq y$. When $n = 1$, F is strictly monotone if and only if F is strictly increasing. A mapping $F : S \to \mathbb{R}^n$ is strongly monotone if there exists $c > 0$ such that $(F(x) - F(y))^\top(x - y) > c(x - y)^\top(x - y)$ for each $x, y \in S$ with $x \neq y$. Clearly, strong monotonicity implies strict monotonicity.

The following sufficient conditions are well-known.[11]

Lemma 3. *Suppose that a mapping $F : S \to \mathbb{R}^n$ is continuously differentiable. If the Jacobian matrix $J_F(x)$ is positive definite for each $x \in S$, then F is strictly monotone. There exists $c > 0$ such that $x^\top J_F(x)x > cx^\top x$ for each $x \in S$ if and only if F is strongly monotone.*

With some abuse of language, we say that the payoff gradient is strictly monotone if the mapping $x \mapsto -\nabla\mathbf{u}(x, \omega)$ is strictly monotone for a.e. $\omega \in \Omega$, i.e.,

$$(\nabla\mathbf{u}(x, \omega) - \nabla\mathbf{u}(x', \omega))^\top(x - x') < 0 \text{ for each } x, x' \in X \text{ with } x \neq x'.$$

We also say that the payoff gradient is strongly monotone if the mapping $x \mapsto -\nabla\mathbf{u}(x, \omega)$ is strongly monotone for a.e. $\omega \in \Omega$ with respect to the same constant $c > 0$, i.e.,

$$(\nabla\mathbf{u}(x, \omega) - \nabla\mathbf{u}(x', \omega))^\top(x - x') < -c(x - x')^\top(x - x') \text{ for each } x, x' \in X \text{ with } x \neq x'.$$

[11]See Facchinei and Pang (2003), for example.

For example, consider a Bayesian game with quadratic payoff functions (4). Because

$$(\nabla \mathbf{u}(x,\omega) - \nabla \mathbf{u}(x',\omega))^\top (x - x') = -2(x - x')^\top Q(x - x'),$$

the payoff gradient is strictly monotone if and only if Q is positive definite for a.e. $\omega \in \Omega$, and it is strongly monotone if the minimum eigenvalue of Q has a strictly positive infimum over $\omega \in \Omega$.

For the general case, we have the following sufficient conditions by Lemma 3.

Lemma 4. *Suppose that* $\nabla \mathbf{u}(\cdot,\omega) : X \to \mathbb{R}^n$ *is continuously differentiable for a.e.* $\omega \in \Omega$. *If the Jacobian matrix of the payoff gradient*

$$F_{\nabla \mathbf{u}}(x,\omega) = \begin{pmatrix} \dfrac{\partial^2 u_1(x,\omega)}{\partial x_1 \partial x_1} & \cdots & \dfrac{\partial^2 u_1(x,\omega)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 u_n(x,\omega)}{\partial x_n \partial x_1} & \cdots & \dfrac{\partial^2 u_n(x,\omega)}{\partial x_n \partial x_n} \end{pmatrix}$$

is negative definite for each $x \in X$ *and a.e.* $\omega \in \Omega$, *then the payoff gradient is strictly monotone. There exists* $c > 0$ *such that* $x^\top F_{\nabla \mathbf{u}}(x,\omega) x < -c x^\top x$ *for each* $x \in X$ *and a.e.* $\omega \in \Omega$ *if and only if the payoff gradient is strongly monotone.*

The next lemma due to Goodman (1980) gives a simple sufficient condition for negative definiteness of $F_{\nabla \mathbf{u}}(x,\omega)$, which we will use in Section 5.2.

Lemma 5. *Assume the following conditions:* (i) $\nabla \mathbf{u}(x,\omega)$ *is continuously differentiable with respect to* x, (ii) $u_i(x,\omega)$ *is strictly concave in* x_i *and convex in* x_{-i}, *and* (iii) $\sum_{i \in N} u_i(x,\omega)$ *is concave in* x. *Then,* $F_{\nabla \mathbf{u}}(x,\omega)$ *is negative definite.*

If $F_{\nabla \mathbf{u}}(x,\omega)$ is negative definite and thus the payoff gradient is strictly monotone, $(\mathbf{u},\eta)$ is a concave game because each diagonal element $\partial^2 u_i / \partial x_i^2$ is negative. More in general (i.e. without the existence of $F_{\nabla \mathbf{u}}(x,\omega)$), if the payoff gradient of $\mathbf{u}$ is strictly monotone, then $(\mathbf{u},\eta)$ is concave (see Ui, 2008).

Lemma 6. *Let* $(\mathbf{u},\eta)$ *be a smooth Bayesian game. If the payoff gradient is strictly monotone, then* $u_i((\cdot, x_{-i}),\omega) : X_i \to \mathbb{R}$ *is strictly concave for each* $x_{-i} \in X_{-i}$, $i \in N$, *and a.e.* $\omega \in \Omega$.

In a complete information game (i.e. $\mathbf{u}$ is independent ω), strict monotonicity implies not only the uniqueness of a Nash equilibrium (Rosen, 1965) but also the uniqueness of a correlated equilibrium (Ui, 2008). In the next section, we show that strict monotonicity implies the uniqueness of a Bayesian Nash equilibrium and strong monotonicity implies the existence as well.

4 Results

In the following main results, we consider a smooth Bayesian game whose payoff gradient is strictly monotone, which is a smooth concave Bayesian game by Lemma 6. First, we show that strict monotonicity is sufficient for the uniqueness of a Bayesian Nash equilibrium.[12]

Proposition 1. *Let* $(\mathbf{u}, \eta)$ *be a smooth Bayesian game. Suppose that the payoff gradient is strictly monotone. Then,* $(\mathbf{u}, \eta)$ *has at most one Bayesian Nash equilibrium.*

Proof. The proof for the uniqueness is the same as the discussion in the end of Section 2. For completeness, we give a proof. Let $\sigma^1, \sigma^2 \in \Sigma$ be Bayesian Nash equilibria. Then, σ^1 and σ^2 are solutions of (3) by Lemmas 2 and 6, which implies that

$$E\left[\nabla\mathbf{u}(\sigma^1(\eta), \omega)^\top(\sigma^2(\eta) - \sigma^1(\eta))\right] \leq 0 \text{ and } E\left[\nabla\mathbf{u}(\sigma^2(\eta), \omega)^\top(\sigma^1(\eta) - \sigma^2(\eta))\right] \leq 0,$$

and thus

$$E\left[(\nabla\mathbf{u}(\sigma^2(\eta), \omega) - \nabla\mathbf{u}(\sigma^1(\eta), \omega))^\top(\sigma^2(\eta) - \sigma^1(\eta))\right] \geq 0. \tag{6}$$

Strict monotonicity implies that

$$(\nabla\mathbf{u}(\sigma^2(\eta), \omega) - \nabla\mathbf{u}(\sigma^1(\eta), \omega))^\top(\sigma^2(\eta) - \sigma^1(\eta)) \begin{cases} < 0 \text{ if } \sigma(\eta) \neq \sigma'(\eta), \\ = 0 \text{ if } \sigma(\eta) = \sigma'(\eta). \end{cases} \tag{7}$$

Therefore, we must have $\sigma(\eta) = \sigma'(\eta)$ almost everywhere. $\square$

Using Proposition 1, we provide sufficient conditions for the existence and uniqueness of a Bayesian Nash equilibrium. In particular, strong monotonicity is a sufficient condition.

[12] If $\mathbf{u}$ is independent of $\omega \in \Omega$, $(\mathbf{u}, \eta)$ is a complete information game with a correlation device η, and a Bayesian Nash equilibrium is a correlated equilibrium. Thus, this result implies that if the payoff gradient is strictly monotone, a unique Nash equilibrium is a unique correlated equilibrium (Ui, 2008).

Proposition 2. *Let* $(\mathbf{u}, \eta)$ *be a smooth Bayesian game. Suppose that the payoff gradient is strictly monotone. If* X *is bounded, or there exists* $\sigma^0 \in \Sigma$ *such that*

$$\lim_{\sigma \in \Sigma,\, E[\sigma^\top \sigma] \to \infty} \frac{E\left[\nabla \mathbf{u}(\sigma(\eta), \omega)^\top (\sigma(\eta) - \sigma^0(\eta))\right]}{\sqrt{E[\sigma(\eta)^\top \sigma(\eta)]}} = -\infty, \tag{8}$$

then a unique Bayesian Nash equilibrium exists. If the payoff gradient is strongly monotone, then (8) *is true.*

To give a proof, we regard a Bayesian Nash equilibrium as a solution of a variational inequality in an infinite-dimensional space (see Stampacchia, 1970; Kinderlehrer and Stampacchia, 1980). As shown by Lions and Stampacchia (1967) and Bensoussan (1974), a Nash equilibrium of a complete information game is a solution of a variational inequality in a finite-dimensional space (see Harker and Pang, 1990; Facchinei and Pang, 2003). In the following proof, we use the fact that a Bayesian Nash equilibrium is a solution of a variational inequality in a Hilbert space.

Let H be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$. Fix a non-empty subset $C \subseteq H$ and let $T : C \to H$ be a mapping. A variational inequality is a problem to seek $\alpha \in C$ satisfying

$$\langle T\alpha, \alpha' - \alpha \rangle \geq 0 \text{ for each } \alpha' \in C. \tag{9}$$

The following result is due to Browder (1965) and Hartman and Stampacchia (1966).[13]

Proposition 3 (Browder-Hartman-Stampacchia)**.** *Assume the following conditions.*

1. *C is a nonempty closed convex subset of H.*
2. $\langle T\alpha - T\beta, \alpha - \beta \rangle \geq 0$ *for all* $\alpha, \beta \in C$.
3. *The mapping* $t \mapsto \langle T((1-t)\alpha + t\beta), \gamma \rangle$ *from* $[0, 1]$ *to* $\mathbb{R}$ *is continuous for all* $\alpha, \beta \in C$ *and* $\gamma \in H$.
4. *C is bounded, or there exists $\alpha^0 \in C$ such that*

$$\lim_{\|\alpha\| \to \infty} \frac{\langle T\alpha, \alpha - \alpha^0 \rangle}{\|\alpha\|} = +\infty.$$

Then, there exists $\alpha \in C$ *satisfying* (9).

[13] They consider a reflexive Banach space.

The first and third conditions are technical assumptions. The second condition is called monotonicity, a weaker version of strict monotonicity. The fourth condition requires either that C is bounded or that if $\|\alpha\|$ is very large then α cannot satisfy (9).

We are ready to prove Proposition 2.

Proof of Proposition 2. Proposition 1 implies the uniqueness. To prove the existence, we show that the first-order condition (3) is a special case of the Browder-Hartman-Stampacchia variational inequalities. Let H be a Hilbert space consisting of (an equivalence class of) a random variable $\alpha : \Omega \to \mathbb{R}^n$ with $\langle \alpha, \beta \rangle \equiv E[\alpha(\omega)^\top \beta(\omega)]$ for $\alpha, \beta \in H$. Let $C = \Sigma \subseteq H$, which is a nonempty closed convex subset of H. Let $T : C \to H$ be such that $T\sigma = -\nabla \mathbf{u}(\sigma(\eta), \omega) \in H$ for each $\sigma \in \Sigma$, which satisfies the conditions in Proposition 3. Because (9) coincides with (3), there exists $\sigma \in \Sigma$ satisfying (3) by Proposition 3, which is a Bayesian Nash equilibrium by Lemmas 2 and 6.

If the payoff gradient is strongly monotone, there exists $c > 0$ such that

$$\begin{aligned}\nabla \mathbf{u}(\sigma, \omega)^\top (\sigma - \sigma^0) &\le \nabla \mathbf{u}(\sigma^0, \omega)^\top (\sigma - \sigma^0) - c(\sigma - \sigma^0)^\top (\sigma - \sigma^0) \\ &= -c\sigma^\top \sigma + \nabla \mathbf{u}(\sigma^0, \omega)^\top \sigma + (c(2\sigma - \sigma^0) - \nabla \mathbf{u}(\sigma^0, \omega))^\top \sigma^0\end{aligned}$$

for each $\sigma, \sigma^0 \in \Sigma$ and $\omega \in \Omega$. Thus,

$$\begin{aligned}&\frac{E\left[\nabla \mathbf{u}(\sigma, \omega)^\top (\sigma - \sigma^0)\right]}{\sqrt{E[\sigma^\top \sigma]}} \\ &\le -c\sqrt{E[\sigma^\top \sigma]} + \frac{E\left[\nabla \mathbf{u}(\sigma^0, \omega)^\top \sigma + (c(2\sigma - \sigma^0) - \nabla \mathbf{u}(\sigma^0, \omega))^\top \sigma^0\right]}{\sqrt{E[\sigma^\top \sigma]}}.\end{aligned}$$

Because the second term in the above is bounded, we have (8). □

In the case of quadratic payoff functions, we need neither (8) nor strong monotonicity. Strict monotonicity suffices, as the next proposition shows. In the proof, we consider another Hilbert space.

Proposition 4. *Let* $(\mathbf{u}, \eta)$ *be a Bayesian game with quadratic payoff functions* (4). *Suppose that* Q *is positive definite for a.e.* $\omega \in \Omega$. *Then, a unique Bayesian Nash equilibrium exists.*

Proof. Proposition 1 implies the uniqueness. To prove the existence, we show that (5) is another special case of the Browder-Hartman-Stampacchia variational inequalities. Let H be a Hilbert space consisting of (an equivalence class of) a random variable $\alpha : \Omega \to \mathbb{R}^n$

with $\langle \alpha, \beta \rangle \equiv E[\alpha^\top Q^\top \beta]$ for $\alpha, \beta \in H$. Let $C = \Sigma \subseteq H$, which is a nonempty closed convex subset of H. Let $T : C \to H$ be such that $T\sigma = \sigma - Q^{-1}\theta \in H$ for each $\sigma \in \Sigma$. Note that Q is invertible because Q is positive definite.[14] Then, (9) is written as

$$\langle T\sigma, \sigma' - \sigma \rangle = \langle \sigma - Q^{-1}\theta, \sigma' - \sigma \rangle = E\left[(Q\sigma(\eta) - \theta)^\top (\sigma'(\eta) - \sigma(\eta))\right] \geq 0 \tag{10}$$

for each $\sigma' \in \Sigma$, which is (5). Thus, a Bayesian Nash equilibrium is a solution of a variational inequality (10). Because

$$\begin{aligned}\lim_{\|\sigma\|\to\infty} \frac{\langle T\sigma, \sigma - \sigma^0 \rangle}{\|\sigma\|} &= \lim_{\|\sigma\|\to\infty} \frac{\langle \sigma - Q^{-1}\theta, \sigma - \sigma^0 \rangle}{\|\sigma\|} \\ &\geq \lim_{\|\sigma\|\to\infty} \left(\|\sigma\| - \|\sigma^0\| - \|Q^{-1}\theta\| + \frac{\langle Q^{-1}\theta, \sigma^0 \rangle}{\|\sigma\|} \right) = \infty,\end{aligned}$$

a Bayesian Nash equilibrium exists by Proposition 3. □

Applying Proposition 4 to linear quadratic Gaussian (LQG) games, where Q is constant and an information structure is Gaussian, we obtain the following existence and uniqueness result.

Proposition 5. *Let* $(\mathbf{u}, \eta)$ *be a Bayesian game with quadratic payoff functions* (4). *Assume the following conditions.*

1. *Q is positive definite and independent of ω.*
2. *$X_i = \mathbb{R}$ and $Y_i = \mathbb{R}^{m_i}$ for each $i \in N$, where $m_i \geq 1$ is an integer.*
3. *$\eta_1(\omega), \ldots, \eta_n(\omega)$ and $\theta_1(\omega), \ldots, \theta_n(\omega)$ are jointly normally distributed with*

$$\mathrm{cov}[\eta_i, \eta_j] = C_{ij}, \ \mathrm{cov}[\eta_i, \theta_i] = G_i,$$

where C_{ii} is positive definite for each $i \in N$.

Then, there exists a unique Bayesian Nash equilibrium obtained as follows:

$$\sigma_i(\eta_i) = b_i^\top (\eta_i - E[\eta_i]) + c_i, \tag{11}$$

where b_i and c_i are determined by the system of linear equations

$$\sum_{j \in N} q_{ij} C_{ij} b_j = G_i \text{ for } i \in N, \tag{12}$$

$$\sum_{j \in N} q_{ij} c_j = E[\theta_i] \text{ for } i \in N. \tag{13}$$

[14]Otherwise, there exists $x \neq \mathbf{0}$ such that $Qx = \mathbf{0}$, which implies $x^\top Q x = 0$, a contradiction.

Proof. Proposition 4 implies the existence and uniqueness. Thus, it is enough to show that the unique Bayesian Nash equilibrium is of the above form. See Appendix B. □

In all the above results, we assume that the payoff gradient is strictly monotone. However, even without strict monotonicity, we can obtain similar results if the best-response correspondence coincides with that of another game whose payoff gradient is strictly monotone. For example, for two games $(\mathbf{u}, \eta)$ and $(\mathbf{u}', \eta)$, if there exist $w_i : Y_i \to \mathbb{R}_{++}$ and $h_i : X_{-i} \times \Omega \to \mathbb{R}$ such that $u'_i(x, \omega) = w_i(\eta_i)u_i(x, \omega) + h_i(x_{-i}, \omega)$ for each $x \in X$, $\omega \in \Omega$, and $i \in N$, then the best-response correspondences coincide. Thus, if the payoff gradient of $\mathbf{u}$ is strictly monotone, $(\mathbf{u}', \eta)$ has at most one Bayesian Nash equilibrium, and if the payoff gradient of $\mathbf{u}$ is strongly monotone, $(\mathbf{u}', \eta)$ has a unique Bayesian Nash equilibrium.[15]

The above discussion leads us to the following weaker concept of strict or strong monotonicity. For $w \equiv (w_i)_{i \in N}$ with $w_i : Y_i \to \mathbb{R}^n_{++}$ for each $i \in N$, we call $w \circ \nabla \mathbf{u} \equiv (w_i \partial u_i / \partial x_i)_{i \in N}$ the w-weighted payoff gradient of $\mathbf{u}$. We say that the w-weighted payoff gradient is strictly monotone if the mapping $x \mapsto -w \circ \nabla \mathbf{u}(x, \omega)$ is strictly monotone for a.e. $\omega \in \Omega$. We also say that the w-weighted payoff gradient is strongly monotone if the mapping $x \mapsto -w \circ \nabla \mathbf{u}(x, \omega)$ is strongly monotone for a.e. $\omega \in \Omega$ with respect to the same constant $c > 0$. Then, strict or strong monotonicity of the payoff gradient in Propositions 1 and 2 can be replaced with strict or strong monotonicity of the w-weighted payoff gradient without any change in the conclusions. The corresponding concept for complete information games is discussed by Rosen (1965). For much weaker concepts, which also work, see Ui (2008).

5 Applications

5.1 Potential games

A Bayesian game $(\mathbf{u}, \eta)$ is a Bayesian potential game if there exists a potential function $v : X \times \Omega \to \mathbb{R}$ such that $u_i((x_i, x_{-i}), \omega) - u_i((x'_i, x_{-i}), \omega) = v((x_i, x_{-i}), \omega) - v((x'_i, x_{-i}), \omega)$ for each $x_i, x'_i \in X_i$, $x_{-i} \in X_{-i}$, $i \in N$, and a.e. $\omega \in \Omega$ (Monderer and Shapley, 1996; van Heumen et al., 1996). A special case is a team, which is an identical interest Bayesian

[15]For more general conditions for best-response equivalence, see Morris and Ui (2004) and Ui (2009).

game with $u_i = u_j$ for each $i, j \in N$ (Marshak, 1955; Radner, 1962; Marshak and Radner, 1972). It is known that the best-response correspondence of a Bayesian potential game is the same as that of the team with $u_i = v$ for each i. This implies that a strategy profile that maximizes the expected value of the potential function is a Bayesian Nash equilibrium.

The following characterization is well known (Monderer and Shapley, 1996).

Lemma 7. *A smooth Bayesian game* $(\mathbf{u}, \eta)$ *is a Bayesian potential game with a potential function* v *if and only if the payoff gradient of* $\mathbf{u}$ *coincides with the gradient of* v*. When the payoff gradient is continuously differentiable,* $(\mathbf{u}, \eta)$ *is a Bayesian potential game with a potential function* v *if and only if the Jacobian matrix of the payoff gradient is symmetric, in which case the Jacobian matrix equals the Hessian matrix of* v*.*

By this lemma, negative definiteness of the Jacobian matrix implies not only strict monotonicity of the payoff gradient but also strict concavity of the potential function. The next lemma shows that strict monotonicity of the payoff gradient is equivalent to strict concavity of the potential function (see Ui, 2008).[16]

Lemma 8. *Let* $(\mathbf{u}, \eta)$ *be a smooth Bayesian potential game with a potential function* v*. The payoff gradient is strictly monotone if and only if* $v(\cdot, \omega) : X \to \mathbb{R}$ *is strictly concave for a.e.* $\omega \in \Omega$*.*

Using this lemma, we can apply our main results in Section 4 to Bayesian potential games with strictly concave potential functions. In particular, a Bayesian Nash equilibrium is unique if the potential function is strictly concave. This implies that the results of Radner (1962) on teams and those of Ui (2009) on Bayesian potential games are special cases of our main results. Radner (1962) considers teams with strictly concave payoff functions and obtains the special cases of Propositions 1,[17] 4, and 5. Ui (2009) considers Bayesian potential games with strictly concave potential functions[18] and obtains the special cases of Propositions 1, 4, and 5 by pointing out that we can apply Radner's results to Bayesian potential games.

[16]This is true even if the payoff gradient is not continuously differentiable.

[17]See also Krainak et al. (1982).

[18]Ui (2009) considers a more general class of Bayesian games, best-response Bayesian potential games whose best-response correspondences coincide with those of teams.

Note that Lemmas 7 and 8 restate the results of Radner (1962) and Ui (2009) as follows: a Bayesian Nash equilibrium is unique if the Jacobian matrix of the payoff gradient is symmetric[19] and negative definite. Our finding is that the symmetry condition is not necessary.

5.2 Aggregative games

In an aggregative game (Selten, 1970), each player's payoff function is a function of the player's own action and the aggregate of all players' actions. Examples include Cournot and Bertrand competition, private provision of public goods, and rent seeking, among others.[20]

To give a simple sufficient condition for strict monotonicity of the payoff gradient, we consider the following special case: $\nabla \mathbf{u}$ is continuously differentiable and there exists $r_i : X_i \times \sum_{j \in N} X_j \times \Omega \to \mathbb{R}$ and $c_i : X_i \times \Omega \to \mathbb{R}$ such that

$$u_i(x, \omega) = r_i(x_i, \bar{x}, \omega) - c_i(x_i, \omega),$$

where $\bar{x} \equiv \sum_{j \in N} x_j \in \sum_{j \in N} X_j \equiv \{\sum_{j \in N} x'_j : x'_j \in X_j \text{ for each } j \in N\}$, and $R(\bar{x}, \omega) \equiv \sum_{i=1}^n r_i(x_i, \bar{x}, \omega)$ is a function of $\bar{x}$ and ω (depends upon each x_i through $\bar{x}$). Using Lemma 5, we obtain the following corollary of Proposition 1.

Corollary 6. *Assume the following conditions: for a.e. $\omega \in \Omega$,* (i) *r_i is strictly concave in x_i and convex in $\bar{x}$,* (ii) *c_i is convex in x_i, and* (iii) *R is concave in $\bar{x}$. Then, $(\mathbf{u}, \eta)$ has at most one Bayesian Nash equilibrium.*

A Cournot game

We apply Corollary 6 to a Cournot game. Player i produces $x_i \in X_i \equiv \mathbb{R}_+$ units of a homogeneous product. Player i's payoff function is

$$u_i(x, \omega) = p(\bar{x}, \omega) x_i - c_i(x_i, \omega),$$

where $p(\bar{x}, \omega)$ is an inverse demand function and $c_i(x_i, \omega)$ is player i's cost function.

[19]Even if a Bayesian game is not smooth and the Jacobian matrix does not exist, symmetry is essential in characterizing potential games. See Ui (2000).

[20]For example, see a recent paper by Acemoglu and Jensen (2013).

Assume the following conditions: for a.e. $\omega \in \Omega$, (i) $\partial p/\partial \bar{x} + x_i \partial^2 p/\partial \bar{x}^2 < 0$ for each $x \in X$,[21] (ii) $p(\bar{x}, \omega)$ is decreasing and convex in $\bar{x}$, and (iii) $c_i(x_i)$ is increasing and convex in x_i. It is straightforward to see that these conditions imply those in Corollary 6, where $r_i(x_i, \bar{x}, \omega) = p(\bar{x}, \omega)x_i$. Thus, the Cournot game has at most one Bayesian Nash equilibrium.

The above conditions are standard except the convex inverse demand function. In the case of complete information, it is known that there exists a unique Nash equilibrium even if the inverse demand function is not convex (Kolstad and Mathiesen, 1987; Gaudet and Salant, 1991; Long and Soubeyran, 2000).

A rent-seeking game

We apply Corollary 6 to a rent-seeking game of Tullock (1967). Player i chooses an effort level $x_i \in X_i \equiv \mathbb{R}_+$ to win a contest. Player i's payoff function is

$$u_i(x, \theta) = v_i(\omega) \cdot x_i/\bar{x} - c_i(x_i, \omega),$$

where $v_i(\omega)$ is player i's valuation of winning, $x_i/\bar{x}$ is player i's probability of winning, and $c_i(x_i, \omega)$ is player i's cost, which is assumed to be convex in x_i. It is straightforward to see that this game satisfies the conditions in Corollary 6, where $r_i(x_i, \bar{x}, \omega) = v_i(\omega) \cdot x_i/\bar{x}$. Thus, it has at most one Bayesian Nash equilibrium.

Ewerhart and Quartieri (2015) consider a more general class of rent-seeking games and obtain a sufficient condition for the existence of a unique Bayesian Nash equilibrium, which is also based upon strict monotonicity of the payoff gradient.

5.3 Network games

Consider a Bayesian game with quadratic payoff functions (4). For each $\omega \in \Omega$, the matrix Q defines a directed graph with a set of nodes N and a set of directed edges $E(\omega) = \{(i, j) : q_{ij}(\omega) \neq 0, \ i, j \in N\}$. Thus, we can regard this game as a Bayesian game played on a random network, or a Bayesian network game for short, where Q is a weighted adjacency matrix of the underlying graph. Proposition 4 states that a Bayesian network

[21] This condition implies strategic substitutabilities.

game has a unique equilibrium if the weighted adjacency matrix is positive definite for a.e. $\omega \in \Omega$.

The network game of Ballester et al. (2006)

Ballester et al. (2006) consider a network game with complete information such that $A_i = [0, \infty)$, $\theta_1 = \cdots = \theta_n = \alpha$, and $Q = \beta I + \gamma U - \lambda G$, where $\alpha, \beta, \gamma, \lambda > 0$ are constant, I is the identity matrix, U is the matrix of ones, and $G = [g_{ij}]$ is a symmetric matrix with $g_{ij} \in [0, 1]$ and $g_{ii} = 0$. Ballester et al. (2006) show that this game has a unique Nash equilibrium if the maximum eigenvalue of G is less than β/λ, which implies that $\beta I + \gamma U - \lambda G$ is positive definite (but not vice versa).

Now consider a Bayesian network game, where $\alpha, \beta, \gamma, \lambda > 0$ are random variables and G is a random asymmetric matrix with $g_{ij} \in [0, 1]$ and $g_{ii} = 0$. By Proposition 4, this game has a unique Bayesian Nash equilibrium if $\beta I + \gamma U - \lambda G$ is positive definite for each state. de Martì and Zenou (2015) consider a special case of this Bayesian network game, where $\beta, \lambda > 0$, $\gamma = 0$, and $g_{ij} \in \{0, 1\}$ are constant, and show that it has a unique Bayesian Nash equilibrium if the maximum eigenvalue of G is less than β/λ.

The network game of Bramoullé et al. (2014)

Bramoullé et al. (2014) consider a network game with complete information such that $A_i = [0, \infty)$, $\theta_i = \bar{x}_i$, $Q = I + \delta G$, where $\bar{x}_i, \delta > 0$ are constant, I is the identity matrix, and $G = [g_{ij}]$ is a symmetric matrix with $g_{ij} \in \{0, 1\}$ and $g_{ii} = 0$. Bramoullé et al. (2014) show that this game has a unique Nash equilibrium if the absolute value of the minimum eigenvalue of G is less than $1/\delta$, which occurs if and only if $I + \delta G$ is positive definite.

Now consider a Bayesian network game, where $\bar{x}_i, \delta > 0$ are random variables and G is a random asymmetric matrix with $g_{ij} \in [0, 1]$ and $g_{ii} = 0$. By Proposition 4, this game has a unique Bayesian Nash equilibrium if the absolute value of the minimum eigenvalue of G is less than $1/\delta$ for each state.

The network game of Blume et al. (2015)

Blume et al. (2015) consider a Bayesian network game such that $A_i = \mathbb{R}$, $\theta_i, \ldots, \theta_n$ are random variables, and $Q = (1+\phi)I - \phi W$ is a constant matrix, where $\phi > 0$, I is the identity matrix, and $W = [w_{ij}]$ is a nonnegative matrix such that $w_{ij} \in [0, 1]$, $\sum_{j \in N} w_{ij} \in \{0, 1\}$, and $w_{ii} = 0$. Blume et al. (2015) show that this game has a unique Bayesian Nash equilibrium.

Now consider another Bayesian network game, where $\phi > 0$ is a random variable and W is a random matrix. We do not require the above condition on W, but assume that $(1 + \phi)I - \phi W$ is positive definite for each state. Then, this game has a unique Bayesian Nash equilibrium by Proposition 4.

The network game of Calvó-Armengol et al. (2015)

Calvó-Armengol et al. (2015) use an LQG framework to study communication in network games. After exchanging information, players play an LQG game with a payoff function

$$u_i(x, \omega) = -d_{ii}(x_i - \theta_i)^2 - \sum_{j \neq i} d_{ij}(x_i - x_j)^2, \tag{14}$$

where θ_i is normally distributed and $d_{ij} \geq 0$ for each $i, j \in N$. Calvó-Armengol et al. (2015) obtain a unique linear equilibrium, but it has been an open question whether the linear equilibrium is a unique equilibrium.

By dividing (14) by $D_i = \sum_{j \in N} d_{ij}$, we obtain (4) with $Q = I - W$, where $W = [w_{ij}]$ is a nonnegative matrix with

$$w_{ij} = \begin{cases} d_{ij}/D_i & \text{if } j \neq i, \\ 0 & \text{if } j = i. \end{cases}$$

Thus, if $I - W$ is positive definite, then the linear equilibrium is a unique equilibrium by Proposition 5.[22]

Appendix

A Proof of Lemmas 1 and 2

Proof of Lemma 1. Because $u_i((\cdot, x_{-i}), \omega) : X_i \to \mathbb{R}$ is concave,

$$\frac{\partial}{\partial x_i} E\left[u_i\left((x_i, \sigma_{-i}), \omega\right) \mid \eta_i\right] = E[\frac{\partial}{\partial x_i} u_i\left((x_i, \sigma_{-i}), \omega\right) \mid \eta_i]$$

[22]The operator norm of W corresponding to the ∞-norm for vectors (i.e. $\max_{i \in N} \sum_{j \in N} |w_{ij}|$) is less than one, so $I - W$ is invertible as shown by Calvó-Armengol et al. (2015). The matrix $I - W$ is positive definite if and only if the operator norm of W corresponding to the 2-norm for vectors is less than one.

by the Lebesgue monotone convergence theorem (see p.863 of Radner (1962)).

For $x'_i \in X_i$, let $f(t) = E[u_i((\sigma_i + t(x'_i - \sigma_i), \sigma_{-i}), \omega) \mid \eta_i]$. If (1) is true, $f(t)$ achieves its maximum at $t = 0$. Thus, it must be true that

$$\begin{aligned} f'(0) &= \frac{\partial}{\partial x_i} E[u_i((x_i, \sigma_{-i}), \omega) \mid \eta_i]\Big|_{x_i=\sigma_i} (x'_i - \sigma_i) \\ &= E[\frac{\partial}{\partial x_i} u_i\,(\sigma, \omega)\ \mid \eta_i](x'_i - \sigma_i) \\ &= E[\frac{\partial}{\partial x_i} u_i\,(\sigma, \omega)\,(x'_i - \sigma_i) \mid \eta_i] \leq 0. \end{aligned}$$

Thus, (2) is true.

Conversely, suppose that (2) is true. Note that $f(t)$ is concave in t because $u_i((\cdot, x_{-i}), \omega) : X_i \to \mathbb{R}$ is concave. Thus, $f(t) \leq f(0) + tf'(0)$ for each t and

$$\begin{aligned} E[u_i((x'_i, \sigma_{-i}), \omega) \mid \eta_i] =& f(1) \\ \leq& f(0) + f'(0) \\ =& E[u_i(\sigma, \omega) \mid \eta_i] + E[\frac{\partial}{\partial x_i} u_i(\sigma, \omega)(x'_i - \sigma_i) \mid \eta_i] \\ \leq& E[u_i(\sigma, \omega) \mid \eta_i] \end{aligned}$$

by (2). Thus, (1) is true. □

Proof of Lemma 2. We have already shown that if σ is a Bayesian Nash equilibrium then (3) is true. We show that if σ is not a Bayesian Nash equilibrium then (3) is not true. Suppose that $\sigma \in \Sigma$ is not a Bayesian Nash equilibrium. Then, by Lemma 1, there exist $j \in N$, $\sigma'_j \in \Sigma_j$, and $E \subseteq \Omega$ with $P(E) > 0$ such that $E[\partial u_j(\sigma, \omega)/\partial x_j \cdot (\sigma'_j - \sigma_j) \mid \eta_j] > 0$ for each $\omega \in E$. Let $\sigma'' \in \Sigma$ be such that

$$\sigma''_i(\eta_i(\omega)) = \begin{cases} \sigma'_i(\eta_i(\omega)) & \text{if } i = j \text{ and } \omega \in E, \\ \sigma_i(\eta_i(\omega)) & \text{otherwise.} \end{cases}$$

Then,

$$E[\nabla \mathbf{u}(\sigma, \omega)^\top (\sigma'' - \sigma)] = \sum_{i \in N} E[\partial u_i\,(\sigma, \omega)\,/\partial x_i \cdot (\sigma''_i - \sigma_i)] > 0,$$

so (3) is not true. □

B Proof of Proposition 5

We use the following lemma (see p.870 of Radner (1962)).

Lemma A. *Suppose that C is a $K \times K$ symmetric positive semi-definite matrix, partitioned symmetrically into blocks C_{ij}, such that C_{ii} is positive definite for every i, and that Q is an $n \times n$ symmetric positive definite matrix with elements q_{ij}. Then, the matrix H composed of blocks $q_{ij}C_{ij}$ is positive definite.*

Proof of Proposition 5. Proposition 4 guarantees the existence and uniqueness of a Bayesian Nash equilibrium. We show that it is given by (11), (12), and (13).

Because $A_i = \mathbb{R}$ for each $i \in N$, the first-order condition for an equilibrium is

$$\sum_{j \in N} q_{ij} E[\sigma_j \mid \eta_i = y_i] = E[\theta_i \mid \eta_i = y_i] \tag{B1}$$

for each $y_i \in Y_i$ and $i \in N$. If an equilibrium is of the form (11), (B1) is calculated as

$$\sum_j q_{ij}(b_j^\top C_{ji} C_{ii}^{-1}(y_i - E[\eta_i]) + c_j) = E[\theta_i] + G_i^\top C_{ii}^{-1}(y_i - E[\eta_i])$$

for each $y_i \in Y_i$ and $i \in N$. Thus, b_i and c_i are determined by

$$\sum_j q_{ij} b_j^\top C_{ji} C_{ii}^{-1} = G_i^\top C_{ii}^{-1} \text{ and } \sum_j q_{ij} c_j = E[\theta_i] \text{ for } i \in N,$$

which is reduced to (12) and (13).

To complete the proof, it is enough to show that the system of linear equations (12) and (13) has a unique solution. Since Q is positive definite, Q is invertible, by which the solvability of (13) follows. Let C be the covariance matrix of η, which satisfies the condition imposed on C in Lemma A. Let R be the matrix composed of blocks $q_{ij}C_{ij}$ and let H be the matrix composed of blocks $(q_{ij} + q_{ji})C_{ij}$. Since C is symmetric, $H = R + R^\top$. By Lemma A, H is positive definite. Therefore, R is also positive definite and thus invertible, by which the solvability of (12) follows. □

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