REMARKS ON THE NONLINEAR BLACK-SCHOLES EQUATIONS WITH THE EFFECT OF TRANSACTION COSTS

NAOYUKI ISHIMURA

Abstract. We deal with the solvablity and a weak formulation of nonlinear partial differential equations (PDEs) of Black-Scholes type for the pricing of options in the presence of transaction costs. Examples include the Hoggard-Whalley-Wilmott equation, which is introduced to model portfolios of European type options with transaction costs based on the idea of Leland. The cost structure gives rise to nonlinear terms. We show the existence and the uniqueness of solutions both in the classical and the weak sense, where the notion of weak solutions is introduced.

1. Introduction

We are concerned with the solvability and a weak formulation of nonlinear partial differential equations (PDEs) of Black-Scholes type for the pricing of options with transaction costs.

$$
\begin{gathered}
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = \kappa F\Big(S^2\Big|\frac{\partial^2 V}{\partial S^2}\Big|\Big), \\
V = V(S,t) \quad \text{in } (S,t) \in (0,\infty)\times(0,T) \\
V(S,T) = V_0(S) \qquad \text{for } S \geq 0,
\end{gathered}
\tag{1}
$$

where $V(S,t)$ denotes the option price written on the underlying asset S. The constants r and σ stand for the risk-free interest rate and the asset volatility, respectively. The maturity data $V_0(S)$ (≥ 0) is assumed at the beginning to fulfill $V_0'(S) \to \alpha$ exponentially as $S \to \infty$ with a nonnegative constant α. We weaken the regularity assumption on V_0 later. Throughout this paper, we assume $V_0(0) = V(0,t) = 0$ for simplicity. That is to say, we confine ourselves to treating the European call type options. The right hand side κF expresses the cost term where κ $(\kappa > 0)$ is a proportionality constant. The function $F(Q)$ $(Q := S^2|\Gamma|)$, which depends on the absolute value of the option gamma $\Gamma := \partial^2 V/\partial S^2$ multiplied by the square of the asset price, is assumed to satisfy the following hypotheses.

$(\mathrm{H1})_k$ $F(Q)$ is nonnegative, $F(0) = 0$, and uniformly C^k-class for $Q \in (0,\infty)$. $(k = 1, 3.)$
(H2) There exists a positive constant M such that

$$
\Big|\frac{\partial F}{\partial Q}(Q)\Big| \leq M \qquad \text{for } Q \in (0,\infty).
$$

We recall background issues concerning the equation (1). In addition to the basic log-normal model for the asset price, it is well conceded that the celebrated Black-Scholes partial differential equation [4, 19], namely $\kappa = 0$ in (1), is derived upon several ideal assumptions. The list of these includes, for example [3, 13], the absence of arbitrage opportunities, the possibility of continuous self-financing trading of the underlying asset, the constant risk-less

Date: 17 February, 2010.
2000 *Mathematics Subject Classification.* 91B28, 35K15.
Key words: transaction costs, nonlinear partial differential equations, solvability.

interest rate as well as the stock volatility, and so on. The transaction costs associated with trading are also excluded in the original Black-Scholes analysis, which, however, is invalid in general; the influence of transaction costs is actually very important for practitioners. Much attention has been paid so far to the effects of transaction costs and many substantial researches have been undertaken. We refer to [5, 6, 11, 17, 18, 21, 24] for instance and the references cited therein. Among others Leland [18] made a pioneering investigation, whose central idea is to modify the dynamic hedging so that transaction costs are considered to be charged to rehedge over a short discrete interval of time. The approach is nowadays classified into the so-called local in time hedging strategy.

Interpreting the essence of Leland [18] in the PDE setting, Hoggard, Whalley, and Wilmott [12] proposed the next model in which $F(Q) = \sigma\sqrt{2/\pi\delta t}Q$. To be precise

$$
\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = \kappa\sigma S^2 \sqrt{\frac{2}{\pi\delta t}}\left|\frac{\partial^2 V}{\partial S^2}\right| \quad \text{in } (S,t) \in (0,\infty)\times(0,T) \\
&V(S,T) = V_0(S) \qquad \text{for } S \geq 0.
\end{aligned} \tag{2}
$$

This is a typical example of (1) where δt is a non-infinitesimal fixed time-step not to be taken $\delta t \to 0$; the portfolio is considered to be revised every δt. We briefly review on the model and the derivation of PDE in §2. We just mention that the equation (2) is claimed as one of the first nonlinear PDEs in finance.

The principal aim of the current article is to solve (1) without assuming the convexity nor the concavity of V in an elementary way. Although the nonlinear right hand side term is essential in (1), its treatment is cumbersome to some degree within the theory of PDE. The foregone literature thus customarily presupposes the convexity of V to remove the absolute value [2, 22]. In the real world, however, this restriction is not appropriate and there are portfolios which are not necessarily convex nor concave.

Now our first result of this paper reads as follows.

Theorem 1. *Assume (H1)$_3$(H2) and suppose $\kappa < \sigma^2/2M$. Then for any uniformly smooth maturity data $V_0(S)$ (≥ 0) with $V_0'(S) \to \alpha$ $(\alpha \geq 0)$ exponentially as $S \to \infty$, there exists a unique classical solution $V(S,t)$ to the equation (1), whose behavior is given by $(\partial V/\partial S)(S,t) \to \alpha$ exponentially as $S \to \infty$.*

The solution asserted in the above theorem is meant in the smooth classical sense. To be precise, $V(\cdot,t) \in C^2(0,\infty) \cap C([0,\infty))$ and $V(S,\cdot) \in C^1(0,T) \cap C([0,T])$. See for example [1]. We also remark once again that the solution is not presumed a priori to be convex nor concave. The condition $\kappa < \sigma^2/2M$, on the other hand, is somewhat stringent to applications. Roughly speaking, this requirement means that the effect of transaction costs is small compared to the one of volatility. We come back to this point again in §5.

The method of proof is performed through an approximation scheme. Transforming into a variant of heat equation and computing somewhat tacitly in the PDE environment, we deduce a priori estimates independent of approximations. The variation of constants formula coupled with the convergence argument then leads to the existence result which we want. This part of the current article generalizes our previous accomplishments [14, 15].

The transformed equation makes it possible to discuss solutions in the weak sense. To be specific the equation for their derivatives is expressed in the divergence form (see (13) in §4 below) and the notion of weak solutions naturally follows. As a result of this formulation, the regularity hypothesis on V_0 can be weakened and the shortcoming in Theorem 1 such as the assumption of smooth maturity data is remedied. Although a weak-solution approach is adopted totally from the mathematical viewpoint, it seems to be new in the mathematical finance and we believe that such extension is interesting in its own right and worth publishing.

To make an advance, we introduce a Hilbert space to keep our description transparent

$$
\begin{aligned}
&E_0 := \{V \in L^1_{\mathrm{loc}}(0,\infty) \mid \|V\|^2_{E_0} := \int_0^\infty V(S)^2 \frac{dS}{S} < \infty\} \\
&E_1 := \{V \in E_0 \mid \|V\|^2_{E_1} := \|V\|^2_{E_0} + \int_0^\infty \Big(S\frac{\partial V(S)}{\partial S}\Big)^2 \frac{dS}{S} < \infty\} \qquad (3)\\
&E_{-1} := E_1^* \text{ the dual space of } E_1 \text{ with respect to the inner product of } E_0.
\end{aligned}
$$

Furthermore we additionally introduce

$$E_1^0 := \{V \in L^1_{\mathrm{loc}}(0,\infty) \mid S\frac{\partial V}{\partial S} \in E_0\}.$$

It is easy to see that E_0, E_1, E_{-1} are respectively equivalent to $L^2(\boldsymbol{R})$, $H^1(\boldsymbol{R})$, $H^{-1}(\boldsymbol{R})$ via the change of variable $u(x) = V(e^x) = V(S)$ $(x = \log S \in \boldsymbol{R})$.

The definition of weak solution is a little involved and we defer it to the next section. Here we just address our result in the following theorem, which is the main achievement of this article.

Theorem 2. *Assume (H1)$_1$(H2) and suppose $\kappa < \sigma^2/2M$. Then for any $V_0 - \alpha S \in E_1^0$ $(\alpha \geq 0)$, there exists a unique weak solution $V(S,t) - \alpha S \in C^1(0,T;E_1^0)$ in the sense of Definition 3 below.*

Definition 3 is given in §2.

It is to be noted that for $V_0 \in E_1^0$ there holds $S(\partial V_0/\partial S) \to 0$ as $S \to \infty$, which apparently restricts the behavior of maturity data. However, since the linear function αS $(\alpha \in \boldsymbol{R})$ solves the equation (1), the existence of solutions should be understood up to the addition of linear functions. In particular, the Lipschitz maturity data $V_0(S) = \max\{S - E, 0\}$ of vanilla call options with exercise E enter our regime.

We further note that for $V_0 \in E_1^0$, the requirement $V_0(0) = 0$ is legitimate. Moreover we hasten to remark that if V_0 is sufficiently regular and suitably bounded, then the uniqueness property for the Cauchy problem (4)(5) implies that our weak solution $V(S,t)$ agrees with the classical solution.

The organization of the paper is as follows. §2 is devoted to recalling the model, the derivation of PDE, and the definition of weak solution. Theorems 1, 2 are proved in §3, §4, respectively. We conclude with discussions in §5.

2. Model Equation and the Definition of Weak Solution

In this section, for completeness of our exposition and for the readers' convenience, we first make a short sketch of the model whose idea originates in the well known work of Leland [18] and of the derivation of corresponding partial differential equations. We reproduce the argument of [12, 22, 23], to which the reader should refer for further details. We then turn our attention to the definition of weak solution for these equations.

We assume that the underlying asset price follows the random walk which is given in discrete time by

$$\delta S = \sigma S \phi \sqrt{\delta t} + \mu S \delta t,$$

where μ denotes the drift coefficient and ϕ is a random number drawn from the standardized normal distribution. Because of the discrete world, we return to the original Bachelier-Einstein law of the square root of time.

The reason why we introduce a non-infinitesimal fixed time-step δt is that if the costs associated with trading are independent of the time-scale of rehedging, then infinite total transaction costs would be resulted in as we take the limit $dt \to 0$. Leland therefore has proposed a modification to the usual Black-Scholes continuous analysis so that the portfolio is now assumed to be revised every discrete δt. Here we remark that a recent nice result of Sekine and Yano [20] is worth mentioning, where time-inhomogeneous rebalancing is discussed.

Let $\Pi = V - \Delta S$ denote the value of the portfolio with typical delta hedging strategy. We consider the change $\delta\Pi$ in Π over a discrete time-step δt.

$$\delta\Pi = \sigma S\Big(\frac{\partial V}{\partial S} - \Delta\Big)\phi\sqrt{\delta t} + \Big(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\phi^2 + \mu S\frac{\partial V}{\partial S} - \mu\Delta S\Big)\delta t - \kappa S|\nu|.$$

In the computation, the effect of transaction costs is assumed to take the form $\kappa S|\nu|$ and has been subtracted off from $\delta\Pi$, where ν is the number of shares which are bought ($\nu > 0$) or sold ($\nu < 0$) at a price S. This form means that a cost is proportional to the value traded with a constant κ depending on the individual investor. It is to be noted that the resulting partial differential equations will then be reduced to (2). Furthermore the square of the random variable ϕ should be saved in this discrete time world.

We follow the same hedging strategy as Black-Scholes concept and thus choose $\Delta = (\partial V/\partial S)(S,t)$, which has been evaluated at time t and asset S. We then compute

$$\nu = \Delta(S+\delta S, t+\delta t) - \Delta(S,t) = \frac{\partial V}{\partial S}(S+\delta S, t+\delta t) - \frac{\partial V}{\partial S}(S,t) = \frac{\partial^2 V}{\partial S^2}(S,t)\sigma S\phi\sqrt{\delta t} + O(\delta t).$$

Consequently, to leading order, the expected change in the value of the portfolio becomes

$$E[\delta\Pi] = \Big(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \kappa\sigma S^2\sqrt{\frac{2}{\pi\delta t}}\,\Big|\frac{\partial^2 V}{\partial S^2}\Big|\Big)\delta t,$$

which, by the principle of non-arbitrage, equals to the return from a risk-free deposit. Precisely stated $E[\delta\Pi] = r(V - \Delta S)\delta t = r(V - S\partial V/\partial S)\delta t$. We thus obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \kappa\sigma S^2\sqrt{\frac{2}{\pi\delta t}}\,\Big|\frac{\partial^2 V}{\partial S^2}\Big| = r\Big(V - S\frac{\partial V}{\partial S}\Big).$$

This is the Hoggard-Whalley-Wilmott equation (2).

Other transaction cost models, even of slightly more general form than (1), are also possible. We return to this issue in §5.

Now we come to the stage of introducing the definition of our weak solution for (1).

Definition 3. We say that $V \in C^1(0,T;E_1^0)$ is a weak solution of (1) with maturity data $V_0 \in E_1^0$ if $V(S,t)$ is a solution of the linear partial differential equation

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = \kappa F\Big(e^{-r(T-t)}\Big|S\frac{\partial W}{\partial S} - W\Big|\Big) \\ &\qquad\qquad \text{in } (S,t) \in (0,\infty)\times(0,T) \\ &V(S,T) = V_0(S) \quad \text{for } S \geq 0, \end{aligned} \tag{4}$$

where $W(S,t)$ satisfies the next conditions.

(W1) $W \in L^\infty(0,T;E_0) \cap L^2(0,T;E_1)$, $\partial W/\partial t \in L^2(0,T;E_{-1})$
(W2) It holds that for each $Z \in E_1$ and almost every $0 \le t \le T$

$$\begin{aligned}&\int_0^\infty \Big(\frac{\partial W}{\partial t} + rS\frac{\partial W}{\partial S}\Big) Z \frac{dS}{S}\\ &\quad = \int_0^\infty \Big\{\frac{\sigma^2}{2}\Big(S\frac{\partial W}{\partial S} - W\Big) - \kappa e^{r(T-t)} F\Big(e^{-r(T-t)}\Big|S\frac{\partial W}{\partial S} - W\Big|\Big)\Big\} S\frac{\partial Z}{\partial S}\frac{dS}{S}.\end{aligned} \tag{5}$$

(W3) $W(S,T) = S(\partial V_0/\partial S)(S)$ in E_0.

An inspection reveals that $W(S,t) = e^{r(T-t)} S(\partial V/\partial S)(S,t)$ should be the relevant relation. It is also readily seen that the solution $V(S,t)$ for (4) can be expressed as

$$\begin{aligned}V(S,t) &= e^{-r(T-t)} \int_0^\infty G(S/R,t) V_0(R) R^{-1} dR\\ &\quad - \kappa \int_t^T ds \int_0^\infty G(S/R, t-s) e^{-r(s-t)} F\Big(e^{-r(T-s)}\Big|R\frac{\partial W}{\partial R} - W\Big|(R,s)\Big) R^{-1} dR,\end{aligned} \tag{6}$$

where the kernel $G(S,t)$ is given by

$$G(S,t) := \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\Big[-\frac{1}{4}\Big(\frac{\log S + r(T-t)}{\sqrt{\sigma^2(T-t)/2}} - \sqrt{\sigma^2(T-t)/2}\Big)^2\Big]. \tag{7}$$

One can clearly checked that if $F \equiv 0$ then (6)(7) gives the expression of the solution to the ordinary Black-Scholes equation with maturity data V_0. We point out, for general F with hypotheses (H1)(H2), that the last integral of (6) is well-defined for W with conditions (W1)(W2)(W3).

Definition 3 involves rather complicated steps. This is partly because the equation for their derivatives of S is in a sense natural from the standpoint of the theory of PDE. Indeed, (W2) is the standard definition of weak solution for the equation of derivatives. We thus deal with the weak solution for derivatives first, which makes the definition tricky. Nevertheless our weak solution is proved to exist for a wide class of initial data.

3. Proof of Theorem 1

This section is devoted to the proof of Theorem 1.

1. We begin with adopting the change of variables

$$x := \log S + k\tau, \qquad \tau := \frac{\sigma^2}{2}(T-t),$$

with $k := 2r/\sigma^2$ and the prices

$$u(x,\tau) := e^{k\tau} V(e^{x-k\tau}, T - \frac{2}{\sigma^2}\tau),$$

so that the equation (1) is transformed into

$$\begin{aligned}-\frac{\partial u}{\partial \tau} + \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} &= \frac{2\kappa}{\sigma^2} e^{k\tau} F\Big(e^{-k\tau}\Big|\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\Big|\Big) && \text{in } (x,\tau) \in \Omega_T := \boldsymbol{R} \times (0,T)\\ u(x,0) = u_0(x) &= V_0(e^x) && \text{on } x \in \boldsymbol{R}.\end{aligned} \tag{8}$$

Taking into account the fact that if V solves (1) then $V - \alpha S$ also does, we understand from the beginning that

$$V(S,t) \text{ converges exponentially to zero as } S \to \infty.$$

Therefore, slightly extending the condition $V(0,t) = 0$, we interpret the boundary condition on (8) as

$$u(x,\tau) \text{ converges exponentially to zero as } x \to -\infty \text{ and as } x \to \infty.$$

In other words we seek a solution u for (8) in a class of these rapidly decaying functions.

2. Now we approximate the equation (8). Let $\varepsilon \in \boldsymbol{R}$ ($\varepsilon \neq 0$) be a small parameter. Consider

$$(9) \qquad \begin{aligned} \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} - \frac{2\kappa}{\sigma^2} e^{k\tau} F\Big(e^{-k\tau}\sqrt{\varepsilon^2 + \Big(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\Big)^2}\Big) \qquad & \text{in } (x,\tau) \in \Omega_T \\ u(x,0) = u_0(x) \qquad & \text{on } x \in \boldsymbol{R}. \end{aligned}$$

The solution u of (9) will be denoted by the same u without involving ε if there arises no confusion. It is easy to ascertain that we recover (8) from (9) by sending $\varepsilon \to 0$.

3. We introduce

$$v(x,\tau) := \frac{\partial u}{\partial x}(x,\tau), \qquad w(x,\tau) := \frac{\partial^2 u}{\partial x^2}(x,\tau),$$

and the solvability for v will be examined. In a sense we had better deal with the derivative function v, which would necessitate the higher regularity on the initial condition u_0. This limitation is weakened in the next section.

After a little computation we find

$$(10) \qquad \begin{aligned} & \frac{\partial v}{\partial \tau} = \Big(1 - \frac{2\kappa}{\sigma^2}\frac{\partial F}{\partial Q}\frac{\partial v/\partial x - v}{\sqrt{\varepsilon^2 + (\partial v/\partial x - v)^2}}\Big)\Big(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\Big) \quad \text{in } (x,\tau) \in \Omega_T \\ & v(x,\tau) \to 0 \quad \text{exponentially as } |x| \to \infty \\ & v(x,0) = u_0'(x) \quad \text{on } x \in \boldsymbol{R}. \end{aligned}$$

The solution v for (10) is provided if the equation is uniformly parabolic and the uniform $C^1(\boldsymbol{R})$ a priori estimates hold [9].

4. The quasilinear equation (10) becomes uniformly parabolic if $\kappa < \sigma^2/2M$. Furthermore the standard maximum principle of PDE is applied to yield a uniform estimate on v independent of ε, namely, $\sup_{\Omega_T} |v| \leq \sup_{\boldsymbol{R}} |u_0'|$.

We next compute

$$\begin{aligned} \frac{\partial (w-v)}{\partial \tau} &= \Big(1 - \frac{2\kappa}{\sigma^2}\frac{\partial F}{\partial Q}\frac{w-v}{\sqrt{\varepsilon^2 + (w-v)^2}}\Big)\Big(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\Big)(w-v) \\ &\quad - \frac{2\kappa}{\sigma^2}\Big\{\frac{\partial F}{\partial Q}\frac{\varepsilon^2}{(\varepsilon^2 + (w-v)^2)^{3/2}} + e^{-k\tau}\frac{\partial^2 F}{\partial Q^2}\frac{(w-v)^2}{\varepsilon^2 + (w-v)^2}\Big\}(\partial w/\partial x - w)\frac{\partial (w-v)}{\partial x}. \end{aligned}$$

This quasilinear equation for $w - v$ is also uniformly parabolic if $\kappa < \sigma^2/2M$, and a uniform estimate on $w = (w - v) + v$ independent of ε follows by the maximum principle. Therefore the existence of solution for v is inferred. Moreover, once the existence of such solution v is obtained, the second derivative $\partial^2 v/\partial x^2$ is bounded uniformly for $\tau \in (0,T)$ on every compact subset of $\boldsymbol{R}$ and the time derivative $\partial v/\partial \tau$ is continuous by the parabolic regularity.

5. The solution u is reconstructed from v on an integration. To be specific, thanks to the fact that the kernel of the operator $\partial/\partial\tau - (\partial^2/\partial x^2 - \partial/\partial x)$ is

$$K(x,\tau) := \frac{1}{\sqrt{4\pi\tau}} \exp\Big[-\frac{1}{4}\Big(\frac{x}{\sqrt{\tau}} - \sqrt{\tau}\Big)^2\Big], \tag{11}$$

we have from the variation of constants formula

$$\begin{aligned} u_\varepsilon(x,\tau) &= \int_{-\infty}^{\infty} K(x-y,\tau)u_0(y)dy \\ &\quad - \frac{2\kappa}{\sigma^2}\int_0^\tau d\eta \int_{-\infty}^{\infty} K(x-y,\tau-\eta)e^{k\eta}F\Big(e^{-k\eta}\sqrt{\varepsilon^2 + \Big(\frac{\partial v_\varepsilon}{\partial x} - v_\varepsilon\Big)^2(y,\eta)}\Big)dy. \end{aligned} \tag{12}$$

Since v_ε and $\partial v_\varepsilon/\partial x$ are uniformly bounded independent of ε and $\partial^2 v_\varepsilon/\partial x^2$ is bounded on every compact subset, we can find a sequence $\varepsilon_j \to 0$ $(j\to\infty)$ such that v_{ε_j} and $\partial v_{\varepsilon_j}/\partial x$ converge on a fixed interval $(-l,l)$ for every $\tau \in (0,T)$ by the theorem of Arzela-Ascoli. The diagonal argument then implies that there exists a subsequence $\varepsilon_n \to 0$ such that v_{ε_n} and $\partial v_{\varepsilon_n}/\partial x$ converge for all $\tau\in(0,T)$ on every compact subset of $\boldsymbol{R}$. Let us denote the limit functions of v_ε and $\partial v_\varepsilon/\partial x$ by v and w, respectively with the abuse of notation. Since $v_\varepsilon(x,\tau) = \int_{-\infty}^x (\partial v_\varepsilon/\partial x)(y,\tau)dy$, we have $w = \partial v/\partial x$ and moreover, we infer that

$$v(x,\tau) = \int_{-\infty}^{\infty} K(x-y,\tau)u_0'(y)dy - \frac{2\kappa}{\sigma^2}\int_0^\tau d\eta \int_{-\infty}^{\infty} \frac{\partial K}{\partial x}(x-y,\tau-\eta)F\Big(e^{-k\eta}\Big|\frac{\partial v}{\partial x} - v\Big|(y,\eta)\Big)dy.$$

Now for every compact set $K\subset \boldsymbol{R}$ and for any small $\gamma>0$, there exists an $l = l(K,\gamma)$ such that

$$\sup_{x\in K, 0\le\tau\le T}\Big|\frac{2\kappa}{\sigma^2}\int_0^\tau d\eta \int_{\boldsymbol{R}\setminus(-l,l)} K(x-y,\tau-\eta)e^{k\eta}F\Big(e^{-k\eta}\sqrt{\varepsilon_n^2 + \Big(\frac{\partial v_{\varepsilon_n}}{\partial x} - v_{\varepsilon_n}\Big)^2(y,\eta)}\Big)dy\Big| < \frac{\gamma}{4}.$$

There then corresponds $n_0 = n_0(K,\gamma,l) = n_0(K,\gamma)$ such that for all $n,m\ge n_0$

$$\begin{aligned} &\sup_{x\in K, 0\le\tau\le T}\Big|u_{\varepsilon_n}(x,\tau) - u_{\varepsilon_m}(x,\tau)\Big| \\ &\quad\le \sup_{x\in K,0\le\tau\le T}\Big|\frac{2\kappa}{\sigma^2}\int_0^\tau d\eta\int_{-l}^{l} K(x-y,\tau-\eta)e^{k\eta} \\ &\qquad\cdot\Big(F\Big(e^{-k\eta}\sqrt{\varepsilon_n^2 + \Big(\frac{\partial v_{\varepsilon_n}}{\partial x} - v_{\varepsilon_n}\Big)^2}\Big) - F\Big(e^{-k\eta}\sqrt{\varepsilon_m^2 + \Big(\frac{\partial v_{\varepsilon_m}}{\partial x} - v_{\varepsilon_m}\Big)^2}\Big)\Big)dy\Big| + \frac{\gamma}{2} \\ &\quad< \gamma, \end{aligned}$$

from which we assert that $\{u_{\varepsilon_n}\}$ converges to a solution u of (8) on every compact subset of $\boldsymbol{R}$ by letting $\varepsilon_n \to 0$ in (12). Transforming back we finally conclude that the solution for (1) is constructed.

The uniqueness property holds true even in a weak setting, and thus we postpone the proof to the next section.

This completes the proof of Theorem 1.

Remark 4. The proof uncovers that the sign of κ or F is actually irrelevant; we are thus able to extend Theorem 1 so that it holds true under the condition $|\kappa| < \sigma^2/2M$. The negative sign induces the short positioned option. The same remark applies to the weak formulation below.

4. Weak formulation

The technique of proof developed in §3 brings us to think about the weak formulation of the problem. As already outlined there, it is convenient to look at v variable; indeed we see that the equation for $v = \partial u/\partial x$ in (8) can be written in the divergence form.

$$\begin{aligned} \frac{\partial v}{\partial \tau} &= \frac{\partial}{\partial x}\Big\{\frac{\partial v}{\partial x} - v - \frac{2\kappa}{\sigma^2} e^{k\tau} F\Big(e^{-k\tau}\Big|\frac{\partial v}{\partial x} - v\Big|\Big)\Big\} \quad \text{in } (x,\tau) \in \Omega_T \\ & \qquad v(x,0) = v_0(x) \quad \text{for } x \in \boldsymbol{R}, \end{aligned} \tag{13}$$

where $v_0(x) = e^x(\partial V_0/\partial S)(e^x)$ now belongs to $L^2(\boldsymbol{R})$. It is to be noted that so long as we are interested in weak solutions there is no need for approximating the absolute value; this owes to the fact that if $v \in H^1(\boldsymbol{R})$ then there holds $|v| \in H^1(\boldsymbol{R})$. Observe [10] for instance.

We come to the point of clarifying the concept of weak solutions for (13).

Definition 5. We say that v is a weak solution of (13) if the next conditions are satisfied.

(W1)′ $v \in L^\infty(0,T; L^2(\boldsymbol{R})) \cap L^2(0,T; H^1(\boldsymbol{R}))$, $\partial v/\partial \tau \in L^2(0,T; H^{-1}(\boldsymbol{R}))$

(W2)′ It holds that for each $\varphi \in H^1(\boldsymbol{R})$ and almost every $0 \le \tau \le T$

$$\int_{-\infty}^{\infty} \frac{\partial v}{\partial \tau}(x,\tau)\varphi(x)dx = -\int_{-\infty}^{\infty} \Big\{\frac{\partial v}{\partial x} - v - \frac{2\kappa}{\sigma^2} e^{k\tau} F\Big(e^{-k\tau}\Big|\frac{\partial v}{\partial x} - v\Big|\Big)\Big\}\frac{\partial \varphi}{\partial x}(x)dx. \tag{14}$$

(W3)′ $v(x,0) = v_0(x)$ in $L^2(\boldsymbol{R})$.

Here $v_0 = \partial u_0/\partial x \in L^2(\boldsymbol{R})$ denotes initial data.

The conditions (W1)′(W2)′(W3)′ parallel to (W1)(W2)(W3) of Definition 3. It will be ascertained by direct calculation that the relation between v and W from Definition 3 is $v(x,\tau) = W(e^{x-k\tau}, T - 2\tau/\sigma^2)$ $(k = 2r/\sigma^2)$.

Grounded on this definition we show the existence of solutions.

Theorem 6. *Assume (H1)$_1$(H2) and suppose $\kappa < \sigma^2/2M$. For any initial data $v_0 \in L^2(\boldsymbol{R})$ there exists a unique weak solution v in the sense of Definition 5.*

Since Theorem 2 is an immediate consequence of Theorem 6 by way of the change of variables, we are now left with the proof of Theorem 6.

1. To start with, we approximate v_0 by a sequence of functions $v_0^n \in L^2(\boldsymbol{R})$ $(n = 1, 2, \cdots)$ whose essential support $\mathrm{spt} v_0^n \subset (-n,n)$. Then the general existence and uniqueness theorem of weak solutions [8] is valid; we infer that there exists a unique weak solution $v^n \in L^2(0,T; H_0^1(-n,n))$ with $\partial v^n/\partial \tau \in L^2(0,T; H^{-1}(-n,n))$ $(n = 1,2,\cdots)$ which satisfies (14) for each $\varphi \in H_0^1(-n,n)$ and almost every $0 \le \tau \le T$, and $v^n(x,0) = v_0^n(x)$ in $L^2(-n,n)$. We extend v^n to be defined on the whole $\boldsymbol{R}$ by setting $v^n = 0$ on $\boldsymbol{R} \setminus (-n,n)$ and we understand that $v^n \in L^2(0,T; H_0^1(\boldsymbol{R}))$ with $\partial v^n/\partial \tau \in L^2(0,T; H^{-1}(\boldsymbol{R}))$ $(n = 1,2,\cdots)$.

We wish to pass to limits as $n \to \infty$ so as to build a desired weak solution on the whole $\boldsymbol{R}$. To justify this aim we appeal to a priori estimates.

2. Multiplying (13) by v and invoking (H1)(H2) we find

$$\frac{1}{2}\frac{d}{d\tau}\|v(\tau)\|_{L^2}^2 + \Big\|\frac{\partial v}{\partial x}(\tau)\Big\|_{L^2}^2 \le \frac{2\kappa}{\sigma^2} M\Big(\Big\|\frac{\partial v}{\partial x}(\tau)\Big\|_{L^2}^2 + \|v(\tau)\|_{L^2}\Big\|\frac{\partial v}{\partial x}(\tau)\Big\|_{L^2}\Big).$$

Choosing γ (> 0) so small that $2\kappa M/\sigma^2 \le 1 - 2\gamma$ we deduce that

$$\frac{1}{2}\frac{d}{d\tau}\|v(\tau)\|_{L^2}^2 + \gamma\Big\|\frac{\partial v}{\partial x}(\tau)\Big\|_{L^2}^2 \le \frac{(1-2\gamma)^2}{4\gamma}\|v(\tau)\|_{L^2}^2, \tag{15}$$

where the use of the inequality $|ab| \leq \gamma a^2 + b^2/4\gamma$ was made. The differential inequality (15) now yields

$$\max_{0\leq\tau\leq T} \|v(\tau)\|^2_{L^2} \leq \|v_0\|^2_{L^2} e^{(1-2\gamma)^2\tau/2\gamma}, \tag{16}$$

and hence we obtain

$$\int_0^\tau \Big\|\frac{\partial v}{\partial x}(\eta)\Big\|^2_{L^2} d\eta \leq \frac{1}{2\gamma} e^{(1-2\gamma)^2\tau/2\gamma} \|v_0\|^2_{L^2}. \tag{17}$$

Next multiplying (13) by any fixed $\varphi \in H^1(\boldsymbol{R})$ with $\|\varphi\|_{H^1} \leq 1$ we calculate

$$\Big|\Big(\frac{\partial v}{\partial \tau}(\tau), \varphi\Big)_{L^2}\Big| \leq \Big(1 + \frac{2\kappa M}{\sigma^2}\Big)\Big|\frac{\partial v}{\partial x} - v\Big|_{L^2}\Big|\frac{\partial \varphi}{\partial x}\Big|_{L^2} \leq 2\|v\|_{H^1},$$

and therefore

$$\int_0^\tau \Big\|\frac{\partial v}{\partial \eta}(\eta)\Big\|^2_{H^{-1}} d\eta \leq \Big(\frac{2}{\gamma} + 4\tau\Big) e^{(1-2\gamma)^2\tau/2\gamma} \|v_0\|^2_{L^2}. \tag{18}$$

Above estimates hold for v^n independently of n.

3. Our task is to show the convergence of $\{v^n\}$. Since $\{v^n\}$ is bounded in $L^2(0,T;H^1(\boldsymbol{R}))$, we can extract a subsequence, which is denoted by the same $\{v^n\}$, such that $\{v^n\}$ converges weakly in $L^2(0,T;H^1(\boldsymbol{R}))$. By the diagonal argument, we may assume that $\{v^n\}$ converges strongly in $L^2(0,T;L^2(-l,l))$ for every $l > 0$. Let us denote a weak limit of $\{v^n\}$ by v^∞.

We want to prove that $\{v^n\}$ converges strongly to v^∞ in $L^\infty(0,T;L^2(\boldsymbol{R})) \cap L^2(0,T;H^1(\boldsymbol{R}))$. To do so, next two preliminary claims are in order.

$$\begin{aligned} &|v^\infty(\pm m,\tau)|^2 \leq 2\|v^\infty(\eta)\|_{L^2(\boldsymbol{R}\setminus(-m,m))}\Big\|\frac{\partial v^\infty}{\partial x}(\eta)\Big\|_{L^2(\boldsymbol{R}\setminus(-m,m))} \to 0 \quad \text{as } m \to \infty, \\ &\int_0^\tau \Big|\frac{\partial v^m}{\partial x}(\pm m,\eta)\Big|^2 d\eta \leq N, \quad \text{where a constant } N \text{ is independent of } m. \end{aligned} \tag{19}$$

For the proof we consider the point $x = m$ only. The point $x = -m$ is treated similarly.

Claim 1 is ascertained as follows.

$$|v^\infty(m,\eta)|^2 = -\int_m^n \frac{\partial}{\partial x}(v^\infty(\eta))^2 dx \leq 2\|v^\infty(\eta)\|_{L^2(m,\infty)}\Big\|\frac{\partial v^\infty}{\partial x}(\eta)\Big\|_{L^2(m,\infty)},$$

from which we particularly deduce that $\int_0^\tau |v^\infty(\pm m,\eta)|^2 d\eta \to 0$ as $m \to \infty$.

As to Claim 2, we see that for any $M > 0$ there corresponds a set $A_M \subset \boldsymbol{R}$ with $\text{meas}|A_M| \leq (2\gamma M)^{-1} e^{(1-2\gamma)^2\tau/2\gamma}\|v_0\|^2_{L^2}$ such that

$$\int_0^\tau \Big|\frac{\partial v}{\partial x}(y,\eta)\Big|^2 d\eta \leq M$$

for every $y \in \boldsymbol{R} \setminus A_M$ by virtue of (17). We multiply (13) by any fixed $\varphi \in H^1(\boldsymbol{R})$ and integrate it over $m - l \leq x \leq m$ with $0 \leq l \leq l_M := 2\text{meas}|A_M|$, where φ is taken to satisfy $\varphi(m) = 1$ and $\|\varphi\|_{H^1} \leq 1$. It follows that

$$\begin{aligned} \int_{m-l}^m \frac{\partial v^m}{\partial \tau}(\tau)\varphi dx &+ \int_{m-l}^m \Big\{\frac{\partial v^m}{\partial x} - v^m - \frac{2\kappa}{\sigma^2}e^{k\tau}F\Big(e^{-k\tau}\Big|\frac{\partial v^m}{\partial x} - v^m\Big|\Big)\Big\}(\tau)\frac{\partial \varphi}{\partial x}dx \\ &= \frac{\partial v^m}{\partial x}(m,\tau) - \frac{2\kappa}{\sigma^2}e^{k\tau}F\Big(e^{-k\tau}\Big|\frac{\partial v^m}{\partial x}\Big|(m,\tau)\Big) + v^m(m-l,\tau)\varphi(m-l) \\ &\quad - \Big\{\frac{\partial v^m}{\partial x}(m-l,\tau) - \frac{2\kappa}{\sigma^2}e^{k\tau}F\Big(e^{-k\tau}\Big|\frac{\partial v^m}{\partial x} - v^m\Big|(m-l,\tau)\Big)\Big\}\varphi(m-l). \end{aligned}$$

Taking account of $|v^m(m-l,\tau)|^2 \le l_M \int_{m-l}^m |\partial v^m/\partial x|^2 dx$ and $|\varphi(m-l)|^2 = (\varphi(m) - \int_{m-l}^m \varphi' dx)^2 \le (1+\sqrt{l_M})^2$, we infer that

(20)
$$\gamma^2 \int_0^\tau \Big|\frac{\partial v^m}{\partial x}(m,\eta)\Big|^2 d\eta \le 4(1+\sqrt{l_M})^2 \int_0^\tau \Big|\frac{\partial v^m}{\partial x}(m-l,\eta)\Big|^2 d\eta + \int_0^\tau \Big\|\frac{\partial v^m}{\partial \eta}(\eta)\Big\|_{H^{-1}}^2 \|\varphi\|_{H^1}^2 d\eta$$
$$+ 4\int_0^\tau \|v^m(\eta)\|_{L^2}^2 d\eta + 4(l_M^2(1+\sqrt{l_M})^2+1)\int_0^\tau \Big\|\frac{\partial v^m}{\partial x}(\eta)\Big\|_{L^2}^2 d\eta.$$

Choosing l appropriately and invoking (16)(17)(18), we obtain

$$\gamma^2 \int_0^\tau \Big|\frac{\partial v^m}{\partial x}(m,\eta)\Big|^2 d\eta \le 4(1+\sqrt{l_M})^2 M + \Big(\frac{2}{\gamma}(2+l_M^2(1+\sqrt{l_M})^2)+8\tau\Big) e^{(1-2\gamma)^2\tau/2\gamma}\|v_0\|_{L^2}^2.$$

It should be noted that although there is no explicit information on A_M, we have the estimate for the measure of A_M. Therefore there does exists $l \le l_M$ $(= 2\text{meas}|A_M|)$ such that $\int_0^\tau |(\partial v^m/\partial x)(m-l,\eta)|^2 d\eta \le M$ in (20). We further remark that l $(\in [0,l_M])$ may be selected differently for every m.

4. Now we plug $v = v^n$ and v^m into the equation (13), respectively $(n > m)$. Subtracting term by term and multiplying by $\varphi := v^n - v^m$, we discover the next estimate after integration by parts. We note that $v^n(\tau)$ and $v^m(\tau)$ are extended to belong to $H_0^1(\boldsymbol{R})$ and hence the care must be paid in the computation.

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\|(v^n-v^m)(\tau)\|_{L^2}^2 + \Big\|\frac{\partial(v^n-v^m)}{\partial x}(\tau)\Big\|_{L^2}^2 \\
&\quad = \frac{2\kappa}{\sigma^2}e^{k\tau}\int_0^1 \frac{d}{ds}\Big\{\int_{-\infty}^\infty F\Big(e^{-k\tau}\Big(s\Big|\frac{\partial v^n}{\partial x}-v^n\Big| + (1-s)\Big|\frac{\partial v^m}{\partial x}-v^m\Big|\Big)\Big)\frac{\partial(v^n-v^m)}{\partial x}dx\Big\}ds \\
&\qquad - \Big\{\frac{\partial v^m}{\partial x} - v^m - \frac{2\kappa}{\sigma^2}e^{k\tau}F\Big(e^{-k\tau}\Big|\frac{\partial v^m}{\partial x}-v^m\Big|\Big)\Big\}(v^n-v^m)(\tau)\Big|_{x=\pm m} \\
&\quad \le (1-2\gamma)\int_{-\infty}^\infty \Big|\Big|\frac{\partial v^n}{\partial x}-v^n\Big| - \Big|\frac{\partial v^m}{\partial x}-v^m\Big|\Big|\cdot\Big|\frac{\partial(v^n-v^m)}{\partial x}\Big|dx \\
&\qquad + \sum\Big|\Big(\frac{\partial v^m}{\partial x} - \frac{2\kappa}{\sigma^2}e^{k\tau}F\Big(e^{-k\tau}\Big|\frac{\partial v^m}{\partial x}\Big|\Big)\Big)v^n(\pm m,\tau)\Big| \\
&\quad \le (1-2\gamma)\int_{-\infty}^\infty \Big|\frac{\partial(v^n-v^m)}{\partial x} - (v^n-v^m)\Big|\cdot\Big|\frac{\partial(v^n-v^m)}{\partial x}\Big|dx + 2\sum\Big|v^n\frac{\partial v^m}{\partial x}\Big|(\pm m,\tau) \\
&\quad \le (1-2\gamma)\Big\{\Big\|\frac{\partial(v^n-v^m)}{\partial x}(\tau)\Big\|_{L^2}^2 + \|(v^n-v^m)(\tau)\|_{L^2}\Big\|\frac{\partial(v^n-v^m)}{\partial x}(\tau)\Big\|_{L^2}\Big\} \\
&\qquad + 2\sum\Big|v^n\frac{\partial v^m}{\partial x}\Big|(\pm m,\tau),
\end{aligned}$$

where γ is chosen as in (15). We thus obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\|(v^n-v^m)(\tau)\|_{L^2}^2 + \gamma\Big\|\frac{\partial(v^n-v^m)}{\partial x}(\tau)\Big\|_{L^2}^2 \\
&\qquad\qquad \le \frac{(1-2\gamma)^2}{4\gamma}\|(v^n-v^m)(\tau)\|_{L^2}^2 + 2\sum\Big|v^n\frac{\partial v^m}{\partial x}\Big|(\pm m,\tau),
\end{aligned}$$

and hence, thanks to the Gronwall lemma,

$$\begin{aligned}\|(v^n - v^m)(\tau)\|_{L^2}^2 &+ 2\gamma \int_0^\tau e^{C(\tau-\eta)} \Big\| \frac{\partial (v^n - v^m)}{\partial x}(\eta) \Big\|_{L^2}^2 d\eta \\ &\le e^{C\tau} \|v_0^n - v_0^m\|_{L^2} + 4 \int_0^\tau e^{C(\tau-\eta)} \sum \Big| v^n \frac{\partial v^m}{\partial x} \Big| (\pm m, \eta) d\eta \\ &\le e^{C\tau} \|v_0^n - v_0^m\|_{L^2} + 4e^{CT} \sqrt{\int_0^\tau \sum |v^n(\pm m, \eta)|^2 d\eta} \sqrt{\int_0^\tau \sum \Big| \frac{\partial v^m}{\partial x}(\pm m, \eta) \Big|^2 d\eta},\end{aligned}$$

where $C := (2\gamma)^{-1}(1-2\gamma)^2$.

We want to send $n \to \infty$ first. Before doing that we notice that we are able to extract a subsequence, which is still denoted by the same $\{v^n\}$, such that

$$\int_0^\tau \sum |v^n(\pm m, \eta)|^2 d\eta \to \int_0^\tau \sum |v^\infty(\pm m, \eta)|^2 d\eta \qquad \text{as } n \to \infty. \tag{21}$$

The demonstration is deferred temporarily and we send $n \to \infty$ first to discover

$$\begin{aligned}\|(v^\infty - v^m)(\tau)\|_{L^2}^2 &+ 2\gamma \int_0^\tau e^{C(\tau-\eta)} \Big\| \frac{\partial (v^\infty - v^m)}{\partial x}(\eta) \Big\|_{L^2}^2 d\eta \\ &\le \liminf_{n\to\infty} \|(v^n - v^m)(\tau)\|_{L^2}^2 + 2\gamma \liminf_{n\to\infty} \int_0^\tau e^{C(\tau-\eta)} \Big\| \frac{\partial (v^n - v^m)}{\partial x}(\eta) \Big\|_{L^2}^2 d\eta \\ &\le e^{C\tau} \|v_0 - v_0^m\|_{L^2} + 4e^{CT} \sqrt{\int_0^\tau \sum |v^\infty(\pm m, \eta)|^2 d\eta} \sqrt{\int_0^\tau \Big| \frac{\partial v^m}{\partial x}(\pm m, \eta) \Big|^2 d\eta}.\end{aligned}$$

Consequently, in light of (19), we learn that $\{v^n\}$ strongly converges in $L^\infty(0, T; L^2(\boldsymbol{R})) \cap L^2(0, T; H^1(\boldsymbol{R}))$.

5. We make sure of (21). We deal with the point $x = m$. Since for every fixed η, $|v^n(m, \eta)|^2 = |\int_m^n (\partial v^n(\eta)^2/\partial x) dx| \le 2\|(\partial v^n/\partial x)(\eta)\|_{L^2} \|v^n(\eta)\|_{L^2}$ is bounded independent of n and m, we select a subsequence $m < n_{11} < \cdots < n_{1j} < \cdots$ such that $\{|v^{n_{1j}}(m, \eta)|^2\}_{j=1}^\infty$ converges. Put $m_2 := n_{11}$. We select a subsequence $m_2 < n_{21} < \cdots < n_{2j} < \cdots$ from $\{n_{1j}\}_{j=1,2\dots}$ such that $\{|v^{n_{2j}}(m_2, \eta)|^2\}_{j=1}^\infty$ converges. Put $m_3 := n_{21}$ and continue the procedure. Define $\{v^n\}_{n=1,2\dots} = \{v^{m_j}\}_{j=1,2,\dots}$ with abuse of notation. This extraction can be made for countable dense point $\{\tau_j\} \subset (0, T)$

Now suppose (21) does not hold and there exists $\varepsilon > 0$ ($\varepsilon \ll 1$) such that

$$\lim_{n\to\infty} \int_0^T |v^n(m, \eta)|^2 d\eta - \int_0^T |v^\infty(m, \eta)|^2 d\eta \ge 4\varepsilon. \tag{22}$$

Here we recall that $\liminf_{n\to\infty} \int_0^T |v^n(m, \eta)|^2 d\eta \ge \int_0^T |v^\infty(m, \eta)|^2 d\eta$. Taking into account that $\{v^n\} \subset L^2(0, T; H^1(\boldsymbol{R}))$ has been extracted to converge strongly in $L^2(0, T; L^2(-l, l))$ for every $l > 0$, there corresponds n_0 such that

$$\int_0^T \Big| \|v^n(\eta)\|_{L^2(-2m,2m)}^2 - \|v^\infty(\eta)\|_{L^2(-2m,2m)}^2 \Big| d\eta \le \frac{\varepsilon^2}{T} \tag{23}$$

for every $n \ge n_0$. By virtue of (22), there exist τ_j and $n > n_0$ such that

$$\Big| \int_{\tau_j}^{\tau_j+\varepsilon} |v^n(m, \eta)|^2 d\eta - \int_{\tau_j}^{\tau_j+\varepsilon} |v^\infty(m, \eta)|^2 d\eta \Big| \ge \frac{3\varepsilon^2}{T},$$

from which we infer that, thanks to $v^n(\tau_j)$, $v^\infty(\tau_j) \in H^1(\boldsymbol{R})$, there is small $\delta > 0$ with

$$\Big| \int_{\tau_j}^{\tau_j+\varepsilon'} \|v^n(\eta)\|^2_{L^2(m-\delta,m+\delta)} d\eta - \int_{\tau_j}^{\tau_j+\varepsilon'} \|v^\infty(\eta)\|^2_{L^2(m-\delta,m+\delta)} d\eta \Big| \geq \frac{2\varepsilon^2}{T}$$

by choosing smaller ε' if necessary. This is in contradiction with (23) and we conclude that (22) is absurd.

6. The uniqueness is provided by a similar inequality as (15). To be precise, suppose v^1 and v^2 are two weak solutions with the same initial data constructed above, then we obtain

$$\frac{1}{2}\frac{d}{d\tau}\|(v^1 - v^2)(\tau)\|^2_{L^2} + \gamma \Big\| \frac{\partial(v^1 - v^2)}{\partial x}(\tau) \Big\|^2_{L^2} \leq \frac{(1-2\gamma)^2}{4\gamma}\|(v^1 - v^2)(\tau)\|^2_{L^2},$$

from which we see that $\|(v^1 - v^2)(\tau)\|^2_{L^2} \equiv 0$.

The proof of Theorem 6 is thereby concluded.

Remark 7. If we adopt the following approximation scheme for (13) similarly as in §3, that is, we deal with

(24)

$$\frac{\partial v}{\partial \tau} = \frac{\partial}{\partial x}\Big\{ \frac{\partial v}{\partial x} - v - \frac{2\kappa}{\sigma^2} e^{k\tau}\Big(F\Big(e^{-k\tau}\sqrt{\varepsilon^2 + \Big(\frac{\partial v}{\partial x} - v\Big)^2}\Big) - F(e^{-k\tau}|\varepsilon|)\Big)\Big\} \quad \text{in } (x,\tau) \in \Omega_T$$

$$v(x,0) = v_0(x) \quad \text{for } x \in \boldsymbol{R},$$

for a small parameter $\varepsilon \in \boldsymbol{R}$ ($\varepsilon \neq 0$). The term $F(e^{-k\tau}|\varepsilon|)$ is for the scaling. The formula corresponding to (14) becomes

$$\int_{-\infty}^{\infty} \frac{\partial v}{\partial \tau}(x,\tau)\varphi(x)dx$$

$$= -\int_{-\infty}^{\infty} \Big\{ \frac{\partial v}{\partial x} - v - \frac{2\kappa}{\sigma^2} e^{k\tau}\Big(F\Big(e^{-k\tau}\sqrt{\varepsilon^2 + \Big(\frac{\partial v}{\partial x} - v\Big)^2}\Big) - F(e^{-k\tau}|\varepsilon|)\Big\} \frac{\partial \varphi}{\partial x}(x)dx.$$

If the initial data v_0 is much regular, then the higher regularity on weak solutions can be expected. Indeed we multiply (24) by $\partial^2 v/\partial x^2$ and integrate to find

$$\frac{1}{2}\frac{d}{d\tau}\Big\|\frac{\partial v}{\partial x}(\tau)\Big\|^2_{L^2} + \Big\|\frac{\partial^2 v}{\partial x^2}(\tau)\Big\|^2_{L^2} = \Big(\frac{2\kappa}{\sigma^2}\frac{\partial F}{\partial Q}\frac{\partial v/\partial x - v}{\sqrt{\varepsilon^2 + (\partial v/\partial x - v)^2}}\Big(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\Big), \frac{\partial^2 v}{\partial x^2}\Big)_{L^2}$$

$$\leq (1-2\gamma)\Big(\Big\|\frac{\partial^2 v}{\partial x^2}(\tau)\Big\|^2_{L^2} + \Big\|\frac{\partial v}{\partial x}(\tau)\Big\|_{L^2}\Big\|\frac{\partial^2 v}{\partial x^2}(\tau)\Big\|_{L^2}\Big),$$

where γ is selected small as before in (15). We assert that

$$\frac{1}{2}\frac{d}{d\tau}\Big\|\frac{\partial v}{\partial x}(\tau)\Big\|^2_{L^2} + \gamma\Big\|\frac{\partial^2 v}{\partial x^2}(\tau)\Big\|^2_{L^2} \leq \frac{(1-2\gamma)^2}{4\gamma}\Big\|\frac{\partial v}{\partial x}(\tau)\Big\|^2_{L^2},$$

and $v \in L^\infty(0,T;H^1(\boldsymbol{R})) \cap L^2(0,T;H^2(\boldsymbol{R}))$ is demonstrated.

5. Discussions

We have established the existence and the uniqueness of solutions, both in the classical sense and in the weak sense, to the model equation which extends the well-known Black-Scholes and incorporates the effects of transaction costs. The solutions we constructed are not assumed a priori to be convex nor concave. We thus believe that the proved solvability result is important still from the viewpoint of applications.

The existence is provided if $\kappa < \sigma^2/2M$ holds for (1). In the case of the Hoggard-Whalley-Wilmott equation (2) this condition corresponds to $2\kappa\sqrt{2/\sigma^2\pi\delta t} < 1$, which means that the proportional rate κ of costs to the traded value is relatively small, or the stock volatility σ is relatively large, or the non-infinitesimal time-step δt of rehedging is relatively large. We hope that these findings may shed light on the criterion of the model itself. Moreover we note that it would be a challenging problem to discuss the equation without imposing this condition.

The nonlinear function F in (1), which reflects the presence of transaction costs, is assumed to depend solely on $Q = S^2|\Gamma| = S^2|\partial^2 V/\partial S^2|$. There may be a concern that this is just a theoretical assumption. To clarify such question we analyze other existing models. We refer to [7, 22] for the background of these models. One is the so-called extended Leland model. This is

$$F(S,|\Gamma|) = \frac{\kappa_1}{\delta t} + \sigma(\kappa_2 + \kappa_3 S)S\sqrt{\frac{2}{\pi\delta t}}|\Gamma| = \frac{\kappa_1}{\delta t} + \sigma\Big(\frac{\kappa_2}{S} + \kappa_3\Big)\sqrt{\frac{2}{\pi\delta t}}Q, \tag{25}$$

where κ_1, κ_2 and κ_3 components describe a fixed cost component, a cost proportional to the number traded, and a cost proportional to the value traded, respectively. The apparent singularity $1/S$ appears if we extract Q variable. We remark that (25) verifies hypotheses (H1)(H2) provided $\kappa_2 = 0$. Therefore our analysis does not work well for the model involving a cost proportional to the number traded, which to some extent restricts the applicability. However it seems that the existence result for the full model of (25) is not known.

Another one is market practice model. This is

$$F(S,|\Gamma|) = \frac{\sigma^2 S^4}{H_0}\Big(\kappa_1 + (\kappa_2 + \kappa_3 S)\frac{\sqrt{H_0}}{S}\Big)|\Gamma|^2 = \frac{\sigma^2 Q^2}{H_0}\Big(\kappa_1 + (\kappa_2 + \kappa_3 S)\frac{\sqrt{H_0}}{S}\Big), \tag{26}$$

where H_0 is a function of S, t as well as V and its derivatives. The parameters κ_1, κ_2 and κ_3 have the same meaning as for (25). The application of our method depends on the choice of H_0. If $H_0 \equiv$constant, then hypotheses (H1)(H2) is verified provided $\kappa_2 = 0$ and additionally Q stays bounded; the last condition is a restrictive requirement. If $H_0 \equiv \sigma^2\delta t S^4|\Gamma|^2$, then (26) reduces to (25) under an adjustment of parameters.

Nevertheless, these two equations indicate that the form (1) is not intended just for the mathematical sophistication. In addition if F depends explicitly on S as in (25)(26) our procedure of handling the partial differential equation would be harder.

One advantage of employing the PDE approach in mathematical finance is that the numerical computation is a common tool for the study of PDE. The situation also applies to a wide class of problems. For the Hoggard-Whalley-Wilmott equation (2) we refer to our accompanying paper [14], where a numerical technique for effectively computing the PDE is pursued. For other related equations including (25), we refer to our recent publication [16]. A numerical implementation combined with the study of real world data is attractive and this would be our future theme for research.

Acknowledgments. This work is partially supported by Grants-in-Aids for Scientific Research (No. 16540184) from the Japan Society for Promotion of Sciences, and by the Inamori Foundation through the Inamori Grants for the fiscal year 2006-2007. Thanks are also due to the referee for careful reading and valuable suggestions, which helps in improving the manuscript.

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Naoyuki ISHIMURA: Department of Mathematics, Graduate School of Economics, Hitotsubashi University, Kunitachi, Tokyo 186-8601, Japan.

E-mail address: ishimura@econ.hit-u.ac.jp