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不確実性下での資産価格

Asset Pricing under Uncertainty

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Contents

Chapter 1

Introduction

The objective of this thesis is

1. to understand what determines so called "hedging demands" of investors (that would give us better understanding of risk-premia) in portfolio selection problems; and
2. to provide examples of pricing under imperfect information that would be clues to solve general problems.

We specifically consider *mutual fund separation* with perfect information in complete markets for the first problem, and consider *speculative trading* and *insurance premium* for the second problem.

The asset pricing theory, as typified by Lucas (1978) and Cox, Ingersoll, and Ross (1985a), derives asset prices and risk-free rate from the preferences of investors and market clearing conditions. For example, Lucas (1978) models the dynamics of the consumption flows, and determines the prices of them, such that its holders consume all the production and such that the market clears. Cox, Ingersoll, and Ross (1985b), whose work Cox et al. (1985a) depend on, model the price dynamics of the production technologies themselves, and determine the risk-free rate such that the investors, who invest in these technologies and the risk-free asset, would have all technologies as a whole market.

In particular, the capital asset pricing model (CAPM, Sharpe (1964)) is one of the most important theories from the point of analyzing stock markets and option prices. According to CAPM, the excess returns on individual stock prices are proportional to the excess return on the market portfolio. An essential property, in order to obtain this result, is that the investors with mean-variance criteria (developed by Markowitz (1952)) invest in the risk-free asset and one identical portfolio of risky assets. This separation property is called the mutual fund separation. Under the CAPM model, such a portfolio maximizes the ratio of excess return to standard deviation of the return—called Sharpe ratio, named after Sharpe (1963), who study a single-factor model.

An important step is taken by Merton (1973), known as intertemporal CAPM (ICAPM). Merton (1973) introduces continuous-time dynamics in CAPM, using an optimization technique of Hamilton–Jacobi–Bellman equation (HJB equation). We also call such an optimization method a Markovian approach because it requires Markovian structure to derive HJB equations. An surprising result is that investors with logarithmic utilities always have (instantaneous) Sharpe ratio maximizing portfolio. In this sense, log-optimal portfolio is called "myopic demand," and is known to be derived relatively easily—even in a general semimartingale model of Goll and Kallsen (2003). Merton (1973) also suggests that the investors' demands are decomposed into "myopic demand" and the remainder called "hedging demands," naturally. Now HJB equations are widely used in dynamic optimization even in presence of transaction costs: Constantinides (1986) and Framstad, Øksendal, and Sulem (2001) for proportional costs on trading volumes; and Lo, Mamaysky, and Wang (2004) and Øksendal (1999) for costs on trades themselves.

At the same time, Black and Scholes (1973) focus on the relation between the CAPM and option prices in continuous time. Assuming a complete market (a market in which all contingent claims can be hedged), they derive the prices of options by replicating their payoffs (no-arbitrage prices), and also find the same prices can be derived by CAPM. This relation has developed as the market prices of risk. It determines the risk-neutral measure, which is used in option pricing. A complete market is often called as an "ideal market" because it admits a unique risk-neutral measure and two prices, derived by this measure and by replication, agree. Being tractable, their asset price process is also used to model defaults of firms (called structure model, e.g. Black & Cox, 1976; Merton, 1974). It is remarkable that option pricing rests on equilibrium based—and thus utility maximization based—CAPM in its origin, although we usually focus on risk-neutral measure and replicability.

Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987) fill the gap between option pricing and utility maximization, introducing an optimization method called the martingale approach. They focused on the relation between optimized portfolio (or consumption) of an investor and market-wide discounted conditional Radon–Nikodym derivative of risk-neutral measure (SDF, stochastic discount factor), which is used in option pricing. The martingale approach is a generalization of the following standard argument in discrete time models (see e.g. Skiadas (2009) for detail): (i) The asset price processes (market) and SDF are orthogonal to each other, in the sense that the multiplication of them becomes a martingale (zero expected return). (ii) At the point of optimized wealth, on the other hand, indifference curve of the utility function touches to the market line, because the optimality implies the investor cannot improve his performance via trades in the market. (iii) The gradient of the utility function at the optimized wealth must be proportional to SDF. They find we can apply this argument also in continuous time. Cvitanić and Karatzas (1992) generalize this method into constrained portfolio optimization problems, which includes short-sale constraint developed by He and Pearson (1991); and untradable asset developed by Karatzas, Lehoczky, Shreve, and Xu (1991). Pham and Touzi (1996) apply both Markovian approach and martingale approach, to characterize risk-neutral measure by the utility function of investors.

Although the martingale approach provides us an explicit representation of the optimized wealth, it is difficult to find an investment strategy that achieve the wealth in general. One prominent method to find the strategy is to apply the Clark–Ocone formula of Malliavin calculus. An idea of Malliavin calculus is to define a differentiation—called Malliavin derivative—with respect to sample paths. In this sense, Malliavin calculus is different from Itô calculus that considers integration (differentiation) in time direction. The Clark–Ocone formula provides an martingale representation of random variables using Malliavin derivative, and Ocone and Karatzas (1991) applied this formula to portfolio selection problem to find optimal investment strategies. Of course, their result also suggests that the demands of investors with logarithmic utility are myopic, and thus the mutual fund separation holds among them. Malliavin calculus is now widely used in financial analysis: Fournié, Lasry, Lebuchoux, Lions, and Touzi (1999) for calculation of option Greeks; Bichuch, Capponi, and Sturm (2017) for valuation of XVA; and Privault and Wei (2004) for sensitivity analysis of insurance risk.

The conditions for the mutual fund separation have been studied. For example, Cass and Stiglitz (1970) and Dybvig and Liu (2018) find the conditions on utility functions of investors for the mutual fund separation in each one period market. A similar result is obtained by Schachermayer, Sîrbu, and Taflin (2009) in continuous-time markets driven by Brownian motion. On the other hand, the market conditions for the mutual fund separation among investors have also been studied. Chamberlain (1988) finds the relation between the mutual fund separation and the hedgeability of the European options of the Radon–Nikodym derivative of the risk-neutral measure. Models with constant coefficients and models with deterministic vector-norm of the market prices of risk are examples of the condition of Chamberlain (1988). We examine the conditions for mutual fund separation in Chapter 2, combining those optimization methods.

An important topic is imperfect information, which appears in various situations in asset pricing. We list some recent results here: Fajgelbaum, Schaal, and Taschereau-Dumouchel (2017) construct a general equilibrium model, in which today's investments provides more information tomorrow, to answer why recession lingers. Dow and Han (2018) focus on the information in the trading volume made by arbitrageurs, and find that it arises so called "lemon" problem if their constraints bind in crisis. Hwang (2018) studies how degree of asymmetry in information varies dynamically. He finds the asymmetry could possibly be relieved at last, because bad assets would be sold in early. Frug (2018) shows fully informative equilibrium can be achieved among information sender and receiver, by choosing appropriate order of experiments. Jeong (2019) demonstrate the usefulness of cheap talk strategies for making an agreement achieved/rejected. Eyster, Rabin, and Vayanos (2018) construct an equilibrium model under which each investor neglects information in price and concentrate on his private information, using cursed (expectations) equilibrium concept introduced by Eyster and Rabin (2005).

One important expression of imperfect information is called heterogeneous beliefs, under which investors have different beliefs (subjective probabilities) although they have common information ("agree to disagree"). Under heterogeneous beliefs, Harrison and Kreps (1978) and Scheinkman and Xiong (2003) show that the asset prices can become higher than the valuation of the most optimistic investor at that time. They explain this is because the current holder of an asset not only has its payoffs, but also has option to resell it (resale option). We focus on this topic in Chapter 3. A possible explanation why they agree to disagree is made by Brunnermeier and Parker (2005). In their equilibrium model, investors maximizes expected time-average indirect utility by choosing their subjective probabilities. They find that investors can have different subjective probabilities in equilibrium endogenously, even if they are homogeneous in advance (one group believes they would win in a lottery, but the other does not believe, in words).

Imperfect information is one of the main concern in insurance (Prescott & Townsend, 1984; Rothschild & Stiglitz, 1976). In discrete-time credibility theory of actuarial science, Bühlmann (1967) provides us a scheme to estimate unobservable risks (accident rates) of policyholders. In continuous-time ruin theory of actuarial science, however, there seems to be few discussions on it. In Chapter 4, we consider a simple model to try to determine an optimal insurance premium rate, from accident rate and surplus of an insurer. It would be a step toward more advanced adverse selection problem under the ruin theory.

This thesis is constructed as follows. Chapter 2 answers two questions in mutual fund separation analytically: (1) under what market conditions does the mutual fund separation hold among investors?; and (2) in which class of utility functions does the mutual fund separation hold in each market? Then it treats asset pricing in incomplete market (specifically, assumption of MEMM) in the view of mutual fund separation. Chapter 3 investigates in the effect of the investors' speculative behavior on the prices in the presence of the differences in bargaining power. Chapter 4 tries to combine ruin theory with imperfect information problem, via seeking an optimal insurance premium rate.

Chapter 2

Mutual Fund Separation and Utility Functions[1] [2]

2.1 Introduction

It is well known that investors with mean-variance preferences hold the Sharpe ratio maximizing portfolio as portfolios of risky assets when there is a risk-free asset. This portfolio plays a critical role in the capital asset pricing model (CAPM). Cass and Stiglitz (1970) and Dybvig and Liu (2018) find analytic utility conditions under which investors' portfolio choice problems can be reduced to dividing their investments between the risk-free asset

[1]A major part of this chapter is electronically published as Igarashi, T. (2018). An Analytic Market Condition for Mutual Fund Separation: Demand for Non-Sharpe Ratio Maximizing Portfolio. *Asia-Pacific Financial Markets.* doi:10.1007/s10690-018-9261-6

[2]The work of this chapter is supported by The Fee Assistance Program for Academic Reviewing of Research Papers (for Graduate Students, Hitotsubashi University).

and some fixed (investor-irrelevant) portfolio of risky assets. If such a reduction can be applied, it is said that *mutual fund separation* holds. It is also known that there exist market models, in which mutual fund separation holds among all investors. To the author's knowledge, however, such market conditions are either hard to verify or too specific. The purpose of this chapter is to find conditions for mutual fund separation, from two perspectives:

1. under what market conditions does the mutual fund separation hold among investors?; and
2. in which class of utility functions does the mutual fund separation hold in each market?

We first provide a market condition for mutual fund separation analytically. Such a condition is obtained in terms of a conditional expectation of an infinitesimal change in the log-optimal portfolio (called the numéraire portfolio). Under the conditions in this study, the numéraire portfolio is characterized as the Sharpe ratio maximizing portfolio. When one decomposes an investor's demand into investment in the Sharpe ratio maximizing portfolio and investment in other portfolios (non-Sharpe ratio maximizing portfolios), this infinitesimal change is related to the latter. To provide a financial interpretation, we also investigate demand for non-Sharpe ratio maximizing portfolios.

Market conditions for mutual fund separation are obtained by two celebrated studies. Chamberlain (1988) finds a market condition for the Brownian motion case, and Schachermayer et al. (2009) study this problem for the general semimartingale case. They find that the stochastic nature of market prices in the risk process prevents mutual fund separation from holding, and find a necessary and sufficient condition for mutual fund separation among all investors. The condition is that any European options for the numéraire portfolio can be replicated by the risk-free asset and one fixed portfolio of risky assets.

However, the hedgeability of European options using some fixed portfolio is not easy to verify unless, for example, deterministic coefficient models are used. Nielsen and Vassalou (2006) tackle this problem, assuming a market driven by Brownian motion. They find that mutual fund separation holds among investors if both the risk-free rate and vector-norm of market prices of risk are deterministic. Dokuchaev (2014) also tackles this problem in a specific incomplete market. He finds that mutual fund separation holds among investors if the parameters (such as the risk-free rate, expected return, and diffusion coefficient) are independent of the Brownian motion that drives the price process. However, there seems to be no comprehensive analytic market condition for mutual fund separation.

This study finds an analytic market condition for mutual fund separation in a complete market driven by Brownian motion. Methodologically, we apply the martingale approach of Karatzas et al. (1987), and the Clark–Ocone formula of Ocone and Karatzas (1991) for the analytic formula to obtain the optimal portfolio strategy for investors. Although the martingale approach of Karatzas et al. (1987) is not applicable for the general semimartingale model, it offers a representation of the optimized terminal wealth of the investor under the existence of a stochastic risk-free rate. In addition to methodological convenience, a closer look at the Clark–Ocone formula gives us an interpretation of investors' demands for non-Sharpe ratio maximizing portfolios.

Mutual fund separation holds if and only if the conditional expectation of the Malliavin derivative of the numéraire portfolio can be hedged by trading the numéraire portfolio, assuming that the vector norm of the market price of risk is bounded away from zero. Such a condition can be rephrased as the condition where an investor's demand for a non-Sharpe ratio maximizing portfolio is fulfilled by trading the numéraire portfolio. Intuitively, the demand for the non-Sharpe ratio maximizing portfolio is demand that reduces the infinitesimal change in the uncertainty of the numéraire portfolio (that is represented by the Malliavin derivative of the numéraire portfolio). The degree of reduction depends on four components: the investor's wealth level, marginal utility and risk tolerance, at the time of consumption, and the shadow price. In Markovian markets, this infinitesimal change is characterized by an infinitesimal change in the terminal value of the numéraire portfolio due to an infinitesimal parallel shift in the initial value of the state variable. This implies that investors use the numéraire portfolio as literally a numéraire, for both instantaneous Sharpe ratio maximization and long-term risk adjustment. It also suggests that securities that hedge the uncertainty of the numéraire portfolio improve investors' performance. We also argue a sufficient condition for $(n+1)$-fund separation.

We also find that mutual fund separation among CRRA utilities implies separation among arbitrary utility functions, which is a clue to give an answer to our second question. It suggests that although investors with a unique risk averseness of CRRA utilities always exhibit mutual fund separation, parameter-beyond separation needs market specification. This leads to a conjecture that there is a market model in which investors must have CRRA utilities of a unique parameter to mutual fund separation holds. We provide a proof of it in a specific Markovian market, which does not need any limit taking, required in Schachermayer et al. (2009). The result is consistent to Cass and Stiglitz (1970) and Dybvig and Liu (2018) in discrete-time models, and Schachermayer et al. (2009) in continuous-time models. We also discuss the relation between minimal equivalent martingale measure (MEMM), which is sometimes exogenously assumed in option pricing, and utility functions in the view of mutual fund separation.

The remainder of this chapter is constructed as follows. Section 2.2 introduces assumptions of this chapter and formal definition of mutual fund separation. Section 2.3 introduces an optimization method called martingale approach. Then it derives an analytic market condition for mutual fund separation among investors. This is one of the main results of this chapter. This section also discusses a financial interpretation of the condition. Section 2.4 shows that there exists a market in which investors must have

CRRA utilities with a unique parameter for mutual fund separation. It also discusses the relation between the mutual fund separation and option pricing with MEMM in incomplete markets, in view of mutual fund separation. Section 2.5 discusses the result. Section 2.6 concludes this chapter.

2.2 The Model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which d-dimensional standard Brownian motion $B = (B^{(1)}, \dots, B^{(d)})^\top$ is defined. Let $\mathbb{F} = \{\mathcal{F}_t\}$ be the augmented filtration generated by B. We consider a market in continuous time with a finite horizon $T < \infty$. There are both a risk-free asset $S^{(0)}$ and d risky assets $S^{(i)}$ for $i = 1, \dots, d$, in the market. These assets solve the following stochastic differential equations (SDEs)

$$\frac{\mathrm{d}S_t^{(0)}}{S_t^{(0)}} = r_t \,\mathrm{d}t, \quad \text{and} \quad \frac{\mathrm{d}S_t^{(i)}}{S_t^{(i)}} = \mu_t^{(i)} \,\mathrm{d}t + \Sigma_t^{(i)} \,\mathrm{d}B_t, \quad \text{for} \quad i = 1, \dots, d,$$

where r is a one-dimensional $\mathbb{F}$-progressively measurable process, and for each i, $\mu^{(i)}$ and $\Sigma^{(i)}$ are one-dimensional and d-dimensional $\mathbb{F}$-progressively measurable processes, respectively. In the sequel, we denote them by

$$\mu_t = \begin{pmatrix} \mu_t^{(1)} \\ \vdots \\ \mu_t^{(d)} \end{pmatrix}, \quad \text{and} \quad \Sigma_t = \begin{pmatrix} \Sigma_t^{(1)} \\ \vdots \\ \Sigma_t^{(d)} \end{pmatrix}.$$

The coefficients r, μ and Σ are assumed to satisfy

Assumption 2.1. *The processes* r, μ *and* Σ *are bounded uniformly in* $(t,\omega) \in [0,T] \times \Omega$. *Furthermore, there exists* $\varepsilon > 0$ *such that*

$$\xi^\top \Sigma_t \Sigma_t^\top \xi \geqslant \varepsilon \xi^\top \xi, \qquad \text{for each} \quad \xi \in \mathbb{R}^d, \quad (t,\omega) \in [0,T] \times \Omega.$$

Under this assumption, there is a unique equivalent martingale measure, and the market prices of risk process is bounded (see Karatzas et al. (1987, p.1562)). Let us denote by $\mathbb{Q}$, the equivalent martingale measure:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left\{ -\int_0^T \lambda_t^\top \,\mathrm{d}B_t - \frac{1}{2}\int_0^T \lambda_t^\top \lambda_t \,\mathrm{d}t \right\},$$

where the market price of the risk process λ is the unique solution of

$$\Sigma_t \lambda_t = \mu_t - r_t \mathbf{1},$$

and $\mathbf{1}$ is a d-dimensional column vector with 1 in each entry. Under the equivalent martingale measure $\mathbb{Q}$, the process $\widetilde{B}$ defined by

$$\mathrm{d}\widetilde{B}_t = \lambda_t \,\mathrm{d}t + \mathrm{d}B_t, \qquad \widetilde{B}_0 = 0$$

is a d-dimensional Brownian motion. Thus, by the Itô formula, the discounted price process $S_t^{(i)}/S_t^{(0)}$ of each risky asset i becomes a martingale under $\mathbb{Q}$. Let us denote the stochastic discount factor of this market by H, which solves the stochastic differential equation

$$\frac{\mathrm{d}H_t}{H_t} = -r_t \mathrm{d}t - \lambda_t^\top \mathrm{d}B_t.$$

In the sequel, we will apply the Clark–Ocone formula under a change of measure of Ocone and Karatzas (1991), which involves Malliavin calculus. For this, we impose additional assumptions on r and λ. First, we introduce some classes of random variables. Let $\mathcal{S}$ be the set of random variables of the form

$$F = f(B_{t_1}^\top, \ldots, B_{t_m}^\top), \qquad t_i \in [0, T], \quad i = 1, \ldots, m,$$

where f is any bounded $C^\infty(\mathbb{R}^{dm})$ function with bounded derivatives of all orders. Such random variables are called smooth functionals. For each $F \in \mathcal{S}$, the Malliavin derivative $\mathrm{D}_t F = (\mathrm{D}_t^{(1)} F, \ldots, \mathrm{D}_t^{(d)} F)^\top$ is defined by

$$\mathrm{D}_t^{(i)} F = \sum_{j=1}^{m} \frac{\partial}{\partial x^{ij}} f(B_{t_1}^\top, \ldots, B_{t_m}^\top) \mathbf{1}_{\{t \leqslant t_j\}}, \qquad \text{for} \quad i = 1, \ldots, d,$$

where $\partial / \partial x^{ij}$ represents the partial derivative with respect to the (i, j)-th variable and $\mathbf{1}_A$ represents the indicator function of event A. Let $\mathbb{D}_{1,1}$ be the closure of $\mathcal{S}$ under the norm $\| \cdot \|_{1,1}$, where

$$\|F\|_{1,1} := \mathbb{E}\Big[|F| + \Big(\sum_{i=1}^{d} \|\mathrm{D}_{\cdot}^{(i)} F\|^2 \Big)^{\frac{1}{2}} \Big],$$

with $L^2([0,T])$ norm $\|\cdot\|$. It is known that the Malliavin derivative is also well-defined on $\mathbb{D}_{1,1}$.

Assumption 2.2 (Ocone and Karatzas (1991)). *The risk-free rate r and the market price of risk λ satisfy the following three conditions*

1. $r_s \in \mathbb{D}_{1,1},\ \lambda_s \in (\mathbb{D}_{1,1})^d$, *for almost every* $s \in [0,T]$;
2. *for each t, the processes $s \mapsto \mathrm{D}_t r_s$ and $s \mapsto \mathrm{D}_t \lambda_s$ admit progressively measurable versions; and*
3. *for some $p > 1$, the following expectations exist*

$$\mathbb{E}\left[\left(\int_0^T r_s^2 \,\mathrm{d}s\right)^{\frac{1}{2}} + \left(\int_0^T \sum_{i=1}^d \|\mathrm{D}_\cdot^{(i)} r_s\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] < \infty,$$

and

$$\mathbb{E}\left[\left(\int_0^T \lambda_s^\top \lambda_s \,\mathrm{d}s\right)^{\frac{1}{2}} + \left(\int_0^T \sum_{i,j=1}^d \|\mathrm{D}_\cdot^{(i)} \lambda_s^{(j)}\|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] < \infty.$$

Specifically, we have $\mathrm{D}_t^{(i)} r_s = 0$ and $\mathrm{D}_t^{(i)} \lambda_s^{(j)} = 0$ for $0 \leqslant s \leqslant t$ and for $i = 1, \ldots, d$ and $j = 1, \ldots, d$, which is used in the sequel.

Now we turn to the assumptions about investors. Each investor can invest in each asset $S^{(i)}$, and his wealth at time $t \in [0,T]$ is denoted by W_t. The wealth of the investor is assumed to be self-financing:

$$\frac{\mathrm{d}W_t}{W_t} = \sum_{i=1}^d \varphi_t^{(i)} \frac{\mathrm{d}S_t^{(i)}}{S_t^{(i)}} + (1 - \varphi_t^\top \mathbf{1}) r_t \,\mathrm{d}t$$

with initial wealth W_0, where $\mathbf{1}$ is a d-dimensional column vector with 1 in each entry and $\varphi = (\varphi^{(1)}, \dots, \varphi^{(d)})^\top$ is an adapted process such that $\varphi_t^{(i)} W_t$ is the amount of money invested in the i-th risky asset. The rest $W_t(1-\varphi_t^\top \mathbf{1})$ is thus the amount of money invested in the risk-free asset. The process φ is assumed to satisfy

$$\int_0^T W_t^2 \varphi_t^\top \varphi_t \, \mathrm{d}t < \infty, \qquad \text{a.s.}$$

We call such a process φ a portfolio strategy and call it admissible if the corresponding wealth process satisfies $W_t \geqslant 0$ for all $t \in [0, T]$ almost surely. The set of admissible strategies given initial wealth W_0 is denoted by $\mathcal{A} = \mathcal{A}(W_0)$.

An important example of a wealth (portfolio) is the numéraire portfolio. Consider a wealth process with portfolio strategy φ:

$$\frac{\mathrm{d}W_t}{W_t} = \big(r_t + \varphi_t^\top \Sigma_t \lambda_t\big)\mathrm{d}t + \varphi_t^\top \Sigma_t \mathrm{d}B_t,$$

here we used $\Sigma\lambda = \mu - r\mathbf{1}$. Fixing the volatility $|\varphi^\top \Sigma|$ of the portfolio process to $|\lambda|$, we can maximize (instantaneous) Sharpe ratio $\varphi_t^\top \Sigma_t \lambda_t / |\varphi_t^\top \Sigma_t|$ by choosing $\varphi_t^\top = \lambda^\top \Sigma^{-1}$. Such a portfolio coincides with the reciprocal $1/H_t$ of the stochastic discount factor. In fact, the Itô formula gives us

$$\mathrm{d}\frac{1}{H_t} = \frac{1}{H_t}\Big((r_t + \lambda^\top \lambda_t)\mathrm{d}t + \lambda_t^\top \mathrm{d}B_t\Big). \tag{2.1}$$

This portfolio is called the numéraire portfolio (see e.g. Schachermayer et al. (2009)), and plays a crucial role in our result. The numéraire portfolio is also related to the optimal portfolio for investors with logarithmic utility: for details, see 2.3.2 in Section 2.3 bellow.

Each investor's performance is evaluated by the expected utility from the terminal wealth $\mathbb{E}[U(W_T)]$, and his optimization problem is

$$\sup_{\varphi \in \mathcal{A}} \mathbb{E}[U(W_T)],$$

where $U : (0, \infty) \to \mathbb{R}$ is the utility function of the investor. The utility function is assumed to satisfy

Assumption 2.3. *The utility function U is a strictly increasing, strictly concave C^2 function with*

$$\lim_{w \downarrow 0} U'(w) = \infty, \qquad \lim_{w \to \infty} U'(w) = 0,$$

and the inverse function $(U')^{-1} =: I$ of its derivative satisfies

$$I(y) + |I'(y)| \leqslant K(y^{\alpha} + y^{-\beta}) \qquad \textit{for} \quad 0 < y < \infty,$$

for some positive constants K, α and β.

We denote the optimal strategy and the optimized wealth process just as φ and W respectively, because we only consider the optimized wealth process.

Mutual fund separation

We are interested in the conditions under which investors have the same portfolio of risky assets, or mutual fund separation holds. Mutual fund separation is formally defined as

Definition 2.1 (Mutual fund separation)**.** *We say that the market admits* $(n+1)$-fund separation *if there exist $n \geqslant 1$ fixed portfolio strategies $\psi_1, \dots, \psi_n$ of risky assets with $\psi_1^\top \mathbf{1} = \cdots = \psi_n^\top \mathbf{1} = 1$, such that the optimal portfolio strategy φ^{U,W_0} of each investor, with utility function U and initial wealth W_0, satisfies $\varphi_t^{U,W_0} = \sum_{i=1}^n a_{i,t}^{U,W_0} \psi_{i,t}$ for some one-dimensional adapted processes $a_1^{U,W_0}, \dots, a_n^{U,W_0}$. A* mutual fund *is defined as each of the following $(n+1)$ processes: portfolio processes with strategies $\psi_1, \dots, \psi_n$ and the risk-free asset $S^{(0)}$. Furthermore, we say that the market admits* mutual fund separation *if* 2*-fund separation holds.*

Remark 2.1. This definition of mutual fund separation assumes that one mutual fund is the risk-free asset. Such a separation is called "monetary separation" in Cass and Stiglitz (1970), "money separation" in Dybvig and Liu (2018), and the "mutual fund theorem" in Schachermayer et al. (2009).

2.3 An Analytic Market Condition for Mutual Fund Separation

2.3.1 Optimal strategy

Using the martingale approach of Karatzas et al. (1987), the investor's optimized discounted terminal wealth is given by

$$\frac{W_T}{S_T^{(0)}} = \frac{1}{S_T^{(0)}} I\big(zH_T\big), \tag{2.2}$$

with $I := (U')^{-1}$ and $z := z^U(W_0)$ where z^U is a positive decreasing function. The constant z is called the *shadow price* (see e.g., Dybvig and Liu (2018)).

A financial interpretation of the martingale approach is provided by Figure 2.1. The two axes represents payoffs for two different scenarios of the future and investors can trade their payoffs along the line named market. The market is characterized by the vector $\frac{1}{S_T^{(0)}}\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ that is orthogonal to the market. The point W_T is a terminal wealth of an investor and the dotted curve is the indifference curve for his endowed wealth. If he trades optimally in the market, the indifference curve for the optimized wealth W_T^* must touch the market line at W_T^*. In other words, the gradient $\nabla U(W_T^*)$, which is related to the first derivative, of his utility at W_T^* must be orthogonal to the market. That is why W_T, $H_T = \frac{1}{S_T^{(0)}}\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ and $I = (U')^{-1}$ appears in (2.2).

Since the discounted wealth process $(W_t/S_t^{(0)})_t$ is a $\mathbb{Q}$-martingale, we need

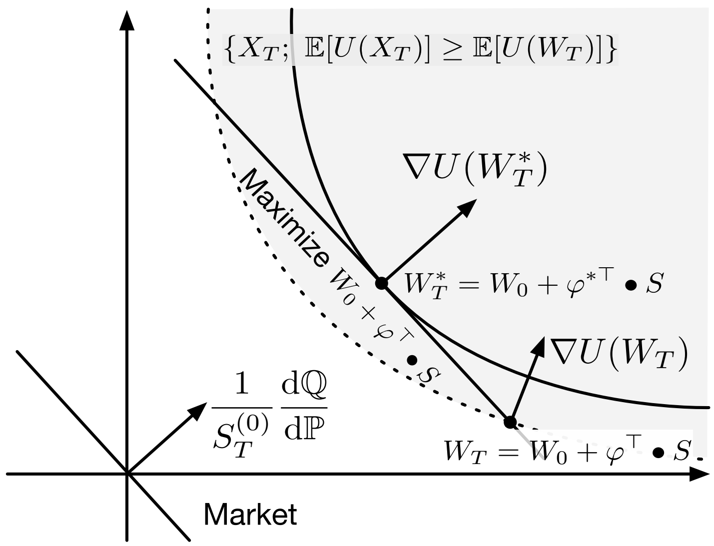

Figure 2.1: Graphical explanation of martingale approach.

the $\mathbb{Q}$-martingale representation to obtain the condition for mutual fund separation. For this, we apply the Clark–Ocone formula of Ocone and Karatzas (1991). They find the $\mathbb{Q}$-martingale representation

$$F = \mathbb{E}\Big[F\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big] + \int_0^T \Big(\widetilde{\mathbb{E}}[\mathrm{D}_t F \mid \mathcal{F}_t] - \widetilde{\mathbb{E}}\Big[F \int_t^T \mathrm{D}_t \lambda_u \mathrm{d}\widetilde{B}_u \Big| \mathcal{F}_t\Big]\Big)^\top \mathrm{d}\widetilde{B}_t$$

of a random variable $F \in \mathbb{D}_{1,1}$ (with additional integrability conditions), where D is the Malliavin derivative operator for B, and $\widetilde{\mathbb{E}}[\cdot]$ is the expectation under $\mathbb{Q}$. According to Ocone and Karatzas (1991, Theorem 4.2), under

Assumptions 2.1–2.3, we have $I(zH_T)/S_T^{(0)} \in \mathbb{D}_{1,1}$, and

$$\frac{W_t}{S_t^{(0)}}\varphi_t = (\Sigma_t^\top)^{-1}\widetilde{\mathbb{E}}\Big[-\frac{1}{S_T^{(0)}}zH_T I'(zH_T)\Big|\mathcal{F}_t\Big]\lambda_t + (\Sigma_t^\top)^{-1}\widetilde{\mathbb{E}}\left[\begin{array}{l} -\frac{1}{S_T^{(0)}}\Big(I(zH_T) + zH_T I'(zH_T)\Big) \\ \times\Big(\int_t^T \mathrm{D}_t r_u \mathrm{d}u + \int_t^T \mathrm{D}_t(\lambda_u^\top)\mathrm{d}\widetilde{B}_u\Big) \end{array}\middle|\mathcal{F}_t\right].$$

We slightly modify this equation, using Bayes' formula, into

$$H_t W_t\varphi_t = (\Sigma_t^\top)^{-1}\mathbb{E}\Big[-H_T zH_T I'(zH_T)\Big|\mathcal{F}_t\Big]\lambda_t + (\Sigma_t^\top)^{-1}\mathbb{E}\left[\begin{array}{l} -H_T\Big(I(zH_T) + zH_T I'(zH_T)\Big) \\ \times\Big(\int_t^T \mathrm{D}_t r_u \mathrm{d}u + \int_t^T \mathrm{D}_t(\lambda_u^\top)\mathrm{d}\widetilde{B}_u\Big) \end{array}\middle|\mathcal{F}_t\right], \tag{2.3}$$

and refer (2.3) as the Clark–Ocone formula of Ocone and Karatzas (1991).

The right hand side of (2.3) illustrates that investors have investor-independent portfolios (the first term) and investor-specific portfolios (the second term), because the expectation in the first term is one-dimensional while the expectation in the second term is d-dimensional. The first term represents the investment in the (instantaneous) Sharpe ratio maximizing portfolio (the numéraire portfolio, see (2.1)).

2.3.2 Market condition for mutual fund separation

It is well known that investors with logarithmic utility $U(w) = \log(w)$ invest in the Sharpe ratio maximizing portfolio. In this case,

$$(I(y) + yI'(y)) = \left(\frac{1}{y} - y\frac{1}{y^2}\right) = 0 \qquad \text{for} \quad y > 0,$$

and thus investors with logarithmic utility invest in the fund $\psi_t = (\Sigma_t^\top)^{-1}\lambda_t$ as the portfolio of risky assets. Therefore, finding the market condition for mutual fund separation can be replaced with finding the condition for an investor-specific portfolio to have the same direction as the numéraire portfolio $1/H$, see (2.1).

The market condition for mutual fund separation is concerned with the Malliavin derivative of the numéraire portfolio.

Proposition 2.1. *Assume $\lambda_t^\top \lambda_t > \varepsilon$ for some constant $\varepsilon > 0$ and let $\mathbb{G} = \{\mathcal{G}_t\}$ be a filtration with $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(1/H_T)$. Mutual fund separation holds among all investors if and only if the conditional expectation of the Malliavin derivative of the terminal wealth of the numéraire portfolio $1/H_T$ satisfies*

$$\mathbb{E}\left[\mathrm{D}_t\left(\frac{1}{H_T}\right)\middle|\mathcal{G}_t\right] = \alpha_{t,T}\lambda_t, \qquad \mathrm{d}t \otimes \mathrm{d}\mathbb{P}\text{-}a.e.,$$

for some one-dimensional $\mathbb{G}$-adapted process $t \mapsto \alpha_{t,T}$.

For the proof of the proposition, we use two lemmas.

Lemma 2.1. *Let $X \in L^p(\mathbb{P})$ be a random variable for some $p > 1$ and let Y be a random variable such that both $e^Y, e^{-Y} \in L^1(\mathbb{P})$. Then, if there exists a constant $0 < \delta < p/(p-1)$ such that $\mathbb{E}[e^{\beta Y} X] = 0$ for each β with $|\beta| < \delta$, we have $\mathbb{E}[X \mid \sigma(Y)] = 0$ almost surely.*

Proof. First we show that the integrability of the random variable $Y^n e^{\beta Y} X$ for $n = 0, 1, 2, \dots$ and $|\beta| < \delta$. Let $q > 1$ and $r > 1$ be constants such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \quad \text{and} \quad \frac{1}{q} > \delta.$$

Then Hölder's inequality yields

$$\mathbb{E}[|Y|^n e^{\beta Y} |X|] \leqslant \mathbb{E}[|Y|^{nr}]^{1/r} \mathbb{E}[e^{\beta Y q}]^{1/q} \mathbb{E}[|X|^p]^{1/p} < \infty,$$

because $|\beta| q < 1$ and both e^Y and e^{-Y} are integrable ($\infty > \mathbb{E}[e^Y + e^{-Y}]$ assures that Y^{2n} is integrable for each $n = 1, 2, \dots$). Then, by the dominated convergence theorem, we have

$$0 = \frac{\mathrm{d}^n}{\mathrm{d}\beta^n} \mathbb{E}[e^{\beta Y} X] = \mathbb{E}[Y^n e^{\beta Y} X], \qquad \text{for} \quad |\beta| < \delta,$$

and thus

$$\mathbb{E}\big[Y^n \mathbb{E}[X \mid \sigma(Y)]^+\big] = \mathbb{E}\big[Y^n \mathbb{E}[X \mid \sigma(Y)]^-\big],$$

where $a^+ = \max\{a, 0\}$ and $a^- = \max\{-a, 0\}$ for $a \in \mathbb{R}$. Since

$$\mathbb{E}\big[\mathbb{E}[X \mid \sigma(Y)]^+\big] = \mathbb{E}\big[\mathbb{E}[X \mid \sigma(Y)]^-\big] < \infty$$

for $n = 0$, we can apply the Cramér condition for moment problem (see e.g. Stoyanov (2013, Section 11)) to obtain

$$\mathbb{E}[X \mid \sigma(Y)]^+ = \mathbb{E}[X \mid \sigma(Y)]^-, \qquad \text{a.s.}$$

Therefore, $\mathbb{E}[X \mid \sigma(Y)] = \mathbb{E}[X \mid \sigma(Y)]^+ - \mathbb{E}[X \mid \sigma(Y)]^- = 0$ almost surely.

$\square$

Lemma 2.2. *We have, for almost every t,*

$$H_t,\ \frac{1}{H_t} \in L^1(\mathbb{P}) \quad \textit{and} \quad \int_t^T \mathrm{D}_t r_u \mathrm{d}u + \int_t^T \mathrm{D}_t(\lambda_u^\top)\mathrm{d}\widetilde{B}_u \in L^p(\mathbb{P}),$$

where $1 < p < 2$ is a constant in Assumption 2.2.

Proof. Because Assumption 2.2 requires the integrability condition to hold for $1 < p$, we can assume $1 < p < 2$ without loss of generality. The first claim is obvious because $dH_t/H_t = -r_t\mathrm{d}t - \lambda_t^\top \mathrm{d}B_t$ and r and λ are bounded. For the second claim, we check the condition for each part

$$\int_0^T \mathrm{D}_t r_u \mathrm{d}u, \qquad \int_0^T \mathrm{D}_t(\lambda_u^\top)\lambda_u \mathrm{d}u \quad \text{and} \quad \int_0^T \mathrm{D}_t(\lambda_u^\top)\mathrm{d}B_u, \tag{2.4}$$

separately (recall that $\mathrm{D^{(i)}}_t r_s = 0$ and $\mathrm{D}_t^{(i)}\lambda_s^{(j)} = 0$ for $0 \leqslant s \leqslant t$ and $i, j =$

$1,\dots,d$). By Assumption 2.2, Fubini's theorem, and Jensen's inequality, we have

$$\infty > \mathbb{E}\Big[\Big(\int_0^T\Big(\sum_{i=1}^d\int_0^T\big(\mathrm{D}_t^{(i)}r_s\big)^2\mathrm{d}t\Big)\mathrm{d}s\Big)^{p/2}\Big] = \mathbb{E}\Big[\Big(\int_0^T\big(\sum_{i=1}^d\int_0^T\big(\mathrm{D}_t^{(i)}r_s\big)^2\mathrm{d}s\Big)\mathrm{d}t\Big)^{p/2}\Big]$$
$$= T^{p/2}\mathbb{E}\Big[\Big(\int_0^T\frac{1}{T}\big(\sum_{i=1}^d\int_0^T\big(\mathrm{D}_t^{(i)}r_s\big)^2\mathrm{d}s\Big)\mathrm{d}t\Big)^{p/2}\Big]$$
$$\geqslant T^{\frac{p}{2}-1}\,\mathbb{E}\Big[\int_0^T\Big(\sum_{i=1}^d\int_0^T\big(\mathrm{D}_t^{(i)}r_s\big)^2\mathrm{d}s\Big)^{p/2}\mathrm{d}t\Big],$$

because $1/2 < p/2 < 1$. Now, it follows that the expectation

$$\mathbb{E}\Big[\Big|\int_0^T\mathrm{D}_t r_u\mathrm{d}u\Big|^p\Big] = \mathbb{E}\Big[\Big(\sum_{i=1}^d\Big(\int_0^T\mathrm{D}_t^{(i)}r_u\mathrm{d}u\Big)^2\Big)^{p/2}\Big]$$

is finite for almost every t by Fubini's theorem, where $|\cdot|$ represents the $\mathbb{R}^d$-vector norm. By the same argument, together with boundedness of λ, the second component in (2.4) is also in $L^p(\mathbb{P})$. For the third component, observe that

$$\mathbb{E}\Big[\Big(\sup_{0\leqslant t\leqslant T}\Big|\int_0^t\mathrm{D}_u(\lambda_s^\top)\mathrm{d}B_s\Big|\Big)^p\Big] \leqslant d^p\mathbb{E}\Big[\Big(\sum_{i=1}^d\frac{1}{d}\sup_{0\leqslant t\leqslant T}\Big|\sum_{j=1}^d\int_0^t\mathrm{D}_u^{(i)}(\lambda_s^{(j)})\mathrm{d}B_s^{(j)}\Big|\Big)^p\Big]$$
$$\leqslant d^p\mathbb{E}\Big[\sum_{i=1}^d\frac{1}{d}\Big(\sup_{0\leqslant t\leqslant T}\Big|\sum_{j=1}^d\int_0^t\mathrm{D}_u^{(i)}(\lambda_s^{(j)})\mathrm{d}B_s^{(j)}\Big|\Big)^p\Big]$$
$$\leqslant d^{p-1}\sum_{i=1}^d\mathbb{E}\Big[\Big(\sup_{0\leqslant t\leqslant T}\Big|\sum_{j=1}^d\int_0^t\mathrm{D}_u^{(i)}(\lambda_t^{(j)})\mathrm{d}B_s^{(j)}\Big|\Big)^p\Big]$$
$$\leqslant d^{p-1}C_p\sum_{i=1}^d\Big[\Big(\sum_{j=1}^d\int_0^T\big(\mathrm{D}_u^{(i)}(\lambda_s^{(j)})\big)^2\mathrm{d}s\Big)^{p/2}\Big]$$

$$
\begin{aligned}
&\leqslant d^p C_p \sum_{i=1}^{d} \frac{1}{d} \Big[\Big(\sum_{j=1}^{d} \int_0^T \big(\mathrm{D}_u^{(i)}(\lambda_s^{(j)})\big)^2 \mathrm{d}s \Big)^{p/2} \Big] \\
&\leqslant d^{p/2} C_p \Big[\Big(\sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^T \big(\mathrm{D}_u^{(i)}(\lambda_s^{(j)})\big)^2 \mathrm{d}s \Big)^{p/2} \Big],
\end{aligned}
$$

by Jensen's inequality (because $1/2 < p/2 < 1 < p$) and the Burkholder–Davis–Gundy inequality, where $C_p > 0$ is a universal constant depending only on p. Thus the third component is also in $L^p(\mathbb{P})$ for almost every t. Finally, we obtain the result by Minkowski's inequality. $\square$

Remark 2.2. The final inequalities in the proof of the previous lemma require $1 < p < 2$ (which does not lose generality in our context, as the author have mentioned). For general $1 < p$ version of this inequality, the author refers Karatzas and Shreve (1991, Remark 3.30).

Proof of Proposition 2.1. First, the chain rule for the Malliavin derivative gives us

$$
\begin{aligned}
\mathrm{D}_t\Big(\frac{1}{H_T}\Big) &= \mathrm{D}_t \exp\Big\{ \int_0^T \big(r_u + \lambda_u^\top \lambda_u\big)\mathrm{d}u + \int_0^T \lambda_u^\top \mathrm{d}B_u \Big\} \\
&= \frac{1}{H_T}\Big(\int_0^T \big(\mathrm{D}_t r_u + (\mathrm{D}_t \lambda_u^\top)\lambda_u\big)\mathrm{d}u + \int_0^T (\mathrm{D}_t \lambda_u^\top)\mathrm{d}B_u + \lambda_t \Big) \\
&= \frac{1}{H_T}\Big(\int_t^T \mathrm{D}_t r_u \mathrm{d}u + \int_t^T \mathrm{D}_t(\lambda_t^\top)\mathrm{d}\widetilde{B}_u + \lambda_t \Big).
\end{aligned} \tag{2.5}
$$

Because both H_T and λ_t are $\mathcal{G}_t$-measurable, the condition of the proposition is equivalent to

$$\mathbb{E}\Big[\Big(\int_t^T \mathrm{D}_t r_u \mathrm{d}u + \int_t^T \mathrm{D}_t(\lambda_u^\top)\mathrm{d}\widetilde{B}_u\Big)\Big|\mathcal{G}_t\Big] = \alpha'_{t,T}\lambda_t, \qquad \forall t \in [0,T], \quad \text{a.s.,} \tag{2.6}$$

for some one-dimensional $\mathbb{G}$-adapted process $t \mapsto \alpha'_{t,T}$.

(If part) Assume that the equation (2.6) holds. Then,

$$\begin{aligned}
&\mathbb{E}\Big[H_T\Big(I(zH_T) + zH_T I'(zH_T)\Big)\Big(\int_t^T \mathrm{D}_t r_u \mathrm{d}u + \int_t^T \mathrm{D}_t(\lambda_u^\top)\mathrm{d}\widetilde{B}_u\Big)\Big|\mathcal{F}_t\Big] \\
&\quad = \mathbb{E}\Big[H_T\Big(I(zH_T) + zH_T I'(zH_T)\Big)\mathbb{E}\Big[\int_t^T \mathrm{D}_t r_u \mathrm{d}u + \int_t^T \mathrm{D}_t(\lambda_u^\top)\mathrm{d}\widetilde{B}_u\Big|\mathcal{G}_t\Big]\Big|\mathcal{F}_t\Big] \\
&\quad = \mathbb{E}\Big[H_T\Big(I(zH_T) + zH_T I'(zH_T)\Big)\alpha'_{t,T}\Big|\mathcal{F}_t\Big]\lambda_t.
\end{aligned}$$

Substituting this into (2.3),

$$\frac{W_t}{S_t^{(0)}}\varphi_t = (\Sigma_t^\top)^{-1}\mathbb{E}\Big[-H_T zH_T I'(zH_T) - H_T\Big(I(zH_T) + zH_T I'(zH_T)\Big)\alpha'_{t,T}\Big|\mathcal{F}_t\Big]\lambda_t,$$

and mutual fund separation always holds because the inside of the expectation is one dimensional.

(Only if part) Let a one-dimensional $\mathbb{G}$-adapted process $t \mapsto \alpha'_{t,T}$ and a d-dimensional $\mathbb{G}$-adapted process $t \mapsto \nu_{t,T}$ be

$$\begin{aligned}
\alpha'_{t,T} &:= \lambda_t^\top \mathbb{E}\Big[\Big(\int_t^T \mathrm{D}_t r_u \mathrm{d}u + \int_t^T \mathrm{D}_t(\lambda_u^\top)\mathrm{d}\widetilde{B}_u\Big)\Big|\mathcal{G}_t\Big]\Big/(\lambda_t^\top\lambda_t), \\
\nu_{t,T} &:= \mathbb{E}\Big[\Big(\int_t^T \mathrm{D}_t r_u \mathrm{d}u + \int_t^T \mathrm{D}_t(\lambda_u^\top)\mathrm{d}\widetilde{B}_u\Big)\Big|\mathcal{G}_t\Big] - \alpha'_{t,T}\lambda_t.
\end{aligned}$$

By construction, we have $\lambda_t^\top \nu_{t,T} = 0$. For investors with CRRA utilities $U(w) = w^{1-\gamma}/(1-\gamma)$ with $\gamma > 0$, we have $I(y) + yI'(y) = (1 - 1/\gamma)y^{1-1/\gamma}$ and

$$
\begin{aligned}
&(\Sigma_t^\top)^{-1}\mathbb{E}_t\Big[H_T^{1-\frac{1}{\gamma}}\Big(\int_t^T \mathrm{D}_t r_u \mathrm{d}u + \int_t^T \mathrm{D}_t(\lambda_u^\top)\mathrm{d}\widetilde{B}_u\Big)\Big|\mathcal{F}_t\Big] \\
&\qquad = (\Sigma_t^\top)^{-1}\mathbb{E}\Big[\mathbb{E}\Big[H_T^{1-\frac{1}{\gamma}}\Big(\int_t^T \mathrm{D}_t r_u \mathrm{d}u + \int_t^T \mathrm{D}_t(\lambda_u^\top)\mathrm{d}\widetilde{B}_u\Big)\Big|\mathcal{G}_t\Big]\Big|\mathcal{F}_t\Big] \\
&\qquad = \mathbb{E}\Big[H_T^{1-\frac{1}{\gamma}}\alpha'_{t,T}\Big|\mathcal{F}_t\Big](\Sigma_t^\top)^{-1}\lambda_t + (\Sigma_t^\top)^{-1}\mathbb{E}\Big[H_T^{1-\frac{1}{\gamma}}\nu_{t,T}\Big|\mathcal{F}_t\Big].
\end{aligned}
$$

Assuming that mutual fund separation holds among investors with power utilities, we must have

$$
\mathbb{E}\Big[H_T^{1-\frac{1}{\gamma}}\nu_{t,T}\Big|\mathcal{F}_t\Big] = \mathsf{0},
$$

because λ_t and $\mathbb{E}[H_T^{1-\frac{1}{\gamma}}\nu_{t,T} \mid \mathcal{F}_t]$ are orthogonal.

Now by Lemma 2.2, we can apply Lemma 2.1 with

$$
X = \nu_{t,T} \quad \text{and} \quad Y = \log\Big(\frac{1}{H_T}\Big),
$$

because λ is bounded and $\lambda^\top\lambda > \varepsilon$ by assumption. This completes the proof. $\square$

A financial interpretation of this proposition can be found in Section 2.3.3. This proof also implies that it suffices to verify mutual fund separation among investors with CRRA utilities to check mutual fund separation among all investors.

Corollary 2.1. *Mutual fund separation holds among all investors if and only if it holds among investors with CRRA utilities.*

The condition (2.6) trivially holds when the coefficients are deterministic. In fact, if both r and λ are deterministic, their Malliavin derivatives vanish and (2.6) holds with $\alpha'_{t,T} \equiv 0$. The condition is, of course, not restricted to the deterministic coefficient case.

Example 2.1 (Deterministic market prices of risk with stochastic interest rate). Let λ_t be a bounded deterministic market price of risk and let r_t be

$$r_t = r\Big(\int_0^t \lambda_s^\top \mathrm{d}B_s \Big)$$

where $r : \mathbb{R} \to \mathbb{R}$ is a bounded C^∞ deterministic function with bounded derivatives of all orders. Then, by the chain rule of the Malliavin derivative, we obtain

$$\mathrm{D}_t r_u = r'\Big(\int_0^u \lambda_s^\top \mathrm{d}B_s \Big) \mathrm{D}_t \Big(\int_0^u \lambda_s^\top \mathrm{d}B_s \Big) = r'\Big(\int_0^u \lambda_s^\top \mathrm{d}B_s \Big) \lambda_t \mathbf{1}_{\{t \leqslant u\}},$$

and

$$\mathbb{E}\Big[\Big(\int_t^T \mathrm{D}_t r_u \mathrm{d}u + \int_t^T \mathrm{D}_t(\lambda_u^\top) \mathrm{d}\widetilde{B}_u \Big) \Big| \mathcal{G}_t \Big] = \mathbb{E}\Big[\int_t^T r'\Big(\int_0^u \lambda_s^\top \mathrm{d}B_s \Big) \mathrm{d}u \Big| \mathcal{G}_t \Big] \lambda_t,$$

because λ is deterministic. This implies that two fund separation always holds.

The argument of the proof of *if part* of Proposition 2.1 also provides a sufficient condition for $(n+1)$-fund separation.

Corollary 2.2. *Assume $\lambda_t^\top \lambda_t > \varepsilon$ for some constant $\varepsilon > 0$ and let $\mathbb{G} = \{\mathcal{G}_t\}$ be a filtration with $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(1/H_T)$. The $(n+1)$-fund separation holds among all investors if*

$$\mathbb{E}\Big[\mathrm{D}_t\Big(\frac{1}{H_T}\Big)\Big|\mathcal{G}_t\Big] = \alpha_{t,T}\lambda_t + \sum_{i=1}^{n-1} \alpha_{i,t,T}\theta_{i,t}, \qquad (t,\omega)\text{-}a.e.,$$

for some d-dimensional $\mathbb{F}$-adapted, bounded processes $\theta_1, \dots, \theta_{n-1}$ such that $\theta_{i,t}^\top \theta_{i,t} > \varepsilon$ and one-dimensional $\mathbb{G}$-adapted processes $t \mapsto \alpha_{t,T}$ and $t \mapsto \alpha_{1,t,T}, \dots, \alpha_{n-1,t,T}$.

A financial interpretation of this corollary is also in Section 2.3.3.

2.3.3 Financial interpretation of Proposition 2.1

In this section we interpret Proposition 2.1 and Corollary 2.2. Specifically, we answer two questions: (i) what is $\mathrm{D}_t(1/H_T)$?; and (ii) why is conditioning on $\mathcal{G}_t$ included? For this, we first investigate the Clark–Ocone formula (2.3), which offers an intuition of the decomposition of demand.

Demand for the non-Sharpe ratio maximizing portfolio

First, we rewrite the Clark–Ocone formula (2.3), to construct a financial interpretation of it. Substituting $W_T = I(zH_T)$, $I'(y) = 1/(U''(I(y)))$ and

(2.5) into (2.3), we obtain

$$
\begin{aligned}
W_t\varphi_t &= (\Sigma_t^\top)^{-1}\mathbb{E}\Big[-\frac{H_T}{H_t}\frac{U'(W_T)}{U''(W_T)}\Big|\mathcal{F}_t\Big]\lambda_t \\
&\qquad + (\Sigma_t^\top)^{-1}\mathbb{E}\Big[-\frac{H_T}{H_t}\Big(W_T+\frac{U'(W_T)}{U''(W_T)}\Big)\Big(H_T\mathrm{D}_t\Big(\frac{1}{H_T}\Big)-\lambda_t\Big)\Big|\mathcal{F}_t\Big] \\
&= W_t(\Sigma_t^\top)^{-1}\lambda_t \\
&\qquad + (\Sigma_t^\top)^{-1}\frac{1}{z}\mathbb{E}\left[\frac{H_T}{H_t}W_TU'(W_T)\Big(-\frac{U'(W_T)}{W_TU''(W_T)}-1\Big)\mathrm{D}_t\Big(\frac{1}{H_T}\Big)\Big|\mathcal{F}_t\right].
\end{aligned}
\tag{2.7}
$$

For the second equation we used that the process $(H_tW_t)_t$ is a $\mathbb{P}$-martingale. By Bayes' formula, we obtain a $\mathbb{Q}$-expectation form of (2.7) as

$$
\varphi_t = (\Sigma_t^\top)^{-1}\lambda_t+(\Sigma_t^\top)^{-1}\frac{1}{z}\widetilde{\mathbb{E}}\Big[\Big(\frac{W_T}{S_T^{(0)}}\Big/\frac{W_t}{S_t^{(0)}}\Big)U'(W_T)\Big(-\frac{U'(W_T)}{W_TU''(W_T)}-1\Big)\mathrm{D}_t\Big(\frac{1}{H_T}\Big)\Big|\mathcal{F}_t\Big].
\tag{2.8}
$$

Here a positive constant z is called the shadow price that depends on initial wealth, and $U'(w)$ is marginal utility. The ratio $-U'(w)/(wU''(w))$ is called risk tolerance, which is the reciprocal of relative risk averseness. If an investor has a logarithmic utility function, $-U'(w)/(wU''(w)) = 1$ and the second term on the right hand side vanishes, it implies that an investor with logarithmic utility invests all her money in the numéraire portfolio $1/H_t$. Since the numéraire portfolio is also characterized as the (instantaneous) Sharpe ratio maximizing portfolio, Equation (2.8) can be rewritten as

$$
(\text{Investment}) = \begin{pmatrix}\text{Sharpe ratio}\\ \text{maximizer}\end{pmatrix} + \begin{pmatrix}\text{Non-Sharpe ratio}\\ \text{maximizing portfolio}\end{pmatrix},
$$

where

$$
\begin{aligned}
&(\text{Non-Sharpe ratio maximizing portfolio})\\
&= (\Sigma_t^\top)^{-1} \begin{pmatrix}\text{Shadow}\\ \text{price}\end{pmatrix}^{-1}\\
&\quad \times \widetilde{\mathbb{E}}\left[\begin{pmatrix}\text{Discounted}\\ \text{Terminal}\\ \text{wealth}\end{pmatrix} \times \begin{pmatrix}\text{Marginal}\\ \text{Utility}\end{pmatrix} \times \begin{pmatrix}\text{Risk}\\ \text{tolerance}\end{pmatrix} \times \mathrm{D}_t \begin{pmatrix}\text{Sharpe}\\ \text{ratio}\\ \text{maximizer}\end{pmatrix} \middle| \mathcal{F}_t \right].
\end{aligned}
$$

This expresses the decomposition of an investor's demand into demand for the Sharpe ratio maximizer and demand for the other portfolio (non-Sharpe ratio maximizing portfolio). Although an investor invests mainly in the Sharpe ratio maximizing portfolio (log-optimal portfolio) myopically, he recognizes an infinitesimal change in such a portfolio as a risk. To hedge this risk, the investor has an additional portfolio, depending on his wealth level, marginal utility, and risk tolerance, at the terminal point. The degree also depends on the shadow price. Among these four components, only the risk tolerance (subtracted by 1) can take both positive and negative values, while others always take positive values. For less risk-tolerant scenarios $-U'(W_T)/(W_T U''(W_T)) < 1$, the investor has additional demand in such a way as to reduce (hedge) his exposure. On the other hand, for risk-tolerant scenarios $-U'(W_T)/(W_T U''(W_T)) > 1$, he has additional demand that increases (levers) the exposure. The investor's additional investment is determined by taking the average of these demands, under the equivalent martingale measure. A further investigation in $D_t(1/H_T)$ also supports this intuition.

Remark 2.3. In this sense, the demand for the non-Sharpe ratio maximizing portfolio can be interpreted as a *hedging demand* of the investor. However, it is not the same as the so-called "hedging demand" in Markovian portfolio optimization problems. They are related, as

$$\begin{pmatrix}\text{Non-Sharpe ratio}\\ \text{maximizing portfolio}\end{pmatrix} = \begin{pmatrix}\text{``Hedging demand''}\\ \text{in the literature}\end{pmatrix} + \Big(- \frac{J_w}{W_t J_{ww}} - 1\Big) W_t (\Sigma_t^\top)^{-1} \lambda_t,$$

where J is the indirect utility function of a Markovian control problem and J_w and J_{ww} are its first and second partial derivatives with respect to wealth level w, respectively. Equation (2.8) shows that the difference between the two demands is held by the investor with the same objective as the non-Sharpe ratio maximizing portfolio of reducing uncertainty due to an infinitesimal change in the numéraire portfolio.

What is $\mathrm{D}_t(1/H_T)$?

Although (2.7) and (2.8) still hold in non-Markovian markets, we consider a Markovian market model to obtain the financial intuition of $\mathrm{D}_t(1/H_T)$.

Markovian market model

Let X^0 be a one-dimensional process and X be an n-dimensional state variable processes such that

$$d \begin{pmatrix} X_t^0 \\ X_t \end{pmatrix} = \begin{pmatrix} (r(X_t) + \lambda^\top \lambda(X_t)) X_t^0 \\ \mu^X(X_t) \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \lambda(X_t)^\top X_t^0 \\ \Sigma^X(X_t) \end{pmatrix} \mathrm{d}B_t, \qquad (2.9)$$

with initial condition $(X_0^0, X_0)^\top = (1, x)^\top \in \mathbb{R}^{1+n}$, where $r : \mathbb{R}^n \to \mathbb{R}$, λ :

$\mathbb{R}^n \to \mathbb{R}^d$, $\mu^X : \mathbb{R}^n \to \mathbb{R}^n$ and $\Sigma^X : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ are deterministic functions with sufficient conditions to obtain Nualart (2006, Equation (2.59)). We denote X^0 by $1/H_t$, because it has the same stochastic integral representation as the numéraire portfolio.

By Nualart (2006, Equation (2.59)), we have

$$\mathrm{D}_t^{(i)}\Big(\frac{1}{H_T}\Big) = \sum_{l=0}^{n} Y_{0,T}^{0} Y_{0,t}^{-1,l} \frac{\lambda^{(i)}(X_t)}{H_t} + \sum_{k=1}^{n}\sum_{l=0}^{n} Y_{l,T}^{0} Y_{k,t}^{-1,l} \Sigma_{k,i}^{X}(X_t).$$

Here $(Y_{j,t}^i)_{i,j=0,\dots,n}$ are often denoted by the partial derivatives of $(1/H, X)$ with respect to their initial values[3]

$$Y_t = (Y_{j,t}^i)_{i,j=0,\dots,n} = \begin{pmatrix} \frac{\partial}{\partial(1/H_0)}\frac{1}{H_t} & \frac{\partial}{\partial x}\frac{1}{H_t} \\ 0 & \frac{\partial}{\partial x}X_t \end{pmatrix}, \qquad Y_0 = (Y_{j,0}^i)_{i,j=0,\dots,n} = E_{1+n}$$

where $Y_t^{-1} = (Y_{j,t}^{-1,i})_{i,j=0,\dots,n}$ denotes the inverse matrix of Y_t, and E_{1+n} denotes the d-dimensional identical matrix.

Interpretation of $\mathrm{D}_t(1/H_T)$

[3] It is defined by

$$\begin{aligned} Y_{j,t}^0 &= 1_{j=0} + \int_0^t Y_{j,s}^0\Big(\lambda^\top(X_s)\mathrm{d}B_s + (r(X_s) + \lambda^\top\lambda(X_s))\mathrm{d}s\Big) \\ &\quad + \sum_{k=1}^{n}\int_0^t Y_{j,s}^k \frac{1}{H_s}\Big(\frac{\partial}{\partial x_k}\lambda(X_s)^\top\mathrm{d}B_s + \frac{\partial}{\partial x_k}(r(X_s) + \lambda^\top\lambda(X_s))\mathrm{d}s\Big), \\ Y_{j,t}^i &= 1_{j=i} + \sum_{k=1}^{n}\int_0^t Y_{j,s}^k\Big(\frac{\partial}{\partial x_k}\Sigma_j^{X\top}(X_s)\mathrm{d}B_s + \frac{\partial}{\partial x_k}\mu_j^X(X_s)\mathrm{d}s\Big) \end{aligned}$$

for $i = 1,\dots,n$ and $j = 0,\dots,n$.

In the Markovian market of (2.9), the Malliavin derivative is written as

$$\mathrm{D}_0\left(\frac{1}{H_T}\right) = \left(\frac{\partial}{\partial(1/H_0)}\frac{1}{H_T}, \quad \frac{\partial}{\partial x}\frac{1}{H_T}\right)\begin{pmatrix}\dfrac{\lambda(X_0)}{H_0}\\ \Sigma^X(X_0)\end{pmatrix}$$

which is interpreted as

$$\mathrm{D}_\cdot\left(\frac{1}{H_T}\right) = \begin{pmatrix}\text{Effect of change}\\ \text{in current state variables}\end{pmatrix} \times \begin{pmatrix}\text{Current exposure}\\ \text{to Brownian motion}\end{pmatrix}.$$

Intuitively, it represents the uncertainty that is produced by a change in the Sharpe ratio maximizer $1/H_T$ due to infinitesimal changes in current state variables.

Why is conditioning on $\mathcal{G}_t$ included?

Each equation (2.7) and (2.8) implies that a sufficient condition for two fund separation is the hedgeability of the infinitesimal change in $1/H_T$ by trading the numéraire portfolio, that is,

$$\mathrm{D}_t\left(\frac{1}{H_T}\right) = \alpha_{t,T}\lambda_t. \tag{2.10}$$

However, it is not a necessary condition because, in view of (2.7), investors determine their additional demands conditioned on their wealths W_T, marginal utilities $U'(W_T)$, and risk tolerances $-U'(W_T)/(W_T U''(W_T))$, at the time of consumption, together with H_T. Since $W_T = I(zH_T)$, it suffices to hold (2.10) conditioned by $1/H_T$ (together with $\mathcal{F}_t$). This is why $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(1/H_T)$ appears in Proposition 2.1.

2.4 A Utility Characterization of Mutual Fund Separation

In previous section, we find a market condition for mutual fund separation. In this section, we find a utility condition under which all investors have the same portfolio as a portfolio of risky assets.

As in previous section, we begin with the Clark–Ocone formula of Ocone and Karatzas (1991):

$$\begin{aligned} H_t W_t \varphi_t =& (\Sigma_t^\top)^{-1} \mathbb{E}\Big[- H_T z H_T I'(zH_T) \Big| \mathcal{F}_t \Big] \lambda_t \\ &+ (\Sigma_t^\top)^{-1} \mathbb{E}\left[\begin{array}{c} -H_T \Big(I(zH_T) + zH_T I'(zH_T) \Big) \\ \times \left(\int_t^T \mathrm{D}_t r_u \mathrm{d}u + \int_t^T \mathrm{D}_t(\lambda_u^\top) \mathrm{d}\widetilde{B}_u \right) \end{array} \middle| \mathcal{F}_t \right]. \end{aligned} \tag{2.11}$$

A closer look in this equation tells us that the mutual fund separation always folds among CRRA utilities with a unique risk-averseness. In fact, if an investor have CRRA utility:

$$U(w) = \begin{cases} \dfrac{w^{1-\gamma} - 1}{1-\gamma}, & 0 < \gamma \neq 1, \\ \log w, & \gamma = 1, \end{cases} \qquad \Longrightarrow \qquad I(y) = y^{-\frac{1}{\gamma}}, \tag{2.12}$$

equation (2.11) can be rewritten as

$$
\begin{aligned}
&H_t W_t \varphi_t \\
&= z^{-\frac{1}{\gamma}} \underbrace{(\Sigma_t^\top)^{-1}\mathbb{E}\left[H_T^{1-\frac{1}{\gamma}}\left\{\frac{1}{\gamma} - \left(1-\frac{1}{\gamma}\right)\left(\int_t^T \mathrm{D}_t r_u \mathrm{d}u + \int_t^T \mathrm{D}_t(\lambda_u^\top)\mathrm{d}\widetilde{B}_u\right)\right\}\middle|\mathcal{F}_t\right]}_{\text{Initial-wealth (investor) independent}}.
\end{aligned}
$$

In this equation, the terms in the left-hand-side are all investor-independent, other than shadow price $z = z(W_0)$ which is a constant. It means that the investors with CRRA utilities with a unique risk-averseness have the same portfolio as a portfolio of risky assets, regardless of initial wealths.

In the remainder of this section, we show that there is a market model in which investors must have CRRA utility in order the mutual fund separation holds, regardless of initial wealths. Such a market model can be found in Markovian markets.

2.4.1 Markovian market

Let us assume that asset prices $S_t^{(0)}$ and $S_t^{(i)}$ solve the following SDE:

$$
\frac{\mathrm{d}S_t^{(0)}}{S_t^{(0)}} = r(X_t)\mathrm{d}t \quad \text{and} \quad \frac{\mathrm{d}S_t^{(i)}}{S_t^{(i)}} = \mu^{(i)}(X_t)\mathrm{d}t + \Sigma^{(i)}(X_t)\mathrm{d}B_t, \quad \text{for } i = 1, \ldots, d, \tag{2.13}
$$

where $r, \mu^{(i)} : \mathbb{R}^n \to \mathbb{R}$ and $\Sigma^{(i)} : \mathbb{R}^{n\times d} \to \mathbb{R}$ are deterministic bounded, C^∞ functions that satisfy Assumption 2.1. An n-dimensional process X solves

$$\mathrm{d}X_t = \mu^X(X_t)\mathrm{d}t + \Sigma^X(X_t)\mathrm{d}B_t, \tag{2.14}$$

where $\mu^X : \mathbb{R}^n \to \mathbb{R}^n$ and $\Sigma^X : \mathbb{R}^{n\times d} \to \mathbb{R}^n$, are deterministic, bounded, C^∞ functions.

Let us denote the indirect utility function by J:

$$J(t, w, x) := \mathbb{E}\big[U(W_T) \mid W_t = w, X_t = x\big], \tag{2.15}$$

for optimal wealth process W with strategy φ. It is well known that the indirect utility function J satisfies Hamilton–Jacobi–Bellman equation (HJB equation) of the form

$$0 = \max_{\varphi\in\mathbb{R}^d} \mathbb{L}^\varphi J(t, w, x), \qquad J(T, w, x) = U(w), \tag{2.16}$$

where $\mathbb{L}^\varphi$ is the generator of (t, W, X)

$$\begin{aligned} \mathbb{L}^\varphi := \frac{\partial}{\partial t} + \frac{1}{2} w^2 \varphi^\top \Sigma\Sigma^\top \varphi \frac{\partial^2}{\partial w^2} + w(\varphi^\top \Sigma\lambda + r)\frac{\partial}{\partial w} \\ + w\varphi^\top \Sigma\Sigma^{X\top} \frac{\partial^2}{\partial w \partial x}^\top + \frac{1}{2}\sum_{i,j=1}^n \Sigma_i^X \Sigma_j^X \frac{\partial^2}{\partial x_i \partial x_j} + \mu^{X\top} \frac{\partial}{\partial x}^\top, \end{aligned} \tag{2.17}$$

and $\Sigma_i^X = \Sigma_i^X(x)$ is the i-th row of the matrix Σ^X. Since φ only appears in

$$\frac{1}{2}w^2\varphi^\top\Sigma\Sigma^\top\varphi\frac{\partial^2}{\partial w^2}J + w\varphi^\top\Sigma\lambda\frac{\partial}{\partial w}J + w\varphi^\top\Sigma\Sigma^{X\top}\frac{\partial^2}{\partial w\partial x}J^\top, \tag{2.18}$$

the first-order condition for optimal φ is

$$\varphi^\top = \underbrace{-\Sigma^{-1}\frac{1}{w}\frac{\partial_w J}{\partial_{ww}J}\lambda^\top}_{\text{"myopic demand"}}\underbrace{-\Sigma^{-1}\frac{1}{w}\frac{\partial_{wx}J}{\partial_{ww}J}\Sigma^X}_{\text{"hedging demand"}}, \tag{2.19}$$

where $\partial_w J$ represents the partial derivative of indirect utility J with respect to w, and so on. Here the first term on the right hand side is called "myopic demand" which represents the investment in instantaneous Sharpe ratio maximizer, and the second term is called "hedging demands" in Markovian portfolio choice problem. As is mentioned above, "hedging demands" are not same as investments in *non-Sharpe ratio maximizing portfolio*.

2.4.2 A utility condition for mutual fund separation

Although Schachermayer et al. (2009, Theorem 3.15) have shown a utility condition for mutual fund separation, their proof requires taking a limit on market models (physical measures). We show that a similar result can be obtained by considering one specific market model. Formally, let $x \mapsto \nu(x)$ be a strictly increasing C^∞ function such that ν is bounded, is bounded away from 0, and has bounded derivatives of all orders. For example $\nu(x) =$

Arctan$(x) + 2$ satisfies such conditions. Let us consider the market of the form:

$$S^{(0)} \equiv 1, \quad \frac{\mathrm{d}S_t^{(1)}}{S_t^{(1)}} = \nu(B_t^{(2)})\,dt + \mathrm{d}B_t^{(1)}, \quad \frac{\mathrm{d}S_t^{(i)}}{S_t^{(i)}} = \mathrm{d}B_t^{(i)}, \quad \text{for} \quad i = 2, \ldots, d. \tag{2.20}$$

Proposition 2.2 (Mutual fund separation and utility function)**.** *Under the market model* (2.20)*, the utility function U of each investor must be an affine of CRRA utility function with a unique relative risk averseness, in order such utility functions to exhibit mutual fund separation.*

Proof. Our proof is in three steps.

Step 1 ($x \mapsto \varphi^{(1)}(T-, w, x)$ is strictly increasing). We first prove that the investor's optimal portfolio weight for the first asset $x \mapsto \varphi^{(1)}(T-, w, x)$ is strictly increasing in x for each w. Assume that $\varphi^{(1)}(T-, w, x) = \varphi^{(1)}(T-, w, y)$ for some pair x, y and for some $w > 0$. Then, by the first order condition of HJB equation, we have

$$\frac{\partial_w J(T-, w, x)}{\partial_{ww} J(T-, w, x)} \nu(x) = \frac{\partial_w J(T-, w, y)}{\partial_{ww} J(T-, w, y)} \nu(y) \tag{2.21}$$

for some w, because $\varphi^{(1)}(T-, w, x) = \varphi^{(1)}(T-, w, y)$ by the assumption. Thus we must have

$$\frac{U'(w)}{U''(w)} \nu(x) = \frac{U'(w)}{U''(w)} \nu(y), \tag{2.22}$$

because $J(T-, w, x) = U(w)$. However, this contradicts to the assumption that U is a strictly increasing and strictly concave function. Therefore $x \mapsto$

$\varphi^{(1)}(T-, w, x)$ must be strictly increasing.

Step 2 (Investors must have CRRA utilities). By the first order condition of the HJB equation,

$$w\varphi(t, w, x) = -\frac{\partial_w J}{\partial_{ww} J}\begin{pmatrix} \nu(x) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \frac{\partial_{wx} J}{\partial_{ww} J}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.23}$$

The mutual fund separation implies

$$\frac{\partial_{wx} J(t, 1, x)}{\partial_w J(t, 1, x)} = \frac{\partial_{wx} J(t, w, x)}{\partial_w J(t, w, x)} =: F(t, x), \tag{2.24}$$

for some function $F(t, x)$ for each t, w, and x. Thus there exist deterministic functions f and g such that

$$\frac{\partial}{\partial w} J(t, w, x) = g(t, w) f(t, x) \tag{2.25}$$

and

$$\frac{\partial_w J(t, w, x)}{\partial_{ww} J(t, w, x)} = \frac{g(t, w)}{\partial_w g(t, w)} =: G(t, w). \tag{2.26}$$

On the other hand, by the martingale approach together with the Clark–

Ocone formula (under change of measure),

$$W_t\varphi_t = \mathbb{E}^{\mathbb{Q}}\Big[-\frac{U'(W_T)}{U''(W_T)}\Big|\mathcal{F}_t\Big]\begin{pmatrix}\nu(X_t)\\0\\0\\\vdots\\0\end{pmatrix} + \mathbb{E}^{\mathbb{Q}}\Big[-\Big(\frac{U'(W_T)}{U''(W_T)}+W_T\Big)\int_t^T\begin{pmatrix}0\\\nu'(X_t)\\0\\\vdots\\0\end{pmatrix}\mathrm{d}\widetilde{B}_t^{(1)}\Big|\mathcal{F}_t\Big], \quad (2.27)$$

which implies

$$G(t, W_t) = \mathbb{E}^{\mathbb{Q}}\Big[\frac{U'(W_T)}{U''(W_T)}\Big|\mathcal{F}_t\Big]. \quad (2.28)$$

Especially, $G(t, W_t)$ is a $\mathbb{Q}$-martingale with terminal condition $G(T, W_T) = U'(W_T)/U''(W_T)$. Now, applying the Feynman–Kac formula, we obtain

$$\frac{\partial}{\partial t}G(t,w) - \frac{1}{2}\frac{\partial^2}{\partial w^2}G(t,w)w^2\varphi^{(1)}(t,w,x)^2\left(1+\Big(\frac{\partial_{wx}J(t,w,x)}{\partial_w J(t,w,x)}\frac{1}{\nu(x)}\Big)^2\right) = 0, \quad (2.29)$$

for each t, w, and x, because $\varphi^{(2)} = \frac{\varphi^{(2)}}{\varphi^{(1)}}\varphi^{(1)}$. Letting $t \to T$, we have

$$\frac{\partial}{\partial t}G(T-,w) - \frac{1}{2}\frac{\partial^2}{\partial w^2}G(T-,w)w^2\varphi^{(1)}(T-,w,x)^2 = 0. \quad (2.30)$$

Since $x \mapsto \varphi^{(1)}(T-,w,x)$ is strictly increasing for each $w > 0$, we obtain

$$\frac{U'(w)}{U''(w)} = G(T-,w) = \alpha w + \beta, \qquad \text{for some} \quad \alpha, \beta \in \mathbb{R}. \quad (2.31)$$

Thus we conclude that U is a utility function with linear risk tolerance; specifically, under our assumption, U is an affine of CRRA utility function.

For the uniqueness of the parameter of utility function, we use the following lemma.

Lemma 2.3. *For m utility functions $U_1, \ldots, U_m$ and positive constants $z_1, \ldots, z_m > 0$, let $(U')^{-1}$ be*

$$(U')^{-1}(y) := \sum_{k=1}^{m} (U_k')^{-1}\Big(\frac{y}{z_k}\Big). \tag{2.32}$$

Then, the inverse function $U' = ((U')^{-1})^{-1}$ exists and the (indefinite) integral U of U' is again a utility function.

Proof. Since $x \mapsto (U_k')^{-1}(x)$ is a strictly decreasing function, the sum $(U')^{-1}$ is also strictly decreasing and has the inverse function $U' := ((U')^{-1})^{-1}$. Thus we need to show that $U' > 0$ and $U'' < 0$. Since each mapping $(U_k')^{-1} : (0,\infty) \to (0,\infty)$ is strictly decreasing, the sum $(U')^{-1}$ is also a strictly decreasing function from $(0,\infty)$ to $(0,\infty)$ and the inverse function U' is also strictly decreasing and is positive. □

Proof of Proposition 2.2 (continued). Step 3 (Uniqueness of the relative risk averseness). Assume that two CRRA utility functions $U_1(w) = w^{1-\gamma_1}/(1-\gamma_1)$ and $U_2(w) = w^{1-\gamma_2}/(1-\gamma_2)$ exhibit mutual fund separation. Let $z_1 = z_2 = 1$. Let a function I be

$$I(y) := (U_1')^{-1}(y) + (U_2')^{-1}(y). \tag{2.33}$$

Then there exists a utility function U with $I = (U')^{-1}$ by the previous lemma.

For each $W_0 > 0$, let $z > 0$ be

$$\mathbb{E}^{\mathbb{Q}}\Big[(U')^{-1}\Big(\frac{M_T}{z}\Big)\Big] = W_0. \tag{2.34}$$

Then, the optimal terminal wealth W_T of the investor with utility function U and initial wealth W_0 satisfies

$$W_T = (U')^{-1}\Big(\frac{M_T}{z}\Big) = (U_1')^{-1}\Big(\frac{M_T}{z}\Big) + (U_2')^{-1}\Big(\frac{M_T}{z}\Big) = W_{1,T} + W_{2,T}, \tag{2.35}$$

which implies that U also exhibits mutual fund separation in each market. By step 2, however, such a utility function must be CRRA-type. For this, we must have $\gamma_1 = \gamma_2$. □

Remark 2.4. This result is related to Cass and Stiglitz (1970) and Dybvig and Liu (2018), who proved the relation between (monetary) separation and utilities with linear risk tolerance in one-period finite-state of either complete or incomplete markets; and it is also related to Schachermayer et al. (2009), who showed an example of a class of markets in which only CRRA utilities are available if the mutual fund separation holds. It is worth noting that the restriction of utility functions by separation was considered as a global relation in previous studies. This result, on the other hand, considers this relation as a *pointwise* restriction.

2.4.3 An application to pricing under the minimal equivalent martingale measure

In complete markets, the market prices of risk λ is uniquely determined by the dynamics of assets, which must be consistent with investor preferences behind market models. In incomplete markets, on the other hand, the market prices of risk cannot determined simply by the dynamics of assets, and thus additional assumptions have to be made to find it out uniquely. Although different investors can estimate different market prices of risks for untradable risks, it is usually assumed that the market admits a market prices of risk of specific types without specifying investor preferences, for option pricing purpose.

In this section we investigate the relation between assumption of some specific market prices of risk and utility function, in incomplete market. Specifically we focus on the minimal equivalent martingale measure (MEMM) from the perspective of mutual fund separation. For simplicity, we assume that investors have the same utility functions, and consider a Markovian market driven by two-dimensional Brownian motion with a risk-free asset and one risky asset. Throughout this section we assume, without loss of generality, an incomplete market of the form

$$\begin{aligned}\frac{\mathrm{d}S_t^{(0)}}{S_t^{(0)}} &= r(X_t)\,\mathrm{d}t,\\ \frac{\mathrm{d}S_t^{(1)}}{S_t^{(1)}} &= \mu^{(1)}(X_t)\,\mathrm{d}t + \sigma^{(1)}(X_t)\,\mathrm{d}B_t^{(1)},\end{aligned} \tag{2.36}$$

where r, $\mu^{(1)}$ and $\sigma^{(1)}$ are C^∞ bounded functions with bounded derivatives and X follows

$$\mathrm{d}X_t = \mu^X(X_t)\,\mathrm{d}t + \Sigma^X(X_t)\begin{pmatrix}\mathrm{d}B_t^{(1)}\\ \mathrm{d}B_t^{(2)}\end{pmatrix}, \tag{2.37}$$

where $B = (B^{(1)}, B^{(2)})^\top$ is a two-dimensional standard Brownian motion. In general, this system determines an incomplete market because the risk driven by $B^{(2)}$ exists and cannot be hedged in the market.

In this case, the market price of risk $\lambda_t^{(1)}$ of the first Brownian motion $B^{(1)}$ is uniquely determined by

$$\lambda^{(1)}(X_t)\sigma^{(1)}(X_t) = \mu^{(1)}(X_t) - r(X_t). \tag{2.38}$$

However, any adapted processes (with sufficient integrability) can serve as a market price of risk $\lambda^{(2)}$ of the second Brownian motion $B^{(2)}$. As a result, the risk-neutral measure $\mathbb{Q}$ can be taken as

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left\{-\int_0^T (\lambda^{(1)}, \lambda^{(2)})\,\mathrm{d}B_t - \frac{1}{2}\int_0^T \|(\lambda^{(1)}, \lambda^{(2)})^\top\|^2\,\mathrm{d}t\right\} \tag{2.39}$$

for any choices of $\lambda^{(2)}$, and thus the prices of contingent claims on $B^{(2)}$ cannot be determined uniquely. The problem of asset pricing in incomplete markets is to determine $\lambda^{(2)}$ (and the corresponding risk-neutral probability measure $\mathbb{Q}$), via characteristics of investors —such as utility functions—.

One usual technique is to impose additional assumptions on λ directly. For example, if one assumes that the investors do not stipulate any premium on $B^{(2)}$, then $\lambda^{(2)}$ equals identically zero. In this case, the risk-neutral measure $\mathbb{Q}$ is determined by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left\{-\int_0^T \lambda_t^{(1)}\,\mathrm{d}B_t^{(1)} - \frac{1}{2}\int_0^T (\lambda_t^{(1)})^2\,\mathrm{d}t\right\}, \tag{2.40}$$

such a risk-neutral measure $\mathbb{Q}$ is called the minimal equivalent martingale measure.

Another technique to determine $\lambda^{(2)}$ in incomplete markets is to introduce additional assets to the model such a way as to make the market complete, such that the investors are prohibited from having positions on these assets (as optimal strategies). In other words, the market price of risk $\lambda^{(2)}$ for $B^{(2)}$ is determined such that each investor have no position on these additional assets optimally. Specifically, we add an asset $S^{(2)}$

$$\frac{\mathrm{d}S_t^{(2)}}{S_t^{(2)}} = \mu^{(2)}(X_t)\,\mathrm{d}t + \Sigma^{(2)}(X_t)\,\mathrm{d}B_t, \tag{2.41}$$

to the market (2.36), where $\mu^{(2)}$ is an $\mathbb{R}$-valued function and $\Sigma^{(2)}$ is an $\mathbb{R}^2$-

valued function. Both are bounded C^∞ functions with bounded derivatives of all orders. we denote by $\Sigma = (\Sigma^{(1)}, \Sigma^{(2)})^\top$, where $\Sigma^{(1)} = (\sigma^{(1)}, 0)^\top$. With this asset, the market becomes complete and the market price of risk $\lambda^{(2)}$ of $B^{(2)}$ is the unique solution of

$$\Sigma^{(2)} \begin{pmatrix} \lambda^{(1)} \\ \lambda^{(2)} \end{pmatrix} = \mu^{(2)} - r. \tag{2.42}$$

The coefficients are assumed to satisfy the conditions of this chapter.

Although applying the first technique —imposing additional assumptions on λ directly— is often easy to calculate option prices, it sometimes difficult to interpret economically. Pham and Touzi (1996) filled this gap by demonstrating that imposing restriction on λ in the second technique —adding assets— is related to imposing restrictions on the utility function of the investor in the framework of representative investor model.

In this section we investigate these relations in the view of mutual fund separation, because the second method is closely related to mutual fund separation. In fact, if one assumes that the risk-neutral measure $\mathbb{Q}$ is identical among investors, there exists a market such that each investor must have the same parameter of CRRA utility by Proposition 2.2, because each investor must have zero position on the additional asset optimally. In this sense, we say that the risk-neutral measure $\mathbb{Q}$ is *viable* for a utility function U, if investors with utility function U have zero position on the additional asset $S^{(2)}$

as their optimal strategies, irrespective of their initial wealth. Specifically, we focus on the relation between minimal equivalent martingale measure and the logarithmic utility.

Minimal equivalent martingale measure

Let us assume the market price of risk $\lambda = \lambda^{\mathrm{MEMM}}$ is of the form

$$\lambda_t^{\mathrm{MEMM}} = \begin{pmatrix} \lambda_t^{(1)} \\ 0 \end{pmatrix}, \qquad \text{where} \quad \lambda_t^{(1)} \sigma_t^{(1)} = \mu_t^{(1)} - r_t \tag{2.43}$$

The minimal equivalent martingale measure $\mathbb{Q}^{\mathrm{MEMM}}$ is the martingale measure defined by the market prices of risk λ^{MEMM} as

$$\frac{\mathrm{d}\mathbb{Q}^{\mathrm{MEMM}}}{\mathrm{d}\mathbb{P}} = \exp\left\{ -\int_0^T \lambda_t^{\mathrm{MEMM}\top} \mathrm{d}B_t - \frac{1}{2}\int_0^T \|\lambda_t^{\mathrm{MEMM}}\|^2 \, \mathrm{d}t \right\}. \tag{2.44}$$

The minimal equivalent martingale measure is often used in incomplete market asset pricing as a benchmark, because this martingale measure always exists if at least one equivalent martingale measure exists.

Under the minimal equivalent martingale measure, the corresponding additional asset $S^{(2)}$ is of the form:

$$\frac{\mathrm{d}S_t^{(2)}}{S_t^{(2)}} = (\sigma_t^{(2,1)}, \sigma_t^{(2,2)}) \big(\lambda_t^{\mathrm{MEMM}} \mathrm{d}t + \mathrm{d}B_t \big). \tag{2.45}$$

By the Clark–Ocone formula under the change of measure, we have

$$\begin{aligned}
&\frac{W_t}{S_t^{(0)}}\begin{pmatrix}\varphi_t^{(1)}\sigma_t^{(1)}+\varphi_t^{(2)}\sigma_t^{(2,1)}\\ \varphi_t^{(2)}\sigma_t^{(2,2)}\end{pmatrix}\\
&\quad=\mathbb{E}^{\mathbb{Q}}\Big[-\frac{1}{S_T^{(0)}}\frac{U'(W_T)}{U''(W_T)}\,\Big|\,\mathcal{F}_t\Big]\begin{pmatrix}\lambda_t^{(1)}\\ 0\end{pmatrix}\\
&\qquad\qquad+\mathbb{E}^{\mathbb{Q}}\left[-\frac{1}{S_T^{(0)}}\Big(\frac{U'(W_T)}{U''(W_T)}+W_T\Big)\Big(\int_t^T \mathrm{D}_t r_s\,\mathrm{d}s+\int_t^T \mathrm{D}_t\lambda_s\,\mathrm{d}\widetilde{B}_s\Big)\,\Big|\,\mathcal{F}_t\right].
\end{aligned}\tag{2.46}$$

Note that investors have zero positions on $S^{(2)}$ if and only if the second part equals to zero. The logarithmic utility is always viable because the second term of the right hand side vanishes when $U(w)=\log w$. Conversely, if one considers markets described in the proof of Proposition 2.2, it can be shown that the investor must have logarithmic utility.

Corollary 2.3. *1. The minimal equivalent martingale measure is always viable if investors have logarithmic utilities.*

2. There exist markets in which the investors must have logarithmic utilities, in order the minimal equivalent martingale measure to be viable.

This can be interpreted as follows: since investors with logarithmic utility do not have hedging demands, they do not stipulate any premia for the risk of coefficients $\mu^{(1)}$, $r^{(1)}$ and $\sigma^{(1)}$. This leads to $\lambda^{(2)}\equiv 0$ or minimal equivalent martingale measure.

Remark 2.5. Of course there is a market model in which investors have different portfolio, even if they have same preferences and information. Endogenous heterogeneous beliefs in optimal expectations equilibria of Brunnermeier and Parker (2005) is one example.

Remark 2.6. For optimization problems in constrained markets, the author refers Cvitanić and Karatzas (1992).

2.5 Discussion

CAPM or APT

In this study, we are motivated to find analytic conditions for mutual fund separation, because it plays critical role in CAPM. Our result implies it is the nature of equivalent martingale measure that determines whether the mutual fund separation holds or not. It suggests us, however, that we should compare our result with no-arbitrage based models such as arbitrage pricing theory introduced by Ross (1976)[4], not equilibrium based CAPM, c.f. Harrison and Kreps (1979). An important example is the interest rate model of Heath, Jarrow, and Morton (1992). Recently, Gehmlich and Schmidt (2018) introduce predictable default of Merton (1974) as well as unpredictable default (intensity models, also known as hazard rate models, see e.g. Jarrow, Lando, & Yu, 2005). In order to carry out empirical tests, however, we need a methodology to estimate Malliavin derivatives from data. This is also a problem worth tackling.

[4]Strictly speaking, his notion of arbitrage seems to be different to pathwise arbitrage concept of mathematical finance.

Jump diffusion model

One natural generalization of our result is to introduce jump risks in the asset price dynamics. The martingale approach for jump-diffusion asset prices (a counterpart to equation (2.2) of this chapter) is found in Kramkov and Schachermayer (1999), whose result is also used in Schachermayer et al. (2009). Malliavin calculus is also applicable to such processes (more precisely, Lévy processes) using chaos expansion technique. An introductory textbook is Di Nunno, Øksendal, and Proske (2009), which also contains a counterpart of Clark–Ocone formula of Ocone and Karatzas (1991) (called "generalized Clark–Ocone theorem under change of measure for Lévy processes," Di Nunno et al. (2009, p.200)). We need to check whether it is applicable to our problem.

Consumptions at non-terminal points

In this study, we assumed that the investor can consume at terminal point T alone. However, the martingale approach as well as the Clark–Ocone formula do not exclude consumptions in $t \in [0, T]$. A difficulty arises because it requires estimation of correlations between some random variables that are related to stochastic integral of future consumptions, to obtain a similar result. This is also a future work.

2.6 Conclusion

This chapter finds an analytic market condition for mutual fund separation using the Malliavin calculus technique. The condition reads: a conditional expectation of an infinitesimal change in the numéraire portfolio can be hedged by the numéraire portfolio itself. In a Markovian market, such an infinitesimal change is characterized as an infinitesimal change in state variables. This is because investors first invest in the numéraire portfolio (Sharpe ratio maximizing portfolio) and then invest in other portfolio to reduce the infinitesimal change in uncertainty produced by an infinitesimal change in the numéraire portfolio (non-Sharpe ratio maximizing portfolio). This implies that investors' performance will improve if securities that hedge the uncertainty of the numéraire portfolio are provided. Such a decomposition is still valid in non-Markovian markets.

It also finds that mutual fund separation among CRRA utilities implies separation among arbitrary utility functions. This leads to a conjecture that there is a market model in which investors must have CRRA utilities of a unique parameter to mutual fund separation holds, and we proved it. The result is consistent to Cass and Stiglitz (1970) and Dybvig and Liu (2018) (discrete-time models), and Schachermayer et al. (2009) (continuous-time model).

Chapter 3

Speculative Trades and Differences in Bargaining Power

3.1 Introduction

It is well known that if investors have different beliefs about future dividends of a stock, the stock price may deviate from investors' valuations. Harrison and Kreps (1978) show that the price of the stock can be even higher than the present value of dividends by the most optimistic investors, because the stock holder has not only the dividend stream but also an option to resell it. They call this phenomenon a *speculation*. Scheinkman and Xiong (2003) show that the speculation occurs even if investors are neither optimistic nor pessimistic in advance.

The speculation is thought to be related to asset bubbles. Previous studies show the effect of speculative trades on stock prices, assuming that the sellers have complete bargaining power: once an investor sells the stock, he will be exploited up to his outside option (zero expected cash flow). However, it is still unsolved how speculative trades effect on stock prices during bubble formation periods (in which sellers are gaining bargaining power), and after bursts of bubbles (in which buyers have bargaining power). The purpose of this chapter is to fill this gap.

In this chapter, we assume that both sellers and buyers can have bargaining power. Unlike previous studies, the buyer's expected payoff (performance) is not necessarily zero and thus the buyer also has an option to buy the stock back. Therefore the price consists of the dividend valuation, the resale option, and the buyback option. Because investors have different beliefs, trades can occur at any price between the reservation price for the seller and for the buyer. When the price gets closer to the reservation price for the buyer, the buyer's expected payoff approaches zero and vice versa. Thus we define the bargaining power as the closeness of the price to the other investor's reservation price. We explore the effect of differences in the bargaining power on the price and its volatility.

We find that there exist equilibria in which prices differ from each other, even though the expected payoffs (performances) of each investor are identical, assuming that the asset is indivisible. This difference comes from the gap between bargaining power as a seller and as a buyer. If each investor has bargaining power as a seller but has less power as a buyer, the buyer would have to put up with high prices. Anticipating this, current holders may boost the price, which forms the resale option. Similarly, a buyback option arises when investors have bargaining powers as buyers. In both cases, investors' expected payoffs are determined by the initial bargaining power, that is the bargaining power as a seller of the initial holder (or equivalently, the bargaining power as a buyer of the other investor). Therefore, no one makes a profit from the rise and fall in the price process caused by the speculative trades.

Cao and Ou-Yang (2005) show some examples in which heterogeneous beliefs can cause both higher prices than the most optimistic investor's valuation and lower prices than the most pessimistic investor's valuation. Although the price may have both resale option value and buyback option value, it is difficult to distinguish them in their model due to the risk averseness of the investors. Assuming risk-neutral investors, we decompose the equilibrium price into four components—seller's valuation, buyer's valuation, resale option, and buyback option—and investigate relations between them.

We find numerically that the speculative trades can both increase and decrease the volatility of the price process. If the current stock holder has bargaining power as a seller (or as a buyer) and the future holder has more power as a seller (or as a buyer), the volatility of stock price tends to increase. On the other hand, if both investors have bargaining power as sellers (or as buyers), the volatility will decrease.

It may be counterintuitive that the speculative trades can decrease the stock volatility (even though the price itself rises due to speculation). If sellers have bargaining power, the stock price gets closer to the reservation price for the buyer, which consists of the dividend value for the buyer and the resale option for him. When the buyer places a high value on the dividend, the valuation by the seller will be relatively lower and thus the resale option value decreases. This negative correlation results in a decrease in the volatility of the price process as a whole. If, on the other hand, sellers are gaining in bargaining power (or as sellers), the current price better reflects the seller's valuation (compared to the future price), and the buyer expects higher value on the resale option than the seller expects on the buyback option. Because the seller's dividend valuation and the buyer's resale option valuation are positively correlated, the volatility can increase due to speculation in this case.

Gallmeyer and Hollifield (2008) find that the volatility can be both higher and lower due to heterogeneous beliefs and a short-sales constraint. Their results largely depend on the parameter of the utility function of the predetermined optimists in comparison with the pessimists'. In contrast, we obtain the result assuming that the investors are not optimistic nor pessimistic in advance and the utilities are identical (risk neutral).

Mathematically, the problem becomes more complicated when the buyer also anticipates profit from the trades. If the buyer's expected payoff is zero, the current stock holder's problem is to choose the time to sell it. If the buyer also expects positive payoff, however, the current holder has to choose the selling time considering the buyback time strategy, the next time to sell, and so on. Such a problem is called optimal multiple stopping problem. Unfortunately, these problems are less studied. In order to grapple with optimal multiple stopping problem, we apply the method of optimal switching developed in, for example, Pham (2007), who show that some optimal switching problems can be reduced to optimal stopping problems with regime-dependent boundary conditions. In this model, investors chooses two regimes—holder and non-holder—to maximize their expected payoff in total.

The remainder of this chapter is constructed as follows. Section 3.2 introduces assumptions of this chapter including the restriction on the price process. Section 3.3 decomposes the price into the dividend value, the resale option value for the buyer and the buyback option value for the seller. Section 3.4 investigates the properties our solution should have. Section 3.5 states the effect of the differences in bargaining power on the equilibrium price and its volatility. This is the main results of this chapter. Section 3.6 discusses the result. Section 3.7 concludes the chapter. Proofs can be found in Appendix.

3.2 The Model

3.2.1 Basic assumptions

In order to investigate the effect of differences in bargaining power, our model follows the basic assumptions of the Scheinkman and Xiong (2003) model, which assumes investors neither optimistic nor pessimistic in advance.

We consider an economy with two risk-neutral investors indexed by 1 and 2 with the discount rate $r > 0$. There is an indivisible risky asset, which generates a flow of dividend. The cumulative dividend is denoted by D and it satisfies

$$\mathrm{d}D_t = \Theta_t \, \mathrm{d}t + \sigma_D \mathrm{d}B_{D,t},$$

where $B_{D,t}$ is a Wiener process and $\sigma_D > 0$ is a constant. The stochastic process Θ_t is a state variable which determines the future dividends. It follows an Ornstein-Uhlenbeck process

$$\mathrm{d}\Theta_t = \lambda\bigl(\mu - \Theta_t\bigr) \, \mathrm{d}t + \sigma_\Theta \, \mathrm{d}B_{\Theta,t},$$

where $B_{\Theta,t}$ is a Wiener process, and $\sigma_\Theta > 0$, $\lambda > 0$, μ are constants. The asset is traded in a stock market and is the only security traded in the market. Short sales in the asset is prohibited, and a seller (resp. buyer) pays $c^S > 0$ (resp. $c^B > 0$) when he sells (resp. buys) the asset. We denote by c the total transaction cost for trading: $c = c^S + c^B$.

The state variable Θ_t gives two signals $S_{1,t}$ and $S_{2,t}$ which satisfy

$$\mathrm{d}S_{i,t} = \Theta_t \mathrm{d}t + \sigma_S\big(\phi_i \mathrm{d}B_{\Theta,t} + \sqrt{1-\phi_i^2}\mathrm{d}B_{i,t}\big), \qquad i = 1, 2,$$

where $B_{1,t}$ and $B_{2,t}$ are Wiener processes, and $\sigma_S > 0$ is a constant. We assume that $\phi_1 = \phi$ and $\phi_2 = 0$ under a probability measure $\mathbb{P}_1$, and $\phi_2 = \phi$ and $\phi_1 = 0$ under another probability measure $\mathbb{P}_2$ for some constant $0 < \phi < 1$. This symmetry of beliefs makes it easy to solve the problem. The four Wiener processes B_D, B_Θ, B_1 and B_2 are independent.

The investors observe the dividend stream D and two signals S_1 and S_2, but cannot observe the state variable Θ directly.

Assumption 3.1. The investors have the same information set $\{\mathcal{F}_t\}$ which is given by

$$\mathcal{F}_t := \sigma(D_s, S_{1,s}, S_{2,s};\ 0 \leqslant s \leqslant t),$$

that is the (augmented) filtration generated by D, S_1 and S_2.

By Assumption 3.1, we have assumed that the investors have the same information about the economy, but the investors cannot tell that a high (resp. low) dividend is due to a high (resp. low) state variable or noise $\mathrm{d}B_D$. In order to maximize their expected discounted cash flows, the investors have to estimate the current value of the state variable Θ. We assume they have different beliefs about the parameter of Θ.

Assumption 3.2. Investor $i \in \{1, 2\}$ believes that the physical probability

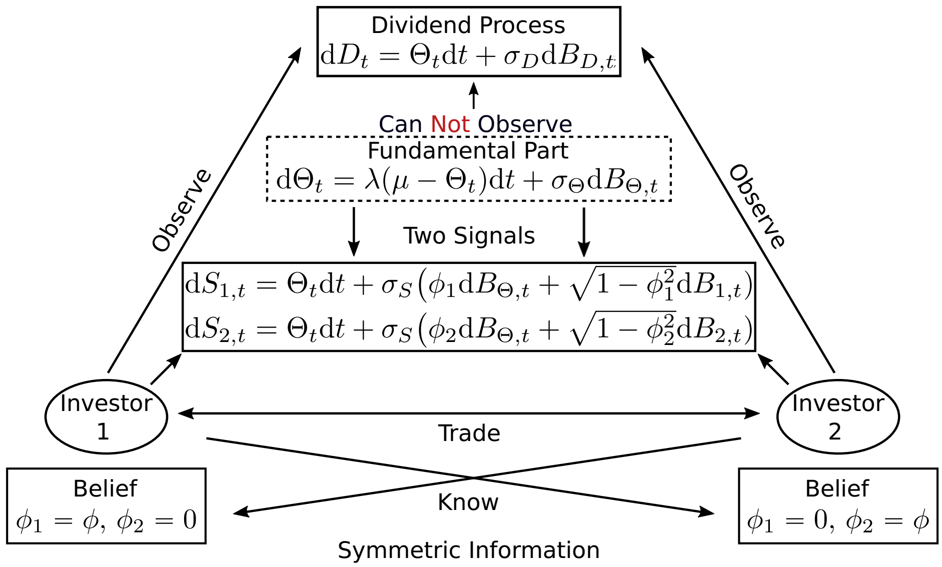

Figure 3.1: A description of the model.

measure is $\mathbb{P}_i$. Investor i knows that Investor $j \neq i$ believes in the probability measure $\mathbb{P}_j$.

This assumption means that Investor i believes that the signal S_i is correlated to the fundamental value Θ and $0 < \phi < 1$ is the correlation parameter. Therefore, Investor i believes that the signal S_i is more informative than the other signal S_j, and the estimations of Θ_t by two investors are different from each other. Figure 3.2.1 shows this model graphically.

For the sake of mathematical tractability, we further assume

Assumption 3.3. Under $\mathbb{P}_i$, the initial value Θ_0 of Θ is independent of the Wiener processes B_D, B_Θ, B_1 and B_2, and normally distributed with mean $\widehat{\Theta}_{i,0} \in \mathbb{R}$ and variance γ, where γ is the unique positive solution of the quadratic equation

$$(1-\phi^2)\sigma_\Theta^2 - 2\Big(\lambda + \frac{\sigma_\Theta\phi}{\sigma_S}\Big)\gamma - \Big(\frac{1}{\sigma_D^2} + \frac{2}{\sigma_S^2}\Big)\gamma^2 = 0.$$

This assumption makes the conditional expectation $\mathbb{E}_i[\Theta_t \mid \mathcal{F}_t] =: \widehat{\Theta}_{i,t}$ time-homogeneous (see Lemma 3.1). Here, γ is the long-run steady-state variance of $\mathbb{E}_i[\Theta_t \mid \mathcal{F}_t]$, that is, $\mathrm{Var}_i(\Theta_t \mid \mathcal{F}_t) \to \gamma$ as $t \to \infty$ if Θ_0 is normally distributed.

3.2.2 Strategies and equilibria

We assume that the asset price may differ according to the holder of the asset because the asset is indivisible and short sales are prohibited.

Definition 3.1 (Price process). *We call a two dimensional $\{\mathcal{F}_t\}$-adapted process $P_t = (P_{1,t}, P_{2,t})^\top$ a price process such that if Investor $o \in \{1,2\}$ has the risky asset then the transaction can take place at price $P_{o,\cdot}$.*

Given a price process $\{P_t\}$ and a sequence $\{\tau_n\}$ of stopping times $0 = \tau_0 \leqslant \tau_1 < \tau_2 < \cdots \to \infty$, which represent the timings of trades, the performance of Investor $i's$ is evaluated by

$$V_i(\{\tau_n\}) = \mathbb{E}_i \left[\begin{aligned} &\int_0^\infty e^{-rs} \sum_{k=0}^\infty 1_{[\tau_{2k},\tau_{2k+1})} \mathrm{d}D_s \\ &+ \sum_{k=0}^\infty e^{-r\tau_{2k+1}} (P_{i,\tau_{2k+1}} - c^S) - \sum_{k=1}^\infty e^{-r\tau_{2k}} (P_{j,\tau_{2k}} + c^B) \end{aligned} \right]$$

if i has the asset initially. In this case, τ_n represents i's selling time for odd n and buying time for even n. Otherwise the performance of Investor i is

$$R_i(\{\tau_n\}) = \mathbb{E}_i \left[\begin{aligned} &\int_0^\infty e^{-rs} \sum_{k=1}^\infty 1_{[\tau_{2k-1}, \tau_{2k})} \mathrm{d}D_s \\ &\quad - \sum_{k=1}^\infty e^{-r\tau_{2k-1}} (P_{j,\tau_{2k-1}} + c^B) + \sum_{k=1}^\infty e^{-r\tau_{2k}} (P_{i,\tau_{2k}} - c^S) \end{aligned} \right]$$

and τ_n represents i's buying time for odd n and selling time for even n, in this case.

Definition 3.2 (Strategy)**.** *We call a sequence of finite $\{\mathcal{F}_t\}$-stopping times $\{\tau_n\}$ such that $0 = \tau_0 \leqslant \tau_1 < \tau_2 < \tau_3 < \cdots \to \infty$ a strategy.*

Figure 3.2.2 shows a strategy τ of Investor j graphically. If Investor j holds the asset initially, he receives dividend stream $\mathrm{d}D_t$ continuously until he sell it at τ_1. At the first selling time τ_1 he sell the asset at the price P_{j,τ_1} and receives $P_{j,\tau_1} - c^S$. Then he will buyback it at τ_2 paying $P_{j,\tau_2} - c^B$. Until the first buyback time τ_2, he receives nothing. This continues infinitely. We denote by $\{\tau_{i,n}\} := \{\tau_{i,n}\}_n$ a strategy of Investor i.

In the sequel, we will investigate the following equilibria:

Definition 3.3 (Equilibrium)**.** *We call a tuple $(\{P_t\}, (\{\tau_{1,n}\}, \{\tau_{2,n}\}))$ of a price process $\{P_t\}$ and a pair of strategies $\{\tau_{1,n}\}$ and $\{\tau_{2,n}\}$ an equilibrium if the following conditions are satisfied:*

1. *(individual optimization) For the initial owner $o \in \{1, 2\}$ of the risky*

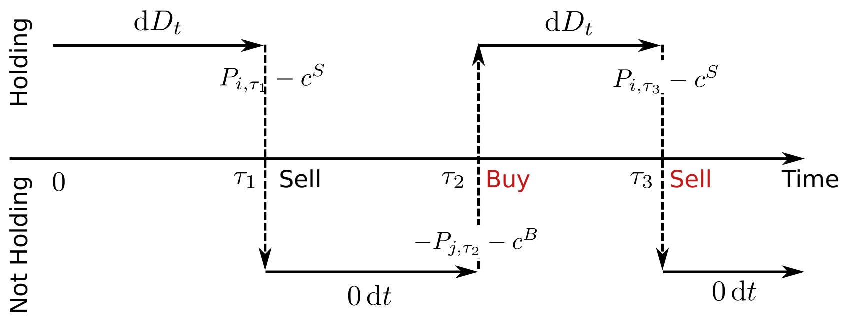

Figure 3.2: A description of a trading strategy.

asset, the strategy $\{\tau_{o,n}\}$ *is optimal:*

$$V_o(\{\tau_{o,n}\}) = \sup V_o(\{\tau_n\}),$$

and for Investor $\widehat{o} \neq o$*, the strategy* $\{\tau_{\widehat{o},n}\}$ *is optimal:*

$$R_{\widehat{o}}(\{\tau_{\widehat{o},n}\}) = \sup R_{\widehat{o}}(\{\tau_n\}),$$

where supremum is taken over strategies.

2. *(market clearing condition) The buying/selling times coincide:*

$$\tau_{1,n} = \tau_{2,n}, \qquad \textit{for all} \quad n = 1, 2, \ldots, \quad \textit{a.s.}$$

3.2.3 Restriction on the price processes

For each price process P, let us denote by $V_{i,t}$ the (optimized) performance for Investor i when i has the asset at t:

$$V_{i,t} = \sup \mathbb{E}_i \left[\begin{aligned} &\int_t^\infty e^{-r(s-t)} \sum_{k=0}^\infty 1_{[\tau_{2k}, \tau_{2k+1})} \mathrm{d}D_s \\ &\qquad + \sum_{k=0}^\infty e^{-r(\tau_{2k+1}-t)} (P_{i,\tau_{2k+1}} - c^S) \\ &\qquad\quad - \sum_{k=1}^\infty e^{-r(\tau_{2k}-t)} (P_{j,\tau_{2k}} + c^B) \end{aligned} \middle| \mathcal{F}_t \right]$$

and let us denote by $R_{i,t}$ the (optimized) performance for Investor i when i does not have the asset at t:

$$R_{i,t} = \sup \mathbb{E}_i \left[\begin{aligned} &\int_t^\infty e^{-r(s-t)} \sum_{k=1}^\infty 1_{[\tau_{2k-1}, \tau_{2k})} \mathrm{d}D_s \\ &\qquad - \sum_{k=1}^\infty e^{-r(\tau_{2k-1}-t)} (P_{j,\tau_{2k-1}} + c^B) \\ &\qquad\quad + \sum_{k=1}^\infty e^{-r(\tau_{2k}-t)} (P_{i,\tau_{2k}} - c^S) \end{aligned} \middle| \mathcal{F}_t \right]$$

where supremum is taken over over all sequences of finite stopping times $\{\tau_n\}$ such that $t = \tau_0 \leqslant \tau_1 < \tau_2 < \cdots \to \infty$.

As it will be shown in the sequel, the equilibria of this model are not unique. For example,

$$P_t^S = \begin{pmatrix} P_{1,t}^S \\ \\ P_{2,t}^S \end{pmatrix} = \begin{pmatrix} V_{2,t} - c^B \\ \\ V_{1,t} - c^B \end{pmatrix}$$

and

$$P_t^B = \begin{pmatrix} P_{1,t}^B \\ \\ P_{2,t}^B \end{pmatrix} = \begin{pmatrix} \mathbb{E}_1[\int_t^\infty e^{-rs} \mathrm{d}D_s \mid \mathcal{F}_t] - R_{1,t} + c^S \\ \\ \mathbb{E}_2[\int_t^\infty e^{-rs} \mathrm{d}D_s \mid \mathcal{F}_t] - R_{2,t} + c^S \end{pmatrix}$$

are two different equilibria price processes. For the price process P_t^S, we have

$$R_{i,t} = R_{i,t}^S \equiv 0, \qquad i = 1, 2,$$

which agrees with the outside option of the buyer (i.e. no trade). In this case, the optimization problem can be reduced to the problem of V. One natural interpretation is that sellers always have the bargaining power in such an equilibrium. Similarly, for the price process P_t^B, we have

$$V_{i,t} = V_{i,t}^B \equiv \mathbb{E}_i\Big[\int_t^\infty e^{-r(s-t)} \mathrm{d}D_s \,\Big|\, \mathcal{F}_t\Big], \qquad i = 1, 2,$$

which coincide with the outside option of the seller (i.e. no trade) and one interpretation is that buyers always have the bargaining power in such an equilibrium.

In these cases, the initial seller or buyer alone can actually make profit. We will investigate non-extreme cases.

Assumption 3.4. *We restrict the price process to the form*

$$P_t = \begin{pmatrix} p_1\big(V_{2,t} - c^B\big) + (1-p_1)\big(\mathbb{E}_1[\int_t^\infty e^{-rs}\mathrm{d}D_s \mid \mathcal{F}_t] - R_{1,t} + c^S\big) \\ p_2\big(V_{1,t} - c^B\big) + (1-p_2)\big(\mathbb{E}_2[\int_t^\infty e^{-rs}\mathrm{d}D_s \mid \mathcal{F}_t] - R_{2,t} + c^S\big) \end{pmatrix}$$

for some constants $0 \leqslant p_1, p_2 \leqslant 1$.

If $p_1 = p_2 = 1$ the price process agrees with P_t^S and if $p_1 = p_2 = 0$ the price process agrees with P_t^B. Therefore we interpret p_i as the degree of bargaining power of Investor i as a seller.

Remark 3.1. When $p_1 = p_2 = 1$, the model coincide with the original model of Scheinkman and Xiong (2003).

3.3 Resale and Buyback Options

Recall that the dividend process $\{D_t\}$ solves the stochastic differential equation $\mathrm{d}D_t = \Theta_t \mathrm{d}t + \sigma_D \mathrm{d}B_{D,t}$, in which the state variable Θ_t cannot be observed directly. In order to evaluate the dividend stream, the investors have to estimate Θ_t using the information $\mathcal{F}_t = \sigma(D_s, S_{1,s}, S_{2,s}; 0 \leqslant s \leqslant t)$.

Because each investor believes that one signal is more informative than the other, the investors estimate the state variable Θ at different values. Let $\widehat{\Theta}_{i,t}$ be the conditional expectation of Θ_t given $\mathcal{F}_t$ under $\mathbb{P}_i$, that is $\mathbb{E}_i[\Theta_t \mid \mathcal{F}_t]$. Applying the Kalman-Bucy filtering (more precisely the theorem of Fujisaki, Kallianpur, and Kunita (1972), cf. Rogers and Williams (2000, p. 325)), we find that $\{\widehat{\Theta}_{i,t}\}$ is again an Ornstein-Uhlenbeck process.

Lemma 3.1. *The process $\{\widehat{\Theta}_{i,t}\}$ is an Ornstein-Uhlenbeck process under $\mathbb{P}_i$ and it satisfies the SDE*

$$\widehat{\Theta}_{i,t} = \widehat{\Theta}_{i,0} + \lambda \int_0^t \big(\mu - \widehat{\Theta}_{i,t}\big)\mathrm{d}t + \widehat{\sigma}_\Theta N^i_{\Theta,t},$$

where $\widehat{\sigma}_\Theta > 0$ is a constant given by

$$\widehat{\sigma}_\Theta = \sqrt{\frac{\gamma^2}{\sigma_D^2} + \Big(\frac{\gamma}{\sigma_S} + \sigma_\Theta\phi\Big)^2 + \frac{\gamma^2}{\sigma_S^2}}.$$

Here γ is the unique positive solution of the quadratic equation

$$(1-\phi^2)\sigma_\Theta^2 - 2\Big(\lambda + \frac{\sigma_\Theta\phi}{\sigma_S}\Big)\gamma - \Big(\frac{1}{\sigma_D^2} + \frac{2}{\sigma_S^2}\Big)\gamma^2 = 0,$$

and $N^i_{\Theta,t}$ is a $(\mathbb{P}_i, \{\mathcal{F}_t\})$-Brownian motion given by

$$\mathrm{d}N^i_{\Theta,t} = \frac{1}{\widehat{\sigma}_\Theta}\frac{\gamma}{\sigma_D}\mathrm{d}N^i_{D,t} + \frac{1}{\widehat{\sigma}_\Theta}\Big(\frac{\gamma}{\sigma_S} + \sigma_\Theta\phi\Big)\mathrm{d}N^i_{i,t} + \frac{1}{\widehat{\sigma}_\Theta}\frac{\gamma}{\sigma_S}\mathrm{d}N^i_{j,t}$$

with three orthogonal $(\mathbb{P}_i, \{\mathcal{F}_t\})$-Brownian motions

$$\mathrm{d}N^i_{D,t} = \frac{1}{\sigma_D}\big(\mathrm{d}D_t - \widehat{\Theta}_{i,t}\mathrm{d}t\big),$$
$$\mathrm{d}N^i_{i,t} = \frac{1}{\sigma_S}\big(\mathrm{d}S_{i,t} - \widehat{\Theta}_{i,t}\mathrm{d}t\big),$$

$$\mathrm{d}N^i_{j,t} = \frac{1}{\sigma_S}\big(\mathrm{d}S_{j,t} - \widehat{\Theta}_{i,t}\mathrm{d}t\big).$$

Proof. See Appendix. □

Now let us denote by $F_{i,t}$ the valuation of the dividend for i, that is

$$F_{i,t} := \mathbb{E}_i\Big[\int_t^\infty e^{-r(s-t)}\mathrm{d}D_s \mid \mathcal{F}_t\Big] = \frac{\widehat{\Theta}_{i,t}}{\lambda + r} + \Big(\frac{1}{r} - \frac{1}{\lambda + r}\Big)\mu.$$

Then the value function $V_{i,t}$ is rewritten as

$$V_{i,t} = \sup \mathbb{E}_i \left[\begin{array}{c|c} \displaystyle\sum_{k=0}^{\infty}(-1)^n e^{-r(\tau_n - t)} F_{i,\tau_n} & \\ \displaystyle + \sum_{k=0}^{\infty} e^{-r(\tau_{2k+1}-t)}(P_{i,\tau_{2k+1}} - c^S) & \mathcal{F}_t \\ \displaystyle - \sum_{k=1}^{\infty} e^{-r(\tau_{2k}-t)}(P_{j,\tau_{2k}} + c^B) & \end{array} \right] =: F_{i,t} + Q_{i,t},$$

where $Q_{i,t}$ is

$$Q_{i,t} = \sup \mathbb{E}_i \left[\begin{array}{c|c} \displaystyle\sum_{k=0}^{\infty} e^{-r(\tau_{2k+1}-t)}(P_{i,\tau_{2k+1}} - F_{i,\tau_n} - c^S) & \\ \displaystyle - \sum_{k=1}^{\infty} e^{-r(\tau_{2k}-t)}(P_{j,\tau_{2k}} - F_{i,\tau_n} + c^B) & \mathcal{F}_t \end{array} \right].$$

Therefore, the price process can be decomposed into

$$P_{i,t} = \underbrace{p_i\big(F_{j,t} - c^B\big) + (1-p_i)\big(F_{i,t} + c^S\big)}_{\text{Dividend Value}} + \underbrace{p_i Q_{j,t} - (1-p_i)R_{i,t}.}_{\text{Resale and Buyback Options Value}} \tag{3.1}$$

The first term represents the effect of the buyer's valuation of the future dividends on the price. The second term, similarly, represents the effect of the seller's valuation on the price. The first and second terms represent the effect of the buyer's valuation and the seller's valuation of the dividend itself on the price. If the price purely reflects the investors' valuation of the dividend process (without speculation), the price should equal to sum of them. Therefore we call these two terms the dividend value part of the price. The third term represents the additional expected payoff for the buyer which arises because the buyer anticipates profit from selling it in the future. We call this term the resale option value part of the price. Similarly, the last term is the additional expected payoff for the seller and we call it the buyback option value part.

3.4 Reduced Form Formulation

In order to solve the problem, we transform the optimal multiple stopping problem $Q_{i,t}$ and $R_{i,t}$ into reduced forms.

Let us assume that the initial owner o of the risky asset has determined $\tau_{o,1}$—the time to sell it to the other investor—, then o's next problem is to find the sequence of finite stopping times $\tau_{o,2} < \tau_{o,3} < \cdots$ which maximizes his expected payoff starting without the risky asset. It means that the optimal multiple stopping problem $Q_{i,t}$ can be reduced to the ordinary optimal stopping problem

$$Q_{i,t} = \sup_{\tau_1} \mathbb{E}_i \left[e^{-r(\tau_1 - t)} \left(P_{i,\tau_1} - F_{i,\tau_1} - c^S + R_{i,\tau_1} \right) \right].$$

Furthermore, substituting $P_{i,t}$ into the equation, we obtain

$$Q_{i,t} = \sup_{\tau \geqslant t} \mathbb{E}_i \left[e^{-r(\tau - t)} \left[p_i \Big(\frac{X_{i,\tau}}{\lambda + r} + R_{i,\tau} + Q_{j,\tau} - c \Big) \right] \,\middle|\, \mathcal{F}_t \right],$$

where $X_{i,t} = \widehat{\Theta}_{j,t} - \widehat{\Theta}_{i,t}$ and $c = c^B + c^S$. Applying a similar argument for the initial non-holder, we obtain

$$R_{i,t} = \sup_{\tau \geqslant t} \mathbb{E}_i \left[e^{-r(\tau - t)} \left[(1 - p_j) \Big(-\frac{X_{i,\tau}}{\lambda + r} + Q_{i,\tau} + R_{j,\tau} - c \Big) \right] \,\middle|\, \mathcal{F}_t \right].$$

We call these expressions for $Q_{i,t}$ and $R_{i,t}$ reduced forms.

Note that $X_{j,t} = -X_{i,t}$, and $Q_{i,t}$ and $R_{i,t}$ depend only on $X_{i,t}$. This leads us to expect that there exist deterministic functions $Q_i : \mathbb{R} \to \mathbb{R}$ and $R_i : \mathbb{R} \to \mathbb{R}$ such that $Q_{i,t} = Q_i(X_{i,t})$ and $R_{i,t} = R_i(X_{i,t})$, by the usual argument for optimal stopping problems.

When Investor j places higher value on the risky asset than i does—that is higher X_i—, Investor i would expect higher profit from speculative trades. Conversely, when X_i is very low, Investor i would not sell the asset soon, and $Q_i(X_i)$ gets closer to 0. Thus the conditions for the function Q_i should be

$$Q_i : \mathbb{R} \to \mathbb{R} \text{ is increasing} \quad \text{and} \quad Q_i(-\infty) = 0.$$

Similarly appropriate R_i should satisfy

$$R_i : \mathbb{R} \to \mathbb{R} \text{ is decreasing} \quad \text{and} \quad R_i(\infty) = 0.$$

Lemma 3.2. *The difference of beliefs $X_{i,t} = \widehat{\Theta}_{j,t} - \widehat{\Theta}_{i,t}$ is the solution of the SDE*

$$\mathrm{d}X_{i,t} = -\rho X_{i,t}\mathrm{d}t + \sigma_X \mathrm{d}N^i_{X,t}$$

with $X_{i,0} = \widehat{\Theta}_{j,0} - \widehat{\Theta}_{i,0}$, where

$$\rho := \lambda + \frac{\gamma}{\sigma_D^2} + \frac{2\gamma + \sigma_S\sigma_\Theta\phi}{\sigma_S^2} \quad \text{and} \quad \sigma_X = \sqrt{2}\sigma_\Theta\phi$$

are positive constants and $N^i_{X,t} = (N^i_{j,t} - N^i_{i,t})/\sqrt{2}$ is a $(\mathbb{P}_i, \mathcal{F}_t)$-Brownian motion. Furthermore, the generator of X_1 and X_2 are identical and is given by

$$\mathbb{L} := -\rho x \frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\sigma_X^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}.$$

This leads us to consider the free-boundary problem for Q_i and R_i:

$$(\mathbb{L} - r)Q_i(x) = 0, \quad \text{for} \quad x < \overline{x}_i, \tag{3.2}$$

$$(\mathbb{L} - r)R_i(x) = 0, \quad \text{for} \quad \underline{x}_i < x, \tag{3.3}$$

$$Q_i(x) = p_i\Big(\frac{x}{\lambda + r} + Q_j(-x) + R_i(x) - c\Big), \quad \text{for} \quad \overline{x}_i \leqslant x, \tag{3.4}$$

$$R_i(x) = (1 - p_j)\Big(-\frac{x}{\lambda + r} + Q_i(x) + R_j(-x) - c\Big), \quad \text{for} \quad x \leqslant \underline{x}_i, \tag{3.5}$$

for some constants $\overline{x}_i$ and $\underline{x}_i$, $j \neq i = 1, 2$. The optimal strategy for i is given by $\tau_0 \equiv 0$, $\tau_{2k+1} = \inf\{t > \tau_{2k} : X_{i,t} > \overline{x}_i\}$ and $\tau_{2k} = \inf\{t > \tau_{2k-1} : X_{i,t} < \underline{x}_i\}$ if i initially holds the asset, and otherwise the optimal strategy is $\tau_0 \equiv 0$, $\tau_{2k+1} = \inf\{t > \tau_{2k} : X_{i,t} < \underline{x}_i\}$ and $\tau_{2k} = \inf\{t > \tau_{2k-1} : X_{i,t} > \overline{x}_i\}$. Therefore in order the market to clear, we must have

$$\overline{x}_1 = -\underline{x}_2 \quad \text{and} \quad \overline{x}_2 = -\underline{x}_1.$$

Furthermore, our solution Q_i must be monotonically increasing and satisfy the boundary condition $Q_i(x) \downarrow 0$ as $x \downarrow -\infty$, and R_i must be monotonically decreasing and $R_i(x) \downarrow 0$ as $x \uparrow \infty$. Fortunately, two symmetric C^∞ general solutions for the ODEs satisfy such conditions. Let us denote them by f and g:

$$f(x) = \begin{cases} U\Big(\frac{r}{2\rho}; \frac{1}{2}; \frac{\rho}{\sigma_X^2} x^2\Big), & \text{for} \quad x \leqslant 0, \\ \frac{2\pi}{\Gamma\big(\frac{1}{2} + \frac{r}{2\rho}\big)\Gamma\big(\frac{1}{2}\big)} {}_1F_1\Big(\frac{r}{2\rho}; \frac{1}{2}; \frac{\rho}{\sigma_X^2} x^2\Big) - U\Big(\frac{r}{2\rho}; \frac{1}{2}; \frac{\rho}{\sigma_X^2} x^2\Big), & \text{for} \quad x > 0, \end{cases}$$

and $g(x) = f(-x)$ such that

$$Q_i(x) = \kappa_i^Q f(x) \qquad \text{for} \quad x < \overline{x}_i, \quad i = 1, 2,$$

and

$$R_i(x) = \kappa_i^R g(x) \qquad \text{for} \quad \underline{x}_i < x, \quad i = 1, 2,$$

for some $\kappa_i^Q, \kappa_i^R \geqslant 0$. Here Γ is the gamma function, ${}_1F_1$ is the Kummer confluent hypergeometric function (of the first kind), and U is the confluent hypergeometric function of the second kind, which are explained in Appendix.

3.5 Main Results

3.5.1 Equilibrium price process

Recall the price process $P_{i,t}$ is decomposed into dividend value part and resale/buyback option value part. The option values are the solution of the optimization problem of X which is the difference of the estimations of the fundamental part of the dividend process between investors.

Theorem 3.1. *The equilibrium price process $P_{i,t}$ is given by*

$$P_{i,t} = p_i(F_{j,t} - c^B) + (1 - p_i)(F_{i,t} + c^S) + p_i Q_j(-X_{i,t}) - (1 - p_i)R_i(X_{i,t}),$$

where functions Q_i and R_i are given by

$$Q_i(x) = \begin{cases} p_i \kappa^* f(x), & \textit{for} \quad x < x^*, \\ p_i\Big(\dfrac{x}{\lambda + r} + \kappa^* f(-x) - c\Big), & \textit{for} \quad x^* \leqslant x, \end{cases} \tag{3.6}$$

and

$$R_i(x) = \begin{cases} (1-p_j)\Big(-\dfrac{x}{\lambda+r} + \kappa^* f(x) - c\Big), & \text{for} \quad x \leqslant -x^*, \\ (1-p_j)\kappa^* f(-x), & \text{for} \quad -x^* < x, \end{cases} \tag{3.7}$$

for $j \neq i = 1, 2$. Two positive constants x^* and κ^* are unique solutions for

$$\begin{cases} \kappa f(x) = \Big(\dfrac{x}{\lambda+r} + \kappa f(-x) - c\Big), \\ \kappa f'(x) = \Big(\dfrac{1}{\lambda+r} - \kappa f'(-x)\Big). \end{cases}$$

Furthermore, the optimal strategy for the initial owner o and Investor $\widehat{o} \neq o$ are given by

$$\begin{aligned} \tau_{o,2k+1} &= \inf\{t > \tau_{2k} : X_{o,t} > x^*\} = \tau_{\widehat{o},2k+1} \\ \tau_{o,2k} &= \inf\{t > \tau_{2k-1} : X_{o,t} < -x^*\} = \tau_{\widehat{o},2k} \end{aligned} \tag{3.8}$$

with $\tau_{o,0} = \tau_{\widehat{o},0} \equiv 0$.

Proof. See Appendix. □

Corollary 3.1. *When Investor i has the asset at t, the price is distorted by $((p_i + p_j) - 1)\kappa^* f(-X_{i,t})$ due to the speculation.*

Proof. See Appendix. □

Here note that the investors' performances (expected payoffs) are determined by the bargaining power as seller p_o of initial holder alone:

$$V_o(\{\tau^*_{o,n}\}) = F_{o,0} + Q_o(X_{o,0}) \quad \text{and} \quad R_{\hat{o}}(\{\tau_{\hat{o},n}\}) = R_{\hat{o}}(X_{\hat{o},0}),$$

for initial holder o and initial non-holder $\hat{o}$ respectively. The option value part of price process, on the other hand, depends on both p_o and $p_{\hat{o}}$ because the resale and buyback options involve the bargaining power in the future. Thus speculation on future trades can distort the price without affecting the expected payoffs (and trading timings). This feature of speculative trades cannot be observed unless one assumes the positive expected payoff for buyers.

Note also that the speculation value can appear both positively and negatively according to the sign of $(p_i + p_j - 1)$. If both investors have bargaining power as sellers, that is the case $p_i = p_j = 1$, as studied in Scheinkman and Xiong, the price will always go up due to speculation.

It may seem counterintuitive that the speculation value in the price vanishes if one investor has bargaining power as *both* a seller and a buyer. It can be explained as follows: If the current holder has complete bargaining power as both seller and buyer, the price will be equal to the non-holder's reservation price. However, it equals to the dividend valuation by the non-holder because he has no bargaining power as seller. Thus the price equals to the buyer's valuation of the dividend and there is no distortion in price due to speculation. It suggests that the distortion in the price is formed by the speculation on the other investor's speculation, not his valuation of dividend.

3.5.2 Additional volatility in the price process

Because we have the explicit form of the price process, we can analyze the effect of the speculation on its quadratic variation (volatility). Recall that we decomposed price process into dividend value and resale/buyback option value as in eq. (3.1). We define the quadratic variation with no speculation as the quadratic variation of the dividend value and the additional quadratic variation as

$$\mathrm{d}[P,P]_t - \mathrm{d}[p_i(F_j - c^B) + (1-p_i)(F_i + c^S), p_i(F_j - c^B) + (1-p_i)(F_i + c^S)]_t,$$

where $[X,Y]$. denotes the quadratic covariation of two stochastic processes X and Y. Because the resale option and buyback option are the parts of solution of the free-boundary problem, they follow different functions according to the value of X_i: $X_i \leqslant -x^*$ or $-x^* < X_i$. Although the function differs, the smooth fit condition assures that the additional quadratic variation is continuous as a function of X_i, unless the stock holder changes.

Proposition 3.1. *The derivative* $\mathrm{d}[P_i,P_i]_t/\mathrm{d}t$ *of quadratic variation process of the price process increases by*

$$2\phi^2\sigma_\Theta^2\Big(\big((p_i+p_j)-1\big)\kappa^* f'(-X_{i,t})\Big)\Big(\frac{1-2p_i}{\lambda+r} + \big((p_i+p_j)-1\big)\kappa^* f'(-X_{i,t})\Big)$$

when $X_{i,t} > -x^*$ *and*

$$2\phi^2\sigma_\Theta^2(p_i + p_j - 1) \\ \times \left(\frac{1}{\lambda + r} - \kappa^* f'(X_{i,t})\right)\left(\frac{p_j - p_i}{\lambda + r} - (p_i + p_j - 1)\kappa^* f'(X_{i,t})\right),$$

when $X_{i,t} \leqslant -x^*$*, due to the speculative trades.*

Proof. See Appendix. □

The quadratic variation may both increase and decrease due to speculation. If the sellers have bargaining power (that is the case of $p_o = p_{\hat{o}} = 0.9$ in Fig. 3.3), the price is close to $(F_{\hat{o}} - c^B) + Q_{\hat{o}}(X_{\hat{o}})$ where o and $\hat{o}$ denote the holder and the non-holder respectively. Because $X_{\hat{o}} = (F_o - F_{\hat{o}})/(\lambda + r)$ and $x \mapsto Q_{\hat{o}}(x)$ is increasing, $F_{\hat{o}}$ (the dividend value for the non-holder) and $Q_{\hat{o}}(X_{\hat{o}})$ (the resale option value for him) are negatively correlated. This decreases the quadratic variation in total. For the similar reason, if the buyers have bargaining power, speculative trades will decrease the quadratic variation.

If the sellers are gaining bargaining power (that is the case of $p_o = 0.6$ and $p_{\hat{o}} = 0.9$ in Fig. 3.3), $p_o Q_{\hat{o}}(X_{\hat{o}})$ (the buyer's buyback option) dominates the resale/buyback option value in the price. The dividend value part, on the other hand, has two non-negligible components $p_o F_{\hat{o}}$ and $(1 - p_o)F_o$ (the dividend value for non-holder and for holder respectively). Therefore the

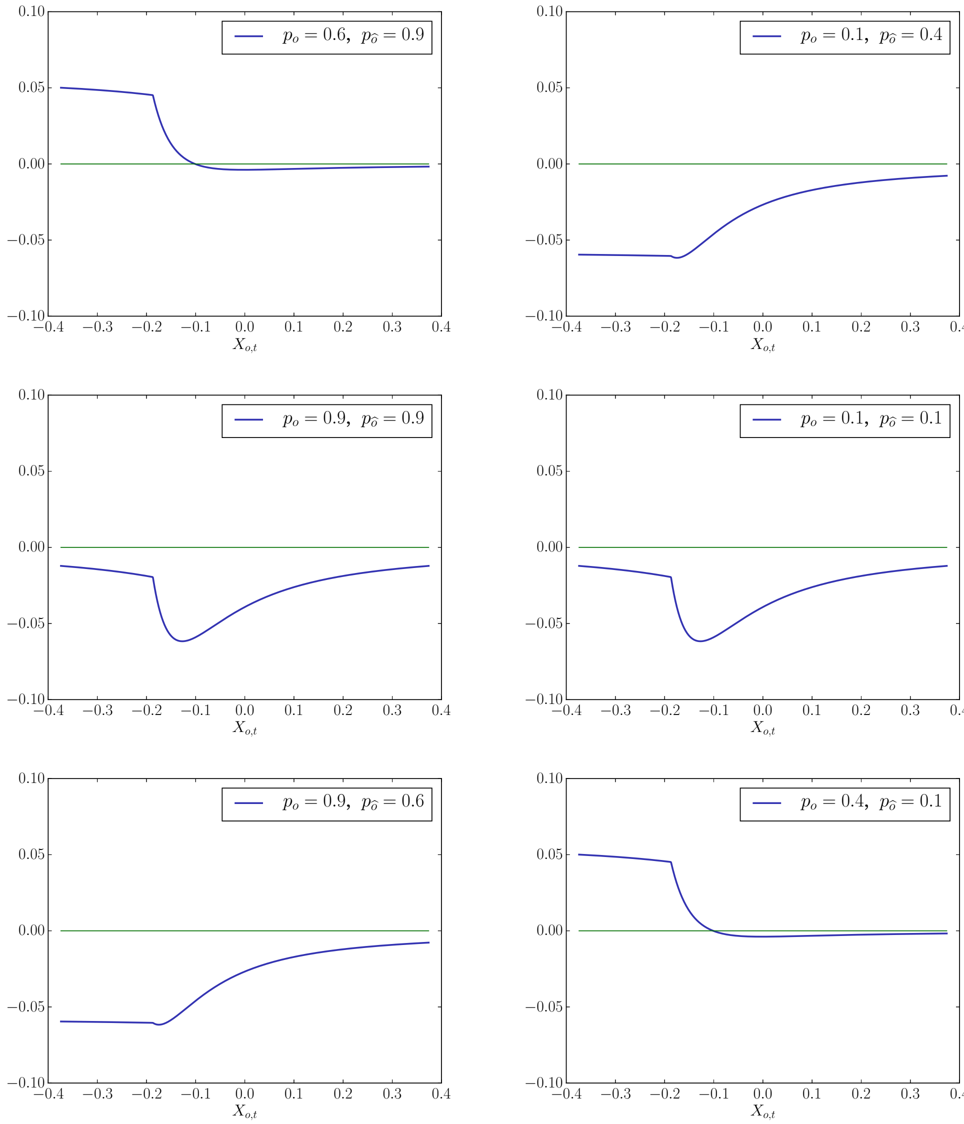

Figure 3.3: Additional quadratic variations $QV(X_{o,t})$.
$r = 0.01$, $\lambda = 0.4$, $\mu = 1.0$, $\sigma_\Theta = 0.3$, $\sigma_S = 0.3$, $\sigma_D = 0.3$ $\phi = 0.6$, $c^B = c^S = 0.1$

speculative trades can increase the quadratic variation in total because of the positive correlation between $p_o Q_{\hat{o}}(X_{\hat{o}})$ and $(1 - p_o)F_o$. Similarly, the quadratic variation decreases if the buyers are gaining bargaining power.

3.6 Discussion

Bargaining power

Unfortunately, our model seems to bring little progress on the results of Scheinkman and Xiong (2003), mathematically. Major reason seems to be due to the definition of bargaining powers, which are exogenously given constants. A development could be made by improving it applying arguments of, for example, search theory. Lester, Shourideh, Venkateswaran, and Zetlin-Jones (2019) focus on how many buyers an seller is offered by, from perspective of search theory, because it determines the outside option of the seller. One might interpret it as a degree of bargaining power of sellers.

Heterogeneous belief

In this chapter, we have assumed that the investors have different beliefs without explanation for the reason. Brunnermeier and Parker (2005) could provide us a possible explanation on it. They show that the investors can have heterogeneous beliefs endogenously, assuming that investors maximize expected time-averaged indirect utility (, which reminds the author so called Pascal's Wager). Cursed (expectations) equilibrium of Eyster and Rabin (2005) and Eyster et al. (2018) (together with Brunnermeier and Parker

(2005)) also could provide some insights.

3.7 Conclusion

This chapter explores the effects of the bargaining power on the speculative trades. If buyers also have bargaining power, the stock price consists of the dividend valuation, resale option, and buyback option. The correlation between dividend valuation and option values determines the additional volatility of the price process.

3.A Appendix

3.A.1 Proof of Lemma 3.1

Proof of Lemma 3.1. For notational convenience, let us denote

$$\widehat{\Theta}_{i,t} = \mathbb{E}[\Theta_t \mid \mathcal{F}_t], \quad \widehat{\Theta^n_{i,t}} = \mathbb{E}[\Theta^n_t \mid \mathcal{F}_t] \quad \text{and} \quad v_{i,t} = \mathrm{Var_i}(\Theta_t \mid \mathcal{F}_t).$$

Because $(\Theta_t, D_t/\sigma_D, S_{i,t}/\sigma_S, S_{j,t}/\sigma_S)$ is a Gaussian, the conditional expectation $\widehat{\Theta}_{i,t} = \mathbb{E}_i[\Theta_t \mid \mathcal{F}_t]$ has normal distribution and thus $\widehat{\Theta^3_{i,t}} = \widehat{\Theta}_{i,t}((\widehat{\Theta}_{i,t})^2 +$

$3v_{i,t}$). By Fujisaki, Kallianpur and Kunita's theorem, we obtain

$$\widehat{\Theta}_{i,t} = \Theta_0 + \int_0^t \lambda\big(\mu - \widehat{\Theta}_{i,t}\big)\,\mathrm{d}t + \int_0^t \begin{pmatrix} v_{i,t}/\sigma_D \\ v_{i,t}/\sigma_S + \sigma_\Theta \phi \\ v_{i,t}/\sigma_S \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}N^i_{D,t} \\ \mathrm{d}N^i_{i,t} \\ \mathrm{d}N^i_{j,t} \end{pmatrix},$$

and

$$\widehat{\Theta^2_{i,t}} = \Theta_0^2 + \int_0^t \Big(\sigma_\Theta^2 + 2\lambda\big(\mu\widehat{\Theta}_{i,t} - \widehat{\Theta^2_{i,t}}\big)\Big)\mathrm{d}t + \int_0^t \begin{pmatrix} 2v_{i,t}\widehat{\Theta}_{i,t}/\sigma_D \\ 2v_{i,t}\widehat{\Theta}_{i,t}/\sigma_S + 2\widehat{\Theta}_{i,t}\sigma_\Theta\phi \\ 2v_{i,t}\widehat{\Theta}_{i,t}/\sigma_S \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}N^i_{D,t} \\ \mathrm{d}N^i_{i,t} \\ \mathrm{d}N^i_{j,t} \end{pmatrix},$$

where each $\{N^i_{\cdot,t}\}$ is the innovation process defined in the statement of the lemma. Now we calculate $v_{i,t}$,

$$\begin{aligned} \mathrm{d}v_{i,t} &= \mathrm{d}(\widehat{\Theta^2_{i,t}} - (\widehat{\Theta}_{i,t})^2) = \mathrm{d}\widehat{\Theta^2_{i,t}} - 2\widehat{\Theta}_{i,t}\mathrm{d}\widehat{\Theta}_{i,t} - \mathrm{d}[\widehat{\Theta}_i, \widehat{\Theta}_i]_t \\ &= \left[(1-\phi^2)\sigma_\Theta^2 - 2\Big(\lambda + \frac{\sigma_\Theta\phi}{\sigma_S}\Big)v_{i,t} - \Big(\frac{1}{\sigma_D^2} + \frac{2}{\sigma_S^2}\Big)v_{i,t}^2\right]\mathrm{d}t \end{aligned}$$

By the assumption of variance of Θ_0, we conclude $v_{i,t} \equiv \gamma$ and the result follows. □

3.A.2 Proof of Theorem 3.1

Recall that the solutions of the ordinary differential equations (3.2) and (3.3) are $Q_i(x) = \kappa_i^Q f(x)$ and $R_i(x) = \kappa_i^R g(x)$, where

$$f(x) = \begin{cases} U\Big(\frac{r}{2\rho}; \frac{1}{2}; \frac{\rho}{\sigma_X^2} x^2\Big), & \text{for} \quad x \leqslant 0, \\ \frac{2\pi}{\Gamma\big(\frac{1}{2} + \frac{r}{2\rho}\big)\Gamma\big(\frac{1}{2}\big)} {}_1F_1\Big(\frac{r}{2\rho}; \frac{1}{2}; \frac{\rho}{\sigma_X^2} x^2\Big) - U\Big(\frac{r}{2\rho}; \frac{1}{2}; \frac{\rho}{\sigma_X^2} x^2\Big), & \text{for} \quad x > 0, \end{cases}$$

and $g(x) = f(-x)$. Here, ${}_1F_1$ is the Kummer confluent hypergeometric function (of the first kind)

$${}_1F_1(a;\, b;\, z) = 1 + \frac{a}{b} z + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}$$

and U is the confluent hypergeometric function of the second kind

$$U(a;\, b;\, z) = \frac{\pi}{\sin \pi b} \Big[\frac{{}_1F_1(a;\, b;\, z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b} \frac{{}_1F_1(1+a-b;\, 2-b;\, z)}{\Gamma(a)\Gamma(2-b)} \Big].$$

In the equations above, Γ is the gamma function and for nonnegative integer n, $(a)_n$ is

$$(a)_n := \begin{cases} a(a+1)\cdots(a+n-1), & \text{for} \quad n = 1, 2, \ldots, \\ 1, & \text{for} \quad n = 0. \end{cases}$$

Our first problem is to find $(\kappa_1^Q, \kappa_2^Q, \kappa_1^R, \kappa_2^R, \overline{x}_1, \overline{x}_2, p_1, p_2)$ which satisfies continuous fit condition

$$\kappa_1^Q f(\overline{x}_1) = p_1\Big(\frac{\overline{x}_1}{\lambda + r} + \kappa_2^Q f(-\overline{x}_1) + \kappa_1^R g(\overline{x}_1) - c\Big), \tag{3.9}$$

$$\kappa_2^Q f(\overline{x}_2) = p_2\Big(\frac{\overline{x}_2}{\lambda + r} + \kappa_1^Q f(-\overline{x}_2) + \kappa_2^R g(\overline{x}_2) - c\Big), \tag{3.10}$$

$$\kappa_1^R g(\underline{x}_1) = (1 - p_2)\Big(- \frac{\underline{x}_1}{\lambda + r} + \kappa_1^Q f(\underline{x}_1) + \kappa_2^R g(-\underline{x}_1) - c\Big), \tag{3.11}$$

$$\kappa_2^R g(\underline{x}_2) = (1 - p_1)\Big(- \frac{\underline{x}_2}{\lambda + r} + \kappa_2^Q f(\underline{x}_2) + \kappa_1^R g(-\underline{x}_2) - c\Big) \tag{3.12}$$

and smooth fit condition

$$\kappa_1^Q f'(\overline{x}_1) = p_1\Big(\frac{1}{\lambda + r} - \kappa_2^Q f'(-\overline{x}_1) + \kappa_1^R g'(\overline{x}_1)\Big), \tag{3.13}$$

$$\kappa_2^Q f'(\overline{x}_2) = p_2\Big(\frac{1}{\lambda + r} - \kappa_1^Q f'(-\overline{x}_2) + \kappa_2^R g'(\overline{x}_2)\Big), \tag{3.14}$$

$$\kappa_1^R g'(\underline{x}_1) = (1 - p_2)\Big(- \frac{1}{\lambda + r} + \kappa_1^Q f'(\underline{x}_1) - \kappa_2^R g'(-\underline{x}_1)\Big), \tag{3.15}$$

$$\kappa_2^R g'(\underline{x}_2) = (1 - p_1)\Big(- \frac{1}{\lambda + r} + \kappa_2^Q f'(\underline{x}_2) - \kappa_1^R g'(-\underline{x}_2)\Big), \tag{3.16}$$

where $\underline{x}_2 = -\overline{x}_1$ and $\underline{x}_1 = -\overline{x}_2$ (the market clearing condition).

If $p_i = 1$, we must have $\kappa_j^R = 0$; and if $p_i = 0$, we must have $\kappa_i^Q = 0$. In these cases, the problem can be solved as in Scheinkman and Xiong (2003). Let us assume $0 < p_1, p_2 < 1$. The market clearing condition $\overline{x}_1 = -\underline{x}_2$ implies

$$- \frac{\underline{x}_2}{\lambda + r} + \kappa_2^Q f(\underline{x}_2) + \kappa_1^R g(-\underline{x}_2) - c = \frac{\overline{x}_1}{\lambda + r} + \kappa_2^Q f(-\overline{x}_1) + \kappa_1^R g(\overline{x}_1) - c,$$

and thus we obtain

$$\frac{\kappa_1^Q}{p_1} f(\overline{x}_1) = \frac{\kappa_2^R}{1-p_1} g(\underline{x}_2) = \frac{\kappa_2^R}{1-p_1} g(-\overline{x}_1),$$

by (3.9) and (3.12). Because $f(x) = g(-x)$, we must have

$$(1-p_1)\kappa_1^Q = p_1 \kappa_2^R.$$

This implies (3.13) and (3.16) are equivalent. In fact, the left-hand-side of (3.16) is

$$\kappa_2^R g'(\underline{x}_2) = \frac{1-p_1}{p_1} \kappa_1^Q g'(-\overline{x}_1) = -\frac{1-p_1}{p_1} \kappa_1^Q f(\overline{x}_1),$$

and the right-hand-side is

$$(1-p_1)\Big(-\frac{1}{\lambda+r} + \kappa_2^Q f'(\underline{x}_2) - \kappa_1^R g'(-\underline{x}_2)\Big) = -(1-p_1)\Big(\frac{1}{\lambda+r} - \kappa_2^Q f'(-\overline{x}_1) + \kappa_1^R g'(\overline{x}_1)\Big).$$

Similarly, we obtain $(1-p_2)\kappa_2^Q = p_2 \kappa_1^R$ and the equivalence of (3.14) and (3.15). Therefore, the simultaneous equation (3.9)–(3.16) in (κ_1^Q, κ_2^Q, κ_1^R, κ_2^R, $\overline{x}_1$, $\overline{x}_2$, p_1, p_2) is reduced to the system

$$\kappa_1^Q f(\overline{x}_1) = p_1 \Big(\frac{\overline{x}_1}{\lambda+r} + \frac{\kappa_2^Q}{p_2} f(-\overline{x}_1) - c\Big), \tag{3.17}$$

$$\kappa_2^Q f(\overline{x}_2) = p_2 \Big(\frac{\overline{x}_2}{\lambda+r} + \frac{\kappa_1^Q}{p_1} f(-\overline{x}_2) - c\Big), \tag{3.18}$$

$$\kappa_1^Q f'(\overline{x}_1) = p_1\Big(\frac{1}{\lambda+r} - \frac{\kappa_2^Q}{p_2} f'(-\overline{x}_1)\Big), \tag{3.19}$$

$$\kappa_2^Q f'(\overline{x}_2) = p_2\Big(\frac{1}{\lambda+r} - \frac{\kappa_1^Q}{p_1} f'(-\overline{x}_2)\Big) \tag{3.20}$$

in $(\kappa_1^Q, \kappa_2^Q, \overline{x}_1, \overline{x}_2)$ for each given pair $0 < p_1, p_2 < 1$. The other parameters can be calculated by

$$\underline{x}_i = -\overline{x}_j \quad \text{and} \quad \kappa_i^R = \kappa_j^Q \frac{1-p_j}{p_j}.$$

In order to verify it, we need the following lemmas.

Lemma 3.3.

$$f(x) > 0, \quad f'(x) > 0, \quad f''(x) > 0, \quad f'''(x) > 0.$$

Proof. See Scheinkman and Xiong (2003). □

Lemma 3.4.

$$Q_i(x) > p_i\Big(\frac{x}{\lambda+r} + Q_j(-x) + R_i(x) - c\Big), \qquad \textit{for} \quad x < x^*,$$
$$R_i(x) > (1-p_j)\Big(-\frac{x}{\lambda+r} + Q_i(x) + R_j(-x) - c\Big), \qquad \textit{for} \quad -x^* < x.$$

Proof. We show the inequality for Q_i. Let us define a function U_i by

$$U_i(x) := Q_i(x) - p_i\Big(\frac{x}{\lambda+r} + Q_j(-x) + R_i(x) - c\Big).$$

Because Q_i and R_i are given by (3.6) and (3.7) respectively, U_i satisfies

$$U_i(x) = \begin{cases} 2p_i c, & \text{if} \quad x \leqslant -x^*, \\ p_i \kappa^* \big(f(x) - f(-x)\big) - p_i \dfrac{x}{\lambda + r} + p_i c, & \text{if} \quad -x^* < x < x^*, \\ 0, & \text{if} \quad x^* \leqslant x. \end{cases}$$

Note that U_i is continuous on $\mathbb{R}$, by the continuous fit condition for Q_i and R_i. It suffices to show that $U_i'(x) < 0$ on $(-x^*, x^*)$. Lemma 3.3 implies

$$f''(x) - f''(-x) < 0, \qquad \text{for} \quad x < 0,$$

and

$$f''(x) - f''(-x) > 0, \qquad \text{for} \quad x > 0.$$

By the smooth fit condition together with equations above, we obtain

$$U_i'(x) < 0, \qquad \text{for} \quad -x^* < x < x^*.$$

The other inequality can be proved by a similar argument. □

Lemma 3.5.

$$\big(\mathbb{L} - r\big) Q_i(x) < 0, \qquad \textit{for} \quad x > x^*,$$
$$\big(\mathbb{L} - r\big) R_i(x) < 0, \qquad \textit{for} \quad x < -x^*.$$

Proof. We show the inequality for Q_i. First, by the (first order) smooth fit condition and continuous fit condition at x^*,

$$
\begin{aligned}
&\big(\mathbb{L}-r\big)Q_i(x^*+) \\
&\quad= \Big(-\rho x\frac{d}{dx}+\frac{1}{2}\sigma_X^2\frac{d^2}{dx^2}-r\Big)Q_i(x^*+) \\
&\quad= \frac{1}{2}\sigma_X^2 p_i\,\frac{d^2}{dx^2}\Big(\frac{x}{\lambda+r}+\kappa^* f(-x)-c\Big)\Big|_{x=x^*+}-\frac{1}{2}\sigma_X^2 Q_i''(x^*-) \\
&\quad= \frac{1}{2}\sigma_X^2 p_i\kappa^* f''(-x^*)-\frac{1}{2}\sigma_X^2 p_i\kappa^* f''(x^*) \\
&\quad= \frac{1}{2}\sigma_X^2 p_i\kappa^*\big(f''(-x^*)-f''(x^*)\big)<0,
\end{aligned}
$$

because $x^*>0$. Now, we prove the claim. For all $x>x^*$,

$$
\begin{aligned}
&\big(\mathbb{L}-r\big)Q_i(x) \\
&\quad= \big(\mathbb{L}-r\big)\Big[p_i\Big(\frac{x}{\lambda+r}+\kappa^* f(-x)-c\Big)\Big] \\
&\quad= -\frac{\rho+r}{\lambda+r}x+\big(\mathbb{L}-r\big)f(-x)+cr \\
&\quad= -\frac{\rho+r}{\lambda+r}(x-x^*)-\frac{\rho+r}{\lambda+r}x^*+cr \\
&\quad= -\frac{\rho+r}{\lambda+r}(x-x^*)+\big(\mathbb{L}-r\big)Q_i(x^*+)<0,
\end{aligned}
$$

because $x>x^*$. The other inequality can be proved by a similar argument.

□

Proof of Theorem 3.1. For $N=1,2,\ldots$, let us define $Q_i^{(N)}(x)$ and $R_i^{(N)}(x)$

by

$$Q_i^{(N)}(x) = \sup \mathbb{E}_i \left[\begin{array}{l} \displaystyle\sum_{k;2k+1\leqslant N} e^{-r\tau_{2k+1}} \left[\begin{array}{r} p_i\Big(\dfrac{X_{i,\tau_{2k+1}}}{\lambda+r} + Q_j(-X_{i,\tau_{2k+1}}) - c\Big) \\ -(1-p_i)R_i(X_{i,\tau_{2k+1}}) \end{array} \right] \\ \displaystyle -\sum_{k;2k\leqslant N} e^{-r\tau_{2k}} \left[\begin{array}{r} (1-p_j)\Big(\dfrac{X_{i,\tau_{2k}}}{\lambda+r} - R_j(-X_{i,\tau_{2k}}) + c\Big) \\ +p_j Q_i(X_{i,\tau_{2k}}) \end{array} \right] \\ + e^{-r\tau_N}\big(\mathbf{1}_{N \text{ is odd}} R_i(X_{i,\tau_N}) + \mathbf{1}_{N \text{ is even}} Q_i(X_{i,\tau_N})\big) \end{array} \middle| \; X_{i,0} = x \right].$$

and

$$R_i^{(N)}(x) = \sup \mathbb{E}_i \left[\begin{array}{l} \displaystyle -\sum_{k;2k+1\leqslant N} e^{-r\tau_{2k+1}} \left[\begin{array}{r} (1-p_j)\Big(\dfrac{X_{i,\tau_{2k+1}}}{\lambda+r} - R_j(-X_{i,\tau_{2k+1}}) + c\Big) \\ +p_j Q_i(X_{i,\tau_{2k+1}}) \end{array} \right] \\ \displaystyle +\sum_{k;2k\leqslant N} e^{-r\tau_{2k}} \left[\begin{array}{r} p_i\Big(\dfrac{X_{i,\tau_{2k}}}{\lambda+r} + Q_j(-X_{i,\tau_{2k}} - c\Big) \\ -(1-p_i)R_i(X_{i,\tau_{2k}}) \end{array} \right] \\ + e^{-r\tau_N}\big(\mathbf{1}_{N \text{ is odd}} Q_i(X_{i,\tau_N}) + \mathbf{1}_{N \text{ is even}} R_i(X_{i,\tau_N})\big) \end{array} \middle| \; X_{i,0} = x \right],$$

where supremum is taken over all finite increasing stopping times $0 \leqslant \tau_1 < \tau_2 < \cdots < \tau_N (< \tau_{N+1} < \cdots)$. Here $Q_i^{(N)}$ (resp. $R_i^{(N)}$) is the value function of the stopping problem in which initial owner (resp. non-owner) i can choose only N stopping times with additional terminal boundary condition Q_i or R_i.

Step 1. We first show $Q_i^{(1)} \equiv Q_i$ and $R_i^{(1)} \equiv R_i$. For $N = 1$, $Q_i^{(N)}$ is reduced to

$$Q_i^{(1)}(x) = \sup_{\tau_1} \mathbb{E}_i \left[e^{-r\tau_1} \left[p_i\Big(\frac{X_{i,\tau_1}}{\lambda+r} + R_i(X_{i,\tau_1}) + Q_j(-X_{i,\tau_1}) - c\Big)\right] \middle| \; X_{i,0} = x \right].$$

We first show $Q_i^{(1)} \leqslant Q_i$. By the Itô formula,

$$\begin{aligned} e^{-rt} Q_i(X_i) = Q_i(x) &+ \int_0^t e^{-rs} \big(\mathbb{L} - r\big) Q_i(X_{i,s}) 1_{X_{i,s} \neq x^*} \mathrm{d}s \\ &+ \int_0^t e^{-rs} \sigma_X Q_i'(X_{i,s}) 1_{X_{i,s} \neq x^*} \mathrm{d}N_{X,s}^i \\ &+ \frac{1}{2} \int_0^t \Big(Q_i'(X_{i,s}+) - Q_i'(X_{i,s}-)\Big) 1_{X_s = x^*} \ell_s^{x*}(X_i), \end{aligned}$$

where $\ell_s^a(X_i)$ is the local time of X_i at $a \in \mathbb{R}$ that is given by

$$\ell_s^a(X_i) = \mathbb{P}\text{-}\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^\varepsilon 1_{a-\varepsilon < X_{i,s'} < a+\varepsilon} d[X_i, X_i]_{s'}.$$

However, the integration with respect to $\ell_s^{x*}(X)$ vanishes by the smoothness of Q_i, and thus

$$\begin{aligned} e^{-rt} Q_i(X_i) = Q_i(x) &+ \int_0^t e^{-rs} \big(\mathbb{L} - r\big) Q_i(X_{i,s}) 1_{X_{i,s} \neq x^*} \mathrm{d}s \\ &+ \int_0^t e^{-rs} \sigma_X Q_i'(X_{i,s}) 1_{X_{i,s} \neq x^*} \mathrm{d}N_{X,s}^i. \end{aligned}$$

Lemma 3.4, and Lemma 3.5 together with the defn of Q_i implies

$$e^{-rt} p_i \Big(\frac{X_{i,t}}{\lambda + r} + R_i(X_{i,t}) + Q_j(-X_{i,t}) - c\Big) \leqslant e^{-rt} Q_i(X_{i,t}) \leqslant Q_i(x) + M_{i,t}$$

where $M_i = \{M_{i,t}\}$ is a continuous $\mathbb{P}_i$-local martingale given by

$$M_{i,t} = \int_0^t e^{-rs} \sigma_X Q_i'(X_{i,s}) 1_{X_{i,s} \neq x*} \mathrm{d}N_{X,s}^i.$$

Let $\{\tau_i^n\}_{n=1,2,\ldots}$ be a localization sequence of stopping times for X_i. Then we have

$$e^{-r\tau\wedge\tau_i^n} p_i\Big(\frac{X_{i,\tau\wedge\tau_i^n}}{\lambda+r} + R_i(X_{i,\tau\wedge\tau_i^n}) + Q_j(-X_{i,\tau\wedge\tau_i^n}) - c\Big) \leqslant Q_i(x) + M_{i,\tau\wedge\tau_i^n} \quad (3.21)$$

for each finite $\{\mathcal{F}_t\}$-stopping time τ. Therefore we obtain

$$\mathbb{E}\Big[e^{-r\tau\wedge\tau_i^n} p_i\Big(\frac{X_{i,\tau\wedge\tau_i^n}}{\lambda+r} + R_i(X_{i,\tau\wedge\tau_i^n}) + Q_j(-X_{i,\tau\wedge\tau_i^n}) - c\Big)\Big] \leqslant Q_i(x)$$

by the optional sampling theorem. Furthermore by Fatou's lemma,

$$\mathbb{E}\Big[e^{-r\tau} p_i\Big(\frac{X_{i,\tau}}{\lambda+r} + R_i(X_{i,\tau}) + Q_j(-X_{i,\tau}) - c\Big)\Big] \leqslant Q_i(x).$$

Because τ is an arbitrary finite stopping time,

$$\sup_{\tau} \mathbb{E}\Big[e^{-r\tau} p_i\Big(\frac{X_{i,\tau}}{\lambda+r} + R_i(X_{i,\tau}) + Q_j(-X_{i,\tau}) - c\Big)\Big] \leqslant Q_i(x).$$

Next, we show the equality holds for $\tau = \tau_{i,1}$, where

$$\tau_{i,1} = \inf\big\{t > 0;\ X_{i,t} > x^*\big\}.$$

Note that

$$(\mathbb{L} - r)(X_{i,t}) = 0, \qquad \text{for} \quad t < \tau_i$$

and

$$p_i\Big(\frac{X_{i,\tau_{i,1}}}{\lambda+r}+R_i(X_{i,\tau_{i,1}})+Q_j(-X_{i,\tau_{i,1}})-c\Big)=Q_i(X_{\tau_{i,1}})$$

holds by the continuity of X_i and properties of Q_i and R_i. Therefore the inequality (3.21) holds with equality:

$$e^{-r\tau\wedge\tau_i^n}p_i\Big(\frac{X_{i,\tau\wedge\tau_i^n}}{\lambda+r}+R_i(X_{i,\tau\wedge\tau_i^n})+Q_j(-X_{i,\tau\wedge\tau_i^n})-c\Big)=Q_i(x)+M_{i,\tau\wedge\tau_i^n}.$$

Therefore

$$\mathbb{E}_i\Big[e^{-r\tau}p_i\Big(\frac{X_{i,\tau}}{\lambda+r}+R_i(X_{i,\tau})+Q_j(-X_{i,\tau})-c\Big)\Big]=Q_i(x).$$

by the optional sampling theorem and Fatou's lemma, again. Thus we have shown $Q_i^{(1)}=Q_i$. The other equality $R_i^{(1)}=R_i$ can be derived similarly.

Step 2. We will show that the strategy $\{\tau_{i,n}\}_{n=1,\dots N}$ in the theorem attains $Q_i^{(N)}$ (or $R_i^{(N)}$) and derive $Q_i^{(N)}\equiv Q_i$ and $R_i^{(N)}\equiv R_i$. For this we use induction. Let us assume that the claim holds for $N=1,\dots,M-1$, then $Q_i^{(M)}$ is reduced to

$$Q_i^{(M)}(x)=\sup_{\tau_1}\mathbb{E}_i\left[\left.\begin{array}{l} e^{-r\tau_1}\left[\begin{array}{r} p_i\Big(\dfrac{X_{i,\tau_1}}{\lambda+r}+Q_j(-X_{i,\tau_1})-c\Big)\\ +(1-p_i)R_i(X_{i,\tau_1})\end{array}\right]\\ +e^{-r\tau_1}R_i^{(M-1)}(X_{i,\tau_1})\end{array}\right| X_{i,0}=x\right].$$

by the strong Markov property of X_i. Because $R_i^{(M-1)}=R_i$, by assumption, we obtain

$$Q_i^{(M)}(x) = \sup_{\tau_1} \mathbb{E}_i \left[e^{-r\tau_1} \left[p_i \Big(\frac{X_{i,\tau_1}}{\lambda + r} + R_i(X_{i,\tau_1}) + Q_j(-X_{i,\tau_1}) - c \Big) \right] \,\middle|\, X_{i,0} = x \right].$$

By the result of step 1, we conclude $Q_i^{(M)} = Q_i$. Furthermore, the sequence of optimal stopping times are the first passage time of X_i for x^* (and $-x^*$), that is $\{\tau_{i,n}\}$. The other equality $R_i^{(M)} = R_i$ can be proved similarly.

Step 3. Now we prove the claim. Recall the strategy $\{\tau_{i,k}\}_{k=0,1,\ldots,N}$ defined by (3.8) attains $Q_i^{(N)} = Q_i$. Because Q_i and R_i are positive,

$$
\begin{aligned}
&Q_i(x) - \mathbb{E}_i\Big[e^{-r\tau_{i,N}}\big(1_{N \text{ is odd}} R_i(X_{i,\tau_{i,N}}) + 1_{N \text{ is even}} Q_i(X_{i,\tau_{i,N}})\big)\Big] \\
&= \mathbb{E}_i \left[\begin{array}{l}
\displaystyle\sum_{k;2k+1 \leqslant N} e^{-r\tau_{i,2k+1}} \left[\begin{array}{r} p_i\Big(\dfrac{X_{i,\tau_{i,2k+1}}}{\lambda + r} + Q_j(-X_{i,\tau_{i,2k+1}}) - c\Big) \\ - (1-p_i) R_i(X_{i,\tau_{i,2k+1}}) \end{array}\right] \\
\displaystyle - \sum_{k;2k \leqslant N} e^{-r\tau_{i,2k}} \left[\begin{array}{r} (1-p_j)\Big(\dfrac{X_{i,\tau_{i,i,2k}}}{\lambda + r} - R_j(-X_{i,\tau_{i,2k}}) + c\Big) \\ + p_j Q_i(X_{i,\tau_{i,2k}}) \end{array}\right]
\end{array}\right] \\
&\leqslant \sup \mathbb{E}_i \left[\begin{array}{l}
\displaystyle\sum_{k;2k+1 \leqslant N} e^{-r\tau_{2k+1}} \left[\begin{array}{r} p_i\Big(\dfrac{X_{i,\tau_{2k+1}}}{\lambda + r} + Q_j(-X_{i,\tau_{2k+1}}) - c\Big) \\ - (1-p_i) R_i(X_{i,\tau_{2k+1}}) \end{array}\right] \\
\displaystyle - \sum_{k;2k \leqslant N} e^{-r\tau_{2k}} \left[\begin{array}{r} (1-p_j)\Big(\dfrac{X_{i,\tau_{2k}}}{\lambda + r} - R_j(-X_{i,\tau_{2k}}) + c\Big) \\ + p_j Q_i(X_{i,\tau_{2k}}) \end{array}\right]
\end{array}\right] \\
&\leqslant \mathbb{E}_i \left[\begin{array}{l}
\displaystyle\sum_{k;2k+1 \leqslant N} e^{-r\tau_{i,2k+1}} \left[\begin{array}{r} p_i\Big(\dfrac{X_{i,\tau_{i,2k+1}}}{\lambda + r} + Q_j(-X_{i,\tau_{i,2k+1}}) - c\Big) \\ - (1-p_i) R_i(X_{i,\tau_{i,2k+1}}) \end{array}\right] \\
\displaystyle - \sum_{k;2k \leqslant N} e^{-r\tau_{i,2k}} \left[\begin{array}{r} (1-p_j)\Big(\dfrac{X_{i,\tau_{i,2k}}}{\lambda + r} - R_j(-X_{i,\tau_{i,2k}}) + c\Big) \\ + p_j Q_i(X_{i,\tau_{i,2k}}) \end{array}\right] \\
+ e^{-r\tau_{i,N}}\big(1_{N \text{ is odd}} R_i(X_{i,\tau_{i,N}}) + 1_{N \text{ is even}} Q_i(X_{i,\tau_{i,N}})\big)
\end{array}\right] \\
&= Q_i(x).
\end{aligned}
$$

Here note that for $\{\tau_{i,n}\}$, $Q_i(X_{i,\tau_{i,2k+1}}) = p_i \kappa^* f(x^*)$ and $R_i(X_{i,\tau_{i,2k}}) = (1 - p_j)\kappa^* f(-x^*)$. Because $\{\tau_{i,n+1} - \tau_{i,n}\}_{n=2,3,\dots}$ is an i.i.d. sequence of interarrival times of hitting times for Ornstein-Uhlenbeck process,

$$\mathbb{E}_i\big[e^{-\lambda\tau_N}\big] = \mathbb{E}_i\big[e^{-\lambda\{(\tau_N-\tau_{N-1})+\cdots+(\tau_3-\tau_2)+\tau_2\}}\big] = \mathbb{E}_i\big[e^{-\lambda(\tau_3-\tau_2)}\big]^{N-2}\mathbb{E}_i\big[e^{-\lambda\tau_2}\big],$$

and this converges to 0 as $N \to \infty$. Therefore

$$\begin{aligned}
&\mathbb{E}_i\Big[e^{-r\tau_{i,N}}\big(1_{N \text{ is odd}}R_i(X_{i,\tau_N}) + 1_{N \text{ is even}}Q_i(X_{i,\tau_N})\big)\Big] \\
&\quad \leqslant \mathbb{E}_i\big[e^{-\lambda(\tau_3-\tau_2)}\big]^{N-2}\mathbb{E}_i\big[e^{-\lambda\tau_2}\big]\kappa^* f(x^*)\max\{p_i, 1-p_j\} \to 0.
\end{aligned}$$

Finally by the squeeze theorem, we obtain

$$Q_i(x) = \lim_{N\to\infty}\sup\mathbb{E}_i\left[\begin{array}{l}
\displaystyle\sum_{k;2k+1\leqslant N} e^{-r\tau_{2k+1}}\left[\begin{array}{r} p_i\Big(\dfrac{X_{i,\tau_{2k+1}}}{\lambda+r} + Q_j(-X_{i,\tau_{2k+1}}) - c\Big) \\ -(1-p_i)R_i(X_{i,\tau_{2k+1}})\end{array}\right] \\
\displaystyle -\sum_{k;2k\leqslant N} e^{-r\tau_{2k}}\left[\begin{array}{r}(1-p_j)\Big(\dfrac{X_{i,\tau_{2k}}}{\lambda+r} - R_j(-X_{i,\tau_{2k}}) + c\Big) \\ + p_j Q_i(X_{i,\tau_{2k}})\end{array}\right]
\end{array}\right].$$

□

3.A.3 Proof of Proposition 3.1

Lemma 3.6.

$$\begin{aligned}
d[F_i, F_i]_t &= \Big(\frac{1}{\lambda+r}\Big)^2\left[\Big(\frac{\phi\sigma_S\sigma_\Theta+\gamma}{\sigma_S}\Big)^2 + \Big(\frac{\gamma}{\sigma_S}\Big)^2 + \Big(\frac{\gamma}{\sigma_D}\Big)^2\right]\mathrm{d}t, \\
d[F_i, X_i]_t &= -\frac{1}{\lambda+r}\phi^2\sigma_\Theta^2\mathrm{d}t, \\
d[X_i, X_i]_t &= 2\phi^2\sigma_\Theta^2\mathrm{d}t,
\end{aligned}$$

for $j \neq i = 1, 2$.

Proof. Recall

$$dF_{i,t} = -\frac{\lambda}{\lambda + r}\big(\widehat{\Theta}_{i,t} - \mu\big)\mathrm{d}t + \frac{1}{\lambda + r}\Big(\frac{\phi\sigma_S\sigma_\Theta + \gamma}{\sigma_S}\mathrm{d}N^i_{i,t} + \frac{\gamma}{\sigma_S}\mathrm{d}N^i_{j,t} + \frac{\gamma}{\sigma_D}\mathrm{d}N^i_{D,t}\Big),$$

and

$$\mathrm{d}X_{i,t} = -\rho X_{i,t}\mathrm{d}t + \phi\sigma_\Theta\big(\mathrm{d}N^i_{j,t} - \mathrm{d}N^i_{i,t}\big).$$

□

Proof of Proposition 3.1. Recall that for $X_{i,t} > -x^*$, the price process $P_{i,t}$ is given by

$$\begin{aligned} P_{i,t} &= \big(p_i(F_{j,t} - c^B) + (1 - p_i)(F_{i,t} + c^S)\big) + \big(p_i + p_j - 1\big)\kappa^* f(-X_{i,t}) \\ &= \Big(F_{i,t} + p_i\frac{X_{i,t}}{\lambda + r}\Big) + \big(p_i + p_j - 1\big)\kappa^* f(-X_{i,t}) + \mathit{constant.} \end{aligned}$$

Therefore the additional quadratic variation in the price process is

$$\begin{aligned} &\frac{1}{\mathrm{d}t}\begin{pmatrix} 2\,d\Big[F_i + p_i\dfrac{X_i}{\lambda + r}, \big((p_i + p_j) - 1\big)\kappa^* f(-X_i)\Big]_t \\ \qquad + d\Big[\big((p_i + p_j) - 1\big)\kappa^* f(-X_i), \big((p_i + p_j) - 1\big)\kappa^* f(-X_i)\Big]_t \end{pmatrix} \\ &= 2\begin{pmatrix} \dfrac{1}{\lambda + r}\phi^2\sigma^2_\Theta\big((p_i + p_j) - 1\big)\kappa^* f'(-X_{i,t}) \\ \qquad - 2\dfrac{1}{\lambda + r}\phi^2\sigma^2_\Theta p_i\big((p_i + p_j) - 1\big)\kappa^* f'(-X_{i,t}) \end{pmatrix} \\ &\quad + 2\phi^2\sigma^2_\Theta\Big(\big((p_i + p_j) - 1\big)\kappa^* f'(-X_{i,t})\Big)^2 \end{aligned}$$

$$
\begin{aligned}
&= 2\frac{1}{\lambda + r}\big(1 - 2p_i\big)\phi^2\sigma_\Theta^2\big((p_i + p_j) - 1\big)\kappa^* f'(-X_{i,t}) \\
&\quad + 2\phi^2\sigma_\Theta^2\Big(\big((p_i + p_j) - 1\big)\kappa^* f'(-X_{i,t})\Big)^2 \\
&= 2\phi^2\sigma_\Theta^2\Big(\big((p_i + p_j) - 1\big)\kappa^* f'(-X_{i,t})\Big)\Big(\frac{1 - 2p_i}{\lambda + r} + \big((p_i + p_j) - 1\big)\kappa^* f'(-X_{i,t})\Big)
\end{aligned}
$$

for $X_{i,t} > -x^*$. Similarly, for $X_{i,t} \leqslant -x^*$, the price process is

$$
P_{i,t} = \big(p_i F_{j,t} + (1 - p_i)F_{i,t}\big) + (p_i + p_j - 1)\Big(-\frac{X_{i,t}}{\lambda + r} + \kappa^* f(X_{i,t})\Big) + \mathit{constant},
$$

and the additional quadratic variation in the price process is

$$
\begin{aligned}
&\frac{1}{\mathrm{d}t}\left(\begin{aligned}&2d\Big[F_i + p_i\frac{X_i}{\lambda + r}, (p_i + p_j - 1)\Big(-\frac{X_{i,t}}{\lambda + r} + \kappa^* f(X_{i,t})\Big)\Big]_t \\ &\quad + d\Big[(p_i + p_j - 1)\Big(-\frac{X_{i,t}}{\lambda + r} + \kappa^* f(X_{i,t})\Big), (p_i + p_j - 1)\Big(-\frac{X_{i,t}}{\lambda + r} + \kappa^* f(X_{i,t})\Big)\Big]_t\end{aligned}\right) \\
&\quad = 2\phi^2\sigma_\Theta^2(p_i + p_j - 1)(1 - 2p_i)\frac{1}{\lambda + r}\Big(\frac{1}{\lambda + r} - \kappa^* f'(X_{i,t})\Big) \\
&\qquad\qquad + 2\phi^2\sigma_\Theta^2(p_i + p_j - 1)^2\Big(\frac{1}{\lambda + r} - \kappa^* f'(X_{i,t})\Big)^2 \\
&\quad = 2\phi^2\sigma_\Theta^2(p_i + p_j - 1)\Big(\frac{1}{\lambda + r} - \kappa^* f'(X_{i,t})\Big)\Big(\frac{p_j - p_i}{\lambda + r} - (p_i + p_j - 1)\kappa^* f'(X_{i,t})\Big).
\end{aligned}
$$

□

Chapter 4

Determining Insurance Premium from Accident Rate and Surplus Level

4.1 Introduction

One of the most typical model in the ruin theory is the Lundberg model. In the Lundberg model, the solvency of an insurer (called surplus) is modeled by compound Poisson process, and it is called ruin if the surplus becomes negative. Typical objects of the model is to calculate the probability of ruin, and the distribution of the time of ruin. Although the basic model is classical, the risk theory get attention again due to Gerber and Shiu (1997, 1998), who

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derive the relation between the distributions of default time, surplus before default, and surplus after default (deficit at ruin). This approach is also applied to the pricing of perpetual put options (Gerber & Shiu, 1999).

A famous problem in non-life insurance mathematics, related to ruin problem, is called De Finetti's problem, in which an insurer maximizes the expected dividend, paid until its ruin, to the policyholders. In recent years, this problem is solved in general surplus processes (called spectrally negative Lévy processes) of insurer (Avram, Palmowski, and Pistorius (2007)). In De Finetti's problem, the optimal solution is known to be a barrier strategy, in which the insurer pays dividends such a way the surplus process not to exceed some fixed level.

If the insurer follows the barrier strategy, however, the surplus process becomes bounded above, and thus the insurer defaults with probability one. After 1990s, De Finetti's problem has developed into *realistic* formulations, in which the surplus process is not bounded and the insurer can survive with non-zero probability (Asmussen and Taksar (1997) and Jeanblanc-Picqué and Shiryaev (1995) for Brownian surplus processes; and Gerber and Shiu (2006) for the Lundberg model). In these studies, the probability of accident (the Poisson intensity in the Lundberg model) is assumed to be constant and to be known by the insurer.

In this chapter, we formulate the optimal dividend problem as the optimal premium rate problem, assuming that the accident probability of the policyholder is two-valued random variable that is not observable by the insurer. This assumption on the accident probability means that the policyholder is of either high risk or low risk, but the insurer does not know the type of him. The insurer's objective is to find an optimal experience rate strategy, observing his surplus level and estimating the accident rate. It must be noted here that we do not consider the optimization problem of the policyholder in this chapter, although the assumption on the policyholder is related to the adverse selection problem of Rothschild and Stiglitz (1976) and Prescott and Townsend (1984).

To solve the optimization problem, the insurer needs to infer whether the policyholder is high risk or low risk. In discrete-time settings, such an inference problem is formulated by Bühlmann (1967). However, this method cannot be directly applied to the continuous-time models we consider, and it is not clear that the relation between discrete-time estimation and continuous-time ruin probability. In this chapter we apply the continuous-time version of estimation method developed by Peskir and Shiryaev (2000), for this problem.

This chapter finds a necessary condition for optimal premium rate policy, by solving the Hamilton–Jacobi–Bellman (HJB) equation numerically. It finds that the insurer should choose the premium rate for the low-risk policyholder, if (i) the conditional probability of high risk is below a certain level, or (ii) the conditional probability exceeds and the surplus level exceeds conditional probability dependent level; otherwise the insurer should choose the premium rate for the high-risk policyholder. We also consider another performance criterion which attaches grater importance to ruin probability.

4.2 The Model

4.2.1 Studies related to our model

Lundberg Model

Our model is based on the Lundberg model. In the Lundberg model, the policyholder's cumulative claim process is defined by a compound Poisson process $S_t = \sum_{n=1}^{N_t} X_n$, where $N = \{N_t\}_{t \geqslant 0}$ is a Poisson process that represents the cumulative number of claims (claim arrival process); and $\{X_n\}_{n=1,2,\ldots}$ is an i.i.d. sequence of positive random variables such that X_n represents the outgoing due to n-th claim. We denote by F_X their (common) distribution function. The intensity $\lambda > 0$ of the Poisson process N represents the policyholder's degree of risk: high λ implies high-risk policyholder, and vice versa. Given an initial surplus $U_0 = u \geqslant 0$, the insurer's surplus process U is defined by $U_t = U_0 + ct - S_t$, where a constant $c > 0$ represents the insurance premium rate. A sample path of the surplus process under the Lundberg model is illustrated in Figure 4.1.

We call a ruin occurs if the surplus process becomes negative, and denote by τ the time of ruin: $\tau := \inf\{t \geqslant 0; U_t < 0\}$. The event that the insurer experiences ruin is thus $\{\tau < \infty\}$. The ruin probability is defined by $\mathbb{P}(\tau < \infty)$.

The premium rate c is often decomposed into three components $c = \lambda\mathbb{E}[X_n](1+\theta)$. Here if θ equals to zero, the premium rate c becomes *fair*. In such a case, however, the ruin occurs with probability one because the surplus process U becomes an unbounded martingale. For positive θ, the insurer can survive with non-zero probability, and the ruin probability becomes lower when higher θ is set. In this sense, θ is called the *safety loading* and usually assumed $\theta > 0$. For the same reason, the Poisson intensity λ of the claim arrival process N can be interpreted as the degree of risk of a policyholder.

Default probability

One major problem under the Lundberg model is calculation of the probability of default. In this chapter, we denote by $\psi(u)$ the default probability of the insurer given initial surplus $U_0 = u$:

$$\psi(u) = \mathbb{P}\big(\tau < \infty \mid U_0 = u\big). \tag{4.1}$$

Under the Lundberg model, the default probability is known to satisfy the following integro-differential equation

$$-\lambda\psi(u) + c\psi'(u) + \lambda\int_0^u \psi(u-y)F_X(\mathrm{d}y) + \lambda(1-F_X(u)) = 0. \tag{4.2}$$

This equation is roughly derived as follows: By the property of the Poisson process, no claim occurs with probability $1 - \lambda \mathrm{d}t + o(\mathrm{d}t)$, and one claim occurs with probability $\lambda \mathrm{d}t + o(\mathrm{d}t)$, in a small time interval $t \in [0, \mathrm{d}t]$. Since we have $\psi(u) = 1$ for $u < 0$, we can approximate the variation in the default probability in $t \in [0, \mathrm{d}t]$ as

$$\begin{aligned}\psi(u) = &\underbrace{(1-\lambda\,\mathrm{d}t)\psi(u+c\,\mathrm{d}t)}_{\text{no accident}} \\ &+ \underbrace{\lambda\,\mathrm{d}t\Big(\int_0^{u+c\,\mathrm{d}t}\psi(u+c\,\mathrm{d}t-y)F_X(\mathrm{d}y) + (1-F_X(u+c\,\mathrm{d}t))\Big)}_{\text{one accident}} \\ &+ \underbrace{o(\mathrm{d}t).}_{\text{more than two accidents}}\end{aligned}$$

Rearranging terms and letting $\mathrm{d}t \downarrow 0$, we obtain (4.2).

In general, explicit solutions for (4.2) is not known. However, if the claim size X has exponential distribution with mean $1/\beta$, the solution is known as

$$\psi(u) = \frac{1}{1+\theta}\exp\Big\{-\frac{\theta\beta}{(1+\theta)}u\Big\},$$

which is used in the second specification of our model.

De Finetti's problem

Another problem under the Lundberg model is determination of premium rate. One formulation of criterion for optimal premium rate is known as De

Finetti's problem[1] in which the insurer maximizes the discounted expected dividend

$$V(u) = \max_{\xi} \mathbb{E}\Big[\int_0^{\tau^\xi} e^{-rt} \,\mathrm{d}\xi_t \Big], \tag{4.3}$$

where a constant r is the discount rate, a $\{\sigma(U_s; s \leqslant t)\}_t$-adapted non-decreasing process ξ is the cumulative dividend process, and a random variable τ^ξ is the default time associated to ξ:

$$\tau^\xi := \inf\{t \geqslant 0;\, U_t^\xi < 0\} \quad \text{with} \quad U_t^\xi = U_t - \int_0^t \mathrm{d}\xi_s.$$

According to Avram et al. (2007), there exists a constant a^* such that the optimal dividend policy is $\xi^* = \min\{a^*, \overline{U}_t\} - a^*$, where $\overline{U}_t = \sup\{U_s;\, s \leqslant t\}$ is the supremum process of pre-dividend surplus U. Intuitively, it is optimal for the insurer to set dividend process such a way the post-dividend surplus not to exceed a certain level a^*. Such a strategy is called the barrier strategy or reflection strategy. A sample path of the surplus process under the barrier strategy is illustrated in Figure 4.2.

For initial surplus $U_0 < a^*$, the dividend process becomes continuous piecewise-linear process, because U is an compound Poisson process with drift and with non-positive jumps. In such cases, we can interpret the choice of dividend strategy as choice of premium rate strategy.

[1] According to Albrecher, Bäuerle, and Bladt (2018), Avram et al. (2007), Gerber and Shiu (2006), Jeanblanc-Picqué and Shiryaev (1995), the problem is named after De Finetti, B. (1957). Su un'impostazion alternativa dll teoria collecttiva del rischio. In *Transactions of the XVth International Congress of Actuarie* (Vol. 2, pp. 433–443).

The barrier strategy, however, does not matches the actual behavior of insurers: the insurer always defaults optimally, under this strategy.

Modified De Finetti's problem

Our model is based on the modified version of De Finetti's problem. Under the Lundberg surplus process, Gerber and Shiu (2006) modify De Finetti's problem to realistic form, imposing a restriction on the dividend strategy (the author refers to Asmussen and Taksar (1997), Gerber and Shiu (2004), Jeanblanc-Picqué and Shiryaev (1995), for Brownian surplus process case).

In their modification, the insurer chooses a dividend strategy ξ, from the form of

$$\xi_t = \int_0^t a_s \, \mathrm{d}s, \qquad \text{for an } [0, \alpha]\text{-valued process } a,$$

to maximize the expected discounted dividend (4.3).

A standard way to solve this kind of problem is to solve the Hamilton–Jacobi–Bellman equation related to (4.3). For compound Poisson surplus processes, however, it is difficult to solve it analytically, because it involves an integro-differential equation. A notable exception is the exponential claim size case. Assuming that each claim size has exponential distribution with mean $1/\beta$, Gerber and Shiu (2006) show that

$$V(u) = -\frac{w}{\beta}\frac{\alpha}{r}\frac{(\beta+\rho)e^{\rho u} - (\beta+\sigma)e^{\sigma u}}{(\rho - w)e^{\rho b^*} - (\sigma - w)e^{\sigma b^*}}, \quad \text{for} \quad 0 \leqslant u \leqslant b^*, \tag{4.4}$$

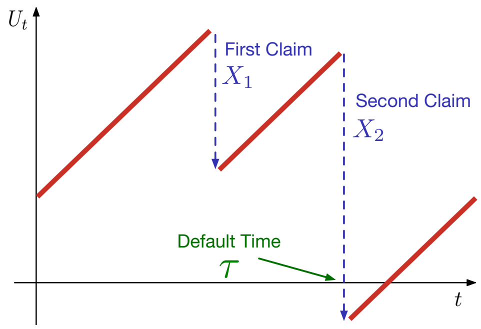

Figure 4.1: A sample path of surplus process under the Lundberg model.

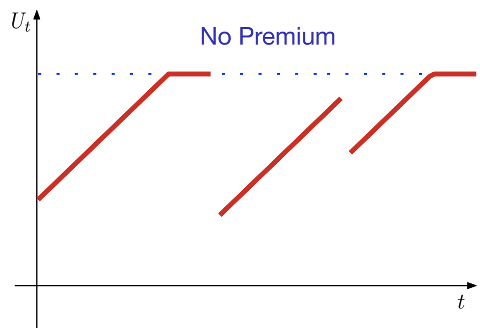

Figure 4.2: A sample path of optimal surplus for De Finetti's problem (barrier strategy).

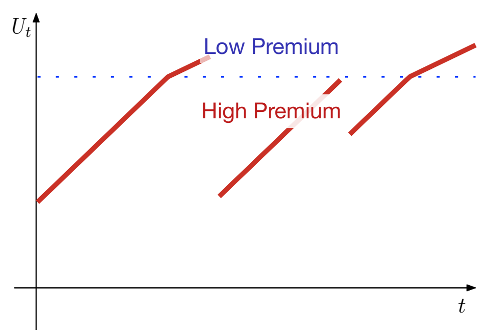

Figure 4.3: A sample path of optimal suplus for modified De Finetti's problem (threshold strategy).

and

$$V(u) = -\frac{\alpha}{r}\left(1 - e^{w(u-b^*)}\right) + f(b^*)e^{w(u-b^*)}, \quad \text{for} \quad b^* < u, \tag{4.5}$$

where $w < 0$ is the negative solution for the quadratic equation

$$(c-\alpha)\zeta^2 + (\beta(c-\alpha) - \lambda - r)\zeta - \beta r = 0 \tag{4.6}$$

in ζ, and $\rho > 0$ and $\sigma < 0$ are the solutions for the quadratic equation

$$c\zeta^2 + (\beta c - \lambda - r)\zeta - \beta r = 0 \tag{4.7}$$

in ζ. Furthermore the threshold b^* is

$$b^* = \frac{1}{\rho - \sigma} \log\left(\frac{\sigma^2 - w\sigma}{\rho^2 - w\rho}\right), \tag{4.8}$$

and the optimal strategy a_t is given by

$$a_t = \begin{cases} 0, & \text{if} \quad U_t \leqslant b^*, \\ \alpha, & \text{if} \quad U_t > b^*. \end{cases} \tag{4.9}$$

Such a dividend strategy is called the threshold strategy or refraction strategy. Again in this case, a dividend strategy can be interpreted as a premium rate strategy because there is no jump in ξ. A sample path for the surplus process under this strategy is illustrated in Figure 4.3. We use their result as a boundary condition of our problem. Recently, Albrecher et al. (2018)

modify these problems further, considering ratcheting effect.

Adverse selection model of Rothschild and Stiglitz (1976)

This topic is related to the discussion on our result, but not related to our main result directly. Readers can skip this topic.

Rothschild and Stiglitz (1976) construct a simple discrete-time model that explains how imperfect information distorts insurance contracts. Under their model, each risk-averse policyholder is endowed with two-valued random cash flow $(W, W - d)$, where W represents the endowment without accident and $W - d$ represents the endowment with accident. They first show that the equilibrium insurance contract is full insurance assuming competitive insurers are risk-neutral (which implies their expected profit π becomes zero), if both policyholder and insurers know the probability p of accident.

Then they turn to adverse selection model, under which each policyholder knows his accident probability is either high or low. Insurers know the fraction of policyholders of high/low accident rate, but do not know whether an policyholder is high type or low type individually. Under this adverse selection model, they show that there is no pooling equilibria, under which insurers offer same insurance contract regardless of policyholder's type. This is because a low risk policyholder can offer better contracts for both insurers and him, proving that he is of low risk. They further demonstrate that in separating equilibria, if exists, high risk policyholders will choose a full insur-

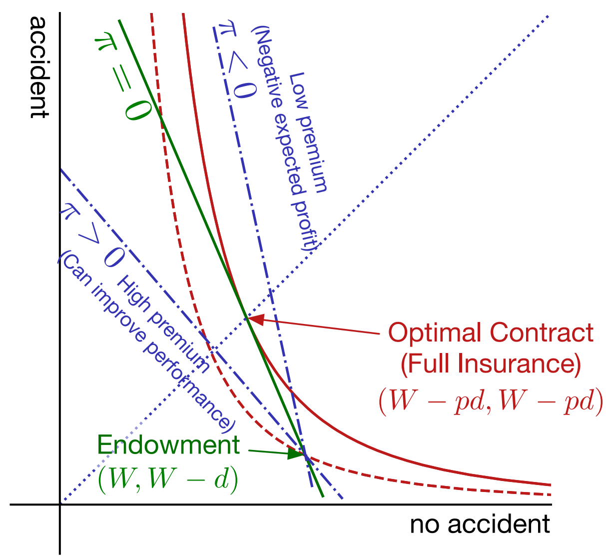

Figure 4.4: Full insurance is an equilibrium contract if policyholders are identical.

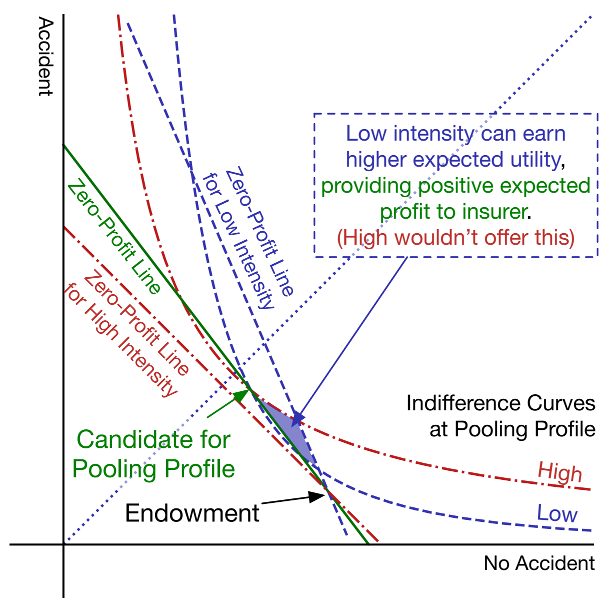

Figure 4.5: No pooling equilibrium exists if there is adverse selection.

ance contract with higher premium, while low risk policyholders will choose a partial insurance contract with lower premium. This menu of contracts prevents high risk policyholders from "free-riding" on inexpensive contract for low risk policyholders. Surprisingly, there can be no separating equilibrium, if there are few high-risk policyholders.

Prescott and Townsend (1984) generalize their model to multi-type model which can also involve moral hazard problem, demonstrating it can be solved linearly for special cases. Motivated by Prescott and Townsend (1984), Bisin and Gottardi (2006) further investigate in this problem, from perspective of equilibrium concepts.

4.2.2 Our model

Our Model

In the Lundberg model, the risk of the policyholder, which is represented by the Poisson intensity λ, is assumed to be known by the insurer. This assumption allows the insurer to set *appropriate* insurance premium c according to the risk of the policyholder. However it misses one of the central problem in insurance: imperfect information (e.g. Bisin & Gottardi, 2006; Lester et al., 2019; Prescott & Townsend, 1984; Rothschild & Stiglitz, 1976).

In our model, we consider two types of policyholders, namely, low-risk and high-risk policyholders, and assume that the insurer do not know whether a policyholder is of high-risk or low-risk. We denote by $0 \leqslant \pi \leqslant 1$ and $1 - \pi$

the fractions of low-risk and high-risk policyholder, respectively. We define the surplus process U of the insurer by

$$U_t = U_0 + c_t - S_t,$$

where $S_t := \sum_{n=1}^{N_t} X_n$ is a compound Poisson process with random intensity λ (a compound mixed Poisson process). We assume that the random intensity λ is a two-valued random variable that takes $\lambda = \lambda_l$ with probability π and $\lambda = \lambda_h (> \lambda_l)$ with probability $1 - \pi$, and assume that $\{X_n\}$ and $\{N_t\}$ are independent. We denote by $\mathbb{F} := \{\mathcal{F}_t\}_t$ the natural filtration generated by S. The $\mathbb{F}$-adapted premium rate process c is determined by the insurer and restricted in each specification of the model. Unlike perfect information cases, it is difficult for the insurer to set *appropriate* premium rate c because the insurer cannot observe the risk λ, directly.

Although the insurer cannot observe λ, he can infer the realization of λ from the realized claim arrival process N, and thus from the realized surplus process U. Let us denote by π_t the conditional probability that the policyholder is of low-risk:

$$\pi_t := \mathbb{P}\big(\lambda = \lambda_l \mid \mathcal{F}_t\big).$$

The performance criteria of the insurer will be described in each specification. We just introduce the forms of them here.

Specification 1 In the first specification, we consider the following maximization problem:

$$J^c(\pi_t, u_t) = \mathbb{E}\Big[\int_t^\tau e^{-r(s-t)}(\widetilde{c} - c_s)\,\mathrm{d}s \Big| \pi_t, U_t\Big]. \qquad \text{(Specification 1)}$$

and $\widetilde{c}$ is the premium rate intended for the type of the policyholder.

Specification 2 In the second specification, we consider a penalty function

$$J^c(\pi_t, U_t) = \mathbb{E}\big[\gamma(\tau_{\mathcal{S}} - t) + \mathbf{1}_{\{\tau_{\mathcal{S}}<\tau\}}(1 - \pi_{\tau_{\mathcal{S}}})\psi(U_{\tau_{\mathcal{S}}}) \mid \pi_t, U_t\big], \qquad \text{(specification 2)}$$

where $\tau_{\mathcal{S}}$ is a stopping time when the insurer reduces insurance premium: $c_t = c_l + (c_h - c_l)\mathbf{1}_{\tau_{\mathcal{S}}<t}$.

In the sequel, we seek an optimal strategy c^* that optimizes the insurer's performance

$$V(\pi, u) := J^{c^*}(\pi, u) = \sup_c(\text{or } \inf_c) J^c(\pi, u), \tag{4.10}$$

solving HJB equation for each problem numerically. We call the optimized performance V the value function. The key variables in evaluating performances are π and U.

Infinitesimal generator of (π, U)

In each specification, we derive the integro-differential equation the value function satisfy. For this, we calculate the infinitesimal generator $\mathbb{L}^c$ for (π, U), given strategy $c_t = c$, which is useful because

$$\left\{g(\pi_t, U_t) - g(\pi_0, U_0) - \int_0^t \mathbb{L}^{c_s} g(\pi_s, U_s)\mathrm{d}s\right\}_t \quad \text{is a martingale,} \tag{4.11}$$

provided that it is well defined.

According to Peskir and Shiryaev (2006), the stochastic differential equation of π_t is given by

$$\mathrm{d}\pi(t) = \frac{(\lambda_h - \lambda_l)\pi(t-)(1-\pi(t-))}{\lambda_l \pi(t-) + \lambda_h(1-\pi(t-))}\Big(\big(\lambda_l \pi(t-) + \lambda_h(1-\pi(t-))\big)\mathrm{d}t - \mathrm{d}N(t)\Big). \tag{4.12}$$

Applying the Itô formula, we obtain

$$\begin{aligned}
&\mathrm{d}g\big(\pi(t), U(t)\big) \\
&\quad = (\lambda_h - \lambda_l)\pi(s-)(1-\pi(s-))\frac{\partial}{\partial \pi} g\big(\pi(s-), U(s-)\big)\,\mathrm{d}t + c\frac{\partial}{\partial u} g\big(\pi(s-), U(s-)\big)\,\mathrm{d}t \\
&\qquad + \int_0^1 \int_{\mathbb{R}} \Big(g\big(\pi(s-) + x, U(s-) + y\big) - g\big(\pi(s-), U(s-)\big)\Big)\big(N(\mathrm{d}t, \mathrm{d}x, \mathrm{d}y) - \nu^\pi(\mathrm{d}t, \mathrm{d}x, \mathrm{d}y)\big) \\
&\qquad + \int_0^1 \int_{\mathbb{R}} \Big(g\big(\pi(s-) + x, U(s-) + y\big) - g\big(\pi(s-), U(s-)\big)\Big)\nu^\pi(\mathrm{d}t, \mathrm{d}x, \mathrm{d}y),
\end{aligned}$$

where $N(\cdot, \cdot, \cdot)$ is a random measure that represents the jump sizes of (π, U), and ν^π is its compensator. Observe that (i) the jump size of π is

$$-\frac{(\lambda_h-\lambda_l)\pi(t-)(1-\pi(t-))}{\lambda_l\pi(t-)+\lambda_h(1-\pi(t-))};$$

(ii) the jump size of U has distribution $F_X(-\mathrm{d}y)$; (iii) the jumps of π and U always occurs simultaneously; and (iv) the conditional intensity of their jumps is $\lambda_l\pi_{t-}+\lambda_h(1-\pi_{t-})$. From them, we obtain the infinitesimal generator $\mathbb{L}^c$ of (π, U) given c as

$$\begin{aligned}
&(\mathbb{L}^c g)(\pi,u)\\
&=(\lambda_h-\lambda_l)\pi(1-\pi)\frac{\partial}{\partial\pi}g(\pi,u)+c\frac{\partial}{\partial u}g(\pi,u)\\
&\quad+\big(\pi\lambda_l+(1-\pi)\lambda_h\big)\int_0^\infty\left(g\left(\frac{\lambda_l\pi}{\lambda_l\pi+\lambda_h(1-\pi)},u-y\right)-g(\pi,u)\right)F_X(\mathrm{d}y).
\end{aligned}$$

4.3 Maximizing Reduction in Insurance Premium (Specification 1)

4.3.1 Performance criterion and strategy

In the first specification, the insurer's object is to maximizes the discounted expectation of difference between the premium rate intended for the true type of policyholder and the actual premium rate. In this specification, we assume that the insurer can choose the premium rate strategy c from $\{\mathcal{F}_t\}$-adapted, $[c_l, c_h]$-valued process. Given a premium rate strategy c, the insurer's performance is evaluated by

$$J^c(\pi_t, u_t) = \mathbb{E}\Big[\int_t^\tau e^{-r(s-t)}(\widetilde{c} - c_s)\,\mathrm{d}s\Big|\pi_t, U_t\Big], \tag{4.13}$$

where $r > 0$ is the constant discount rate, and $\widetilde{c}$ is a $\{c_l, c_h\}$-valued random variable that represents the premium rate intended for the type of the policyholder. We denote by V, the value function for (4.13) that is

$$V(\pi, u) := J^{c^*}(\pi, u) = \sup_c J^c(\pi, u).$$

Since $c_l \leqslant c(t) \leqslant c_h$, the insurer can improve his performance by lowering the premium rate if the policyholder is high-risk preventing him from ruin soon.

Remark 4.1. This model coincides with Gerber and Shiu (2006) if the policyholders is high risk certainly, that is $\pi = 0$. In fact, in such a case, we have

$$\mathbb{E}\Big[\int_t^\tau e^{-r(s-t)}(c_h - c_s)\mathrm{d}s\Big|\pi_t = 0, U_t\Big],$$

and the difference $(c_h - c_t)$ represents the dividend rate in Gerber and Shiu (2006). We use their result as a boundary condition at $\pi = 0$, in numerical experiment.

4.3.2 HJB equation

We apply the dynamic programing principle to obtain the Hamilton–Jacobi–Bellman equation (HJB equation) of the value function V (see e.g. Øksendal & Sulem, 2007):

$$\begin{cases} \max_{c_l \leqslant c \leqslant c_h} \Big\{ -rV(\pi,u) + \mathbb{L}^c V + \pi c_l + (1-\pi)c_h - c \Big\} = 0, & u \geqslant 0, \\ V(\pi,u) = 0, & u < 0. \end{cases} \tag{4.14}$$

The optimal strategy becomes $c_t = c$ in this equation given $\pi_t = \pi$ and $U_t = u$. Because the strategy $c_t \equiv c_l$ yields nonnegative performance regardless of π, we assume that V is a nonnegative function without loss of generality.

The relation between value function and HJB equation (4.14) is derived in two steps.

Step 1 (Integro-differential equation). Let $V = J^{c^*}$ be the performance for the insurer for the optimized strategy c^*. Since

$$\underbrace{\Big\{ e^{-rt} V(\pi_t, U_t) + \int_0^t e^{-rs} \big(\underbrace{\pi_s c_l + (1-\pi_s)c_h}_{=\mathbb{E}[\tilde{c}|\pi_t, U_t]} - c_s^* \big) \mathrm{d}s \Big\}}_{=\mathbb{E}\left[\int_0^\tau e^{-rs}(\tilde{c}-c_s^*)\,\mathrm{d}s \middle| \mathcal{F}_t\right]}$$

is a $(\mathbb{P}, \{\mathcal{F}_t\})$-martingale for optimal c^*, we have (provided that the ruin have not occurred yet)

$$-rV(\pi,u) + \mathbb{L}^{c^*} V + \pi c_l + (1-\pi)c_h - c^* = 0,$$

by (4.11). Furthermore, by the optimality of c^*, we must have

$$\max_{c_l \leqslant c \leqslant c_h} \Big\{ - rV(\pi, u) + \mathbb{L}^c V + \pi c_l + (1-\pi)c_h - c \Big\} = 0$$

for $u \geqslant 0$. For $u < 0$, on the other hand, we have $V(\pi, u) = 0$ from (4.13), directly.

Step 2 (Optimality). Let V be a solution of (4.14). First, observe that

$$\begin{aligned} \mathbb{E}\Big[\int_0^\tau e^{-rt}\big(\widetilde{c} - c_t\big)\mathrm{d}t\Big] &= \mathbb{E}\Big[\mathbb{E}\Big[\int_0^\tau e^{-rt}\big(\widetilde{c} - c_t\big)\mathrm{d}t \,\Big|\, \mathcal{F}_t\Big]\Big] \\ &= \mathbb{E}\Big[\int_0^\tau e^{-rt}\Big(\pi_t c_l + (1-\pi_t)c_h - c_t\Big)\mathrm{d}t\Big], \end{aligned} \tag{4.15}$$

by the definition of π_t. For each fixed insurance premium strategy c, let us define $A_t^c := -rV(\pi_t, U_t) + \mathbb{L}^{c_t}V(\pi_t, U_t)$. Then, the process

$$\Big\{ e^{-rt}V(\pi_t, U_t) - \int_0^t e^{-rt}A_t^c \mathrm{d}t \Big\}_t$$

is a martingale and thus we have

$$\mathbb{E}\Big[e^{-rt}V(\pi_t, U_t) - \int_0^t e^{-rt}A_t^c \mathrm{d}t\Big] = V(\pi, u).$$

Recalling that we have assumed $V \geqslant 0$, we obtain.

$$- \mathbb{E}\Big[\int_0^\tau e^{-rt}A_t^c \mathrm{d}t\Big] \leqslant V(\pi, u)$$

Since V satisfies (4.14), we obtain

$$A_t^c + \pi_t c_l + (1-\pi_t)c_h + c_t \leqslant 0$$

for each strategy c_t. This inequality, together with (4.15), we obtain

$$J^c(\pi,u) = \mathbb{E}\Big[\int_0^\tau e^{-rt}\Big(\pi_t c_l + (1-\pi_t)c_h - c\Big)\mathrm{d}t\Big] \leqslant V(\pi,u).$$

4.3.3 Numerical example

In this subsection, we calculate (4.14) numerically, assuming that the claim size X has exponential distribution with expectation $1/\beta$.

In the HJB equation (4.14), c appears only in

$$\Big(\frac{\partial}{\partial u}V(\pi,u) - 1\Big)c.$$

It means that the optimal premium rate strategy is $\{c_l, c_h\}$-valued: (4.14) is maximized by choosing $c = c_l$ if $\frac{\partial}{\partial u}V(\pi,u) < 1$ and $c = c_h$ if $\frac{\partial}{\partial u}V(\pi,u) > 1$, which makes it easy to find optimal boundary. Such a structure is found also in Gerber and Shiu (2006). Furthermore, as we mentioned, the result of Gerber and Shiu (2006) (for constant intensity) provides us the boundary condition for $\pi = 0$ as (4.4)–(4.9) with $\alpha = c_h - c_l$.

Using these conditions, we calculate the value function V numerically by

1. divide $[0,1]$ and $[0,\overline{U}]$ into $0 = \pi_1 < \pi_2 < \cdots < \pi_{M_\pi} = 1$ equally and

$0 = u_1 < u_2 < \cdots < u_{M_U} = \overline{U}$ equally,

2. set $V(0,u) = V(\pi_2, u)$ by (4.4)–(4.9),

3. check $(V(\pi_i, u_{j+1}) - V(\pi_i, u_j))/(u_{j+1} - u_j)$ to determine the infinitesimal generator $\mathbb{L}^c$, and

4. calculate $V(\pi_{i+1}, u_j)$ by finite-difference method.

Figure 4.6 shows the result. The optimal premium rate strategy becomes again a threshold strategy as in Gerber and Shiu (2006). However the threshold in the present model is two-dimensional. The insurer will choose high premium rate c_h only when the surplus is low and there is strong possibility that the policyholder is of high-risk.

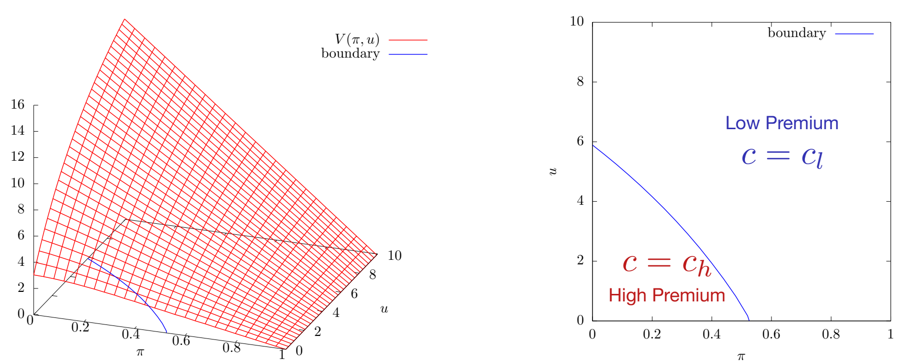

Figure 4.6: The value function V (left) and the boundary (right). The parameters are $\lambda_1 = 1.2, \lambda_0 = 1.0, c_1 = 1.5, c_0 = 1.3, r = 0.01, \beta = 1.0$.

4.4 Minimizing Default Probability after Reduction in Premium Rate (Specification 2)[2]

4.4.1 Performance criterion and strategy

As the title of the book of Asmussen and Albrecher (2010) indicates, ruin probability is one of the main interests of insurers. In this section, we assume that the insurer can reduce the insurance fee once, minimizing the default probability after reduction and considering the expected time until it. In this specification, the premium rate strategy c is restricted to

$$c_t = \begin{cases} c_h, & \text{if} \quad \tau_{\mathcal{S}} < t, \\ c_l, & \text{if} \quad \tau_{\mathcal{S}} \geqslant t, \end{cases}$$

for some $\{\mathcal{F}_t\}$-stopping time $\tau_{\mathcal{S}}$ that represents the time of reduction in insurance premium. The insurer's problem is to minimize the penalty evaluated by

$$J^c(\pi_t, U_t) = \mathbb{E}\big[\gamma(\tau_{\mathcal{S}} - t) + 1_{\{\tau_{\mathcal{S}} < \tau\}}(1 - \pi_{\tau_{\mathcal{S}}})\psi(U_{\tau_{\mathcal{S}}}) \mid \pi_t, U_t\big], \tag{4.16}$$

[2] The result of Section 4.4 is in the master's thesis of the author: 五十嵐徹. (2015). 最適停止の手法を用いた保険料引き下げ時刻の決定問題 (Master's thesis, 一橋大学大学院 商学研究科).

where $\gamma > 0$ is a constant and $\mathbf{1}_A$ is an indicator function of event A. The function ψ represents the default probability after $\tau_{\mathcal{S}}$:

$$\psi(u) := \mathbb{E}[\tau < \infty \mid U_{\tau_{\mathcal{S}}} = u].$$

Throughout this section, we assume $\psi(u)$ is C^1 function on $\mathbb{R}_{>0}$.

Equation (4.16) represents a trade-off between premium rate and default probability. Although its beneficial for insurer to wait to decrease the default probability, it means that the policyholder is imposed higher premium, which is considered in De Finetti's problem. If the policyholder is of low-risk, the insurer should reduce the premium to intended rate as soon as possible ($1 - \pi_{\tau_{\mathcal{S}}}$ in the second term). Note that we should not give penalties if the default occurs before $\tau_{\mathcal{S}}$, because such defaults occurs regardless of reduction in premium rate. This is why $\mathbf{1}_{\{\tau_{\mathcal{S}} < \tau\}}$ appears. We assume $\tau_{\mathcal{S}} \leqslant \tau$ without loss of generality.

4.4.2 Free-boundary problem

Let us denote by $\tau_{\mathcal{S}*}$ and V, the optimal stopping time and the minimized penalty, respectively:

$$V(\pi, u) := J^{\tau^*_{\mathcal{S}}}(\pi, u) = \min_{\tau_{\mathcal{S}}} J^{\tau_{\mathcal{S}}}(\pi, u).$$

Such a problem is called an *optimal stopping problem*, and is known to be related to a free-boundary problem (see e.g. Peskir & Shiryaev, 2006):

$$\begin{aligned} \mathbb{L}^{c_h} V(\pi,u) &= -\gamma, && \text{if} \quad (\pi,u) \in C, \\ V(\pi,u) &= (1-\pi)\psi(u), && \text{if} \quad (\pi,u) \in D, \\ V(\pi,u) &= 0 && \text{if} \quad u < 0, \\ V(\pi,u) &\leqslant (1-\pi)\psi(u), \end{aligned} \tag{4.17}$$

for some $C \subset [0,1] \times \mathbb{R}_{\geqslant 0}$ and $D = [0,1] \times \mathbb{R}_{\geqslant 0} \backslash C$. The stopping time $\tau_{\mathcal{S}}^* := \inf\{t > 0; (\pi_t, U_t) \in D\}$ is a candidate of the optimal stopping time. We provide a verification of it for large γ, later. The second and third equations are trivial because $\tau_{\mathcal{S}} = 0$ if $(\pi_0, U_0) \in D$ in (4.16). The first equation is derived by the argument for HJB equation in the previous section, using that

$$\Big\{V(\pi_{t\wedge\tau_{\mathcal{S}}}, U_{t\wedge\tau_{\mathcal{S}}}) + \gamma(t \wedge \tau_{\mathcal{S}})\Big\} = \Big\{\mathbb{E}\big[\gamma\tau_{\mathcal{S}} + 1_{\{\tau_{\mathcal{S}} < \tau\}}(1-\pi_{\tau_{\mathcal{S}}})\psi(U_{\tau_{\mathcal{S}}}) \mid \pi_t, U_t\big]\Big\},$$

is a $\{\mathcal{F}_t\}$-martingale.

The following conditions are called continuous fit and smooth fit conditions:

$$\begin{cases} V(\pi-, u-) = (1-\pi)\psi(u), & (\pi, u) \in \overline{C} \cap \overline{D}, \\ \dfrac{\partial}{\partial \pi} V(\pi-, u-) = -\psi(u), & (\pi, u) \in \overline{C} \cap \overline{D}, \\ \dfrac{\partial}{\partial u} V(\pi-, u-) = (1-\pi)\psi'(u), & (\pi, u) \in \overline{C} \cap \overline{D}. \end{cases} \tag{4.18}$$

These conditions are used to characterize the value function. However, to our setting (without Brownian motion), smooth fit condition is known as rather strong condition. We argue, in Appendix, that V of the present problem actually satisfies these conditions.

Verification of optimality

We show that a function V that satisfies (4.17) and (4.18) actually minimizes the expected penalty (4.16), for a large γ. For each stopping time $\tau'_{\mathcal{S}} \leqslant \tau$, we have

$$\mathbb{E}\big[(1-\pi_{\tau'_{\mathcal{S}}})\psi(U_{\tau'_{\mathcal{S}}})\big] \geqslant \mathbb{E}\big[V(\pi_{\tau'_{\mathcal{S}}}, U_{\tau'_{\mathcal{S}}})\big] = V(\pi, u) + \mathbb{E}\Big[\int_0^{\tau'_{\mathcal{S}}} \mathbb{L}^{c_h} V(\pi_s, U_s)\mathrm{d}s\Big],$$

where $\mathbb{L}^{c_h} V(\pi, u)$ is well defined on $[0,1] \times \mathbb{R}_{\geqslant 0}$ thanks to smooth and continuous fit conditions. If

$$\mathbb{L}^{c_h} V(\pi, u) \geqslant -\gamma, \qquad \text{on} \quad [0,1] \times \mathbb{R}_{\geqslant 0}, \tag{4.19}$$

we obtain

$$\mathbb{E}\big[\gamma\tau'_{\mathcal{S}} + (1-\pi_{\tau'_{\mathcal{S}}})\psi(U_{\tau'_{\mathcal{S}}})\big] \geqslant V(\pi, u),$$

for each stopping time $\tau'_{\mathcal{S}}$, where the equality holds for $\tau'_{\mathcal{S}} = \tau^*_{\mathcal{S}}$. Thus it suffices to show (4.19). However, on C, we have (4.19) from (4.17); and on D, a direct calculation shows that

$$\begin{aligned}
&\mathbb{L}^{c_h} V(\pi, u) = \mathbb{L}^{c_h}((1-\pi)\psi(u)) \\
&\quad = -(\lambda_h - \lambda_l)\pi(1-\pi)\psi(u) + c_h(1-\pi)\psi'(u) \\
&\qquad + \lambda_h(1-\pi)\int_0^u \psi(u-y)F_X(\mathrm{d}y) - (1-\pi)(\pi\lambda_l + (1-\pi)\lambda_h)\psi(u),
\end{aligned} \tag{4.20}$$

which is bounded on $[0,1]\times\mathbb{R}_{\geqslant 0}$. It implies (4.19) holds for some $\gamma > 0$.

4.4.3 Numerical example

In this subsection, we calculate (4.17)–(4.18) numerically, again assuming that the claim size X has exponential distribution with expectation $1/\beta$. We follow the procedure of

1. divide $[0,1]$ and $[0,\overline{U}]$ into $0 = \pi_1 < \pi_2 < \cdots < \pi_{M_\pi} = 1$ equally and $0 = u_1 < u_2 < \cdots < u_{M_U} = \overline{U}$ equally,
2. for $\pi = 0$, find initial condition $V(0,0)$ that satisfies (4.17) and (4.18), using finite-difference method,
3. set $V(0,u) = V(\pi_2, u)$, and

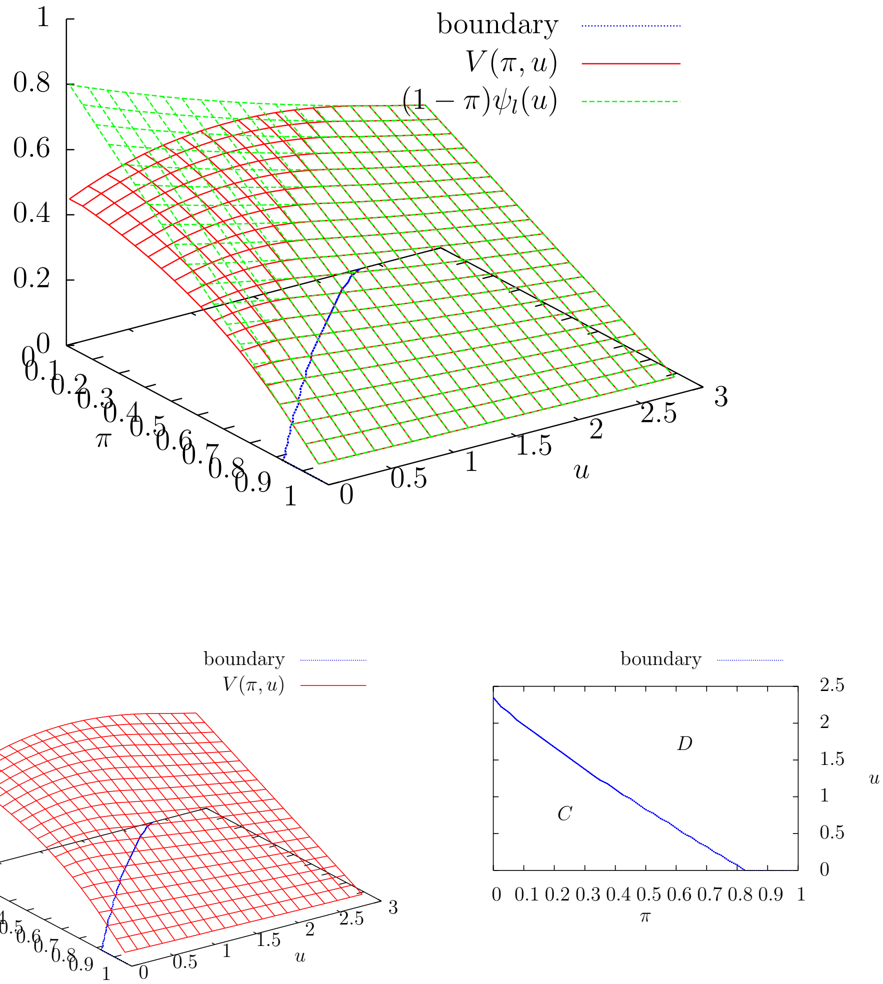

Figure 4.7: The value function and boundary condition (top), the value function (bottom left), and the boundary (bottom right). The parameters are $\gamma = 0.20,\ \beta = 1.0,\ \lambda_h = 1.2,\ \lambda_l = 1.0,\ c_h = 1.8,\ c_l = 1.5$.

4. calculate $V(\pi_{i+1}, u_j)$ by finite-difference method.

Figure 4.7 shows the result. Although we set a different performance criterion to our first specification, the results are similar to each other. Again the boundary is two-dimensional. The insurer keeps high premium rate c_h only when the default probability is high (low surplus) and there is strong possibility that the policyholder is of high-risk.

4.5 Discussion

Sequential testing problem and Poisson disorder problem

In our present model, we assumed that the policyholder's type does not change over time, to derive conditional distribution π of the type. Peskir and Shiryaev (2000) call such a problem "sequential testing problem." In insurance modeling, the change in accident rate over time is also important (e.g. elderly drivers in car insurance). For such a modeling, "Poisson disorder problem" of Peskir and Shiryaev (2002) could be used.

Adverse selection

As we have mentioned, there are few studies that combines ruin theory with imperfect information. One exception is the master thesis of Tomita[3], who studies the ruin probability under mixed Poisson driven Lundberg model

[3] 冨田昌 (2018), in Japanese.

with experience rate. Both Tomita and our present observation do not considers adverse selection problem, under which policyholder acts strategically. The difficulty is that asymmetry in information will distorts insurance contract itself, as shown in Rothschild and Stiglitz (1976). However, one may speculate that there are separating equilibria in which a design of insurance contract screens the types of policyholders. This problem should be worth tackling.

4.A Appendix

Continuous fit and smooth fit

In this appendix, we show that optimal V satisfies continuous and smooth fit conditions (4.18). First, we have

$$\frac{V(\pi,u)-V(\pi-\varepsilon,u)}{\varepsilon} \geqslant \frac{(1-\pi)\psi(u)-(1-(\pi-\varepsilon))\psi(u)}{\varepsilon}, \quad \text{on} \quad (\pi,u)\in \overline{C}\cap\overline{D},$$

because $V(\pi,u)\leqslant(1-\pi)\psi(u)$ on C and $V(\pi,u)=(1-\pi)\psi(u)$ on D. Thus

$$\liminf_{\varepsilon\downarrow 0}\frac{V(\pi,u)-V(\pi-\varepsilon,u)}{\varepsilon} \geqslant -\psi(u).$$

For the opposite inequality, let $\mathbb{P}_\pi = \pi\mathbb{P}_l + (1-\pi)\mathbb{P}_h$, where $\mathbb{P}_l = \mathbb{P}(\cdot \mid$

$\lambda = \lambda_l)$ and $\mathbb{P}_h = \mathbb{P}(\cdot \mid \lambda = \lambda_h)$. Let $(\pi_0, U_0) = (\pi, u) \in \overline{C} \cap \overline{D}$ and let $\tau_{\mathcal{S}}^{\varepsilon}$ be

$$\tau_{\mathcal{S}}^{\varepsilon} = \inf \Big\{ t \geqslant 0;\ (\pi_t^{\varepsilon}, U_t) \in D \Big\}.$$

Here π_t^{ε} is a stochastic process that solves the same stochastic differential equation (4.12) as π_t, with different initial condition $\pi_0^{\varepsilon} = \pi - \varepsilon$. The strategy $\tau_{\mathcal{S}}$ is the optimal strategy associated to $V(\pi - \varepsilon, u)$ but is not optimal for $V(\pi, u)$. Then, for $A_t = \mathbb{L}^{c_h}((1-\pi_t)\psi(U_t))$ and $A_t^{\varepsilon} = \mathbb{L}^{c_h}((1-\pi_t^{\varepsilon})\psi(U_t))$, which are given in (4.20), we obtain

$$\begin{aligned}
&V(\pi, u) - V(\pi - \varepsilon, u)\\
&\quad \leqslant \mathbb{E}_{\pi}\Big[\gamma\tau^{\varepsilon} + (1-\pi_{\tau^{\varepsilon}})\psi(U_{\tau^{\varepsilon}})\Big] - \mathbb{E}_{\pi-\varepsilon}\Big[\gamma\tau^{\varepsilon} + (1-\pi_{\tau^{\varepsilon}}^{\varepsilon})\psi(U_{\tau^{\varepsilon}})\Big]\\
&\quad = (1-\pi)\psi(u) + \gamma\mathbb{E}_{\pi}[\tau^{\varepsilon}] + \mathbb{E}_{\pi}\Big[\int_0^{\tau^{\varepsilon}} A_t\,\mathrm{d}t\Big]\\
&\qquad\qquad - (1-(\pi-\varepsilon))\psi(u) - \gamma\mathbb{E}_{\pi-\varepsilon}[\tau^{\varepsilon}] - \mathbb{E}_{\pi-\varepsilon}\Big[\int_0^{\tau^{\varepsilon}} A_t^{\varepsilon}\mathrm{d}t\Big]\\
&\quad = -\varepsilon\psi(u) + \gamma\varepsilon\big(\mathbb{E}_l[\tau^{\varepsilon}] - \mathbb{E}_h[\tau^{\varepsilon}]\big)\\
&\qquad + \mathbb{E}_{\pi}\Big[\int_0^{\tau^{\varepsilon}} (A_t - A_t^{\varepsilon})\mathrm{d}t\Big] + \varepsilon\Big(\mathbb{E}_l\Big[\int_0^{\tau^{\varepsilon}} A_t^{\varepsilon}\mathrm{d}t\Big] - \mathbb{E}_h\Big[\int_0^{\tau^{\varepsilon}} A_t^{\varepsilon}\mathrm{d}t\Big]\Big).
\end{aligned}$$

If there is no jump, $(\pi_t^{\varepsilon}, U_t)$ reaches to D at least π^{ε} increases by ε. To evaluate the time for this happens, observe

$$\varepsilon = \int_0^t (\lambda_h - \lambda_l)\pi_s(1-\pi_s)\mathrm{d}s, \qquad \text{on} \quad \{N_{\tau^{\varepsilon}} = 0\},$$

which comes from dt-term in (4.12). Solving this equation, we obtain

$$\pi_t = \frac{e^{(\lambda_h-\lambda_l)t}}{\frac{1}{\pi_0} - 1 + e^{(\lambda_h-\lambda_l)t}},$$

and thus

$$\tau^\varepsilon \leqslant \frac{1}{\lambda_h-\lambda_l}\log\frac{(1-\pi)(\varepsilon/\pi+1)}{1-\pi-\varepsilon} \in O(\varepsilon), \qquad \text{on} \quad \{N^\varepsilon_\tau = 0\}.$$

This together with $\mathbb{P}(N_t \geqslant 1) = O(\varepsilon)$ show

$$V(\pi,u) - V(\pi-\varepsilon,u) \leqslant -\varepsilon\psi(u) + o(\varepsilon).$$

The conditions for u is due to similar argument.

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