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Asymmetric Majority Pillage Games

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Abstract

This paper studies pillage games (Jordan in J Econ Theory 131.1:26-44, 2006, "Pillage and property"), which are well suited to modelling unstructured power contests. To enable empirical test of pillage games' predictions, it relaxes a symmetry assumption that agents' intrinsic contributions to a coalition's power is identical. In the three-agent game studied: (i) only eight configurations are possible for the core, which contains at most six allocations; (ii) for each core configuration, the stable set is either unique or fails to exist; (iii) the linear power function creates a tension between a stable set's existence and the interiority of its allocations, so that only special cases contain strictly interior allocations. Our analysis suggests that non-linear power functions may offer better empirical tests of pillage game theory.

Key words: power contests, core, stable sets

JEL classification numbers: C71; D51; P14

1 Introduction

This paper studies unstructured power contests between sophisticated, well-informed agents — which we regard as pervasive even in the presence of strong institutions.

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Consider: a government that shelves legislation on realising that it would lose a vote; *or* an employee who chooses not to whistle-blow a policy violation, knowing that management would find a way of tarnishing and firing her if she did; *or* powerful states that sell weapons to subordinate allies to maintain both their allegiance and their effectiveness.

Such situations are often marked by: (i) actors who find ways to exercise power through formal institutional structures; (ii) power that depends both on coalitions' intrinsic characteristics and the resources available to them; (iii) agents who can predict the outcome of power contests, so do not spend resources on costly signalling, but costlessly accept their defeat.

To formalise analysis, this paper studies *pillage games*, a class of game introduced by Jordan (2006). As cooperative games, they do not rely on a game form — which might be contested in an unstructured power contest.[1] Instead, a dominance relation is defined directly on possible outcomes.

In pillage games, the dominance relation is represented by a power function that monotonically associates a power to coalitions and their resources: as coalitions gain members or resources, they become more powerful. Thus, pillage games provide a richer treatment of resources than do games in characteristic function form (the best known class of cooperative games), which rule out both the possibility of conflict between coalitions "from the beginning" (Maskin, 2003) and the possibility that resources might help produce power.

Not only do pillage games seem well suited, then, to study unstructured power contests, but their stable sets — the solution concept most analysed — have at least three appealing theoretical properties. First, they are equivalent to (farsighted) cores in expectation (Jordan, 2006), allowing them to be interpreted as undominated allocations for farsighted agents. Second, they are small.[2] Third, in contrast to the possibly infinite number of stable sets found in games in characteristic function form (q.v. the famous 'signature' example of Shapley (1959)), multiplicity seems less of a problem in pillage games.[3]

In spite of these appealing modelling and theoretical properties, the pillage

[1]As Aumann (2008) observed, "procedures are not really all that relevant; that it is the possibilities for coalition forming, promising and threatening that are decisive, rather than whose turn it is to speak".

[2]Jordan (2006) proved that they were finite; Kerber and Rowat (2011) tightened Jordan's bound to a Ramsey number; Saxton (2011) tightened the bound yet further; Rowat and Kerber (2014) derived a tight bound of 15 allocations for three-agent pillage games satisfying three additional axioms on their power functions; in a multi-good extension of Jordan (2006), Beardon and Rowat (2013) both proved that stable sets have measure zero and bounded their Hausdorff dimension.

[3]Jordan (2006) proved that his leading class of pillage games had a unique stable set, and did not find examples of multiple stable sets; Rowat and Kerber (2014) proved uniqueness in three agent pillage games satisfying their three additional axioms; however, MacKenzie, Kerber, and Rowat (2015) constructed examples of pillage games with multiple (finite) stable sets.

games literature has been entirely theoretical, leaving open the question of whether pillage games can deliver insights into actual power contests. Further, the theory has focussed almost exclusively on cases in which agents' intrinsic contributions to a coalition's power are symmetric.[4] As symmetry is a very strong property, it tends not to be satisfied in empirical settings: political parties' bases (whether socio-economic or ethnic) vary in power; employees' inalienable characteristics (inc. race and gender) influence their ability to effect outcomes; nation states are even more heterogeneous, with populations varying by five orders of magnitude.

This paper therefore relaxes symmetry, allowing us to derive predictions of pillage game theory in the more empirically interesting environment of asymmetric agents.[5] For the sake of tractability, it works with an asymmetric version of the three-agent *majority pillage game*.[6] In the classic majority game (in characteristic function form), one allocation dominates another if and only if it is preferred by a strictly larger coalition. In the majority pillage game (Jordan and Obadia, 2015), an allocation may also dominate another if the coalitions favouring and opposing it are the same size, as long as the former holds more resources. This paper generalises further, allowing agents' intrinsic contributions to be asymmetric, and for the resource contribution to overpower the intrinsic contributions. Explicitly, the class of power functions considered is

$$\pi\left(C, \boldsymbol{x}\right) = \sum_{i \in C} \left(x_i + v_i\right); \tag{1}$$

where C is a non-empty coalition of agents, indexed by $i \in I \equiv \{1, 2, 3\}$, $\boldsymbol{x} \equiv (x_1, x_2, x_3)$ such that $\sum_{i \in I} x_i = 1$ and $x_i \in \mathbb{R}_+$ is an allocation, and $v_i \in \mathbb{R}_+$ allows agents' intrinsic power to differ. Power function (1) provides one of the simplest possible specifications of an asymmetric pillage game, introducing one parameter per agent.

We are particularly interested in two questions. First, over what range of the v_i do stable sets exist? We already have answers for the symmetric special cases:

1. when $v_i = 0$ for all three agents, the unique stable set consists of the allocations $(1, 0, 0)$, $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ and their permutations (Jordan, 2006);

[4]The exception to this is MacKenzie, Kerber, and Rowat (2015), which violated symmetry for theoretical purposes, constructing examples in which pillage games might have multiple stable sets.

[5]A more general factor impeding empirical tests of pillage game theory is its reliance on the von Neumann and Morgenstern stable set, which von Neumann himself seemed to oppose testing (Leonard, 2010, p.244, pp.260–262). Yet, although cooperative game theory has disappeared from textbooks on empirical game theory, the earliest empirical game theory addressed cooperative games (q.v. Kalisch et al., 1954).

[6]Given cooperative game theory's combinatorial possibilities for coalition formation, the three-agent case has often been the starting point for analysis (von Neumann and Morgenstern, 1953; Thrall and Lucas, 1963).

2. when $v_i = v \in (0, 1]$, no stable set exists (Rowat and Kerber, 2014);

3. when $v_i = v > 1$ for all three agents, $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ and its permutations form the unique stable set (Jordan and Obadia, 2015).

Second, can allocations in stable sets split the unit endowment between all three agents? If not, this limits applicability of the theory to situations in which the winner(s) 'take all'.

The rest of the paper is structured as follows. Section 2 introduces pillage games.[7] Theorem 1 identifies six possible allocations that may belong to the asymmetric three-agent majority pillage game's core; Corollary 1 identifies the corresponding values of (v_1, v_2, v_3) for each of the eight possible cores.

Section 3 analyses the empty core case, in which no agent is ever more powerful than the other two. Theorem 2 is a necessary and sufficient condition for the existence of a unique stable set comprising three allocations, each splitting the resource equally between two of the agents. Perhaps surprisingly, this is identical to the stable set in the symmetric special case. Intuitively, power struggles over these 50/50 allocations pit one agent (without resources) against another agent (with half the resources); as long as the former's intrinsic power is not too much greater than the latter's, the 50/50 split is defensible; otherwise, the stable set fails to exist.

Section 4 addresses the non-empty core case. As any stable set must contain the core, this yields a simple algorithm for deciding and computing stable sets. First, if a stable set exists, the core must belong to it. Second, allocations dominated by the core must be excluded. Third, the remaining allocations — typically loci within which one agent is as powerful as the other two — can be analysed using techniques from the empty core case.

These last allocations induce a tension between a stable set's existence and the possibility of interior allocations. Along the loci, power contests again reduce to those setting one agent against one other. For a strictly interior allocation to belong to a stable set, there must be a point on such loci at which the two contestants are equally powerful. As this allocation does not dominate its neighbouring allocations, some way of dominating those neighbours must be found. Typically (as will be seen below) this requires including the extremal elements of the locus, which lie on the simplex's edge. When the relevant $v_i > 0$, these extremal allocations are themselves dominated by core allocations, preventing existence.[8] If, though, no interior point balances the two contestants' power, one of the core allocations may also be extremal, and dominate all other allocations along the locus, establishing existence – but precluding interior allocations.

[7] More thorough introductions are found in Jordan (2006) and Rowat and Kerber (2014).

[8] Rowat and Kerber (2014) analysed this effect in detail in the symmetric case.

Section 5 concludes the paper. To answer our two primary questions: non-existence of stable sets is pervasive in this class of pillage games; strictly interior allocations only occur in special cases — including both new configurations and the symmetric case already known to the literature (Jordan, 2006, Theorem 3.3); when stable sets do exist, they are unique. Thus, for multiple stable sets in asymmetric majority pillage games, at least four agents are required (MacKenzie, Kerber, and Rowat, 2015). The link between the linear power function used here and the non-existence mechanism suggests that non-linear power functions may by better candidates for empirical tests of the theory.

2 Pillage games

Let I be a finite set of *agents*, indexed by i. An *allocation* is a division of a unit resource among them, so that the *feasible set of allocations* is

$$X \equiv \left\{ (x_i)_{i \in I} \middle| x_i \geq 0, \sum_{i \in I} x_i = 1 \right\}. \tag{2}$$

We index agents in two different ways. First, if we merely wish to label them, we use i, j and k or $1, 2$ and 3. Second, if we wish to order agents by their intrinsic contributions to their coalitions' power, we use a, b and c such that

$$v_a \geq v_b \geq v_c \geq 0. \tag{3}$$

If we display an allocation's constituent coordinates, we do so in the natural order implied by our choice of index, whether $\boldsymbol{x} = (x_1, x_2, x_3)$ or — if we wish to order agents by intrinsic power — $\boldsymbol{x} = (x_a, x_b, x_c)$. This does not imply that, for example, $x_a = x_1$.

Let $\subset$ denote a proper set inclusion, and use $\subseteq$ to allow the possibility of equality. Jordan (2006) defined a *power function* over subsets of agents and allocations, so that $\pi : 2^I \times X \to \mathbb{R}$ satisfies:

(WC) if $C \subset C' \subseteq I$ then $\pi(C', \boldsymbol{x}) \geq \pi(C, \boldsymbol{x}) \, \forall \boldsymbol{x} \in X$;

(WR) if $y_i \geq x_i$ for all $i \in C \subseteq I$ then $\pi(C, \boldsymbol{y}) \geq \pi(C, \boldsymbol{x})$; and

(SR) if $\emptyset \neq C \subseteq I$ and $y_i > x_i$ for all $i \in C$ then $\pi(C, \boldsymbol{y}) > \pi(C, \boldsymbol{x})$.

Simply, the axioms imply that more is better: by axiom WC, adding agents makes a coalition more powerful; by axiom WR, weakly adding resources makes a coalition weakly more powerful; axiom SR is a strict version of axiom WR. It is easily verified that function (1) is a power function.

The symmetry condition broken by this paper was introduced as an anonymity axiom in Rowat and Kerber (2014):

(AN) let $\sigma : I \to I$ be a bijective function permuting the agent set; if $i \in C \Leftrightarrow \sigma(i) \in C'$ and $x_i = x'_{\sigma(i)}$, then $\pi(C, \boldsymbol{x}) = \pi(C', \boldsymbol{x}')$.

This prevents power depending on the identity of agents, allowing it to depend merely on their cardinality and resources. As noted in the introduction, prior to MacKenzie, Kerber, and Rowat (2015), all previous examples of power functions studied by the literature satisfied it.

The *wealth is power* (WIP) function defined in Jordan (2006),

$$\pi_w(C, \boldsymbol{x}) = \sum_{i \in C} x_i; \tag{4}$$

is a special case of power function (1) in which $v_i = 0$ for all i.

An allocation $\boldsymbol{y}$ *dominates* an allocation $\boldsymbol{x}$, written $\boldsymbol{y} \succ \boldsymbol{x}$, iff

$$\pi(W, \boldsymbol{x}) > \pi(L, \boldsymbol{x});$$

where $W \equiv \{i \,|y_i > x_i\}$ and $L \equiv \{i \,|x_i > y_i\}$ are called the *win set* and *lose set*, respectively. Thus, allocation $\boldsymbol{y}$ dominates allocation $\boldsymbol{x}$ if and only if the set of agents who benefit in a re-allocation from $\boldsymbol{x}$ to $\boldsymbol{y}$ (the win set, W) are more powerful at the original allocation $\boldsymbol{y}$ than is the set of agents who lose from that re-allocation (the lose set, L). [9]

By the strict inequality, domination is irreflexive; by axiom SR, it is asymmetric. As in the general case (von Neumann and Morgenstern, 1953), dominance is not generally transitive.

For $Y \subset X$, let

$$D(Y) \equiv \{\boldsymbol{x} \in X \,|\exists \boldsymbol{y} \in Y \text{ s.t. } \boldsymbol{y} \succ \boldsymbol{x}\} \tag{5}$$

be the *dominion* of Y, the set of allocations dominated by an allocation in Y.

This paper studies asymmetric three player majority pillage games, defined as follows:

Definition 1. *An* asymmetric $n = 3$ majority pillage game *is a profile* $\langle I, X, (v_i)_{i \in I}, \pi \rangle$, *where* $I = \{1, 2, 3\}$, *X is defined by equation* (2), $(v_i)_{i \in I}$ *are non-negative reals, and the power function* π *is defined by* (1).

2.1 The core

The *core* is the set of undominated allocations, $K \equiv X \backslash D(X)$.

Now define two types of allocation:

[9] When referring to two allocations in the following, we may use W and L as a shorthand to indicate the agents benefiting and losing, respectively, from a move between them, even if we do not explicitly define them as such.

Definition 2. *(Jordan, 2006): let $\boldsymbol{t}^i \in X$ be a* tyrannical *allocation such that $t^i_i = 1$ and $t^i_j = 0$ for all $j \neq i \in I$.*

Let $\boldsymbol{b}^{ij}$ bilaterally balance *power, so that $\pi\left(\{i\}, \boldsymbol{b}^{ij}\right) = \pi\left(\{j\}, \boldsymbol{b}^{ij}\right)$ when $b^{ij}_k = 0$ for distinct agents i, j and k.*

Thus, given power function (1), we have:

$$b^{ij}_i = \tfrac{1}{2}\left(1 - v_i + v_j\right). \tag{6}$$

Then:

Theorem 1. *In the asymmetric $n = 3$ majority pillage game, the core can contain no allocations other than $\boldsymbol{t}^1, \boldsymbol{t}^2, \boldsymbol{t}^3, \boldsymbol{b}^{12}, \boldsymbol{b}^{13}$ and $\boldsymbol{b}^{23}$:*

1. *$\boldsymbol{t}^i$ belongs to the core iff $v_i \geq v_j + v_k - 1$; and*
2. *$\boldsymbol{b}^{ij}$ belongs to the core iff $v_k = 0$ and $v_j \in (v_i - 1, v_i + 1)$.*

Proof. First, consider the allocations that assign the resource exclusively to a single agent. These are, by definition, $\boldsymbol{t}^1, \boldsymbol{t}^2$ and $\boldsymbol{t}^3$. Each $\boldsymbol{t}^i$ belongs to the core when $v_i \geq v_j + v_k - 1$.

Now consider the allocations that split the resource between two agents, say i and j, so that $x_k = 0$. For these agents to defend their holdings against the other two, the following inequalities must be satisfied:

$$x_i + v_i \geq x_j + v_j + v_k;$$
$$x_j + v_j \geq x_i + v_i + v_k.$$

Combining these requires $x_j + v_j + 2v_k \leq x_i + v_i + v_k \leq x_j + v_j$. As all the numbers are non-negative, this forces $v_k = 0$. Thus, $x_i + v_i = x_j + v_j > x_k = 0$ which, with the unit endowment constraint, yields $x_i = \frac{1}{2}\left(1 - v_i + v_j\right)$ and $x_j = \frac{1}{2}\left(1 + v_i - v_j\right)$, which yield the $\boldsymbol{b}^{ij}$. Constraining b^{ij}_i and b^{ij}_j to lie in $(0, 1)$ yields the interval condition for v_j in the statement of the Theorem.

Finally consider allocations that split the resource between all three agents. By the same reasoning as above, these now require

$$2x_i + x_j + 2v_i + v_j \leq x_i + v_i + x_k + v_k \leq x_j + v_j;$$

for all distinct agents i, j and k. This requires, in the case above, $x_i = v_i = 0$; by permutation, these conditions must apply to all agents, an impossibility by the unit endowment.

The conditions under which the allocations belong to the core are direct consequences of the requirement that each agent holding resources must be at least as powerful as the other two combined. □

As a $\boldsymbol{b}^{ij}$ can belong to the core only if both $\boldsymbol{t}^i$ and $\boldsymbol{t}^j$ do, the following correspondences between cores and parameters holds:

Corollary 1. *1. $K = \emptyset$:*

$$v_a < v_b + v_c - 1. \tag{7}$$

2. $K = \{\boldsymbol{t}^a\}$:

$$v_a \geq v_b + v_c - 1, v_b < v_a + v_c - 1. \tag{8}$$

3. $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b\right\}$:

$$(v_b \geq v_a + v_c - 1) \textit{ and } (0 < v_c < v_a + v_b - 1); \textit{ or} \tag{9}$$

$$v_c = 0 < v_b = v_a - 1. \tag{10}$$

4. $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{b}^{ab}\right\}$:

$$v_c = 0, v_b \in (v_a - 1, v_a]. \tag{11}$$

5. $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c\right\}$:

$$v_c \geq v_a + v_b - 1 \textit{ and } v_c > 0; \textit{or} \tag{12}$$

$$v_a = 1 > v_b = v_c = 0. \tag{13}$$

6. $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}\right\}$:

$$v_c = 0 < v_b \leq 1 - v_a. \tag{14}$$

7. $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}, \boldsymbol{b}^{ac}\right\}$:

$$0 = v_c = v_b < v_a \leq 1. \tag{15}$$

8. $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}, \boldsymbol{b}^{ac}, \boldsymbol{b}^{bc}\right\}$:

$$v_c = v_b = v_a = 0. \tag{16}$$

The corollary follows directly from the theorem's requirement that each agent holding resources must be at least as powerful as the other two combined. Each case enumerated simply instantiates the theorem's conditions for the inclusion or exclusion of the $\boldsymbol{t}^i$ and $\boldsymbol{b}^{ij}$. Case 4 modifies the condition for $\boldsymbol{b}^{ab}$'s inclusion in light of $v_a \geq v_b$.

Note that the core excludes the following configurations:

$$\left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ac}\right\}, \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{bc}\right\}, \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}, \boldsymbol{b}^{bc}\right\}, \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ac}, \boldsymbol{b}^{bc}\right\}.$$

Further note that, under the conditions of inequalities and equalities in (10), $\boldsymbol{b}^{ab} = \boldsymbol{t}^b$.

2.2 Stable sets

The *stable set* is the original von Neumann and Morgenstern (1953) solution concept, initially just called the 'solution'. Unlike the core, which is defined pointwise, it is defined setwise, making it harder to compute. Intuitively, a set of allocations is stable if they satisfy internal stability (no allocation in a stable set dominates another) and external stability (every allocation outside a stable set is dominated by at least one allocation in a stable set).

More precisely, a set of allocations, $S \subseteq X$, is a *stable set* iff it satisfies *internal stability*,

$$S \cap D(S) = \emptyset; \tag{IS}$$

and *external stability*,

$$S \cup D(S) = X. \tag{ES}$$

The conditions combine to yield $S \equiv X \backslash D(S)$.

While stable sets may not exist, or may be non-unique, the core necessarily belongs to any stable set; when the core also satisfies external stability, it is the unique stable set. Jordan (2006) proved that a pillage game's stable set has the property of being the set of allocations that are undominated given a consistent set of expectations about what subsequent domination operation would be attempted following the initial one.

3 The empty core

The analysis in this section extends that in Jordan and Obadia (2015) to the asymmetric case. The core is empty when inequality (7) holds, so that even $\boldsymbol{t}^a$ is not defensible.

The following lemma generalises Jordan and Obadia (2015, Lemma 3.4(i)) beyond its symmetric case of $v_1 = v_2 = v_3$:

Lemma 1. *Suppose that S is internally stable in an $n = 3$ asymmetric majority pillage game with an empty core. If $\boldsymbol{x}$ and $\boldsymbol{x}'$ belong to S, then there exists an $i \in I$ such that $x_i = x'_i$. Equivalently, an empty core implies that, if distinct $\boldsymbol{x}$ and $\boldsymbol{x}'$ belong to S, then $W \equiv \left\{i \in I \middle| x_i > x'_i\right\}$ and $L \equiv \left\{i \in I \middle| x_i < x'_i\right\}$ are necessarily singletons.*

Proof. Assume, contrary to the lemma, that there exist $\boldsymbol{x}, \boldsymbol{x}' \in S$ such that $x_i \neq x'_i$ for all $i \in I$. Then, without loss of generality, we generically have that $x_i > x'_i$, $x_j < x'_j$ and $x_k < x'_k$. This yields a contradiction:

1. the empty core property requires that $x_i + v_i < 1 - x_i + v_j + v_k$ for all $\boldsymbol{x} \in X$ and distinct $i, j, k \in I$;

2. internal stability requires that $\boldsymbol{x}' \not\leftarrow \boldsymbol{x}$, so that $x_i + v_i \geq 1 - x_i + v_j + v_k$.

Thus, for two internally stable allocations there must, therefore, exist at least one agent for whom $x_i = x_i'$. When these allocations are distinct, $x_j - x_j' = -\left(x_k - x_k'\right) \neq 0$, so that W and L as defined above are singletons that exclude i. □

Intuitively, the empty core property ensures that the loci of allocations at which one agent is just as powerful as the other two lies outside the set of feasible allocations, X. Internal stability requires that all pairs of allocations in an internally stable set lie on opposite sides of the balance of power locus equating the power of W with that of L (q.v. Beardon and Rowat, 2013). Thus, the empty core means that pairs of allocation that align two agents against one cannot belong to an internally stable set. The relevant balances of power can therefore only involve two agents, so that each bilateral comparison requires the indifference of a third agent.

The following lemma generalises Jordan and Obadia (2015, Lemma 3.5(i)) beyond the symmetric case of $v_1 = v_2 = v_3$:

Lemma 2. *Suppose that S is internally stable in an $n = 3$ asymmetric majority pillage game with an empty core. Then S has no more than three elements.*

The proof establishes that, under the lemma's conditions, an internally stable set cannot contain four allocations. This prevents it having more than four, establishing the result. Specifically, the proof rules out the possibility of more than three allocations for which agent i's share is constant: as the win and lose sets W and L (both singletons), are constant over such allocations, there cannot – by axiom SR – be two allocations on one side of the locus of allocations such that $\pi(W, \boldsymbol{x}) = \pi(L, \boldsymbol{x})$. If there were, one of the sets would be able to enforce a move from one to the other.[10]

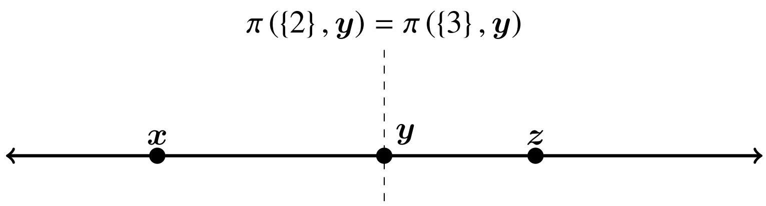

Figure 1: Collinear, internally stable $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ with $x_1 = y_1 = z_1$

[10]The proofs of Theorem 2.9 in Jordan (2006) and Theorem 1 of Kerber and Rowat (2011) used similar reasoning.

Proof. Consider three points, $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$, in an internally stable set S. By Lemma 1, the line running between any two of these must be parallel to the edge of the simplex X. Without loss of generality, there are two possible configurations:

1. $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ are collinear, as depicted in Figure 1, with (wlog) $x_1 = y_1 = z_1, x_2 > y_2 > z_2$ and $x_3 < y_3 < z_3$. As $\pi(\{2\}, \boldsymbol{x}) > \pi(\{2\}, \boldsymbol{y}) > \pi(\{2\}, \boldsymbol{z})$ and $\pi(\{3\}, \boldsymbol{x}) < \pi(\{3\}, \boldsymbol{y}) < \pi(\{3\}, \boldsymbol{z})$, internal stability requires that $\pi(\{2\}, \boldsymbol{y}) = \pi(\{3\}, \boldsymbol{y})$. Now seek to place a distinct $\boldsymbol{w} \in S$:
   (a) it cannot be that $w_1 = x_1$ as this would leave $\boldsymbol{w}$ with another allocation on the same side of the balance of power locus between agents 2 and 3 (q.v. Figure 1): the more extreme of those two allocations would dominate the other (Beardon and Rowat, 2013).
   (b) it cannot be that $w_1 \neq x_1$. By Lemma 1, this would require both that $w_2 = x_2$ and $w_3 = y_3$ and that $\boldsymbol{w}$ and $\boldsymbol{z}$ share a coordinate. This latter condition cannot be fulfilled: $w_1 \neq x_1 = z_1$; $w_2 = z_2 \Rightarrow x_2 = z_2$, a contradiction; $w_3 = z_3 \Rightarrow y_3 = z_3$, again a contradiction.

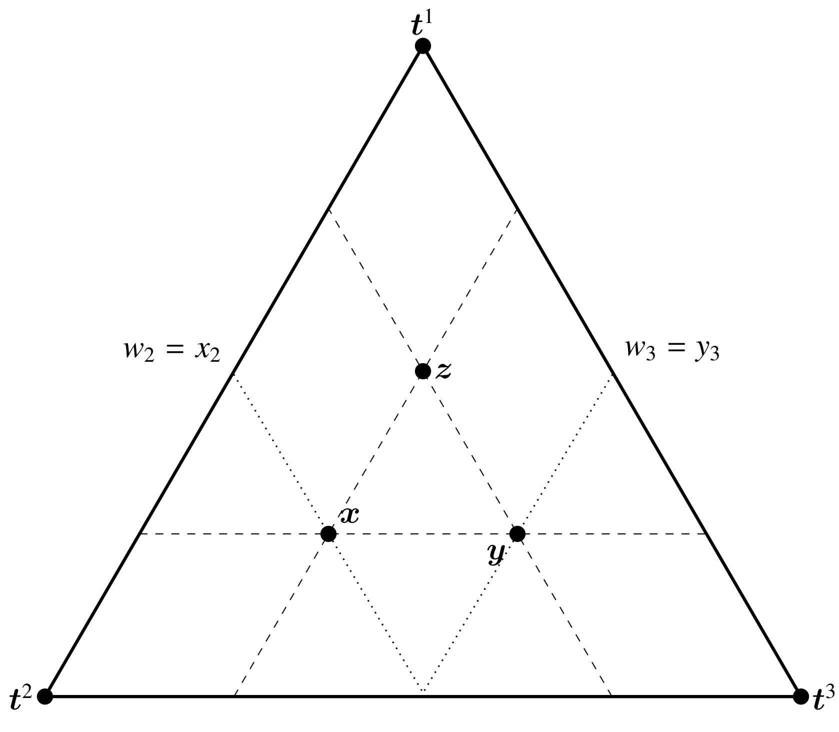

Figure 2: Triangular, internally stable $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ with $x_1 = y_1, y_2 = z_2$ and $z_3 = x_3$

2. $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ are triangular, as depicted in Figure 2, with $x_1 = y_1, y_2 = z_2, z_3 = x_3$. Again, seek to place a distinct $\boldsymbol{w} \in S$:

(a) $\boldsymbol{w}$ cannot lie on any of the three lines passing through any two of $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ as this would return us to the second subcase, above. This rules out $w_1 = x_1$, $w_2 = y_2$ or $w_3 = z_3$.

(b) by the previous subcase, we are only left with $w_2 = x_2$, the dotted line passing through $\boldsymbol{x}$ in Figure 2. By Lemma 1, there must be an agent $i \neq 2$ such that $w_i = y_i$. It cannot be $i = 1$: if $w_1 = x_1$ then, as $x_1 = y_1$ in this case, it would follow that $\boldsymbol{w} = \boldsymbol{x}$. This requires $w_3 = y_3$, the dotted line through $\boldsymbol{y}$ in Figure 2. Finally, Lemma 1 forces $w_1 = z_1$. We shall now see that this forces a contradiction:

$$\begin{aligned} w_1 + w_2 &= z_1 + x_2 && \text{by the implications of Lemma 1 derived above;} \\ &= 1 - w_3 && \text{by feasibility of } \boldsymbol{w}; \\ &= 1 - y_3 && \text{by the implication of Lemma 1 derived above;} \\ &= y_1 + y_2 && \text{by feasibility of } \boldsymbol{y}; \\ &= x_1 + z_2 && \text{by the definition of this case.} \end{aligned}$$

The right hand sides of the first and last lines allow us to define

$$k \equiv x_1 - x_2 = z_1 - z_2.$$

As $\boldsymbol{w}$ and $\boldsymbol{z}$ are feasible,

$$x_1 + x_2 + x_3 - k = z_1 + z_2 + z_3 - k \Rightarrow 2x_2 + x_3 = 2z_2 + z_3$$

where the implication follows from the definition of k. As this case requires $x_3 = z_3$, the previous equality forces $x_2 = z_2$, so that $\boldsymbol{x} = \boldsymbol{z}$, the contradiction.

We have therefore eliminated the possible configurations for four allocations in an internally stable set. This also disallows more than four such allocations, establishing the result. □

Figure 2 illustrates the lemma's second case: once $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ are placed, in a triangle, there is no place for a distinct $\boldsymbol{w}$ that shares a coordinate with each of them.

Definition 3. *Let $\boldsymbol{s}^{ij}$ be the allocation that* splits *the resource equally between agents i and j, so that $s_i^{ij} = s_j^{ij} = \frac{1}{2}$.*

Thus, unlike the $\boldsymbol{b}^{ij}$ previously defined, the $\boldsymbol{s}^{ij}$ need not balance power.

Then:

Lemma 3. *Suppose that S is stable in an $n = 3$ asymmetric majority pillage game with an empty core. Then $S = \left\{\boldsymbol{s}^{12}, \boldsymbol{s}^{13}, \boldsymbol{s}^{23}\right\}$.*

Proof. The largest possible S contains, by Lemma 2, three elements, $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$. Consider that possibility first.

By Lemma 1, there are again two possible configurations of three internally stable allocations:

1. *linear*, so that (wlog) $\bar{x} \equiv x_1 = y_1 = z_1$, as in Figure 1. Further, let $x_2 > y_2 > z_2$, so that $x_3 < y_3 < z_3$. It follows that $\boldsymbol{x} = (\bar{x}, 1 - \bar{x}, 0)$. Were it otherwise, there would exist a feasible $\boldsymbol{x}'$ with $x_2' > x_2$, $x_3' < x_3$ and $x_1' = \bar{x}$. External stability requires it to be dominated by one of $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$. All three possibilities require the same conditions:

$$\pi\left(\{3\}, \boldsymbol{x}'\right) > \pi\left(\{2\}, \boldsymbol{x}'\right) \Leftrightarrow x_3' + v_3 > x_2' + v_2.$$

   However, internal stability's requirement that $\boldsymbol{y} \not\succ \boldsymbol{x}$ implies $x_3 + v_3 \leq x_2 + v_2$, contradicting — by definition of $\boldsymbol{x}'$ — the above inequality.

   Similar reasoning establishes that $\boldsymbol{z} = (\bar{x}, 0, 1 - \bar{x})$.

   Now consider $\boldsymbol{x}'' \equiv \frac{1}{2}(\boldsymbol{y} + \boldsymbol{z}) + (\varepsilon, 0, -\varepsilon)$ for small $\varepsilon > 0$. (As $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ are distinct, $x_1 < 1$ so that $\boldsymbol{x}'' \in X$.) None of $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ dominate $\boldsymbol{x}''$:

   (a) for $\boldsymbol{z}$ to dominate $\boldsymbol{x}''$, it must be that $\pi\left(\{3\}, \boldsymbol{x}''\right) > \pi\left(\{1,2\}, \boldsymbol{x}''\right)$. As, by the empty core property, this inequality does not hold even at $\boldsymbol{t}^2$, it cannot – by axiom SR – hold at $\boldsymbol{x}''$ either.

   (b) for $\boldsymbol{x}$ or $\boldsymbol{y}$ to dominate $\boldsymbol{x}''$, it must be that $\pi\left(\{2\}, \boldsymbol{x}''\right) > \pi\left(\{1,3\}, \boldsymbol{x}''\right)$. The same reasoning as in the previous case excludes this.

   Thus, this case fails to satisfy external stability.

2. *triangular*. We first establish that each of the three allocations must set one component to zero. Suppose otherwise, so that $\boldsymbol{x} > \boldsymbol{0}$, with $x_1 = y_1, y_2 = z_2$, and $z_3 = x_3$, as in Figure 2. Then there exists a feasible $\boldsymbol{x}''' \equiv \boldsymbol{x} + (\varepsilon, -2\varepsilon, \varepsilon)$ for small $\varepsilon > 0$. This is undominated by $\boldsymbol{x}, \boldsymbol{y}$ or $\boldsymbol{z}$.

   First consider $\boldsymbol{x} \not\succ \boldsymbol{x}'''$: let $W \equiv \left\{i \middle| x_i > x_i'''\right\} = \{2\}$ and $L \equiv \left\{i \middle| x_i''' > x_i\right\} = \{1, 3\}$, so that $\boldsymbol{x} \succ \boldsymbol{x}'''$ requires

$$x_2''' + v_2 > x_1''' + v_1 + x_3''' + v_3.$$

   An empty core implies that $1 + v_2 < v_1 + v_3$ which — as $x_2''' \leq 1$ and $x_1''' + x_3''' \geq 0$ — contradicts the previous domination inequality.

   Establishing that neither $\boldsymbol{y} \succ \boldsymbol{x}'''$ nor $\boldsymbol{z} \succ \boldsymbol{x}'''$ makes use of a case distinction:

(a) In addition to the above, let $x_2 < y_2 = z_2$. In this case, demonstration that $\boldsymbol{y} \not\succ \boldsymbol{x}'''$ and $\boldsymbol{z} \not\succ \boldsymbol{x}'''$ is identical: both set $W = \{2\}$ and $L = \{1, 3\}$, and both lead – following calculations like those for $\boldsymbol{x} \not\succ \boldsymbol{x}'''$ – to contradiction.

(b) Now let $x_2 > y_2 = z_2$. Then, for $\boldsymbol{y} \not\succ \boldsymbol{x}'''$, $W = \{3\}$ and $L = \{1, 2\}$, yielding the same type of contradiction as before. For $\boldsymbol{z} \not\succ \boldsymbol{x}'''$, $W = \{1\}$ and $L = \{2, 3\}$, but a contradiction is again produced in the same way.

By Lemma 1, we need only consider two types (without loss of generality) of triangular configurations; as just established, at least one component of each allocation must be zero, leaving:

(a) $\boldsymbol{x} = (a, 0, 1 - a)$, $\boldsymbol{y} = (a, 1 - a, 0)$, and $\boldsymbol{z} = (0, 1 - a, 1 - a)$, so that – by the resource constraint – $a = \frac{1}{2}$. This is $\left\{\boldsymbol{s}^{12}, \boldsymbol{s}^{13}, \boldsymbol{s}^{23}\right\}$, the lemma's result.

(b) $\boldsymbol{x} = \boldsymbol{t}^1, \boldsymbol{y} = \boldsymbol{t}^2$ and $\boldsymbol{z} = \boldsymbol{t}^3$. This, however, cannot satisfy external stability as, for any internal feasible allocation, only one agent benefits from a move to one of the tyrannical allocations, while two lose. As the core is empty, this cannot allow dominance.

Now consider the possibility that S contains only two elements. This suffers from the same failure of external stability as the linear case considered above, so may be eliminated.

Finally, an S that contains only a single element is even less able to satisfy external stability than one containing that element and a second, a case which we have just eliminated. □

To summarise the proof, Lemma 1 leaves two possible configurations for a three-element stable set, the linear and the triangular. Linear sets of three elements necessarily fail to dominate some allocations lying off their line, violating external stability. Each allocation in a triangular configuration must set at least one term to zero: if not, the allocations will leave undominated some more extremal allocations; the equal split configuration then also ensures that the allocations between them are dominated.

The main result of this section generalises Jordan and Obadia (2015, Theorem 3.7) beyond the symmetric case:

Theorem 2. *For an $n = 3$ asymmetric majority pillage game with an empty core, $v_a \leq \frac{1}{2} + v_c$ is necessary and sufficient for internal stability of $S = \left\{\boldsymbol{s}^{12}, \boldsymbol{s}^{13}, \boldsymbol{s}^{23}\right\}$. Empty core condition (7) suffices for the external stability of S, leaving it the unique stable set when it is internally stable.*

Proof. By Lemma 3, the only candidate stable set is $S = \left\{\boldsymbol{s}^{12}, \boldsymbol{s}^{13}, \boldsymbol{s}^{23}\right\}$.

Internal stability requires that no allocation in S dominates any other. Thus, internal stability implies $v_i \leq \frac{1}{2} + v_j$ for all i and j. By inequality (3)'s ordering of v_a, v_b and v_c, this is equivalent to $v_a \leq \frac{1}{2} + v_c$, the lemma's stated inequality. Now establish the other direction, that this inequality implies internal stability, thus preventing even the easiest possible domination fails within S fails. By the resource monotonicity axioms, it must be that $\boldsymbol{s}^{ab} \not\succ \boldsymbol{s}^{bc}$: the agent with the greatest intrinsic strength (but no resources) cannot defeat that with the least intrinsic strength (but with half the resources); this is equivalent to the lemma's inequality.

External stability requires that each $\boldsymbol{x} \in X \backslash S$ is dominated by an $\boldsymbol{s} \in S$. For any such $\boldsymbol{x}$ there are distinct agents such that $x_j, x_k < \frac{1}{2}$. Further, by inequality (3), empty core inequality (7) implies that $1 + v_i < v_j + v_k$ for these agents j and k. As $x_i \leq 1$ and $x_j, x_k \geq 0$, it follows that $x_i + v_i < x_j + v_j + x_k + v_k$, so that $\boldsymbol{s}^{jk} \succ \boldsymbol{x}$, establishing external stability. □

This result generalises that in Jordan and Obadia (2015), the symmetric case, when $n = 3$. It may be surprising that the allocations in the stable set do not vary with v_i until the set ceases to exist. Intuitively, the empty core condition limits the relative strength of the intrinsically strongest agent. This prevents any single agent overpowering the other two for any allocation. As a result, the only power contests that must be considered are those between singleton coalitions. The theorem's bound on relative power ensures that the intrinsically least powerful agent with half of the resource can defend itself against the most powerful with none of the resource. Thus, power need not be equally balanced at the $\boldsymbol{s}^{ij}$ allocations: while v_a increases relative to v_c, there is no distortion in S until the asymmetry becomes too large, and internal stability fails completely.

4 The non-empty core

This section derives the stable sets (if any) corresponding to each of the possible non-empty cores identified in Corollary 1.

As the core must be included in any stable set, the proofs follow a similar pattern: the core seeds a candidate stable set; the allocations dominated by the core are excluded from consideration; over the remaining allocations (often just loci) it is as if the core is empty, allowing the use of techniques from the previous section. Of particular importance is the locus of allocations along which the most intrinsically powerful agent, a, is just as powerful as the other two, b, c; we prove that the existence of a stable set on X requires existence a stable set along that locus. When non-existence arises, it is for the same reason identified in the symmetric case in Rowat and Kerber (2014): allocations on these loci are only dominated by more extreme ones; thus, external stability requires including the most extreme allocations in a stable set; when these extreme allocations are themselves dominated by a core allocation, existence fails.

Before proceeding, we present versions of Lemmas 1 and 2 that apply to the non-empty core case:

Lemma 4. *Suppose that S is internally stable in an $n = 3$ asymmetric majority pillage game with a non-empty core K. If $\boldsymbol{x}$ belongs to $S \backslash K$ and $\boldsymbol{x}'$ belongs to S, then there exists an $i \in I$ such that $x_i = x'_i$.*

Proof. For all $\boldsymbol{x} \in S \backslash K$, it holds that $x_i + v_i < 1 - x_i + v_j + v_k$ for any labeling of agents i, j and k. The rest of the proof follows that of Lemma 1. □

Lemma 5. *Suppose that S is internally stable in an $n = 3$ asymmetric majority pillage game. Then the maximal number of allocations in S is three plus the number of allocations in K, the core.*

Proof. For all the elements in $S \backslash K$, the proof of Lemma 2 applies, setting an upper bound of three non-core allocations in S. As inclusion of core elements makes the internal stability requirement on S more demanding, the upper bound on S cannot be more than three plus the number of core allocations. □

The following lemma helps establish the result that existence of a stable set on X requires the existence of a stable set on the balance of power locus

$$B(i) \equiv \{\boldsymbol{x} \in X \,|\pi(\{i\}, \boldsymbol{x}) = \pi(\{j, k\}, \boldsymbol{x})\} \tag{17}$$

when $B(a) \neq \emptyset$:

Lemma 6. *Consider an $n = 3$ asymmetric majority pillage game with non-empty core, and $B(a) \neq \emptyset$. Then $\boldsymbol{x} \in B(a) \Rightarrow \nexists \boldsymbol{y} \in X \backslash B(a)$ such that $y_a < x_a$ and $\boldsymbol{y} \succcurlyeq \boldsymbol{x}$.*

Proof. Suppose that such a $\boldsymbol{y}$ does exist and define $W \equiv \{i \in I \,|y_i > x_i\}$ and $L \equiv \{i \in I \,|y_i < x_i\}$ in the usual way.

As $y_a < x_a$, it follows that $\{a\} \subseteq L$, which implies $\pi(\{a\}, \boldsymbol{x}) \leq \pi(L, \boldsymbol{x})$. Similarly, $W \subseteq \{b, c\}$ implies that $\pi(W, \boldsymbol{x}) \leq \pi(\{b, c\}, \boldsymbol{x})$. Together, these imply (by equality (17) defining $B(a)$, to which $\boldsymbol{x}$ belongs) that $\pi(W, \boldsymbol{x}) \leq \pi(L, \boldsymbol{x}) \Rightarrow \boldsymbol{y} \not\succcurlyeq \boldsymbol{x}$, establishing the result. □

Corollary 2. *In an $n = 3$ asymmetric majority pillage game with non-empty core, and $B(a) \neq \emptyset$, the existence of a stable set on X requires the existence of a stable set on $B(a)$.*

Proof. A non-empty core must, by Theorem 1, contain $\boldsymbol{t}^a$. This, as previously argued, must be part of any stable set that exists, preventing allocations in $D(\boldsymbol{t}^a)$ belonging to such a set. This leaves the allocations in $\boldsymbol{x} \in B(a)$ and those $\boldsymbol{y}$ such that $y_a < x_a$ to consider. By Lemma 6, the $\boldsymbol{x} \in B(a)$ are undominated by the $\boldsymbol{y}$.

Thus, for a stable set on X to satisfy external stability, it must be that external stability holds on $B(a)$: for any $\boldsymbol{x}' \in B(a)$ that does not belong to S, there must be an $\boldsymbol{x} \in B(a)$ such that $\boldsymbol{x} \mathrel{\in\!\!-} \boldsymbol{x}'$. As stability of S also requires internal stability, then internal stability must also hold over the elements of S that are also in $B(a)$. □

Define

$$\begin{aligned} x_a &\equiv \frac{1}{2}(1 - v_a + v_b + v_c); \\ \boldsymbol{e}_b^a &\equiv (x_a, 1 - x_a, 0) \end{aligned} \tag{18}$$

so that any $\boldsymbol{x}$ with a^{th} component x_a belongs to $B(a)$, and $\boldsymbol{e}_b^a$ is an extremal member of $B(a)$, maximising x_b by setting $x_c = 0$.

Corollary 3. *Consider an $n = 3$ asymmetric majority pillage game with non-empty core and $B(a) \neq \emptyset$. Then, if a stable set S exists, $\boldsymbol{e}_b^a \in S$.*

Proof. Under the stated conditions, there does not exist an $\boldsymbol{x}' \in B(a)$ such that $\boldsymbol{x}' \mathrel{\in\!\!-} \boldsymbol{e}_b^a$. Consider otherwise: as $\boldsymbol{x}' \mathrel{\in\!\!-} \boldsymbol{e}_b^a$ is equivalent to $\pi\left(\{c\}, \boldsymbol{e}_b^a\right) > \pi\left(\{b\}, \boldsymbol{e}_b^a\right)$, it is equivalent to

$$0 + v_c > 1 - x_a + v_b.$$

This cannot hold with strict inequality: $x_a \in [0, 1)$ ensures that $1 - x_a \geq 0$ and inequality (3) ensures that $v_b \geq v_c$.

As no other allocation on $B(a)$ dominates $\boldsymbol{e}_b^a$, Corollary 2 requires that, if S exists, it must contain $\boldsymbol{e}_b^a$. □

Corollary 4. *Consider an $n = 3$ asymmetric majority pillage game with non-empty core and $B(a) \neq \emptyset$. Then $x_a < 1$ and $v_c > 0$ imply that no stable set exists.*

Proof. By Corollary 3, $\boldsymbol{e}_b^a$ must belong to any stable set under the conditions of this corollary.

As $\boldsymbol{e}_b^a \in B(a)$ it satisfies $\pi\left(\{a\}, \boldsymbol{e}_b^a\right) = \pi\left(\{b, c\}, \boldsymbol{e}_b^a\right)$ or, equivalently, $x_a + v_a = (1 - x_a) + v_b + v_c > (1 - x_a) + v_b$ where the inequality results from the corollary's condition that $v_c > 0$.

If the core is non-empty, then — by Theorem 1, it must contain $\boldsymbol{t}^a$. However, $\boldsymbol{t}^a \mathrel{\in\!\!-} \boldsymbol{e}_b^a$ as

$$x_a + v_a = (1 - x_a) + v_b + v_c > (1 - x_a) + v_b;$$

where the equality follows from the definition of $B(a)$, to which $\boldsymbol{e}_b^a$ belongs, and the inequality follows from the corollary's condition that $v_c > 0$. Internal stability of S cannot, therefore, be satisfied, yielding the result.

The corollary's inequality restriction on x_a ensures that $\boldsymbol{e}_b^a \neq \boldsymbol{t}^a$. □

Proofs for the following results are largely moved to the appendix.

4.1 The singleton core, $K = \{\boldsymbol{t}^a\}$

When the inequalities in (8) hold, the core consists of $\boldsymbol{t}^a$ only.

Theorem 3. *Consider an $n = 3$ asymmetric majority pillage game satisfying inequalities (8):*

1. *if $v_a > v_b + v_c + 1$, then the unique stable set is $S = \{\boldsymbol{t}^a\}$.*
2. *if $v_a \in v_b + v_c + (-1, 1]$, then no stable set exists.*
3. *if $v_a = v_b + v_c - 1$, then no stable set exists.*

4.2 $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b\right\}$

This case is defined by the parameter restrictions in (9) and (10).

Theorem 4. *Consider an $n = 3$ asymmetric majority pillage game satisfying conditions (9) or (10). Then:*

1. *if condition (10) holds and $v_a \geq 2$, then the unique stable set is $S = K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b\right\}$.*
2. *if condition (10) holds and $v_a \in (1, 2)$, then no stable set exists.*
3. *if condition (9) holds then no stable set exists.*

4.3 $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{b}^{ab}\right\}$

This case is defined by the parameter restrictions in (11).

Theorem 5. *Consider an $n = 3$ asymmetric majority pillage game satisfying conditions (11):*

1. *when $v_b \geq \frac{1}{3}(1 + v_a)$, the unique stable set is the core itself;*
2. *otherwise no stable set exists.*

4.4 The tyrannical core, $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c\right\}$

This case is defined by the parameter restrictions in (12) and (13).

Theorem 6. *Consider an $n = 3$ asymmetric majority pillage game satisfying either condition* (12) *or condition* (13)*.*

1. *when condition* (12) *holds, no stable set exists unless $v_a = v_b = v_c = 1$, in which case the unique stable set is $S = K \cup \left\{\boldsymbol{b}^{ab}, \boldsymbol{b}^{ac}, \boldsymbol{b}^{bc}\right\} = K \cup \left\{\boldsymbol{s}^{ab}, \boldsymbol{s}^{ac}, \boldsymbol{s}^{bc}\right\}$.*

2. *when condition* (13) *holds, the unique stable set is $S = K \cup \left\{\boldsymbol{b}^{bc}\right\} = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{bc}\right\} = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{s}^{bc}\right\}$.*

4.5 $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}\right\}$

This case is defined by the parameter restrictions in (14).

Theorem 7. *In an $n = 3$ asymmetric majority pillage game satisfying conditions* (14)*, no stable set exists unless $v_a = v_b = \frac{1}{2}$. In that remaining case, the unique stable set is the core.*

4.6 $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}, \boldsymbol{b}^{ac}\right\}$

This case is defined by the parameter restrictions in (15).

In these cases, define the midpoint of a balance of power locus, $\boldsymbol{m}^i \in B(i)$ so that

$$m_i^i = \frac{1}{2}(1 - v_i), m_j^i = m_k^i = \frac{1}{4}(1 + v_i). \tag{19}$$

Theorem 8. *In an $n = 3$ asymmetric majority pillage game satisfying conditions* (15)*:*

1. *when $v_a \geq \frac{1}{3}$, the unique stable set is $K \cup \{\boldsymbol{m}^a\}$;*

2. *otherwise no stable set exists.*

4.7 $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}, \boldsymbol{b}^{ac}, \boldsymbol{b}^{bc}\right\}$

This case is defined by the equalities in (16), which set $\boldsymbol{b}^{ij} = \boldsymbol{s}^{ij}$. Thus, this is the symmetric *wealth is power* case defined in Jordan (2006), which also derived its stable set for any n. In the present $n = 3$ case, the result — illustrated in Figure 3 — reduces to:

Theorem 9. *Jordan (2006, Theorem 3.3) In the $n = 3$ wealth is power pillage game with $v_i = 0$ for all $i = 1, 2, 3$, the unique stable set is*

$$S = \left\{\boldsymbol{t}^1, \boldsymbol{t}^2, \boldsymbol{t}^3, \boldsymbol{b}^{12}, \boldsymbol{b}^{13}, \boldsymbol{b}^{23}, \boldsymbol{m}^1, \boldsymbol{m}^2, \boldsymbol{m}^3\right\}.$$

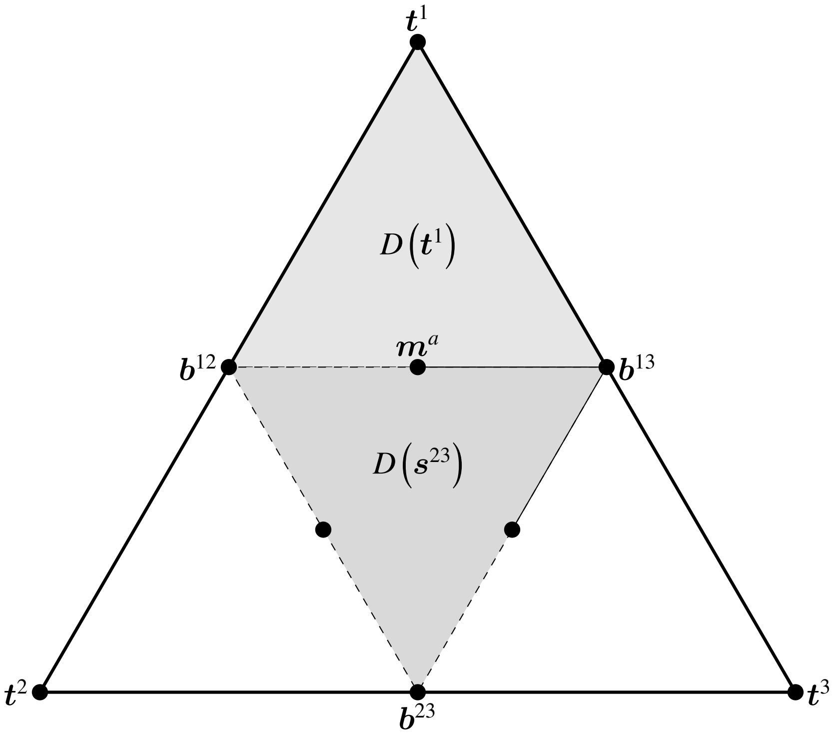

Figure 3: Wealth is power: $\pi(C, \boldsymbol{x}) = \sum_{i \in C} x_i$

5 Discussion

This paper removes the symmetry (or anonymity) assumption usually assumed in analysis of pillage games. It is motivated by a hope of identifying a tractable class of pillage games to allow empirical tests of this theory of unstructured power contests.

It finds, first, that stable sets are unique — when they exist — in three agent asymmetric majority pillage games. MacKenzie, Kerber, and Rowat (2015) demonstrated that violating the anonymity axiom could lead to multiple stable sets in majority pillage games when the number of agents was at least four. This paper therefore establishes that as a lower bound for multiplicity in asymmetric majority pillage games.

Second, it suggests a link between linear power functions and winner(s) 'take all' contests: the only case in which the stable set contains interior allocations, $\boldsymbol{x} > \mathbf{0}$, are the special cases of $1 \geq v_a \geq v_b = v_c = 0$. This reflects the tension between existence of a stable set and strictly interior allocations along balance of power loci: when there is an interior allocation at which the two contestants' power is balanced along such a locus, its extremal elements must also belong to a stable set; if, though, the relevant $v_i > 0$, these elements may be dominated by a tyrannical allocation, preventing existence.

Third, it finds that non-existence is pervasive. Thus, the theory fails to deliver testable predictions for non-negligible sets of values of v_1, v_2 and v_3.

Tantalisingly, the theory yields predictions for almost the full range of cases in which $v_a \geq v_b = v_c = 0$ — so that agent a has some source of intrinsic power that b and c do not. This power could arise from greater internal cohesion, a structural advantage (e.g. perhaps being located more centrally). Figure 4 illustrates, with v_a increasing from 0 in the leftmost diagram to $v_a > 1$ in the rightmost. Filled dots represent allocations in a stable set.

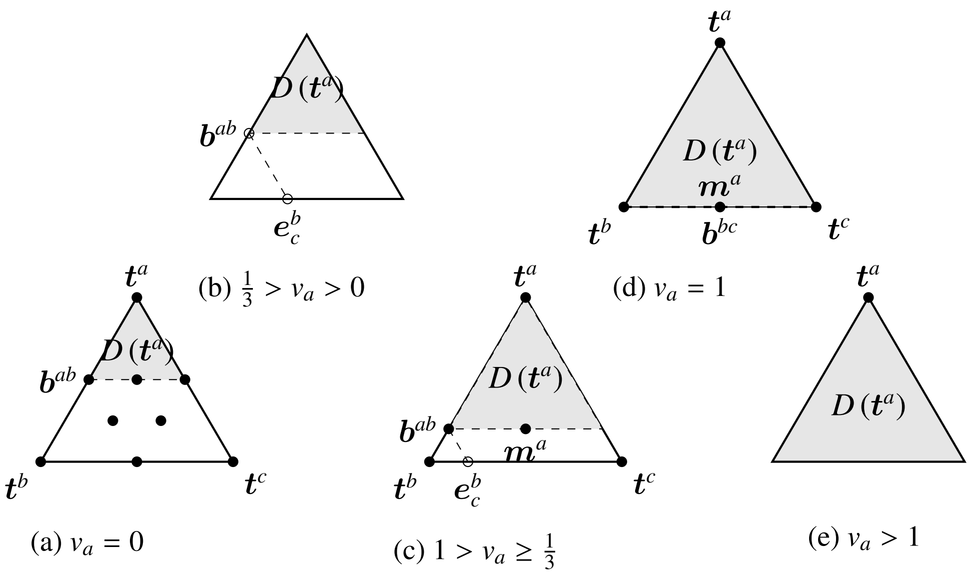

Figure 4: Stable sets when $v_a \geq v_b = v_c = 0$

When $v_a = 0$, the fully symmetric case depicted in diagram 4a, a stable set exists with interior allocations, as per Theorem 3.3 of Jordan (2006). As v_a increases, $D(\boldsymbol{t}^a)$ reaches further into the simplex, as depicted in diagram 4b, so that $\boldsymbol{b}^{ab}$ no longer dominates the extremal $\boldsymbol{e}_c^b$; no stable set exists, as per the second case in Theorem 8. When v_a grows beyond $\frac{1}{3}$ (q.v. diagram 4c), $\boldsymbol{b}^{ab}$ comes to dominate all allocations to the extremal $\boldsymbol{e}_c^b$; this allows the stable set in the first case of Theorem 8. When $v_a = 1$, $D(\boldsymbol{t}^a)$ pushes further down, dominating the whole simplex except its bottom margin (as depicted in diagram 4d); now the stable set in the second case of Theorem 6 exists. Finally, when $v_a > 1$, depicted in diagram 4e, $\boldsymbol{t}^a$ dominates the whole simplex, leaving it the singleton stable set, as per the first case of Theorem 3.

In conclusion, we mention three possible ways forward for an empirically-testable theory of pillage games.

First, production could be considered, so that assigning an agent nothing inefficiently excludes its production function from society. In the terminology of Olson (1993), production provides an incentive for bandits to be stationary rather than roving. Any study of production in pillage games should build on Jordan (2009), which gave each agent in a pillage game a production function for converting wealth into consumption goods. While production could be expected to favour interior allocations, it is less obvious that they would rescue existence.

Second, the tension between existence and interior allocations arises in part because the balance of power loci (such as the dashed lines in Figure 7) are linear: interior allocations like $\boldsymbol{x}$ and $\boldsymbol{y}$ do not dominate their neighbours as the loci restrict power contests to those pitting one agent against another. However, a power function like

$$\pi(C, \boldsymbol{x}) = \sum_{i \in C} \left(\sqrt{x_i} + v_i \right);$$

generates curved balance of power loci which allow the third agent (who would be indifferent along a linear balance of power locus) to benefit from moves towards the centre of the locus. Its involvement allows central allocations to dominate the extremes of the locus, rescuing existence and yielding strictly interior solutions. Figure 5 illustrates the symmetric special case in which $v_i = 0$ for all agents.

Third, solution concepts other than the stable set can be experimented with. In addition to the usual cooperative solution concepts, it would make sense to study the *legitimate set*, which extended the stable set to production pillage games (Jordan, 2009). Chaturvedi (2016) has applied farsighted concepts from Chwe (1994) to pillage games.

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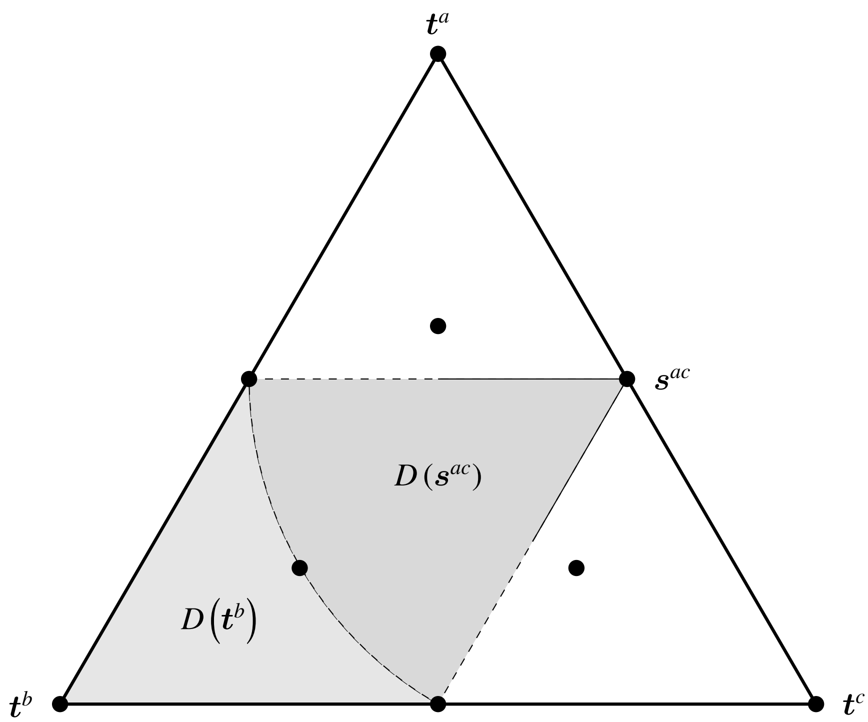

Figure 5: Strictly concave power: $\pi(C, \boldsymbol{x}) = \sum_{i \in C} \sqrt{x_i}$

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A Proofs

Proof of Theorem 3. Consider each case in turn:

1. The case's characterising inequality implies that $x_a < 0$. By inequalities (8), the game's core is the singleton $\boldsymbol{t}^a$. The additional inequality ensures that $\boldsymbol{t}^a$ dominates any allocation splitting the entire resource between the other two agents which – by axiom SR – is the most demanding allocation for $\boldsymbol{t}^a$ to dominate. Thus, $D(\boldsymbol{t}^a) = X \setminus \{\boldsymbol{t}^a\}$. This ensures that $\{\boldsymbol{t}^a\}$ is externally stable.

   Internal stability is trivial as $\{\boldsymbol{t}^a\}$ is a singleton.

   The ensuing stable set is unique as $\boldsymbol{t}^a$ must – as a core allocation – belong to any candidate stable set; once it is included, there is no further room for discretion.

2. The case's characterising inequality implies that $x_a \in [0, 1)$. It further implies $v_b > 0$: if $v_b = 0$, then this would force $v_c = 0$, reducing the case condition to $v_a \leq 1$, while the second inequality in (8) requires $v_a > 1$, a contradiction.

   Now consider the locus of allocations, $B(a)$, defined in equation (17) under the case's parameter restrictions. The parameter restrictions ensure that

$x_a \geq 0$ for those $\boldsymbol{x} \in B(a)$, with x_a attaining its lower bound when the case's inequality in v_a holds with equality.

By Corollary 2, existence of an S depends on the existence of a stable set on $B(a)$. Let the allocation at the other extreme of $B(a)$ than $\boldsymbol{e}_b^a$ be $\boldsymbol{e}_c^a = (x_a, 0, 1 - x_a)$. Then $\boldsymbol{e}_b^a \mathrel{\varepsilon\!\!-} \boldsymbol{e}_c^a \Leftrightarrow v_a < 3v_b - v_c - 1$, yielding two subcases to consider:

(a) $v_a \geq 3v_b - v_c - 1$. In this case, no allocation $\boldsymbol{y} \in B(a)$ dominates $\boldsymbol{e}_c^a$. Thus, the reasoning that held for $\boldsymbol{e}_b^a$ in Corollary 3 holds here for $\boldsymbol{e}_c^a$: $\boldsymbol{e}_c^a$ must belong to any candidate stable set, but the case's implication that $v_b > 0$ ensures that $\boldsymbol{t}^a \mathrel{\varepsilon\!\!-} \boldsymbol{e}_c^a$, ruling out existence.

(b) $v_a < 3v_b - v_c - 1$. Now $\boldsymbol{e}_b^a$ dominates all other allocations on $B(a)$, so that $\left\{\boldsymbol{e}_b^a\right\}$ is a stable set on $B(a)$. There are two sub-cases to address:

   i. $v_a = v_b + v_c + 1 \Leftrightarrow x_a = 0$, which implies — by definition of x_a — that $v_a = v_b + v_c + 1$. Together with the second inequality (8), this implies that $v_c > 0$ which, by Corollary 4 rules out existence.

   ii. $v_a \in v_b + v_c + (-1, 1) \Leftrightarrow x_a \in (0, 1)$. When $v_c > 0$, Corollary 4 again rules out existence, leaving the case of $v_c = 0$. This reduces the second inequality (8) to $v_a > v_b + 1$ while $x_a > 0$ implies $v_a < v_b + 1$. This contradiction rules out existence here as well.

3. $v_a = v_b + v_c - 1 \Leftrightarrow x_a = 1$, so that $B(a) = \{\boldsymbol{t}^a\}$. In this case, analysis proceeds as in the 'empty core' case, but over $X \backslash \{\boldsymbol{t}^a\}$ rather than over X. There are two sub-cases:

   (a) $v_b > \frac{3}{2}$, in which case the only candidate stable set over the empty core (by Lemma 3) fails to be internally stable (by Theorem 2). Thus, no stable set exists.

   (b) $v_b \leq \frac{3}{2}$, in which case the only candidate stable set over the empty core (by Lemma 3) is both internally and externally stable (by Theorem 2). Thus, the unique candidate stable set is $\left\{\boldsymbol{t}^a, \boldsymbol{s}^{ab}, \boldsymbol{s}^{ac}, \boldsymbol{s}^{bc}\right\}$. However, internal stability also requires that $\boldsymbol{t}^a \not\mathrel{\varepsilon\!\!-} \boldsymbol{s}^{ac}$, which reduces to $v_a \leq v_c$, only possible when $v_a = v_b = v_c$ — contradicting inequalities (8).

□

Proof of Theorem 4. Before considering the cases, note that

$$D(\boldsymbol{t}^a) \supseteq \{\boldsymbol{y} \in X \,|\pi(\{a\}, \boldsymbol{y}) > \pi(\{b, c\}, \boldsymbol{y})\} = \{\boldsymbol{y} \in X \,| y_a > x_a\}. \tag{20}$$

Additionally, recall that core allocations, here $\boldsymbol{t}^a$ and $\boldsymbol{t}^b$, must belong to any stable set.

Now consider the cases in turn:

1. Under the conditions in (10), $B(a) = \{\boldsymbol{x} \in X \,|x_a = 0\}$ so that $D(\boldsymbol{t}^a)$ includes $\{\boldsymbol{y} \in X \,|y_a > 0\}$. As $\boldsymbol{t}^a$ must belong to any stable set, and $D(\boldsymbol{t}^a)$ cannot (by internal stability), we may restrict our attention to those $\boldsymbol{y} \in X$ such that $y_a = 0$. One such allocation, $\boldsymbol{t}^b$, must also belong to any stable set. Of the remaining allocations with $y_a = 0$:

$$\begin{aligned} \boldsymbol{t}^b \succ \boldsymbol{y} &\Leftrightarrow y_b + v_b > 1 - y_b \\ &\Leftrightarrow 2y_b + v_a > 2; \end{aligned}$$

   where the second equivalence follows from inequalities (10). The $v_a \geq 2$ condition assumed in this case ensures this holds for all $y_b > 0$. When $y_b = 0$, then $\boldsymbol{y} = \boldsymbol{t}^c$; as $v_a \geq 2$ ensures that $\boldsymbol{t}^a \succ \boldsymbol{t}^c$, all allocations such that $y_a = 0$ are dominated either by $\boldsymbol{t}^a$ or $\boldsymbol{t}^b$, concluding the proof for this case.

2. As in the previous case: $B(a) = \{\boldsymbol{x} \in X \,|x_a = 0\}$. By Corollary 2, existence of a stable set on X requires existence of such a set on $B(a)$. As $\boldsymbol{t}^b \in B(a)$ belongs to the core, it must also belong to any stable set on $B(a)$. By the case's assumption that $v_a \in (0,1)$, $\boldsymbol{b}^{bc} = \left(0, 1 - \frac{1}{2}v_a, \frac{1}{2}v_a\right)$ lies strictly between $\boldsymbol{t}^b$ and $\boldsymbol{t}^c$, as depicted in Figure 6. As, by definition, $b^{bc}_b + v_b = b^{bc}_c + v_c$, $\boldsymbol{t}^b$ dominates all allocations in the interior of the interval between $\boldsymbol{t}^b$ and $\boldsymbol{b}^{bc}$. Thus, whether allocations in the interval $\left[\boldsymbol{b}^{bc}, \boldsymbol{t}^c\right]$ belong to S must be decided.

   As $v_a > 1 > v_c = 0$, $\boldsymbol{t}^a \succ \boldsymbol{t}^c$, leaving membership of allocations in the interval $\left[\boldsymbol{b}^{bc}, \boldsymbol{t}^c\right)$ to be decided. As any such allocation, say $\boldsymbol{x} = (0, x_b, 1 - x_b)$ is dominated by an allocation to its right, $\boldsymbol{x} + (0, -\varepsilon, \varepsilon)$ for small $\varepsilon > 0$, the rightmost such allocation must belong to S to satisfy external stability. As the interval is open at $\boldsymbol{t}^c$, this is impossible, yielding — by Corollary 2— the case's non-existence result.

3. To analyse this case, recall the locus of allocations $\boldsymbol{x} \in B(a)$ with x_a as defined in equation (18). Then:

   (a) $x_a = 0$ is impossible, as it requires $v_c \leq 0$ and $v_c > 0$ from inequalities (9).

   (b) $x_a = 1$ yields the 'empty core' analysis of case 3 of Theorem 3, and the same contradictory requirement that $v_a = v_b = v_c$. Again, this yields non-existence.

   (c) $x_a > 1$ is also impossible, as it implies that $\pi(\{a\}, \boldsymbol{t}^a) < \pi(\{b, c\}, \boldsymbol{t}^a)$, which is incompatible with $\boldsymbol{t}^a \in K$.

   The only remaining case to consider is therefore $x_a \in (0,1)$. As inequalities (9) require $v_c > 0$, non-existence of S is immediate by Corollary 4.

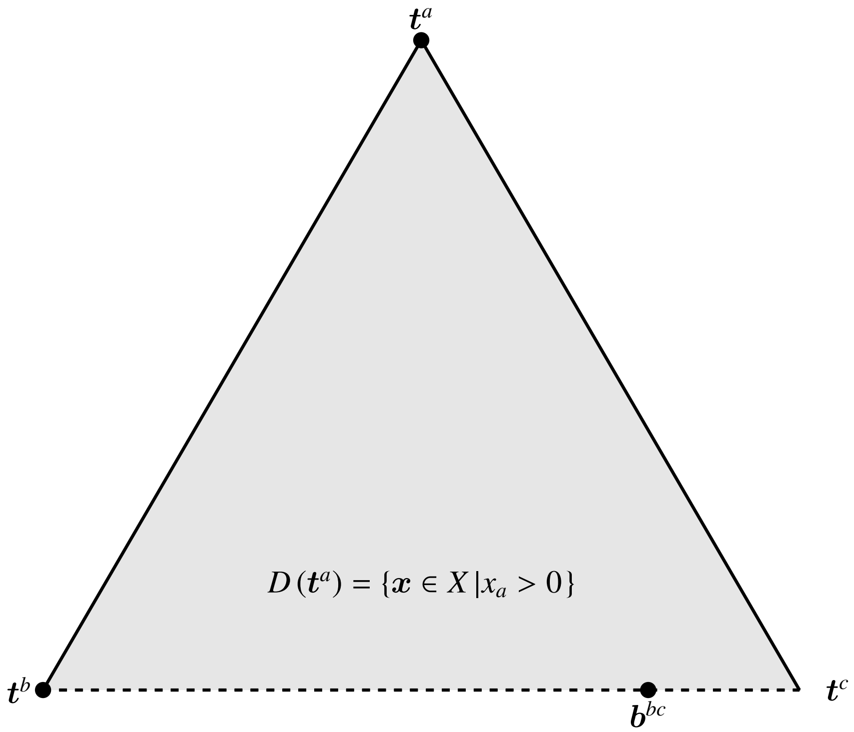

Figure 6: Constructing the stable set when $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b\right\}$

These cases are exhaustive: when conditions (10) hold, v_a must be strictly greater than one as $v_b = v_a - 1$ is strictly positive. □

Proof of Theorem 5. Recall that, by equation (6), $b_a^{ab} = \frac{1}{2}(1 - v_a + v_b)$. The inequalities in v_b in (11) ensure that $\boldsymbol{b}^{ab}$ is distinct from both $\boldsymbol{t}^a$ and $\boldsymbol{t}^b$, so that $b_a^{ab} \in (0, 1)$. As the core must belong to any stable set, we can exclude the following from any stable set:

$$D(\boldsymbol{t}^a) \supseteq \left\{\boldsymbol{y} \in X \,\middle|\, y_a \in \left(b_a^{ab}, 1\right)\right\};$$
$$D\left(\boldsymbol{t}^b\right) \supseteq \left\{\boldsymbol{y} \in X \,\middle|\, y_b \in \left(1 - b_a^{ab}, 1\right)\right\};$$
$$D\left(\boldsymbol{b}^{ab}\right) \supseteq \left\{\boldsymbol{y} \in X \,\middle|\, y_a < b_a^{ab} \text{ and } y_b < 1 - b_a^{ab}\right\}.$$

Only two loci remain to be considered, namely $B(a) \setminus \left\{\boldsymbol{b}^{ab}\right\}$ and $B(b) \setminus \left\{\boldsymbol{b}^{ab}\right\}$. Along each of these, the power contests induced oppose a single agent against another single agent: agent b versus c in the first case, and a versus c in the second. If the former is strictly more powerful than the latter for any such allocation, then those allocations are all dominated by $\boldsymbol{b}^{ab}$, so that the core itself satisfies external stability, and is the unique stable set.

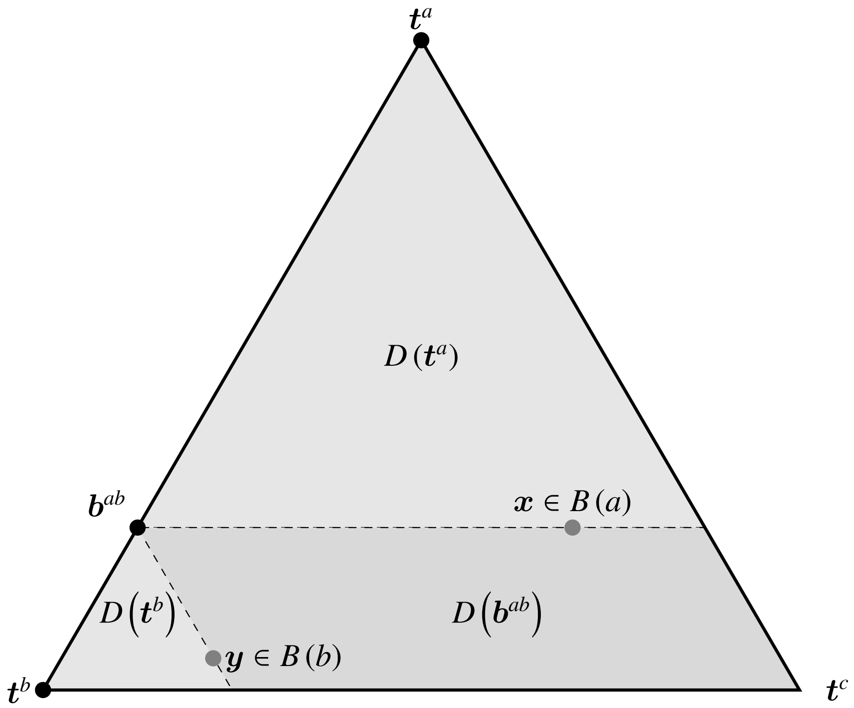

Figure 7: Constructing the stable set when $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{b}^{ab}\right\}$

We proceed by case analysis on the relative size of v_b and $\frac{1}{3}(1 + v_a)$:

1a. $v_b > \frac{1}{3}(1 + v_a)$, as illustrated in Figure 7. Let $\boldsymbol{x}$ be the allocation in $B(a)$ such that b and c are equally powerful. As $x_a = b_a^{ab}$, the resource constraint forces $x_b + x_c = 1 - b_a^{ab}$. The balance of power requirement itself is that $x_b + v_b = x_c$. Together with the definition of b_a^{ab} from equation (6), these allow solution for $x_b = \frac{1}{4}(1 + v_a - 3v_b)$. Thus, in this case, $x_b < 0$, so that $\boldsymbol{b}^{ab}$ dominates all allocations in this locus.

Now let $\boldsymbol{y}$ be the allocation in $B(b)$ such that a and c are equally powerful. The resource constraint forces $y_a + y_c = b_a^{ab}$ while the balance of power requirement is that $y_a + v_a = y_c$. Together, these allow solution for $y_a = \frac{1}{4}(1 - 3v_a + v_b)$, so that $\boldsymbol{b}^{ab}$ dominates all allocations in this locus iff $v_a > \frac{1}{3}(1 + v_b)$. This is automatic when the case's characterising inequality holds:

$$v_b > \frac{1}{3}(1 + v_a) \geq \frac{1}{3}(1 + v_b) \Rightarrow v_b > \frac{1}{2};$$

where the second inequality follows from (3). By the same inequality and the preceding, it follows that $v_a > \frac{1}{2} \Rightarrow v_a > \frac{1}{3}(1 + v_b)$, so that $\boldsymbol{b}^{ab}$ dominates all other allocations in $B(b)$.

In conclusion, when the case's characterising inequality holds, $\boldsymbol{b}^{ab}$ is the unique stable set on $B(a)$ so that, by Corollary 2, it must also belong to any candidate stable set on X. As it also dominates all allocations on $B(b) \setminus \left\{\boldsymbol{b}^{ab}\right\}$, the core is the uniquely stable set under these conditions.

1b. $v_b = \frac{1}{3}(1 + v_a) > 0$. Now the $\boldsymbol{x}$ such that b and c are equally powerful sets $x_b = 0$. Thus, this allocation is not dominated by $\boldsymbol{b}^{ab}$ but by $\boldsymbol{t}^a$ as

$$v_a + x_a = v_b + v_c + x_b + x_c = v_b + 0 + 0 + (1 - x_a) > 1 - x_a;$$

where the inequality follows from $v_b > 0$.

Again, let $\boldsymbol{y} \in B(b)$ such that a and c are equally powerful. As in the previous case, we may derive

$$y_a = \frac{1}{4}(1 - 3v_a + v_b) = \frac{1}{3}(1 - 2v_a) \leq 0.$$

where the last equality derives from the case's characterising condition. The inequality derives from $v_a \geq v_b = \frac{1}{3}(1 + v_a)$. Then:

(a) $y_a < 0$ means that $\boldsymbol{b}^{ab}$ dominates all other allocations in $B(b)$. As this accounts for the last allocations in X, the core is the unique stable set.

(b) $y_a = 0$, which forces $v_a = v_b = \frac{1}{2}$, so that $b_a^{ab} = \frac{1}{2}$. Then

$$\boldsymbol{t}^b \multimapinv \boldsymbol{y} \Leftrightarrow \frac{1}{2} + y_b > y_c.$$

As $y_a = 0$, $y_b = y_c = \frac{1}{2}$, so that this holds. Again, all remaining allocations in X have been accounted for, leaving the core as the unique stable set.

2. $v_b < \frac{1}{3}(1 + v_a)$ implies that the $\boldsymbol{x} \in B(a)$ such that agents b and c are equally powerful is strictly in the interior of X. As $\boldsymbol{x}$ is not dominated by any core allocation, nor by any other allocation setting $x_a = b_a^{ab}$, it must belong to any stable set that exists. Allocations $\boldsymbol{z}$ along the locus but with $z_b < x_b$ are undominated by $\boldsymbol{x}$ or any core allocation, but are dominated by allocations along the locus granting agent b even less than z_b. However, the end-point of that locus, which grants agent b nothing, is dominated by $\boldsymbol{t}^a$. By Corollary 4, as no stable set exists on $B(a)$, no stable set can exist on X.

$\square$

Proof of Theorem 6. Consider the cases in turn:

1. when condition (12) holds, the cases in x_a — as defined in equation (18) — to consider are:

   (a) $x_a = 0$, which implies $v_a = 1 + v_b + v_c$; as the first inequality in (12) requires $v_a \leq 1 - v_b + v_c$ they together require that $v_b \leq 0$, which — as $v_b \geq v_c > 0$ — contradicts the second inequality in (12).

   (b) $x_a > 1$: inequalities (3) and the first inequality in (12) imply that $v_a \geq v_b + v_c - 1$. As $x_a > 1$ implies $v_a < v_b + v_c - 1$, this yields a contradiction.

   (c) $x_a = 1$, which implies that $v_a = v_b + v_c - 1$. As $v_a \geq v_b \geq v_a + v_c - 1$, it follows that: $v_c \geq 1$ and $v_c \leq 1$ (so that $v_c = 1$) and $v_b \geq v_a$ (so that $v_b = v_a$). Finally, the preceding and the first inequality (12) imply that $v_a \leq 1$ so that $v_a = v_b = v_c = 1$. This returns us to the 'empty core' analysis, now on $X \backslash \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c\right\}$. By Lemma 3) and Theorem 2, the unique candidate stable set is $\left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}, \boldsymbol{b}^{ac}, \boldsymbol{b}^{bc}\right\}$, as $\boldsymbol{b}^{ij} = \boldsymbol{s}^{ij}$ in this case; as $v_a \leq \frac{1}{2} + v_c$, it is the unique stable set.

   (d) $x_a \in (0, 1)$, so that Corollary 4 applies, ensuring non-existence of S.

2. when condition (13) holds, $D(\boldsymbol{t}^a) \supseteq \{\boldsymbol{y} \in X \,|y_a \in (0, 1)\}$. As $\boldsymbol{t}^a$ must be included in a candidate stable set, and $D(\boldsymbol{t}^a)$ must be excluded, we need only consider the remaining allocations, such that $x_b, x_c > 0 = x_a$. As $v_b = v_c = 0$, it follows that $\boldsymbol{b}^{bc} = \left(0, \frac{1}{2}, \frac{1}{2}\right) = \boldsymbol{s}^{bc}$. Further, all the remaining $\boldsymbol{y}$ other than $\boldsymbol{s}^{bc}$ are dominated by either $\boldsymbol{t}^b$ or $\boldsymbol{t}^c$, so are excluded from a candidate stable set. Thus, the only allocation not accounted for is $\boldsymbol{s}^{bc}$ which must therefore be added to the stable set to satisfy external stability.

□

Proof of Theorem 7. As the proof closely follows that of Theorem 5, we present it here by explaining how it differs from its predecessor.

Now four sets can be excluded from any stable set: the previous three and $D(\boldsymbol{t}^c)$ as well. This leaves a third locus that remains to be considered, namely $B(c) \backslash \boldsymbol{b}^{ab}$.

The same case distinction helps analyse allocations in $B(a) \backslash \left\{\boldsymbol{b}^{ab}\right\}$:

1a. $v_b > \frac{1}{3}(1 + v_a)$. This case is now impossible:

$$v_a \geq v_b > \frac{1}{3}(1 + v_a) \Rightarrow v_a > \frac{1}{2};$$
$$1 - v_a \geq v_b > \frac{1}{3}(1 + v_a) \Rightarrow v_a < \frac{1}{2};$$

with the first line's weak inequality coming from inequalities (3) and the second line's coming from conditions (14).

1b. $v_b = \frac{1}{3}(1 + v_a)$. This reduces the two sets of inequalities in the previous case to $v_a = v_b = \frac{1}{2}$, which forces $y_a = 0$. That, in turn, implies $\boldsymbol{t}^b \mathrel{\varepsilon\!-} \boldsymbol{y}$. It remains to account for the allocations in $B(c)$, which we do now. Let x_c play the same role in $\boldsymbol{x} \in B(c)$ as x_a did in $\boldsymbol{x} \in B(a)$:

$$B(c) = \{(x_a, x_b, x_c) \in X \,|\, \pi(\{c\}, \boldsymbol{x}) = \pi(\{a, b\}, \boldsymbol{x})\}$$

so that $x_c = 1 - x_c + v_a + v_b \Rightarrow x_c \leq 1$. Consider each sub-case in turn:

(a) $x_c = 1$, so that $v_a + v_b = 1 \Rightarrow v_a = v_b = \frac{1}{2}$ and $B(c) = \{\boldsymbol{t}^c\}$. As $\boldsymbol{t}^c$ already belongs to the core, this accounts for the remaining allocations: the unique stable set is the core.

2. $x_c < 1$, so that $v_a + v_b < 1$. By the case's equality in v_b, this reduces to $v_a < \frac{1}{2}$. However, the case's equality and inequality (3) yield $v_b = \frac{1}{3}(1 + v_a) \leq v_a \Rightarrow v_a \geq \frac{1}{2}$, a contradiction which eliminates this case.

1. $v_b < \frac{1}{3}(1 + v_a)$. This case produces non-existence for the same reason as in Theorem 5.

□

Proof of Theorem 8. Figure 8 illustrates the proof's construction.

As before, the core allocations belong to any candidate stable set; the strict interiors of their dominions cannot; thus, we need only consider the $B(a)$ and $B(b)$ loci (by symmetry).

First consider $B(b)$. By the usual arguments (including Corollary 2), allocations on $B(b)$ must either be members of any stable set, or be dominated by other allocations on $B(b)$.

As dominance along $B(b)$ is determined by conflict between agents a and c, then — by resource monotonicity — $\boldsymbol{b}^a_b \mathrel{\varepsilon\!-} \boldsymbol{e}^b_c$ implies that $\boldsymbol{b}^a_b$ dominates all other allocations along $B(b)$. This dominance condition is equivalent to $v_a > \frac{1}{2}(1 - v_a) \Leftrightarrow v_a > \frac{1}{3}$. Thus, when $v_a > \frac{1}{3}$, the core allocation $\boldsymbol{b}^a_b$ is the unique stable set on $B(b)$.

The case of $v_a = \frac{1}{3}$ is similar: now $\boldsymbol{b}^{ab}$ dominates all other allocations in $B(b)$ except for $\boldsymbol{e}^b_c$. However, this last allocation is dominated by the core allocation $\boldsymbol{t}^b$:

$$\boldsymbol{t}^b \mathrel{\varepsilon\!-} \boldsymbol{e}^b_c \Leftrightarrow x_b > x_c \Leftrightarrow \frac{1}{2}(1 + v_a) > \frac{1}{2}(1 - v_a) \Leftrightarrow v_a > 0;$$

which is guaranteed by inequalities (15).

Thus, when $v_a \geq \frac{1}{3}$, all non-core allocations along $B(b)$ (and, by symmetry, $B(c)$) are dominated by a core allocation. This leaves $B(a)$. Again, by Corollary 2, for a stable set to exist on X, its restriction to $B(a)$ must also be stable; by Corollary 3, this must include $\boldsymbol{e}^a_b = \boldsymbol{b}^{ab}$ and $\boldsymbol{e}^a_c = \boldsymbol{b}^{ac}$. Their inclusion dominates all other allocations in $B(a)$ except for $\boldsymbol{m}^a$. Thus, the unique stable set on $B(a)$ is $\left\{\boldsymbol{b}^{ab}, \boldsymbol{b}^{ac}, \boldsymbol{m}^a\right\}$.

This concludes the theorem's first case, in which $v_a \geq \frac{1}{3}$ yields a unique stable set, $K \cup \{\boldsymbol{m}^a\}$.

In the theorem's second case, $v_a < \frac{1}{3}$ so that core allocations (e.g. $\boldsymbol{b}^{ab}$ and $\boldsymbol{t}^b$) do not dominate all other allocations on $B(b)$. Thus, by the same reasoning as in Corollary 3, existence fails: the allocations in $B(b)$ undominated by $\boldsymbol{b}^{ab}$ must be dominated by a maximal allocation at the other extreme of $B(b)$, that is itself undominated by an allocation in a candidate stable set; as $\boldsymbol{t}^b \varepsilon\!- \boldsymbol{e}^b_c$, such an allocation does not exist, concluding the proof. □

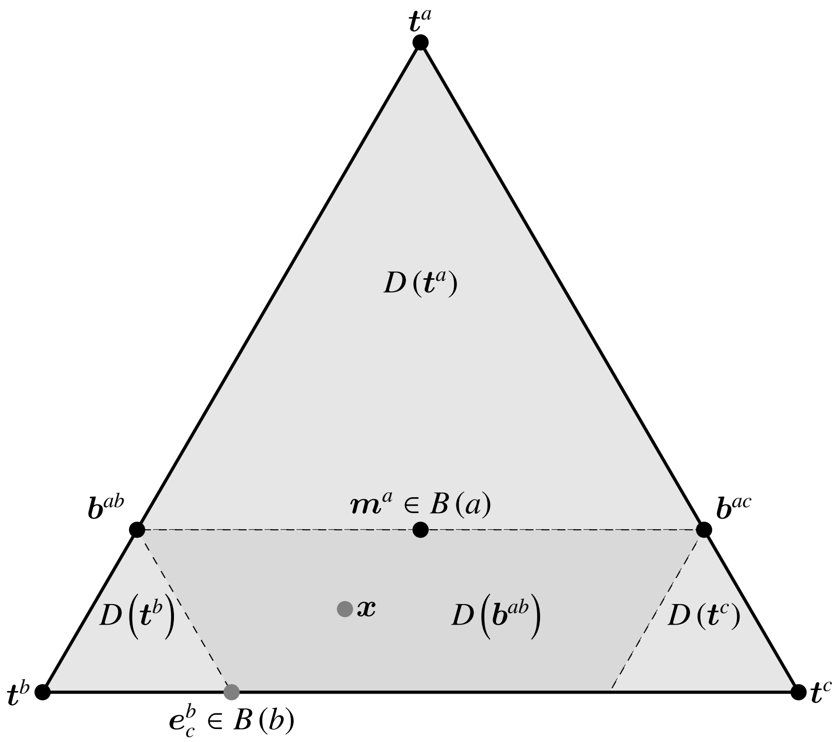

Figure 8: Constructing the stable set when $K = \left\{\boldsymbol{t}^a, \boldsymbol{t}^b, \boldsymbol{t}^c, \boldsymbol{b}^{ab}, \boldsymbol{b}^{ac}\right\}$