ON A CERTAIN CLASS OF RECURSIVE FUNCTIONS

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The main purpose of this paper is to prove that, in Kleene's normal form theorem and enumeration theorem, the primitive recursive predicates T_n and the primitive recursive function U can be replaced by ones which are defined explicitly from the following number-theoretic functions: S, $\lambda xy(x \dot{-} y)$, $\lambda xy(xy)$, and $\lambda x[\sqrt{x}]$. This improves my previous result [11] that the predicates T_n and the function U can be replaced by ones defined explicitly from the functions $\lambda xy(x+y)$, $\lambda xy(x \dot{-} y)$, $\lambda xy(xy)$, $\lambda xy[x/y]$ and $\lambda x[\sqrt{x}]$. The latter was obtained by applying Matijasevič's theorem to the present author's theorem on diophantine predicates. The present construction of substitutes for the predicates T_n is likewise carried out by applying Matijasevič's theorem. As mentioned above, the function $\lambda xy[x/y]$ is dispensable in constructing the substitutes for T_n and U. On the contrary, the function $\lambda x[\sqrt{x}]$ can not be dispensed with, as shown later. Theorem 4 below can be interpreted as follows. Consider a computer capable of handling any natural number as a unit. Suppose the machine has the addition, the subtraction, the multiplication and the extraction of square root as its basic operations. Then, for each n, there is a universal program, which can compute every computable function of n arguments, with a single loop.

As to notations and terminologies, we refer mainly to [6]. Let N denote the set of natural numbers. Thus $N' = \{x' | x \in N\}$ is the set of positive integers. For any set $\mathscr{X}$ of functions, let $\mathscr{X}_n$ denote the set of n-argument functions in $\mathscr{X}$ (cf. [4], § 1). For any set $\mathscr{X}$ of functions, an *$\mathscr{X}$-predicate* is defined to be a predicate which can be expressed as $\lambda x_1 \ldots x_n(f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n))$ where $f, g \in \mathscr{X}$. For a function f and a one-argument function g, let gf denote the composition $\lambda x_1 \ldots x_n(g(f(x_1, \ldots, x_n)))$. For a one-argument function f, the functions f^n are defined thus: $f^0 = U_1^1$, $f^{n+1} = ff^n$. Sometimes we spare parentheses and write fx in place of $f(x)$. If $\mathscr{X}$ is a set of number-theoretic functions, the set of functions explicit (cf. [6], § 44) in $\mathscr{X}$ and the constants 0, 1, 2, ... is denoted as *expl*($\mathscr{X}$). If $\mathscr{X} \subset \mathscr{Y}$, then expl($\mathscr{X}$)$\subset$expl($\mathscr{Y}$). For any $\mathscr{X}$, $\mathscr{X} \subset$ expl($\mathscr{X}$) and expl(expl($\mathscr{X}$))=expl($\mathscr{X}$). If $f \in$expl($\mathscr{X} \cup \{g\}$) and $g \in$expl($\mathscr{Y}$), then $f \in$expl($\mathscr{X} \cup \mathscr{Y}$). Furthermore, $(\text{expl}(\emptyset))_n = \{U_i^n | 1 \leq i \leq n\} \cup \{C_k^n | k \in N\}$. A set $\mathscr{X}$ of functions is *closed under the operations of substitutions* (cf. [4], § 1) if expl($\mathscr{X}$)$=\mathscr{X}$. If $\mathscr{X}$ is closed under the operations of substitutions, then $\lambda xy(x=y)$ is an $\mathscr{X}$-predicate. If f belongs to a set $\mathscr{X}$ which is closed under the operations of substitutions, the predicate

$$\lambda x_1 \ldots x_n y(f(x_1, \ldots, x_n) = y),$$

namely the graph of f, is an $\mathscr{X}$-predicate. Under the assumptions that $\mathscr{X}$ contains the function $\lambda xy\,\text{sg}|x-y|$ and that $\mathscr{X}$ is closed with respect to the operations of substitutions, a predicate is an $\mathscr{X}$-predicate if and only if its representing function belongs to $\mathscr{X}$.

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In [4], Grzegorczyk defined the subsets $\mathscr{E}^n$ of the set of recursive functions. The sequence $\mathscr{E}^0$, $\mathscr{E}^1$, $\mathscr{E}^2$, ... is strictly increasing, and the sum of all the sets $\mathscr{E}^n$ is equal to the set of primitive recursive functions. The set $\mathscr{E}=\mathscr{E}^3$ is the set of Kalmár's elementary functions (cf. [5], § 1 or [6], § 46). We define the sets $\mathscr{A}$, $\mathscr{B}$, $\mathscr{F}$, $\mathscr{P}$ as follows:

$$\mathscr{A}=\mathrm{expl}(\{\lambda xy(x+y),\ \lambda xy(x\dot{-}y),\ \lambda xy(xy),\ \lambda xy[x/y],\ \lambda x[\sqrt{x}]\}),$$
$$\mathscr{B}=\mathrm{expl}(\{S,\ \lambda xy(x\dot{-}y),\ \lambda xy(xy),\ \lambda xy[x/y]\}),$$
$$\mathscr{F}=\mathrm{expl}(\{S,\ \lambda xy(x\dot{-}y),\ \lambda xy(xy),\ \lambda x[\sqrt{x}]\}),$$
$$\mathscr{P}=\mathrm{expl}(\{\lambda xy(x+y),\ \lambda xy(xy)\}).$$

Thus $\mathscr{P}$ is the set of polynomials with coefficients in $\mathbf{N}$. A $\mathscr{P}$-predicate is a *polynomial predicate* (cf. [1], p. 103). Since $a+b=a'b'\dot{-}(ab)'$, the sets $\mathscr{B}$ and $\mathscr{F}$ contain the function $\lambda xy(x+y)$, and

$$\mathscr{A}=\mathrm{expl}(\{S,\ \lambda xy(x\dot{-}y),\ \lambda xy(xy),\ \lambda xy[x/y],\ \lambda x[\sqrt{x}]\}).$$

The inclusions $\mathscr{A}\subset\mathscr{E}^2$, $\mathscr{P}\subset\mathscr{B}\subset\mathscr{A}$ and $\mathscr{P}\subset\mathscr{F}\subset\mathscr{A}$ are evident. Because $\lambda x(1\dot{-}x)\notin\mathscr{P}$, neither $\mathscr{B}$ nor $\mathscr{F}$ is equal to $\mathscr{P}$. $\mathscr{B}\neq\mathscr{F}$ and $\mathscr{B}\neq\mathscr{A}$ will be shown later.

Prior to Matijasevič's negative solution of Hilbert's tenth problem, Kurata and Hirai [7] proved that whether a function in $\mathscr{A}_1$ is surjective or not is undecidable and they further proposed a conjecture that for any fixed strictly monotone primitive recursive function g, whether the range of a function $f\in\mathscr{A}_1$ equals the range of g or not is undecidable. The undecidability result of Kurata-Hirai was deduced from the author's theorem ([10], Theorem 5) that every nonempty diophantine set is enumerable by a function in $\mathscr{A}_1$. The latter follows from the author's theorem ([10], Theorem 3) that every diophantine predicate can be expressed in the form of an $\mathscr{A}$-predicate with a single existential quantifier prefixed to it. This theorem implies, according to Matijasevič's theorem, that every recursively enumerable predicate is expressible in the above-mentioned form. In [11], the construction of the predicates corresponding to T_n was based upon the latter expressibility theorem. The present construction of the predicates F_n, which correspond to T_n, is parallel to that in [11]. On the other hand, a function corresponding to U is obtained by a similar argument as in Markov's theorem on primitive recursive functions of large oscillation.

For any function $f\colon \mathbf{N}\to\mathbf{N}$, the function f' is defined as $f'(a)=\sum_{x<a}\mathrm{sg}|f(x)-f(a)|$. The operation which corresponds f' to f is due to Markov (cf. [8], pp. 136-137). The remainder function rm, usually regarded as a function of natural numbers, is extended to $\mathrm{rm}\colon \mathbf{Q}\times\mathbf{Z}\to\mathbf{Q}$ as follows: Let $\mathrm{rm}(\alpha,\beta)=\alpha$ if $\beta=0$, otherwise let $\mathrm{rm}(\alpha,\beta)$ be the rational number ρ such that $(\exists\gamma\in\mathbf{Q})(\alpha=\beta\gamma+\rho\wedge 0\le\rho<|\beta|)$. The signum function sg is extended to $\mathrm{sg}\colon \mathbf{Q}\to\mathbf{Z}$ in a self-explanatory manner. For any $\alpha\in\mathbf{Q}$, let $\alpha^+=\max(\alpha, 0)$ and $\alpha^-=\max(-\alpha, 0)$. Thus $\alpha=\alpha^+-\alpha^-$ and $|\alpha|=\alpha^++\alpha^-$. Now consider a polynomial $\varphi(x, y, \dots)=\sum_{i,j,\dots}\alpha_{i,j,\dots}x^iy^j\dots$ with rational coefficients. Then we define the polynomials φ^+ and φ^- as follows:

$$\varphi^+(x, y, \dots)=\textstyle\sum_{i,j,\dots}\alpha^+_{i,j,\dots}x^iy^j\dots,$$
$$\varphi^-(x, y, \dots)=\textstyle\sum_{i,j,\dots}\alpha^-_{i,j,\dots}x^iy^j\dots.$$

The polynomial $\varphi^+(x, y, \dots)$ should not be confused with $(\varphi(x, y, \dots))^+$; the latter is not necessarily a polynomial. If φ is a polynomial with integer coefficients, then $\varphi^+\in\mathscr{P}$ and $\varphi^-\in\mathscr{P}$. A number-theoretic predicate is *diophantine* (cf. [1], p. 103) if it is expressible in the form of

$$\lambda x_1\dots x_n\exists y_1\dots\exists y_m(\varphi(x_1, \dots, x_n, y_1, \dots, y_m)=0)$$

where φ is a polynomial with integer coefficients. Hence, a predicate is diophantine if and only if it is expressible in the form of a $\mathscr{P}$-predicate with some (possibly none) existential quantifiers prefixed to it. As above, we use lower-case Greek letters (except λ, μ and ι, which are reserved for the specific meanings) to denote either rationals, integers or functions of rationals and/or integers.

LEMMA 1. $\overline{\mathrm{sg}}$, sg, $\lambda xy|x-y|$, max, min$\in\mathscr{F}$.

Proof. $\overline{\mathrm{sg}}(a)=1\dot{-}a$, $\mathrm{sg}(a)=1\dot{-}(1\dot{-}a)$, $|a\dot{-}b|=(a\dot{-}b)+(b\dot{-}a)$, $\max(a, b)=(a\dot{-}b)+b$, $\min(a, b)=a\dot{-}(a\dot{-}b)$.

COROLLARY. *A predicate is an $\mathscr{F}$-predicate if and only if its representing function belongs to $\mathscr{F}$.*

LEMMA 2. *The set of $\mathscr{F}$-predicates is closed under the logical operations $\neg$, $\wedge$ and $\vee$.*

Proof. $\overline{\mathrm{sg}}$, max, min$\in\mathscr{F}$.

LEMMA 3. *If a function f is defined as*

$$f(a_1, \ldots, a_n)=\begin{cases} g(a_1, \ldots, a_n) & \text{if } R(a_1, \ldots, a_n),\\ h(a_1, \ldots, a_n) & \text{otherwise,}\end{cases}$$

from functions g, $h\in\mathscr{F}$ and an $\mathscr{F}$-predicate R, then $f\in\mathscr{F}$.

Proof. Let r be the representing function of R. Then r is in $\mathscr{F}$ and f can be expressed explicitly by means of g, h and r, hence $f\in\mathscr{F}$.

LEMMA 4. *Every $\mathscr{F}$-predicate is diophantine.*

Proof. It suffices to show that any function in $\mathscr{F}$ has a diophantine predicate as its graph. If functions f and g have diophantine graphs, then the graph of gf is diophantine since

$$gf(a)=b\equiv\exists x(f(a)=x\wedge g(x)=b),$$

and similarly for functions of two or more arguments. The graphs of the initial functions of $\mathscr{F}$ are diophantine. Hence, by induction, any function in $\mathscr{F}$ has a diophantine graph.

The functions P and Q are defined as follows:

$$P(a, b)=\begin{cases} a+b^2 & \text{if } a<b,\\ a^2+a+b & \text{otherwise,}\end{cases}$$

$$Q(a)=\mu y\exists x(P(x, y)=a).$$

P is a bijection from $\mathbf{N}^2$ to $\mathbf{N}$. Save for the order of arguments, P defined here equals Gödel's pairing function P (cf. [2], 7.9) restricted to the set of natural numbers. For any a, the functions $\lambda xP(x, a)$ and $\lambda xP(a, x)$ are strictly monotone. Hence Q is totally defined, $Q''=Q$ and

$$Q'P(a, b)=a,$$
$$QP(a, b)=b,$$
$$P(Q'a, Qa)=a.$$

LEMMA 5. *P, Q', $Q\in\mathscr{F}$.*

Proof. $P\in\mathscr{F}$ since $\lambda xy(x<y)$ is an $\mathscr{F}$-predicate.

$$\max(a, b)^2\leq P(a, b)<(\max(a, b)+1)^2,$$

hence $[\sqrt{P(a, b)}]=\max(a, b)$. We set $c=P(a, b)$. If $a<b$ then

$$[\sqrt{c}]=b\wedge c\dot{-}[\sqrt{c}]^2=a,$$

and if $a\geq b$ then

$$[\sqrt{c}]=a\wedge c\dot{-}[\sqrt{c}]^2=a+b.$$

Hence

$$Q'c=a=\min([\sqrt{c}], c\dot{-}[\sqrt{c}]^2),$$

$$Qc=b=\begin{cases}[\sqrt{c}] \text{ if } [\sqrt{c}]>c\dot{-}[\sqrt{c}]^2,\\ (c\dot{-}[\sqrt{c}]^2)\dot{-}[\sqrt{c}] \text{ otherwise,}\end{cases}$$

therefore Q', $Q\in\mathscr{F}$.

LEMMA 6. *A predicate D is diophantine if and only if there exists an $\mathscr{F}$-predicate R such that*

$$D(a_1, \ldots, a_n)\equiv\exists xR(a_1, \ldots, a_n, x).$$

Proof. If D is a diophantine predicate, then there exist functions g_1, $g_2\in\mathscr{P}$ such that

$$D(a_1, \ldots, a_n)\equiv\exists x_1\ldots\exists x_m(g_1(a_1, \ldots, a_n, x_1, \ldots, x_m)=g_2(a_1, \ldots, a_n, x_1, \ldots, x_m)).$$

Without any loss of generality, we can assume that $m>0$. Let

$$g(a_1, \ldots, a_n, b_1, \ldots, b_m)=\mathrm{sg}|g_1(a_1, \ldots, a_n, b_1, \ldots, b_m)-g_2(a_1, \ldots, a_n, b_1, \ldots, b_m)|$$

and let

$$f(a_1, \ldots, a_n, b)=g(a_1, \ldots, a_n, Q'b, Q'Qb, \ldots, Q'Q^{m-1}b),$$

then $f\in\mathscr{F}$ and

$$D(a_1, \ldots, a_n)\equiv\exists x(f(a_1, \ldots, a_n, x)=0).$$

The converse follows immediately from Lemma 4.

THEOREM 1. *A predicate E is recursively enumerable if and only if there exists an $\mathscr{F}$-predicate R such that*

$$E(a_1, \ldots, a_n)\equiv\exists xR(a_1, \ldots, a_n, x).$$

Proof. By Matijasevič's theorem, every recursively enumerable predicate is diophantine.

COROLLARY. *A predicate C is recursive if and only if there exist $\mathscr{F}$-predicates R_1 and R_2 such that*

$$C(a_1, \ldots, a_n)\equiv\exists xR_1(a_1, \ldots, a_n, x)\equiv\forall xR_2(a_1, \ldots, a_n, x).$$

For each n, let V_n^* be an $\mathscr{F}$-predicate such that

$$T_n(c, a_1, \ldots, a_n, b)\equiv\exists xV_n^*(c, a_1, \ldots, a_n, b, x)$$

and let V_n be the $\mathscr{F}$-predicate $\lambda zx_1\ldots x_nyV_n^*(z, x_1, \ldots, x_n, Q'y, Qy)$.

LEMMA 7. $\exists yT_n(c, a_1, \ldots, a_n, y)\equiv\exists yV_n(c, a_1, \ldots, a_n, y)$.

Proof.
$$\begin{aligned}\exists yT_n(c, a_1, \ldots, a_n, y)&\equiv\exists y\exists zV_n^*(c, a_1, \ldots, a_n, z, y)\\ &\equiv\exists yV_n^*(c, a_1, \ldots, a_n, Q'y, Qy)\\ &\equiv\exists yV_n(c, a_1, \ldots, a_n, y).\end{aligned}$$

THEOREM 2. *A predicate E is recursively enumerable if and only if there exists a number e such that*

$$E(a_1, \ldots, a_n)\equiv\exists yV_n(e, a_1, \ldots, a_n, y).$$

Proof. By Lemma 7, this follows immediately from Kleene's enumeration theorem.

THEOREM 3. *A nonempty set is recursively enumerable if and only if it can be enumerated by a funcation in $\mathscr{F}$.*

Proof. Let A be a nonempty recursively enumerable set and let k be an element of A. By Theorem 1, there is an $\mathscr{F}$-predicate R such that $A=\{x|\exists yR(x, y)\}$. Let

$$f(a)=\begin{cases}Q'a \text{ if } R(Q'a, Qa),\\ k \text{ otherwise,}\end{cases}$$

then $f\in\mathscr{F}$ and f enumerates A. The converse is evident.

For each n, the $\mathscr{F}$-predicate F_n is defined as $\lambda zx_1\ldots x_nyV_{n+1}(z, x_1, \ldots, x_n, Q'y, Qy)$.

LEMMA 8. $\exists yF_n(c, a_1, \ldots, a_n, y)\equiv\exists x\exists yV_{n+1}(c, a_1, \ldots, a_n, x, y)$.

THEOREM 4. *For any recursive function f, there exists a number e such that*

$$\forall x_1\ldots\forall x_n\exists yF_n(e, x_1, \ldots, x_n, y),$$
$$f(a_1, \ldots, a_n)=Q'(\mu yF_n(e, a_1, \ldots, a_n, y))$$

and

$$F_n(e, a_1, \ldots, a_n, b) \rightarrow Q'b = f(a_1, \ldots, a_n).$$

Proof. This is a special case of the next Theorem.

THEOREM 5. *For any partial recursive function f, there exists a number e such that*

$$(a_1, \ldots, a_n) \in \mathrm{dom} f \equiv \exists y F_n(e, a_1, \ldots, a_n, y),$$
$$f(a_1, \ldots, a_n) \simeq Q'(\mu y F_n(e, a_1, \ldots, a_n, y))$$

and

$$F_n(e, a_1, \ldots, a_n, b) \rightarrow Q'b \simeq f(a_1, \ldots, a_n).$$

Proof. Let f be a partial recursive function and e_0 be the Gödel number of f. The predicate $\lambda x_1 \ldots x_n y(f(x_1, \ldots, x_n) \simeq y)$ is recursively enumerable since

$$f(a_1, \ldots, a_n) \simeq b \equiv \exists y (T_n(e_0, a_1, \ldots, a_n, y) \wedge U(y) = b).$$

By Theorem 2, there is a number e such that

$$f(a_1, \ldots, a_n) \simeq b \equiv \exists y V_{n+1}(e, a_1, \ldots, a_n, b, y).$$

Hence

$$\begin{aligned}(a_1, \ldots, a_n) \in \mathrm{dom} f &\equiv \exists x (f(a_1, \ldots, a_n) \simeq x) \\ &\equiv \exists x \exists y V_{n+1}(e, a_1, \ldots, a_n, x, y) \\ &\equiv \exists y F_n(e, a_1, \ldots, a_n, y).\end{aligned}$$

Now we assume $F_n(e, a_1, \ldots, a_n, b)$. Then $V_{n+1}(e, a_1, \ldots, a_n, Q'b, Qb)$, hence

$$\exists y V_{n+1}(e, a_1, \ldots, a_n, Q'b, y).$$

The last formula is equivalent to $f(a_1, \ldots, a_n) \simeq Q'b$, therefore

$$F_n(e, a_1, \ldots, a_n, b) \rightarrow Q'b \simeq f(a_1, \ldots, a_n).$$

Thence it follows that

$$f(a_1, \ldots, a_n) \simeq Q'(\mu y F_n(e, a_1, \ldots, a_n, y)).$$

THEOREM 6. *A predicate E is recursively enumerable if and only if there exists a number e such that*

$$E(a_1, \ldots, a_n) \equiv \exists y F_n(e, a_1, \ldots, a_n, y).$$

Proof. If E is recursively enumerable then there exists a partial recursive function f whose domain is the set $\{(x_1, \ldots, x_n) | E(x_1, \ldots, x_n)\}$. By Theorem 5, there is a number e such that

$$(a_1, \ldots, a_n) \in \mathrm{dom} f \equiv \exists y F_n(e, a_1, \ldots, a_n, y).$$

THEOREM 7. *For any recursive function f, there exists an $\mathscr{F}$-predicate R such that*

$$\forall x_1 \ldots \forall x_n \exists y R(x_1, \ldots, x_n, y)$$

and

$$R(a_1, \ldots, a_n, b) \rightarrow f(a_1, \ldots, a_n) \leq b.$$

Proof. Let f be a recursive function. By Theorem 4, there is a number e such that $f(a_1, \ldots, a_n) \simeq Q'(\mu y F_n(e, a_1, \ldots, a_n, y))$ and $\forall x_1 \ldots \forall x_n \exists y F_n(e, x_1, \ldots, x_n, y)$. We define R as $\lambda x_1 \ldots x_n y F_n(e, x_1, \ldots, x_n, y)$. Since $\forall x(Q'x \leq x)$, $R(a_1, \ldots, a_n, b)$ implies $f(a_1, \ldots, a_n) \leq b$.

A set A of natural numbers is an *$\mathscr{F}$-set* if and only if the predicate $\lambda x(x \in A)$ is an $\mathscr{F}$-predicate. If A and B are $\mathscr{F}$-sets then $A \cap B$, $A \cup B$ and $\mathrm{N} - A$ are $\mathscr{F}$-sets. Every finite set is an $\mathscr{F}$-set.

THEOREM 8. *There is a strictly monotone function $f \notin \mathscr{F}$ whose range is an $\mathscr{F}$-set.*

Proof. Let fib be the Fibonacci sequence: $\mathrm{fib}(0)=0$, $\mathrm{fib}(1)=1$, $\mathrm{fib}(a+2)=\mathrm{fib}(a)+\mathrm{fib}(a+1)$. Then

$$a^2 = ab + b^2 + 1 \equiv \exists x (a = \mathrm{fib}(2x+1) \wedge b = \mathrm{fib}(2x)).$$

Let $f = \lambda x P(\mathrm{fib}(2x+1), \mathrm{fib}(2x))$. Then f is strictly monotone and $f(a) \geq 4^a$, hence $f \notin \mathscr{F}$.

The range of f is an $\mathscr{F}$-set since

$$\exists x(a=f(x))\equiv\exists x(a=P(\mathrm{fib}(2x+1),\ \mathrm{fib}(2x)))$$
$$\equiv\exists x(Q'a=\mathrm{fib}(2x+1)\wedge Qa=\mathrm{fib}(2x))$$
$$\equiv(Q'a)^2=(Q'a)(Qa)+(Qa)^2+1.$$

Now we will prove that there is an $\mathscr{F}$-predicate which is not a polynomial predicate.

LEMMA 9. *Let $\varphi(x)$ be a polynomial with integer coefficients. Then either $a<\varphi^+(1)+\varphi^-(1)$ for any natural number a such that $\varphi(a)=0$ or else $\varphi(a)=0$ for all a.*

Proof. Let φ be $\lambda x\sum_{i\leq n}\alpha_i x^i$ ($\alpha_0, \ldots, \alpha_n\in\mathbf{Z}$). If φ is not the constant zero, we can assume $\alpha_n\neq 0$. Case $n=0$: $\varphi(a)=0\rightarrow a<\varphi^+(1)+\varphi^-(1)$ is vacuously true. Case $n>0$: Let $a\geq\varphi^+(1)+\varphi^-(1)$. Then

$$a\geq\sum_{i\leq n}|\alpha_i|>\sum_{i<n}|\alpha_i|\geq 0,$$

hence

$$|\alpha_n a^n|=|\alpha_n|a^n\geq a^n>a^{n-1}\sum_{i<n}|\alpha_i|\geq\sum_{i<n}|\alpha_i|a^i\geq|\sum_{i<n}\alpha_i a^i|,$$

hence

$$|\varphi(a)|\geq|\alpha_n a^n|-|\sum_{i<n}\alpha_i a^i|>0.$$

LEMMA 10. *For any $\mathscr{P}$-predicate $R(a_1, \ldots, a_n, b)$, there exists a $g\in\mathscr{P}$ such that for any $a_1, \ldots, a_n$ either*

$$R(a_1, \ldots, a_n, b)\rightarrow b<g(a_1, \ldots, a_n)$$

or

$$\forall yR(a_1, \ldots, a_n, y).$$

Proof. By the assumption, a given predicate R can be expressed as $f_1(a_1, \ldots, a_n, b)=f_2(a_1, \ldots, a_n, b)$ where $f_1, f_2\in\mathscr{P}$. Let

$$\varphi(a_1, \ldots, a_n, b)=f_1(a_1, \ldots, a_n, b)-f_2(a_1, \ldots, a_n, b).$$

Then φ can be expressed as $\sum_{i\leq m}\psi_i(a_1, \ldots, a_n)b^i$ where $\psi_0, \ldots, \psi_m$ are polynomials with integer coefficients. We define g as

$$g(a_1, \ldots, a_n)=\sum_{i\leq m}(\psi_i^+(a_1, \ldots, a_n)+\psi_i^-(a_1, \ldots, a_n)).$$

Then for any fixed $a_1, \ldots, a_n$, either

$$\forall y(\varphi(a_1, \ldots, a_n, y)=0)$$

or else $\varphi(a_1, \ldots, a_n, b)=0$ implies

$$b<\sum_{i\leq m}|\psi_i(a_1, \ldots, a_n)|\leq g(a_1, \ldots, a_n)$$

The above proof is due to Goodstein [3]. He pointed out and corrected an error in the author's proof of a theorem (cf. [9], Theorem 16) stating that a predicate expressible in a form of a single existential quantifier prefixed to a polynomial predicate is an elementary predicate.

LEMMA 11. *For any $\mathscr{P}$-predicate $R(a_1, \ldots, a_n, b)$, there exists a $g\in\mathscr{P}$ such that*

$$\exists yR(a_1, \ldots, a_n, y)\equiv(\exists y<g(a_1, \ldots, a_n))R(a_1, \ldots, a_n, y).$$

Proof. Immediate consequence of Lemma 10.

COROLLARY. *For any $\mathscr{P}$-predicate R, the predicate $\lambda x_1\ldots x_n\exists yR(x_1, \ldots, x_n, y)$ is an $\mathscr{E}^2$-predicate.*

Proof. By Theorem 4.6 of [4].

THEOREM 9. *The set of $\mathscr{P}$-predicates is a proper subset of the set of $\mathscr{F}$-predicates.*

Proof. Since $\mathscr{P}\subset\mathscr{F}$, the set of $\mathscr{P}$-predicates is a subset of the set of $\mathscr{F}$-predicates. By Theorem 7, there is an $\mathscr{F}$-predicate R such that

$$\forall x\exists yR(x, y)\wedge\forall x\forall y(R(x, y)\rightarrow 2^x\leq y).$$

By Lemma 11, R is not a $\mathscr{P}$-predicate.

Now we will prove that the functions in $\mathscr{B}$ are insufficient to obtain a proposition corresponding to Theorem 4. Let $\mathbf{Z}_1$ be the set $\{\varphi|\varphi(x)\in\mathbf{Z}[x]\}$, i.e. the set of polynomials of one variable with integer coefficients. Let $\mathbf{Q}_1$ be the set $\{\varphi|\varphi(x)\in\mathbf{Q}[x]\}$. Let $\mathscr{Q}$ be the set of functions φ: $\mathbf{N}\to\mathbf{Q}$ such that for some functions $\varphi_0, \ldots, \varphi_{p-1}\in\mathbf{Q}_1$ $(p>0)$,

$$\exists u(\forall x\geq u)(\forall i<p)(x\equiv i\,(\mathrm{mod}\,p)\to\varphi(x)=\varphi_i(x)).$$

It will be shown that $\mathscr{B}_1\subset\mathscr{Q}$. From this fact it follows that no function corresponding to Kleene's U belongs to $\mathscr{B}$.

LEMMA **12.** *If* $\varphi\in\mathbf{Q}_1$ *then* $\lambda x[\varphi(x)]\in\mathscr{Q}$.

Proof. Let $\varphi\in\mathbf{Q}_1$. Then $\lambda x(p\varphi(x))\in\mathbf{Z}_1$ for some $p\in\mathbf{N}'$. For each i $(0\leq i<p)$, let $\varphi_i(x)=\varphi(x)-\mathrm{rm}(p\varphi(i), p)/p$. If $0\leq i<p$ and $x\equiv i$ $(\mathrm{mod}\,p)$ then

$$p\varphi(x)-p[\varphi(x)]=\mathrm{rm}(p\varphi(x), p)=\mathrm{rm}(p\varphi(i), p)$$

since $p\varphi(x)\equiv p\varphi(i)$ $(\mathrm{mod}\,p)$, hence $\varphi(x)=\varphi_i(x)$.

LEMMA **13.** *If* φ, $\psi\in\mathbf{Q}_1$ *then* $\lambda x[\varphi(x)/\psi(x)]\in\mathscr{Q}$.

Proof. Let θ be $\lambda x[\varphi(x)/\psi(x)]$. If ψ is constant then $\theta\in\mathbf{Q}_1\subset\mathscr{Q}$ is evident. Now suppose ψ is not constant. Let ξ and ρ be respectively the quotient and the remainder of the division of φ by ψ. Then

$$\varphi(x)=\psi(x)\xi(x)+\rho(x).$$

Let p be a positive integer such that $\lambda x(p\xi(x))\in\mathbf{Z}_1$. Let u be a natural number such that $\lambda x\mathrm{sg}\psi(x)$ and $\lambda x\mathrm{sg}\rho(x)$ are constant on the set $\{x|x\geq u\}$ and that $|\psi(x)|>|p\rho(x)|$ for any $x\geq u$. For any $x\geq u$, since $\psi(x)\neq0$,

$$-p\rho(x)/\psi(x)\leq p\xi(x)-p\theta(x)<p-p\rho(x)/\psi(x).$$

Case 1. $0\leq p\rho(u)/\psi(u)<1$. For each $i<p$, let $\theta_i(x)=\xi(x)-\mathrm{rm}(p\xi(i), p)/p$. If $i<p$ and $x\geq u$ and $x\equiv i\,(\mathrm{mod}\,p)$ then

$$p\xi(x)-p\theta(x)=\mathrm{rm}(p\xi(x), p)=\mathrm{rm}(p\xi(i), p),$$

hence $\theta(x)=\theta_i(x)$.

Case 2. $-1<p\rho(u)/\psi(u)<0$. For each $i<p$, let $\theta_i(x)=\xi(x)+\mathrm{rm}(-p\xi(i), p)/p-1$. If $i<p$ and $x\geq u$ and $x\equiv i$ $(\mathrm{mod}\,p)$ then

$$p\theta(x)-p\xi(x)=\mathrm{rm}(-p\xi(x), p)-p=\mathrm{rm}(-p\xi(i), p)-p,$$

hence $\theta(x)=\theta_i(x)$.

LEMMA **14.** *If* φ, $\psi\in\mathscr{Q}$, *then* $\lambda x(\varphi(x)+\psi(x))$, $\lambda x(\varphi(x)\psi(x))$, $\lambda x(\psi(x)\dot{-}\psi(x))$, $\lambda x[\varphi(x)/\psi(x)]\in\mathscr{Q}$.

Proof. By the assumption, there exist $\varphi_0, \ldots, \varphi_{p-1}, \psi_0, \ldots, \psi_{q-1}\in\mathbf{Q}_1$ and $u, v\in\mathbf{N}$ such that

$$(\forall x\geq u)(\forall i<p)(x\equiv i\,(\mathrm{mod}\,p)\to\varphi(x)=\varphi_i(x))$$

and

$$(\forall x\geq v)(\forall i<q)(x\equiv i\,(\mathrm{mod}\,q)\to\psi(x)=\psi_i(x)).$$

Let r be the least common multiple of p and q. Let χ be one of the functions mentioned in the conclusion.

Case 1. $\chi=\lambda x(\varphi(x)+\psi(x))$. Let $w=\max(u, v)$. For any k $(k<r)$, define as

$$\chi_k=\lambda x(\varphi_{\mathrm{rm}(k, p)}(x)+\psi_{\mathrm{rm}(k, q)}(x)).$$

Then $\chi(x)=\chi_{\mathrm{rm}(x, r)}(x)$ for any $x\geq w$.

Case 2. $\chi=\lambda x(\varphi(x)\psi(x))$. Similar as Case 1.

Case 3. $\chi=\lambda x(\varphi(x)\dot{-}\psi(x))$. For a sufficiently large w, the function $\lambda x\mathrm{sg}(\varphi_{\mathrm{rm}(k, p)}(x)-\psi_{\mathrm{rm}(k, q)}(x))$ is constant on the set $\{x|x\geq w\}$ for every k $(k<r)$. We may suppose that $w\geq\max(u, v)$. For each k $(k<r)$, if $\varphi_{\mathrm{rm}(k, p)}(w)\geq\psi_{\mathrm{rm}(k, q)}(w)$ then define as

$$\chi_k = \lambda x \varphi_{\mathrm{rm}(k,\,p)}(x)$$

and otherwise define as

$$\chi_k = \lambda x(0).$$

Then $\chi(x) = \chi_{\mathrm{rm}(x,\,r)}(x)$ for any $x \geq w$.

Case 4. $\chi = \lambda x[\varphi(x)/\psi(x)]$. Suppose that a number k $(k<r)$ is fixed. Consider the functions $\lambda x \varphi_{\mathrm{rm}(k,\,p)}(rx+k)$ and $\lambda x \psi_{\mathrm{rm}(k,\,q)}(rx+k)$. By Lemma 13, there exist $\theta_{kh} \in \mathbf{Q}_1$ $(0 \leq h < t_k)$ and $z_k \in \mathbf{N}$ such that

$$(\forall x \geq z_k)\forall h(\mathrm{rm}(x,\ t_k)=h \rightarrow [\varphi_{\mathrm{rm}(k,\,p)}(rx+k)/\psi_{\mathrm{rm}(k,\,q)}(rx+k)] = \theta_{kh}(x)).$$

Let $z = \max(z_0,\ \ldots,\ z_{r-1})$. Let t be the least common multiple of $t_0,\ \ldots,\ t_{r-1}$. For each h such that $t_k \leq h < t$ (if any), define θ_{kh} as θ_{kl} where $l = \mathrm{rm}(h,\ t_k)$. Then

$$(\forall x \geq z)(\forall k < r)\forall h(\mathrm{rm}(x,\ t)=h \rightarrow [\varphi_{\mathrm{rm}(k,\,p)}(rx+k)/\psi_{\mathrm{rm}(k,\,q)}(rx+k)] = \theta_{kh}(x)).$$

Let $s = rt$. For each i $(i<s)$, define as

$$\chi_i = \lambda x \theta_{kh}((x-k)/r)$$

where $i = k+rh$, $k<r$ and $h<t$. Let $w = \max(u,\ v,\ rz+r)$. Then $\chi(x) = \chi_{\mathrm{rm}(x,\,s)}(x)$ for any $x \geq w$.

LEMMA 15. $\mathscr{B}_1 \subset \mathscr{Q}$.

Proof. Since $C_k^1 \in \mathscr{Q}$ for every k and $U_1^1 \in \mathscr{Q}$, this follows immediately from Lemma 14.

A function f: $\mathbf{N} \rightarrow \mathbf{N}$ is called *extensive* (umfangreich) or a function *of large oscillation* (bol'šogo razmaha) if for every y there exist infinitely many x such that $f(x)=y$. If g_1 and g_2 satisfy

$$g_1(f(a,\ b)) = a \wedge g_2(f(a,\ b)) = b$$

for an injection f: $\mathbf{N}^2 \rightarrow \mathbf{N}$ then g_1 and g_2 are extensive. Conversely, if g is an extensive function then g' is also extensive and the function

$$\lambda xy \mu z(g'(z) = x \wedge g(z) = y)$$

is an injection from $\mathbf{N}^2$ to $\mathbf{N}$. Under the supposition that g is primitive recursive, for every recursive function f there exists a primitive recursive predicate R such that

$$f(a_1,\ \ldots,\ a_n) \simeq g(\mu y R(a_1,\ \ldots,\ a_n,\ y))$$

if and only if g is extensive (Markov's theorem, cf. [8], pp. 136-137).

LEMMA 16. *No function in* $\mathscr{B}$ *is extensive.*

Proof. Let $f \in \mathscr{B}$. By Lemma 15, there exist $\varphi_0,\ \ldots,\ \varphi_{p-1} \in \mathbf{Q}_1$ and $u \in \mathbf{N}$ such that $(\forall x \geq u)(f(x) = \varphi_{\mathrm{rm}(x,\,p)}(x))$. For a sufficiently large v, each φ_i is either strictly increasing or else constant on the set $\{x | x \geq v\}$. We may suppose $v \geq u$. Let C be the set of i $(i<p)$ such that φ_i is constant. Let m be the maximal element of the union of the sets $\{f(x) | x < v\}$ and $\{\varphi_i(0) | i \in C\}$. Then $f(x) = m+1$ for at most p values of x.

THEOREM 10. $\lambda x[\sqrt{x}] \notin \mathscr{B}$.

Proof. Suppose $\lambda x[\sqrt{x}] \in \mathscr{B}$. It follows that $\lambda x(x \dot{-} [\sqrt{x}]^2) \in \mathscr{B}$, which contradicts Lemma 16.

COROLLARY. $\mathscr{B} \neq \mathscr{F}$, $\mathscr{B} \neq \mathscr{A}$.

THEOREM 11. $\mathscr{B}$ *is closed under none of the following operations: the bounded* μ-*operator, the operation* $\sum$ *and Grzegorczyk's limited recursion.*

Proof. $\mathscr{B}$ is not closed under bounded μ-operator since

$$[\sqrt{a}] = (\mu x \leq a)(a \dot{-} (x^2+2x) = 0).$$

The bounded μ-operator is expressible in terms of $\sum$ and $\overline{\mathrm{sg}}$ as

$$(\mu x < a)(f(x)=0) = \textstyle\sum_{x<a} \overline{\mathrm{sg}}(\sum_{y \leq x} \overline{\mathrm{sg}}(f(y))),$$

hence $\mathscr{B}$ is not closed under $\sum$. $\mathscr{B}$ is not closed under the limited recursion because $f=\lambda x[\sqrt{x}]$ can be defined thus:

$$f(0)=0,$$
$$f(a')=f(a)+\overline{\mathrm{sg}}((f(a)^2+2f(a))\dot{-}a),$$
$$f(a)\leq a.$$

THEOREM **12.** *Given any fixed $g\in\mathscr{B}$, not every recursive function is expressible in the form of $\lambda x_1...x_n g(\mu yR(x_1, ..., x_n, y))$ with a $\mathscr{B}$-predicate R.*

Proof. By Markov's theorem, if every recursive function is expressible in the above-mentioned form then g would be extensive.

THEOREM **13.** *There exists a recursively enumerable set which can not be enumerated by any function in $\mathscr{B}$.*

Proof. Let $f\in\mathscr{B}$. By Lemma 15, there exist $\varphi_0, ..., \varphi_{p-1}\in\mathbf{Q}_1$ and $u\in\mathbf{N}$ such that $(\forall x\geq u)(f(x)=\varphi_{\mathrm{rm}(x, p)}(x))$. Hence the range of the values of f can not be the recursively enumerable (indeed elementary) set $\{2^x|x\in\mathbf{N}\}$.

We conclude with a remark on Grzegorczyk's $\mathscr{E}^0$. The functions sgmax$=\lambda xy\mathrm{sg}(\max(x, y))$, Q and Q' belong to $\mathscr{E}^0$. Suppose that R is a recursive predicate. Let f be the representing function of R. By theorem 5.1 of [4], there exist functions $g, h\in\mathscr{E}^0$ such that

$$\forall x_1...\forall x_n\exists!y(g(x_1, ..., x_n, y)=0)$$

and

$$f(a_1, ..., a_n)=h(\iota y(g(a_1, ..., a_n, y)=0)),$$

hence

$$R(a_1, ..., a_n)\equiv\exists y(\mathrm{sgmax}(g(a_1, ..., a_n, y), h(y))=0).$$

Thus it is proved that for any recursive predicate R there is a function $r\in\mathscr{E}^0$ such that

$$R(a_1, ..., a_n)\equiv\exists y(r(a_1, ..., a_n, y)=0).$$

For each n, let t_n be a function in $\mathscr{E}^0$ such that

$$T_n(c, a_1, ..., a_n, b)\equiv\exists y(t_n(c, a_1, ..., a_n, b, y)=0)$$

and let E_n be $\lambda zx_1...x_ny(t_n(z, x_1, ..., x_n, Q'y, Q'Qy, Q^2y)=0)$. By arguments similar to proofs of Theorems 5 and 6, the following propositions can be shown.

(1) For every partial recursive function f there exists a number e such that

$$(a_1, ..., a_n)\in\mathrm{dom}\,f\equiv\exists yE_n(e, a_1, ..., a_n, y),$$
$$f(a_1, ..., a_n)\simeq Q'(\mu yE_n(e, a_1, ..., a_n, y)),$$
$$E_n(e, a_1, ..., a_n, b)\rightarrow f(a_1, ..., a_n)\simeq Q'b.$$

(2) A predicate R is recursively enumerable if and only if there is a number e such that

$$R(a_1, ..., a_n)\equiv\exists yE_n(e, a_1, ..., a_n, y).$$

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