A NOTE ON STATISTICAL SYSTEM DYNAMICS

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In the previous paper[1], we pointed out that there are two aspects of an optimization principle in its application to the economic phenomena: one for a leading principle of a behavioral subject, and another for that of a system including a number of behavioral subjects which interact to each other.

It is well known that the former provides us a fundamental method for the theory of consumers behavior (utility maximization) and production planning (sales maximization). However the latter would not be so familiar to the readers as the former. In this paper we shall discuss a method of finding macroscopic properties of a system consisted of many behavioral subjects which possess and exchange resources, goods and money to each other under a rule of transaction. Here it is assumed that when a system has several attainable states, the most probable one is realized. As well known in physics, the above-mentioned assumption is a basic hypothesis for constructing statistical mechanics, which is proved true in the natural phenomena. Although we are not convinced that it holds in the real economic system, it would be worthwhile for us to study what the nature of an abstractive system is and how parallel discussions both on the natural and the economic systems might be possible through a way similar to statistical mechanics.[2],[3]

I. *Definitions and Assumptions*

Let us begin with defining an abstractive system in the following way.

D[1]1: A subject which behaves under a rule of behavior is called a " unit ".

D2: A set of units is called a " system ".

D3: Each unit possesses " assets " measured by a common scale (here we use the term " assets " in an abstractive sense: for example, they mean energy of a moving particle in a material system, and goods and money of a person in an economic system).

D4: Units in a system exchange their assets to each other under some rules of payoff structure of the system. We call the above-mentioned behavior of the units " interaction " (in this paper, the rules of the payoff are not necessary expressed by an explicit formula).

D5: Suppose there exist two systems A and A', and units of both systems interact to each other. Then A and A' are said " open "; especially, if A' is comparatively greater than A, A' is called an " environmental system " of A.

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[1] The notation D stands for *Definition*, and A for *Assumption.*

D6: If a system is not open, it is called " closed " or " isolated ".

For an isolated system two fundamental assumptions are needed for extending our statistical system dynamics as the following Al and A2.

Al: If a system is isolated, total amount of assets of the system is always kept constant.

This assumption Al means that when two or more units exchange their assets under a rule of their behavior, even though shares of their assets may change after the interaction, the total amount is kept invariant if the system is isolated.

In order to describe the properties of a system, we introduce a concept of a " microscopic state " of the system, which is represented by a set of f microscopic variables, $q_1, q_2, \ldots, q_f$. For instance, if the system is consisted of n persons, we consider one system variable is assigned to every economic status of each person (amount of assets, annual income, number of members of his family, level of education, etc.). Thus one can represent one microscopic state by a point $P(q_1, q_2, \ldots, q_f)$ in an f dimentional space R^f.

Suppose the total amount of assets E of an isolated system A, described variables $q_1, q_2, \ldots, q_f$, is given by

$$E=E(q_1, q_2, \ldots, q_f). \tag{1}$$

From the assumption Al, we see that all microscopic states of the isolated system A have equal values of E, and their corresponding points are placed on an $f-1$ dimensional surface $E=$const. in R^f. Hereafter we call the total amount of assets of a microscopic state a " level ".

II. *Statistical Description of System*

As mentioned above, since the system which we are concerned with is composed of many units, it would be impossible to describe the behavior of the system completely by any sophisticated mathematics. However it seems that the complexity of the system or the greatness of the number of the units may afford us a way to solve this problem conversely, just as physics has done it in the analysis of material.

Let us consider an isloated system A composed of n units, of which total amount of assets is denoted by E. Then we see that, in the lapse of time, the units in the system exchanges their assets to each other many times, and the system passes microscopic states from one to another successively.

For the sake of computational conveniency, we consider an interval $[E, E+\delta E]$, instead of the single value of E, where δE, is a sufficiently small value comparing with E, but still contains many levels of microscopic states. Now denote a set of microscopic states of which levels lie in $[E, E+\delta E]$ with $R(E)$, and their number with $\Omega(E)$.

In order to proceed to the statistical description of the system, we have to define a probability concept for the system. Let us introduce a probability P_j of finding the system in a microscopic state $j \in R(E)$. In general the probability P_j may differ from P_k for $k \neq j$. However we could assume the system would pass these microscopic states with an equal probability, if we observe the system in a sufficiently long time.

A2: An isolated system is found in every microscopic state with an equal probability in the long run.

D7: If a system satisfies the assumption A2, it is said that it attained to an "equilibrium state".

Of course an inequilibrium state is much more interesting to us than the equilibrium, since it does represent the real system in our world, and our final goal of this statistical system dynamics should be the study of the inequilibrium one. In this paper, accordingly, we study and discuss the properties of the equilibrium state as a preliminary step to the goal.

In equilibrium it is seen that the system is characterized by the function $\Omega(E)$, which is calculated by a rule of interaction between units. Since, however, it is quite difficult to get the function $\Omega(E)$, in an explicit form, let us assume the following qualitative property for the function $\Omega(E)$.

A3: $\Omega(E)$ is a monotone increasing and differentiable function of E.

Let us now proceed to explore the interaction between systems. Suppose there are two isolated systems A and A', which are separated initially with total assets E_0 and E_0', and the number of microscopic states $\Omega(E_0)$, and $\Omega'(E'_0)$ respectively. Contacting these two systems, we allow them to interact and compose a new isolated system A^* with total assets $E^*=E_0+E_0'$. Let us compute the number of microscopic states $\Omega^*(E)$ of the compound system A^* in $[E^*, E^*+\delta E]$, giving E to the partial system A. Then we have

$$\Omega^*(E)=\Omega(E)\Omega'(E')=\Omega(E)\Omega'(E^*-E). \tag{2}$$

As was already discussed at the beginning of this paper, assuming the maximum principle to this system, let us express it in the following way.

A4: If two systems are allowed to contact and interact to each other, the most probable state of the compound system is realized in a sufficiently long time.

Using the above assumption A4 and (2), we can obtain the most probable state at the maximum point of $\Omega^*(E)$. Differentiating $\log \Omega^*(E)$ with respect to E, we have

$$\frac{\partial \log \Omega(E)}{\partial E}+\frac{\partial \log \Omega'(E')}{\partial E'}(-1)=0. \tag{3}$$

Defining $\beta(E)$ and $\beta'(E')$ in the following,

$$\beta(E')=\frac{\partial \log \Omega(E)}{\partial E}, \qquad \beta'(E')=\frac{\partial \log \Omega'(E')}{\partial E'}, \tag{4}$$

we have

$$\beta(E)=\beta'(E') \quad \text{or} \quad \beta(E)=\beta'(E^*-E) \tag{5}$$

for solving an optimal value of E in the equilibrium state. Thus we have the following property.

P[2]1: If one system is in contact with another and attains an equilibrium state, the functions β's of the both systems are equal to each other.

Later we call $\beta(E)$ "system temperature", which is analogous to the thermal temperature in physics.

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[2] P stands for *Property*.

III. *Distribution Function and Entropy*

The next step we are going to proceed to is to obtain a distribution function of the total amount of assets E for the partial system A defined in the preceding section. From the discussion made above, it is seen that the number of microscopic states of the compound system A^* is given by $\Omega'(E^*-E_i)$ for fixed values of E, E_i (i-th level of E). Suppose now E' is much greater than E_i, i.e., the system A' is an environmental one to A. Then we have

$$\log \Omega'(E^*-E_i) \doteqdot \log \Omega'(E^*) - \left(\frac{\partial \log \Omega'}{\partial E'}\right)_{E'=E^*} E_i$$
$$= \log \Omega'(E^*) - \beta E_i, \tag{6}$$

subsequently,

$$\Omega'(E^*-E_i) = \Omega'(E^*) e^{-\beta E_i}, \tag{7}$$

where $\beta = \beta'(E^*) = (\partial \log \Omega'(E')/\partial E')_{E'=E^*}$ and $E^* \gg E_i$ are assumed, where β is a system temperature of the environmental system A'.

Now since the fundamental requirement for our statistical analysis asserts that every microscopic state of the isolated system A^* is attained with an equal probability in the equilibrium state, the probability P_i that the partial system A in A^* has a fixed amount of E_i is expressed by

$$P_i \propto \Omega'(E^*-E_i) \quad \text{or} \quad P_i = c e^{-\beta E_i}, \tag{8}$$

where c is a constant independent of E_i and given by

$$c = 1/\sum_i e^{-\beta E_i}. \tag{9}$$

Then we have

$$P_i = \frac{e^{-\beta E_i}}{\sum_i e^{-\beta E_i}}. \tag{10}$$

Defining a function of β,

$$Z = \sum_i e^{-\beta E_i}, \tag{11}$$

we call it a " state sum function ". Using it, for the average value of E_i, we have

$$\bar{E} = \sum_i E_i P_i = -\frac{d}{d\beta}(\log Z). \tag{12}$$

Suppose now the state of the partial system A is also described by k macroscopic parameters $\lambda_1, \lambda_2 \ldots, \lambda_k$ besides β. Taking total differentials for $\log Z$ and $\bar{E}$ with respect to β and λ_j ($j=1, 2, \ldots, k$), we have

$$d\log Z = \frac{\partial \log Z}{\partial \beta} d\beta + \sum_j \frac{\partial \log Z}{\partial \lambda_j} d\lambda_j \tag{13}$$

$$d\bar{E} = \frac{\partial \bar{E}}{\partial \beta} d\beta + \sum_j \frac{\partial \bar{E}}{\partial \lambda_j} d\lambda_j. \tag{14}$$

Using (12), from (13), (14) we have

$$\beta\Big(d\bar{E}+\frac{1}{\beta}\sum_j\frac{\partial \log Z}{\partial\lambda_j}d\lambda_j\Big)=\beta\frac{\partial\bar{E}}{\partial\beta}d\beta+\beta\sum_j\frac{\partial\bar{E}}{\partial\lambda_j}d\lambda_j+d\log Z+\bar{E}d\beta$$

$$=\frac{\partial(\beta\bar{E})}{\partial\beta}d\beta+\sum_j\frac{\partial(\beta\bar{E})}{\partial\lambda_j}d\lambda_j+d\log Z$$

$$=d(\beta\bar{E}+\log Z). \tag{15}$$

Let us consider the meaning of the second term in the left hand side:

$$\frac{\partial \log Z}{\partial\lambda_j}=\frac{1}{Z}\frac{\partial Z}{\partial\lambda_j}=-\frac{\beta}{Z}(\sum_j e^{-\beta E_j})\frac{\partial E_i}{\partial\lambda_j}. \tag{16}$$

Since the term $\partial E_i/\partial\lambda_j$ means the increment of i-th level of the assets for the unit increment of the parameter λ_j, the term

$$-\frac{1}{\beta}\sum_j\frac{\partial \log Z}{\partial\lambda_j}d\lambda_j=\sum_j\sum_i\frac{\partial E_i}{\partial\lambda_j}e^{-\beta E_i}d\lambda_j/Z \tag{17}$$

is considered as an average increase in the total amount of assets for the differential increment of the parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$. Therefore the left hand side of (15) except β,

$$d\bar{E}-\sum_j\sum_i\frac{\partial E_i}{\partial\lambda_j}e^{-\beta E_i}/Zd\lambda_j \tag{18}$$

means the difference between the total differential increment of the average total assets due to the small changes $d\beta, d\lambda_1, \ldots, d\lambda_k$ and the average increment of total assets due to $d\lambda_1, \ldots, d\lambda_k$. Let us denote the term (18) dQ and call it " indirect increment of assets ". Then from (15), it is seen that βdQ is a total differential of the function $\beta\bar{E}+\log Z$. We call this function " entropy " S of the system A:

$$S=\beta\bar{E}+\log Z, \tag{19}$$

and we have

$$dS=\beta dQ. \tag{20}$$

Suppose the state of a partial system A is changed from M_1 ($\beta^{(1)}, \lambda_1^{(1)}, \lambda_2^{(1)}, \ldots, \lambda_k^{(1)}$) to M_2 ($\beta^{(2)}, \lambda_1^{(2)}, \lambda_2^{(2)}, \ldots, \lambda_k^{(2)}$) very slowly, integrating (20) from M_1 to M_2 we have

$$\int_{M_1}^{M_2}\beta dQ=\int_{M_1}^{M_2}dS=S_2-S_1 \tag{21}$$

where S_1 and S_2 are entropies computed at the states M_1 and M_2 respectively. Let us state the above-mentioned results in the following way.

P2: If a partial system, surrounded by an environmental system, changes its state from M_1 to M_2, the integrated indirect increment of assets multiplied by the system temperature at each differential stage is equal to the difference between the entropies of two states M_1 and M_2, irrespectively of the path from M_1 to M_2.

Finally we discuss the relationship between the entropy defined by (19) and the function $\Omega(E)$. Since the summation in (11) which defines the state sum function is taken over all possible microscopic states of the partial system A, instead of (11) we use

$$Z=\int_0^\infty e^{-\beta E}d\Phi(E), \tag{22}$$

where the function $\Phi(E)$ means the number of levels of assets less than E (cumulative distribution function). Assuming $\Phi(E)$ is differentiable, $d\Phi(E)=G(E)dE$, we have

$$Z=\int_0^\infty e^{-\beta E}G(E)dE, \tag{23}$$

where the function $G(E)$, the density distribution function, is found to be almost equal to $\Omega(E)/\delta E$ from the definition of $\Omega(E)$. We have already assumed that $\Omega(E)$ is a monotone increasing function of E (A3). Furthermore let us make an additional assumption.
A5: $\Omega(E)$ (or $G(E)$) is such a rapid increasing function that the following relation approximately holds,

$$\Omega(E)e^{-\beta E}\doteqdot\begin{cases}\Omega(\bar{E})e^{-\beta\bar{E}} & \bar{E}\leqq E\leqq\bar{E}+\delta E\\ 0 & \text{otherwise.}\end{cases} \tag{24}$$

Then for the intergation in (23) we have

$$Z=\int_0^\infty e^{-\beta E}G(E)dE=\Omega(\bar{E})e^{-\beta\bar{E}}. \tag{25}$$

Substituting (25) into (19), we obtain approximately

$$S=\beta\bar{E}+\log\Omega(\bar{E})-\beta\bar{E}=\log\Omega(\bar{E}). \tag{26}$$

Therefore we have the following property of the partial system A.
P3: If $\Omega(E)$ of a partial system satisfies the assumption A5, the entropy of the system, derived from (19), is equal to natural logarithm of the number of microscopic states in the interval $[\bar{E},\ \bar{E}+\delta E]$, $\Omega(\bar{E})$.

Let us show an example of this statistical system dynamics. Suppose a partial system A is composed of n units, and surrounded by an environmental system A' of which system temperature is β. Farther consider that each unit has a same structure of the microscopic states $\varphi(E)$ (the number of microscopic states less than E) and is independent to the others. Then we have, using a technique of convolution,

$$Z=\int_0^\infty e^{-\beta E}d\Phi(E)=z^n, \tag{27}$$

where

$$z=\int_0^\infty e^{-\beta E}d\varphi(E),\quad \Phi(E)=\{\varphi(E)\}^{(n)}. \tag{28}$$

Furthermore if we assume

$$\varphi(E)=cE, \tag{29}$$

where c is a constant, we obtain from (27) and (28)

$$Z=\frac{c^n}{\beta^n},\quad \bar{E}=\frac{n}{\beta}, \tag{30}$$

and from (19)

$$S=n+\log\frac{c^n}{\beta^n}. \tag{31}$$

On the other hand, from (28) and (29) we can get

$$\Phi(E)=\frac{c^n}{n!}E^n \tag{32}$$

$$\Omega(E)=\frac{d\Phi}{dE}\delta E=\frac{c^n}{(n-1)!}E^{n-1}\delta E. \tag{33}$$

Subsequently, we have

$$\begin{aligned}\log\Omega(\bar{E})&=\log\frac{n^{n-1}}{(n-1)!}\frac{c^n}{\beta^{n-1}}+\log\delta E\\ &\simeq\log\frac{c^n}{\beta^{n-1}}+(n-1)\{\log n-\log(n+1)+1\}+\log\delta E,\end{aligned} \tag{34}$$

where Stirling's approximation of the first order for $(n-1)!$ is used. Seeing (31) and (34), we could find that, for a sufficiently large n, S is equal to $\log\Omega(\bar{E})$, excepting a trivial constant $\log\delta E$.

IV. *Conclusions*

Introducing an abstractive system which satisfies several assumptions, we have constructed a framework for studying fundamental properties of a complex real system. It seems that the most essential assumptions are the existence of something, called " assets " here, which is invariant for an isolated system, and that of the equilibrium state. Both of them are not satisfied by the real economic system which is growing dynamically, producing and consuming materials. However after clarifying the properties of the equilibrium states, it would be possible to develop a theory of an inequilibrium or dynamic system on the basis of the equilibrium theory of the system, although, of course, it would not be so easy as that done in this paper.

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